STATISTICAL PROPERTIES AND PARAMETER ESTIMATION OF THE TWO-SIDED LENGTH BIASED INVERSE GAUSSIAN DISTRIBUTION

BY

MR. TEERAWAT SIMMACHAN

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN STATISTICS (INTERNATIONAL PROGRAM)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
ACADEMIC YEAR 2014
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DISSERTATION

BY

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ENTITLED

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was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy in Statistics (International Program)
on August 3, 2015

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ABSTRACT

The dissertation consists of two parts: the theoretical part and the computational part. The aims of the theoretical part are to introduce the new lifetime distribution based on non-classical parametrization under the situation when a crack develops from two sides, to investigate some statistical properties of the proposed distribution, and to apply a method of moment estimation for estimating the parameters of the distribution.

The new distribution called the two-sided length biased inverse Gaussian distribution with two parameters $\lambda$ and $\theta$ is proposed. It is shortly denoted as $TS-LBIG(\lambda, \theta)$ distribution. However, its density function involves an integral sign. Consequently, it may be difficult to directly find important functions such as a characteristic function and a moment generating function. Reciprocal properties and Maclaurin expansion are necessarily needed for finding the important properties such as the first four cumulants (semi-invariants), moments, and raw moments. The first four cumulants, moments, and raw moments for the $TS-LBIG(\lambda, \theta)$ distribution exist when the condition $|t| < \frac{1}{2\theta}$ is satisfied. The
method of moment point estimation is obtained to estimate the parameters of
the distribution by equating the first two moments and the corresponding sample
moments. General approach to asymptotic analysis of the estimate by the method
of moment is provided by applying the delta method also called the Taylor series
approximation. Then, the asymptotic variances and covariance of the suggested
estimators are given.

The goal of the computational part is to assess the performance of the
proposed estimators of the two-sided length biased inverse Gaussian distribution.
Initially, the simulation results indicate that the proposed generator performs
better than the generator created by package in R program. The criteria for
evaluating the performance of the method of moment estimators obtained from
simulation studies are variance, coefficient of variation, bias and mean square
error. The numerical study suggests that the method of moment estimators are
consistent and asymptotically unbiased since variance, coefficient of variation, bias
and mean square error are decreasing functions as sample sizes $n$ increase. In other
words, when sample sizes are sufficiently large, variance, coefficient of variation,
bias and mean square error are small. Correspondingly, the result of asymptotic
analysis of the estimates by the method of moment indicates that the asymptotic
variances decrease as $n$ increases.

**Keywords:** method of moment estimate, lifetime distribution, parametrization,
reciprocal property, asymptotic property.
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Mr. Teerawat Simmachan
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CHAPTER 1

INTRODUCTION

1.1 Statement of the Problem and Importance of the Study

Generally, the choice of distribution is made on the basis of how fine the data are fitted by the distribution. In reliability studies, however, that of distribution is frequently operated on the basis of understanding about the failure mechanism (Chhikara and Folks, 1977). It handles with analysis of time duration to until one or more events or failures happen. Analogous studies are extensively used in many fields of research such as biology, physics, statistics, economics, engineering, environmental science, epidemiology, etc. Furthermore, this topic is called survival analysis being a branch of statistics, event history analysis in sociology, or duration analysis in economics. For example, in survival analysis, practitioners attempt to answer questions such as what the proportion of a population will survive past a certain time, what rate will they die or fail, and how do particular characteristics increase or decrease the probability of survival. In order to answer such questions, a lifetime will be necessarily defined. Consequently, a lifetime or failure time distribution has an important role in the areas. In this research, nevertheless, we only focus on a lifetime distribution in reliability aspects.

In reliability theory, it is easy to consider a lifetime or failure time of physical objects such as valves, metallic plates, electric light bulbs, etc. The lifetime is the age of the component at which some clearly defined event occurs. The object of primary interest is the reliability function or survival function, traditionally
denoted by $R(x)$, which is defined as

$$R(x) = P(X > x) = 1 - P(X \leq x) = 1 - F(x),$$

where $x$ is certain time, $X$ is a positive continuous random variable, and $F(x)$ is a cumulative distribution function (c.d.f.) also called distribution function. Another name of $R(x)$ is a complementary of cumulative distribution function (c.c.d.f.). Thus, the lifetime distribution function, $F(x)$, is the complement of the reliability function. Additionally, lifetime distributions provide useful information on some practical problems. Since some machines or systems are very important and extremely expensive, this information motivates practitioners to prevent the industrial or financial damages occurring after the lifetime or failure time is terminated. Importantly, this may also protect life safety.

There are numerous families of distributions commonly used to describe a lifetime in the area. For instance, exponential, gamma, weibull, normal, lognormal, Birnbaum-Saunders, inverse Gaussian, length biased inverse Gaussian etc. One of the interesting views of lifetime distributions in models of reliability analysis is the case when a failure of the object under consideration appears to be due to fatigue crack development. The common distributions used in practical applications of reliability theory for modeling lifetime of products with failure due to a development of fatigue cracks are Birnbaum-Saunders, inverse Gaussian, and length biased version of inverse Gaussian distribution called length biased inverse Gaussian (Lisawadi, 2009; Gupta and Akman, 1995a). These distributions had been studied in various cases as follows.

Birnbaum and Saunders (1969a) introduced the two-parameter Birnbaum-Saunders distribution as a failure time distribution for fatigue failure caused by cyclic or periodic loading. They also considered a probability model of such phenomena in the area of renewal theory. Desmond (1985, 1986) proposed more general derivation based on a biological model and strengthened the physical justification for the use of this distribution. This proof follows from consideration of renewal theory for the number of cycles needed to force a fatigue crack extension to exceed a critical value. Birnbaum and Saunders (1969b) provided a
comprehensive review for both theoretical and practical aspects of the fitting of the distribution to the solution of the problem of crack development. Desmond (1986) established the parameter estimation for censored data. Ahmad (1988) presented the estimation of the scale parameter by the jackknife method to eliminate first-order bias. Some recent studies on Birnbaum-Saunders distribution were presented as follows. Ng et al. (2003) proposed the modified moment estimators to solve the problems of difficulty for finding the conventional estimators, i.e., they may not always exist and if they exist, they may not be unique. Nevertheless, Wu and Wong (2004) suggested that those expressions for the interval estimation for $\beta$ presented by Ng et al. (2003) are incorrect. Moreover, there is no guarantee that the upper bounds of those intervals are always positive. Ahmed et al. (2008) introduced the new parametrization of Birnbaum-Saunders distribution based on the recurrence relations given in Birnbaum and Saunders (1969a). Importantly, the physics of the phenomena under the study is fitted by this re-parametrization since the suggested parameters correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively. The original shape and scale parameters of the distribution do not give this physical interpretation. They also provided the relationship between the usual parameters and the proposed parameters. Balakrishnan et al. (2009) developed parameter estimation of the Birnbaum-Saunders distribution based on scale-mixture of normal distribution. Also, they applied the EM-algorithm for maximum likelihood estimation of the parameters. Lisawadi (2009) presented two new classes of distributions called the two-sided Birnbaum-Saunders and inverse Gaussian lifetime distributions for fatigue crack development. These distributions are considered in the case when a crack develops from two sides. Based upon the non-classical parametrization provided in Ahmed et al. (2008), the method of moments of parameter estimation for the new distributions is investigated. Also, asymptotic statistical properties of the suggested estimators are developed. Kundu et al. (2010) subsequently introduced the bivariate Birnbaum-Saunders distribution being an absolutely continuous and its marginals are univariate Birnbaum-Saunders distributions. The different prop-
erties of the new distribution were given. Furthermore, statistical inferences were provided including the maximum likelihood estimators obtained by solving two non-linear equations, the modified moment estimators, confident intervals constructed by the asymptotic distributions of the maximum likelihood estimators, and the likelihood ratio tests for some hypotheses of interest.

Another distribution which is frequently used as a life time distribution is the inverse Gaussian distribution shortly denoted as IG distribution. The name of the inverse Gaussian distribution was defined by Tweedie (1957) who found the inverse relation between the cumulant generating functions of the IG and normal distribution. Nevertheless, the IG model was initially derived by Schrödinger (1915) as the first passage time distribution in the Brownian motion with a positive drift taken to reach a fixed positive level. Furthermore, the IG distribution is a positively skewed distribution, which offers an interesting and useful alternative in this framework, to the weibull, lognormal, gamma, and so forth (Gupta and Akman, 1995b). This distribution is very relevant for reliability analysis and life-testing problems. Besides, the IG distribution has widely been used in many applications; for example, tracer dynamics, emptiness of dam, a purchase incidence model, the distribution of strike duration, a word frequency distribution, and other applications. The IG distribution has been considered by authors and researchers for the last 20 years. Several books in this framework are often referred to Chhikara and Folks (1989), Seshadri (1993, 1999), and Johnson et al. (1994). Recent works on the inverse Gaussian distribution were proposed as follows. Vladimirescu and Tunaru (2003) developed estimation functions by confident regions for the two-parameter IG distribution and the statistical tests were also constructed for hypotheses testing concerning the scale parameter when the mean parameter was known. Singh and Bundit (2008) then presented the estimation of the reciprocal mean of inverse Gaussian distribution based on prior information. The mean square error is used to evaluate the performance of the proposed estimator.

The length biased version of the inverted Gaussian distribution called length biased inversed Gaussian distribution which is conventionally denoted by
LBIG distribution has also received considerable attention from researchers due to its various applications. In addition, the length biased inversed Gaussian distribution is the reciprocal IG distribution and also called complementary reciprocal of IG distribution. To describe a length biased random variable, the definition presented in Ahsanullah and Kirmani (1984) and Khattree (1989) is given below. Let $X$ be a non-negative random variable having an absolutely continuous distribution with probability density function (pdf.) or $f(\cdot)$. If $X$ has finite first moment, $E(X)$, then one can define another p.d.f. saying $h(\cdot)$ as:

$$h(\cdot) = \frac{xf(x)}{E(X)}, \quad x > 0.$$ 

The random variable $Y$ with p.d.f. $h(\cdot)$ is known as length biased random variable associated with $X$. It also called length biased sampling of $f(x)$. It is a special case of weighted distributions (Patil and Rao, 1977). Cox (1962) gave the following interpretation of pdf. $h(\cdot)$. Consider a sample of failure times with pdf. $f(x)$ and let the probability of selecting any individual unit in the population be proportional to its size or length $x$. Then, the failure time selected has the pdf. $h(x)$.

The researches associated with the LBIG distribution are displayed as follows. Ahsanullah and Kirmani (1984) offered a characterization of the Wald distribution, a special case of IG distribution when the mean parameter ($\mu$) is equal to one, by using the relation between the pdf. of Wald distribution and pdf. of its reciprocal. Khattree (1989) then presented a characterization of IG and gamma distributions through their length biased distributions and the Wald distribution was finally characterized. Moreover, a review of applications of length biased distributions was given in Gupta and Kirmani (1990). This kind of distributions was wildly used in many areas such as biomedical studies, life length studies seen in Blumenthal (1967) and Scheaffer (1972), etiological studies seen in Simon (1980), study of human families and wildlife populations seen in Patil and Rao (1977, 1978), etc. Jörgensen et al. (1991) proposed a generalizations of the inverse Gaussian distribution. It is the mixture of the IG distribution and its complementary reciprocal, and it is called the inverse Gaussian mixture distribution. The new three-parameter generalized inverse Gaussian distribution was formed by combin-
ing two independent probability density functions: the pdf. of IG distribution with the weight $1 - p$ and pdf. of LBIG distribution with the weight $p$ where $0 \leq p \leq 1$, and it was denoted as M-IG($\mu, \sigma^2, p$). Subsequently, the length biased inverse Gaussian distribution had been studied by Akman and Gupta (1992). They provided comparative simulation studies of various estimators for the mean of the IG distribution. The variance of IG distribution was assumed to be proportional to the mean, the data were postulated to be available from the IG distribution and its length biased version, and the MVUE and MLE were compared in terms of their variances and mean square errors from both kinds of data. Gupta and Akman (1995a) studied the mixture of IG distribution and LBIG distribution, given in Jörgensen et al. (1991) and here called JSW distribution, in a reliability view. The failure rate and mean square residual life functions were derived and their behaviors were also investigated. The MLE of the parameters and that of reliability function were examined. It followed that the inverse Gaussian distribution, the length biased inverse Gaussian distribution, and the Birnbaum-Saunders distribution were special cases of the JSW distribution. Gupta and Akman (1995b) also studied the mixture of IG distribution and LBIG distribution, provided in Jörgensen et al. (1991) but here called mixture inverse Gaussian distribution (MIG), in the view of Bayes estimation. Then, Gupta and Akman (1998) developed confident intervals and tests regarding the mean and the coefficient of variation of the IG distribution based on length biased data. Recent works on the inverse Gaussian distribution were proposed as follows. Balakrishnan et al. (2009) discussed the usefulness of the mixture of inverse Gaussian distribution for modeling different types of data. Transformations, the derivation of the moments, fitting of models, and a shape analysis of transformations were investigated. Leiva et al. (2010) subsequently introduced the new family of mixture models based on the IG distribution and it’s reciprocal. The new class of models is created from symmetric distributions by using relationship between the inversed Gaussian and standard normal distributions. Recently, Gupta and Kundu (2011) provided parameter estimation for a three-parameter generalized inverse Gaussian distribution which
is a mixture of inverse Gaussian distribution and length biased inverse Gaussian distribution by using the Expectation-Maximization (EM) algorithm.

Importantly, as seen in the review of literature, all of them are considered in the term of usual parameters except the studies of Ahmed et al. (2008) and Lisawadi (2009). Thus, our contribution in this research is to suggest a new lifetime distribution based on non-classical parametrization presented in Ahmed et al. (2008). The new distribution is called the two-sided length biased inverted Gaussian lifetime distribution and it is denoted as TS-LBIG distribution. Let $\lambda > 0$ and $\theta > 0$ be their proposed parameters, which correspond to a thickness of a specified object and a nominal treatment loading on the object, respectively. The original scale and shape parameters ($\mu$ and $\beta$) lack these characteristics. Hence, if a random variable $X$ is two-sided length biased inverted Gaussian distributed with parameters $\lambda$ and $\theta$, it is shortly denoted as $X \sim TS-LBIG(\lambda, \theta)$.

Interestingly, based on the reviewed literature, this kind of study has not been considered before. Therefore, probability model of the two-sided length Biased inverse Gaussian distribution will be formed by applying the approach of Lisawadi (2009). The new lifetime distribution is considered in the case when a crack develops from two sides. For example, on a metallic sample, which has a rectangular form with fixed on two sides, a crack development is applied to both upper and lower sides of the plate with the same speed or distribution. Also, the traditional parameter point estimation, method of moment, will be developed in the TS-LBIG distribution together with the asymptotic analysis of the proposed estimators. Additionally, a Monte Carlo simulation study is also conducted to evaluate the performance of the proposed parameter estimators for given sample sizes.
1.2 Research Objectives

The objectives of this research are as follows:

1. To establish and investigate statistical properties of the two-sided length biased inverse Gaussian distribution.

2. To apply the conventional point estimation, method of moment, in order to estimate the parameters of the two-sided length biased inverse Gaussian distribution.

3. To evaluate the performance of the proposed estimators of the two-sided length biased inverse Gaussian distribution.

1.3 Research Scope

The scope of this research is partitioned into two parts as follows:

1.3.1 Theoretical Part

1. The probability model of the two-sided length biased inverse Gaussian distribution will be constructed based on the non-ordinary parametrization presented by Ahmed et al. (2008).

2. The important functions will be only determined on the cumulative distribution function (c.d.f.) and probability density function (p.d.f.).

3. The statistical properties of the new distribution will be investigated, e.g., reciprocal properties, the first four cumulants, moments, and raw moments.

4. Parameter estimation by method of moment for the two-sided length biased inverse Gaussian distribution will be developed.

5. Asymptotic variances and covariance of the suggested estimates by the method of moment for the two-sided length biased inverse Gaussian distribution will be examined.
1.3.2 Computational Part

1. The parameter estimates by the method of moment for the two-sided length biased inverse Gaussian distribution will be computed.

2. Bias, variance, coefficient of variation (C.V.) and mean square error (MSE) of the proposed estimators will be calculated.

3. The asymptotic variances of the proposed estimators will also be calculated and compared with the simulated variances.

4. Histograms of error (difference between the true and estimated values) will be constructed.

5. The practical applications of the suggested estimators are also given. The two real illustrative examples are provided by Lieblein and Zelen (1956) for Example 1 and Nichols and Padgett (2006) for Example 2.

For parameter estimation, we consider the sets of parameters and sample sizes as follows: $\lambda = 2, 5, 10, 20, 50$, $\theta = 1, 5, 10, 50$, and $n = 10, 50, 100, 200, 500$.

1.4 Research Advantages

This research contains new contributions as follows:

1. The new lifetime distribution based on non-classical parametrization is introduced for modeling a lifetime for objects or products with failure due to a development of fatigue cracks. It may be useful in other research areas such as physics, economics, and statistics.

2. There is valuable motivation of practitioners to protect the industrial or financial damages before the lifetime terminates and this, importantly, may protect life safety.

3. The two-sided length biased inverse Gaussian distribution is developed from a practical application and it is also the inspiration for researchers to improve statistical knowledge of related lifetime models.
1.5 Abbreviations

\( f_{IG} \): Inverse Gaussian probability density function with the classical parameters

\( fn_{IG} \): Inverse Gaussian probability density function with the proposed parameters

\( f_{RIG} \): Reciprocal inverse Gaussian probability density function with the classical parameters

\( fn_{LBIG} \): Length biased inverse Gaussian probability density function with the proposed parameters

p.d.f. : Probability density function

p.m.f. : Probability mass function

c.d.f. : Cumulative distribution function

c.c.d.f. : Complementary cumulative distribution function

\( F(x) \): Distribution function or c.d.f.

\( f(x) \): Density function or p.d.f.

\( R(x) \): Reliability function

\( S(x) \): Survival function

ADPM : Absolute difference-percentage of population and sample raw moments

IG : Inverse Gaussian distribution

RIG : Reciprocal inverse Gaussian distribution

LBIG : Length biased inverse Gaussian distribution

TS-LBIG : Two-sided length biased inverse Gaussian distribution

BS : Birnbaum-Saunders distribution

CR : Crack Lifetime distribution

MLE : Maximum likelihood estimator

MME : Method of moment(s) estimator

r.v. : Random variable

LHS : Left-hand side

RHS : Right-hand side

i.i.d. : Independent and identically distributed
1.6 Notations

\(\alpha\) : Shape parameter
\(\beta\) : Scale parameter
\(\mu\) : Location parameter or mean of a random variable
\(\lambda\) : Parameter corresponds to the thickness of the object under consideration
\(\theta\) : Parameter stands for the nominal treatment loading on the object
\(P\) : Probability
\(\tau\) : A moment of the object under consideration (block) break down
\(E(X)\) : Mathematical expectation or expected value of the first raw moment of a random variable \(X\)
\(\mu_X\) (or \(\mu\)) : The same as \(E(X)\)
\(V(X)\) : Variance or the second central moment of a random variable \(X\), i.e., \(E[(X - E(X))^2]\)
\(\sigma_X^2\) (or \(\sigma^2\)) : The same as \(V(X)\)
\(\sigma_X\) (or \(\sigma\)) : Standard deviation of a random variable \(X\) also denoted by \(STD(X)\) or \(\sqrt{\sigma_X^2}\)
\(C.V.(X)\) : Coefficient of variation of a random variable \(X\), i.e., \(\sigma_X/\mu_X\)
\(N(\mu, \sigma^2)\) : Normal distribution with mean \(\mu\) and variance \(\sigma^2\)
\(N(0, 1)\) : Standard normal distribution with mean 0 and variance 1
\(\Phi(\cdot)\) : The c.d.f. of a standard normal random variable
\(IG(\mu, \beta)\) : Inverse Gaussian distribution with parameters \(\mu\) and \(\beta\)
\(IG(\lambda, \theta)\) : Inverse Gaussian distribution with proposed parameters \(\lambda\) and \(\theta\)
\(RIG(\mu, \beta)\) : Reciprocal inverse Gaussian distribution with parameters \(\mu\) and \(\beta\)
$LBIG(\mu, \beta)$ : Length biased inverse Gaussian distribution with parameters $\mu$ and $\beta$

$LBIG(\lambda, \theta)$ : Length biased inverse Gaussian distribution with proposed parameters $\lambda$ and $\theta$

$TS-LBIG(\lambda, \theta)$ : Two-sided length biased inverse Gaussian distribution with parameters $\lambda$ and $\theta$

$CR(\lambda, \theta, p)$ : Crack Lifetime distribution with parameters $\lambda$, $\theta$, and $p$

$\sim$ : Distributed as

$n$ : Sample size

$\varphi_X(t)$ : Characteristic function of a random variable $X$

$\phi_X(t)$ : Moment generating function of a random variable $X$

$\psi_X(t)$ : Cumulant generating function of a random variable $X$

$k_n$ : The $n^{th}$ cumulant

$\mu_k'$ : The $k^{th}$ population raw moment, i.e., $E(X^k)$, expected value of the $k^{th}$ power of a random variable $X$

$\mu_k$ : The $k^{th}$ population central moment, $E\left[(X - E(X))^k\right]$, or expected value of the $k^{th}$ power of $(X - E(X))$

$m_k'$ : The $k^{th}$ sample raw moment

$m_k$ : The $k^{th}$ sample central moment

$i$ : Imaginary unit or unit imaginary element, $(i = \sqrt{-1})$

$Re(z)$ : Real part of complex number $z$

$Im(z)$ : Imaginary part of complex number $z$

$\mathbb{R}$ : The set of all real numbers

$\hat{\theta}$ : Maximum likelihood estimator for parameter $\theta$

$\tilde{\theta}$ : Method of moment estimator for parameter $\theta$

$B(\hat{\theta})$ : Bias of an estimator $\hat{\theta}$ for parameter $\theta$, i.e., $E(\hat{\theta}) - \theta$

$MSE(\hat{\theta})$ : Mean square error of an estimator $\hat{\theta}$, i.e.,

$$E\left[(\hat{\theta} - \theta)^2\right] = Var(\hat{\theta}) + \left(B(\hat{\theta})\right)^2$$

$\square$ : The notation used to denote the end of proof
CHAPTER 2

BACKGROUND AND REVIEW OF LITERATURE

2.1 Background

In this part, the theoretical background are presented. The important statistical topics related to this research are provided as follows.

2.1.1 Inverse Gaussian Distribution

The inverse Gaussian distribution shortly denoted as IG distribution received the attention from authors and researchers for 20 years. Many contemporary statistical methods are involved and derived with the expansive use of IG distribution. The name of the inverse Gaussian distribution was defined by Tweedie (1957) who found the inverse relation between the cumulant generating functions of the inverse Gaussian and Gaussian (normal) distributions. However, the IG model was initially derived by Schrödinger (1915) as the first passage time distribution in the Brownian motion with a positive drift taken to reach a fixed positive level. Furthermore, the IG distribution is a positively skewed distribution, which offers an interesting and useful alternative in this framework, to the weibull, lognormal, gamma, and so fourth (Gupta and Akman, 1995b). It is commonly analyze observed data using statistical methods based upon skewed distribution; consequently, the IG distribution is the most appropriate choice for the need of skewed data analysis. This distribution is very relevant in reliability analysis and life-testing problems. Besides the framework of reliability theory, the IG distribution has widely been used in many applications; for example, tracer dynamics,
emptiness of dam, a purchase incidence model, the distribution of strike duration, and a word frequency distribution.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size $n$ drawn from an inverse Gaussian distribution denoted by $IG(\mu, \beta)$. The probability density function (p.d.f.) of the IG distribution of a random variable $X$ is given in standard form as:

$$f_{IG}(x; \mu, \beta) = \begin{cases} \sqrt{\frac{\beta}{2\pi x^3}} & \exp \left\{ -\frac{\beta(x-\mu)^2}{2\mu^2x} \right\} & x > 0, \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (2.1)

where $\mu > 0$ and $\beta > 0$. The parameter $\mu$ stands for the mean and $\beta$ represents the scale parameter. Both $\mu$ and $\beta$ have the same physical dimension as the random variable $X$ itself, but the shape parameter $\alpha = \beta/\mu$ is invariant under a scale transformation of $X$. Their relationships are (Chhikara & Folks, 1989) as follows:

$$f_{IG}(x; \mu, \beta) = \mu^{-1}f_{IG}\left(\frac{x}{\mu}; 1, \alpha\right) = \beta^{-1}f_{IG}\left(\frac{x}{\beta}; \alpha, 1\right).$$  \hspace{1cm} (2.2)

![Figure 2.1: Density functions of $IG(\mu, \beta)$ for fixed $\mu = 10$ and increasing $\beta$](image-url)
Figure 2.2: Density functions of \( IG(\mu, \beta) \) for fixed \( \beta = 1 \) and increasing \( \mu \)

Probability density functions of IG distribution are shown in Figures 2.1 and 2.2 for various values of \( \beta(\mu = 10) \) and \( \mu(\lambda = 1) \), respectively. These present classes of asymmetric distributions which are useful for skewed data analysis. The cumulative distribution function (c.d.f.) of an inverse Gaussian random variable \( X \) can be written as

\[
F_{IG}(x; \mu, \beta) = \Phi\left(\sqrt{\frac{\beta}{x}} \left(\frac{x}{\mu} - 1\right)\right) + \exp\left\{\frac{2\beta}{\mu} \Phi\left(-\sqrt{\frac{\beta}{x}} \left(\frac{x}{\mu} + 1\right)\right)\right\},
\]

where \( \Phi(\cdot) \) denotes the c.d.f. of a standard normal random variable. The characteristic function of \( IG(\mu, \beta) \) is

\[
\varphi_{IG}(t; \mu, \beta) = \exp\left\{\frac{\beta}{\mu} \left[1 - \left(1 - \frac{2\mu^2 t}{\beta}\right)^{1/2}\right]\right\},
\]

the corresponding moment generating function is given by

\[
\phi_{IG}(t; \mu, \beta) = \exp\left\{\frac{\beta}{\mu} \left[1 - \left(1 - \frac{2\mu^2 t}{\beta}\right)^{1/2}\right]\right\},
\]

and its cumulant generating function is defined by

\[
\psi_{IG}(t; \mu, \beta) = \frac{\beta}{\mu} \left[1 - \left(1 - \frac{2\mu^2 t}{\beta}\right)^{1/2}\right].
\]
Hence, for any integer \( r \geq 1 \), the \( r^{th} \) cumulant of \( IG(\mu, \beta) \) is given as

\[
k_r = [1 \times 3 \times 5 \times \cdots \times (2r - 3)] \frac{\mu_{2r-1}}{\beta^{r-1}}.
\] (2.7)

Therefore, the first four cumulants of the \( IG(\mu, \beta) \) are

\[
k_1 = \mu, \quad k_2 = \frac{\mu^3}{\beta}, \quad k_3 = \frac{3\mu^5}{\beta^2}, \quad k_4 = \frac{15\mu^7}{\beta^3}.
\] (2.8)

The coefficients of skewness and kurtosis of the distribution are respectively

\[
\sqrt{\beta_1} = 3 \sqrt{\frac{\mu}{\beta}}, \quad \beta_2 = 3 + \frac{15\mu}{\beta}.
\] (2.9)

The Laplace transform of an \( IG(\mu, \beta) \) random variable is

\[
\exp \left\{ \frac{\beta}{\mu} \left[ 1 - \left( 1 + \frac{2\mu^2 t}{\beta} \right)^{1/2} \right] \right\}.
\] (2.10)

Moreover, Shuster (1968) found that if \( X \) is an inverse Gaussian random variable then \( \beta(X - \mu)^2/\mu^2X \) is \( \chi^2_1 \) distributed. Shuster also proposed the method to calculate probability by using standard normal tables and Log tables.

Subsequently, Ahmed et al. (2008) introduced the new parametrization on Birnbaum-Saunders (BS) distribution, and this parametrization gives meaningful interpretation in the area of fatigue crack development. Accordingly, the parametrization was also extended to the IG model as presented in the monograph of Lisawadi (2009). The proposed parameters, \( \lambda > 0 \) and \( \theta > 0 \), stand for the thickness of the object under consideration and the nominal treatment on the object, respectively. The interrelations between the classical parameters \( (\mu, \beta) \) and the proposed parameters \( (\lambda, \theta) \) are as follows:

\[
\lambda = \frac{\beta}{\mu}, \quad \theta = \frac{\mu^2}{\beta}, \quad \mu = \lambda \theta \text{ and } \beta = \lambda^2 \theta.
\] (2.11)

Hence, the non-classical parameterization of the inverse Gaussian probability density function is given by

\[
f_{nIG}(x; \lambda, \theta) = \begin{cases} 
\lambda \sqrt{\frac{\theta}{2\pi}} x^{-3/2} \exp \left\{ -\frac{(x-\lambda \theta)^2}{2\theta x} \right\} & x > 0, \\
0 & \text{otherwise},
\end{cases}
\] (2.12)
and another form is

\[ f_{nIG}(x; \lambda, \theta) = \begin{cases} \frac{\lambda}{\theta \sqrt{2\pi}} \left( \frac{\theta}{x} \right)^{3/2} \exp \left\{ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right\} & x > 0, \\ 0 & \text{otherwise}, \end{cases} \]  

(2.13)

where \( \lambda > 0 \) and \( \theta > 0 \). The corresponding distribution function is given as

\[ F_{nIG}(x; \lambda, \theta) = \Phi \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right) + \exp \{ 2\theta \} \Phi \left( -\sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right), \]  

(2.14)

where \( \Phi(\cdot) \) is the c.d.f. of a standard normal random variable, and its mean and variance are respectively:

\[ E(X) = \mu = \lambda \theta \text{ and } Var(X) = \lambda \theta^2. \]  

(2.15)

Some density functions of IG distribution under non-classical parametrization are presented in Figures 2.3 and 2.4 for different values of \( \theta (\lambda = 2) \) and \( \lambda (\theta = 3) \), respectively. This re-parametrization still gives families of asymmetric distributions which are useful for skewed data analysis.

Figure 2.3: Density functions of \( IG(\lambda, \theta) \) for fixed \( \lambda = 2 \) and increasing \( \theta \)
The characteristic function of $IG(\lambda, \theta)$ is
\[
\varphi_{IG}(t; \lambda, \theta) = \exp \left\{ \lambda \left[ 1 - (1 - 2\theta it)^{1/2} \right] \right\},
\]
the corresponding moment generating function is given by
\[
\phi_{IG}(t; \lambda, \theta) = \exp \left\{ \lambda \left[ 1 - (1 - 2\theta t)^{1/2} \right] \right\},
\]
and the cumulant generating function is defined by
\[
\psi_{IG}(t; \lambda, \theta) = \lambda \left[ 1 - (1 - 2\theta t)^{1/2} \right].
\]

(Lisawadi, 2009)

### 2.1.2 Reciprocal Inverse Gaussian Distribution

Generally, a reciprocal inverse Gaussian distribution has been considered together with the IG distribution as seen in Chhikara and Folks (1989) and Se- shadri (1993, 1999). If $X \sim IG(\mu, \beta)$, then $Y = 1/X$ is said to follow a reciprocal inverse Gaussian distribution with the original parameters $\mu$ and $\beta$, and is abbreviated as $Y \sim RIG(\mu, \beta)$. The probability density function (p.d.f.) of $Y$ is given
by

\[ f_{RIG}(y; \mu, \beta) = \begin{cases} \sqrt{\frac{\beta}{2 \pi y}} \exp \left\{ -\frac{\beta (1 - \mu y)^2}{2 \mu^2 y} \right\} & y > 0, \\ 0 & \text{otherwise}, \end{cases} \]  

(2.19)

where \( \mu > 0 \) and \( \beta > 0 \). The mean of random variable \( Y \) is

\[ E(Y) = E(1/X) = \frac{1}{\mu} + \frac{1}{\beta}, \]  

(2.20)

and the variance of \( Y \) is

\[ Var(Y) = Var(1/X) = \frac{1}{\mu \beta} + \frac{2}{\beta^2}, \]  

(2.21)

Probability density functions of the reciprocal inverse Gaussian distribution are illustrated in Figures 2.5 and 2.6 for different values of \( \beta (\mu = 10) \) and \( \mu (\beta = 10) \), respectively. It is indicated that the RIG distribution is a class of asymmetric distributions.

Figure 2.5: Density functions of \( RIG(\mu, \beta) \) for fixed \( \mu = 10 \) and increasing \( \beta \)

The Laplace transform has the following form:

\[ \left(1 + \frac{2t}{\beta}\right)^{-1/2} \exp \left\{ \frac{\beta}{\mu} \left[ 1 - \left( 1 + \frac{2t}{\beta} \right)^{1/2} \right] \right\}, \]  

(2.22)
Figure 2.6: Density functions of $RIG(\mu, \beta)$ for fixed $\beta = 10$ and increasing $\mu$

which is a product of two Laplace transforms; i.e., that of a gamma random variable, $\Gamma \left( \frac{1}{2} \beta \right)$, and that of an inverse Gaussian random variable, $IG \left( \frac{1}{\mu}, \frac{\beta}{\mu^2} \right)$. Additionally, the IG random variable has both positive and negative moments. It can be presented that the positive and negative moments share the following relation (Natarajan, 2007):

$$E \left[ \left( \frac{X}{\mu} \right)^{-r} \right] = E \left[ \left( \frac{X}{\mu} \right)^{r+1} \right], \quad r = 1, 2, \cdots$$  \hspace{1cm} (2.23)

2.1.3 Lenght Biased Inverse Gaussian Distribution

Length biased distributions have been applied in various areas such as biomedical studies, life length studies, etiological studies, study of human families and wildlife populations, labeling of cells by radioactive tracers, etc (Akman and Gupta, 1992). The length biased version of the inversed Gaussian distribution called length biased inversed Gaussian distribution which is conventionally denoted by LBIG distribution has also received considerable attention from researchers due to its various applications. To describe a length biased random variable, the

**Definition 1.** The probability density function of length biased random variable. Let $X$ be a non-negative random variable having an absolutely continuous distribution with probability density function (p.d.f.) or $f(x)$. If $X$ have a finite first moment, $E(X)$, then the length biased p.d.f. of $X$ is given by

$$h(y) = \frac{yf(y)}{E(X)} , y > 0.$$  \(2.24\)

The p.d.f. of the random variable $Y$ is the length biased version of the original distribution. The random variable $Y$ with p.d.f., $h(\cdot)$, is known as length biased random variable associated with $X$.

Additionally, it is also called length biased sampling of $f(x)$. It is a special case of weighted distributions (Patil and Rao, 1977). Cox (1962) gave the following interpretation of p.d.f., $h(\cdot)$. Consider a sample of failure times with p.d.f., $f(x)$, and let the probability of selecting any individual unit in the population be proportional to its size or length $x$. Then, the failure time selected has the p.d.f. in the form of (2.24). The random variable $Y$ is applied in the studies of lifetime models (Ahsanullah and Kirmani, 1984 and Chhikara and Folks, 1977).

The distribution under consideration here is the length biased inverse Gaussian distribution. Let $X$ be an inverse Gaussian random variable with parameters $\mu$ and $\beta$ or $X \sim IG(\mu, \beta)$. Based on the equation (2.24), the p.d.f. of the length biased inverse Gaussian distribution is defined by

$$f_{LBIG}(x; \mu, \beta) = \begin{cases} \sqrt{\frac{\beta}{2\pi x^3 \mu^2}} \exp \left\{ \frac{-\beta(x-\mu)^2}{2\mu^2x} \right\} & x > 0, \\ 0 & \text{otherwise,} \end{cases}$$ \(2.25\)

where $\mu > 0$ and $\beta > 0$. It can be seen that the length biased inversed Gaussian distribution is the same as that of the reciprocal of IG variate ($\mu^2/X$) and is also called complementary reciprocal of IG distribution. Some statistical properties of the $LBIG(\mu, \beta)$ distribution were provided in Wasan (1969); Chhikara and Folks (1989); Seshadri (1993) and Johnson et al. (1994).
In the new parametrization proposed by Ahmed et al. (2008), the length biased inverse Gaussian distribution is denoted as $LBIG(\lambda, \theta)$. By equations (2.13) and (2.24) and the finite first moment being $E(X) = \mu = \lambda \theta$, the probability density function of the LBIG distribution with the proposed parameters is defined by

$$fn_{LBIG}(x; \lambda, \theta) = \frac{x}{\lambda \theta} \cdot \frac{\lambda}{\theta \sqrt{2\pi}} \left( \frac{\theta}{x} \right)^{3/2} \exp \left\{ -\frac{1}{2} \left( \sqrt{\frac{x}{\theta}} - \lambda \sqrt{\frac{\theta}{x}} \right)^2 \right\}, \ x > 0.$$  

Finally, the probability density function of the length biased inverse Gaussian distribution is given by

$$fn_{LBIG}(x; \lambda, \theta) = \begin{cases} \frac{1}{\theta \sqrt{2\pi}} \left( \frac{\theta}{x} \right)^{1/2} \exp \left\{ -\frac{1}{2} \left( \lambda \sqrt{\frac{\theta}{x}} - \sqrt{\frac{x}{\theta}} \right)^2 \right\} & x > 0, \\ 0 & \text{otherwise}, \end{cases}$$  

where $\lambda > 0$ and $\theta > 0$. Figures 2.7 and 2.8 show variety of probability density functions of $LBIG(\lambda, \theta)$.

![Density functions of $LBIG(\lambda, \theta)$ for fixed $\lambda = 5$ and increasing $\theta$](image)

Figure 2.7: Density functions of $LBIG(\lambda, \theta)$ for fixed $\lambda = 5$ and increasing $\theta$

Its mean and variance are respectively:

$$E(X) = \mu = \theta (1 + \lambda) \quad \text{and} \quad Var(X) = \theta^2 (2 + \lambda).$$  

(2.28)
Figure 2.8: Density functions of $LBIG(\lambda, \theta)$ for fixed $\theta = 10$ and increasing $\lambda$

The characteristic function of $LBIG(\lambda, \theta)$ provided by Lisawadi (2009) is

$$\varphi_{LBIG}(t; \lambda, \theta) = (1 - 2\theta it)^{-1/2} \exp \left\{ \lambda \left[ 1 - (1 - 2\theta it)^{1/2} \right] \right\}, \quad (2.29)$$

the corresponding moment generating function is given by

$$\phi_{LBIG}(t; \lambda, \theta) = (1 - 2\theta t)^{-1/2} \exp \left\{ \lambda \left[ 1 - (1 - 2\theta t)^{1/2} \right] \right\}, \quad (2.30)$$

and the cumulant generating function is defined by

$$\psi_{LBIG}(t; \lambda, \theta) = -\frac{1}{2} \ln (1 - 2\theta it) \lambda \left[ 1 - (1 - 2\theta t)^{1/2} \right]. \quad (2.31)$$

### 2.1.4 The Useful Integral

For two complex numbers $p$ and $q$ such that $Re(p) > 0$ and $Re(q) > 0$:

$$(I_*) = \int_0^\infty x^{-3/2} \exp \left\{ -px - \frac{q}{x} \right\} dx = \sqrt{\pi} \frac{e^{-2\sqrt{pq}}}{q}. \quad (2.32)$$

**Proof.** According to formula from Gradshteyn-Ryzhik (2007), p. 369,

$$\int_0^\infty x^{-n-1/2} \exp \left\{ -px - \frac{q}{x} \right\} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{pq}} \quad (2.32)$$
Take \( n = 1 \) to obtain \((I_4)\) as follows:

\[
\int_0^\infty x^{-1-1/2} \exp \left\{-px - \frac{q}{x} \right\} \, dx = -\sqrt{\frac{\pi}{p}} \frac{\partial q}{\partial q} e^{-2p^{1/2}q^{1/2}}
\]

\[
= -\sqrt{\frac{\pi}{p}} (-2p^{1/2}) \left(\frac{1}{2} q^{-1/2}\right) e^{-2p^{1/2}q^{1/2}}
\]

\[
= \sqrt{\frac{\pi}{q}} e^{-2\sqrt{pq}}.
\]

Thus,

\[
\int_0^\infty x^{-3/2} \exp \left\{-px - \frac{q}{x} \right\} \, dx = \sqrt{\frac{\pi}{q}} e^{-2\sqrt{pq}} \quad \square
\]

### 2.1.5 Moments

The first four moments represent some important properties of the distributions including mean, variance, skewness, and kurtosis. Additionally, Carleman’s condition states that a set of moments uniquely determines a distribution if \( \sum_{j=0}^\infty (\mu_2)^{-\frac{j}{2}} \) diverges.

#### Population Moments

Let \( X \) be a real-value random variable. The \( k \text{th} \) moment about the origin (order \( k \) raw moment) of the random variable \( X \) is given by

\[
\mu'_k = E \left( X^k \right), \quad k = 0, 1, 2, \ldots
\]

The \( k \text{th} \) moment about the mean (the \( k \text{th} \) central moment) of the random variable \( X \) is given as

\[
\mu_k = E \left[ (X - E(X))^k \right], \quad k = 0, 1, 2, \ldots
\]

- **Mean** \( \mu_X = \mu = \mu'_1 = E(X) \) (the first raw moment)

The mean of distribution of a random variable \( X \) or mathematical expectation of \( X \) and also called the mean of \( X \) is a weighted average of its possible values. It measures location of all possible values of \( X \). It corresponds to the center of gravity of the distribution.
• **Variance** \( \text{Var}(X) = \sigma^2_X = \mu_2 \) (the second central moment)

The variance measures variability of all possible values of \( X \) about its mean. It is the moment of inertia of the distribution. The \( \text{Var}(X) \) is strictly positive, unless \( X \) is a constant and it is affected by the units of measurement.

Additionally, a positive square root of a variance is called a standard deviation, \( \text{STD}(X) = \sigma_X = \sigma = \sqrt{\text{Var}(X)} \). The unit of measure of random variable \( X \) and that of its standard deviation are the same.

Furthermore, the ratio of the standard deviation and the mean is called the coefficient of variation, \( C.V.(X) = \sigma_X / \mu_X \). It is measurement of variability independent of the scale. The coefficient of variation is not affected by the units of measurement.

• **Coefficient of Skewness** \( \beta_1 = \mu_3 / \sigma^3 = k_3 / k_1^{3/2} \)

The coefficient of skewness measures the degree of asymmetry of distribution of \( X \). Its values represent the following meaning.

- If \( \beta_1 > 0 \), the distribution is skewed to the right (a long right tail).
- If \( \beta_1 = 0 \), the distribution is symmetric (the Gaussian distribution).
- If \( \beta_1 < 0 \), the distribution is skewed to the left (a long left tail).

• **Coefficient of Kurtosis** \( \beta_2 = \mu_4 / \sigma^4 \) or \( \beta_2 - 3 = k_4 / k_2^2 \)

The coefficient of kurtosis also called the coefficient of excess measures the degree of flatness of the distribution of \( X \). It is a scale and location invariant measure of degree of peakedness of the probability density curve. The values of the coefficient of kurtosis indicate the following meaning.

- If \( \beta_2 > 3 \), the probability density curve is leptokurtic.
- If \( \beta_2 = 3 \), the probability density curve is mesokurtic (the Gaussian distribution).
- If \( \beta_2 < 3 \), the probability density curve is platykurtic.
Sample Moments

Let \( X_1, X_2, \ldots, X_n \) be a sample from a population. The sample \( k^{th} \) moment about the origin (sample \( k^{th} \) raw moment) is given by

\[
m'_k = \frac{1}{n} \sum_{j=1}^{n} X_j^k, \quad k = 0, 1, 2, \ldots
\]

The sample \( k^{th} \) moment about the mean (sample \( k^{th} \) central moment) is given as

\[
m_k = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X})^k, \quad k = 0, 1, 2, \ldots;
\]

where \( \bar{X} = m'_1 \).

- Skewness of the sample

The standardized 3\textsuperscript{rd} sample central moment is

\[
\sqrt{b_1} = \frac{m_3}{m_2^{3/2}} = \sqrt{\frac{n}{\sum_{j=1}^{n} (X_j - \bar{X})^3}} \left[ \frac{\sum_{j=1}^{n} (X_j - \bar{X})^2}{\sum_{j=1}^{n} (X_j - \bar{X})^2} \right]^{3/2}.
\]

It has been studied rather than the 3\textsuperscript{rd} sample central moment, \( m_3 = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X})^3 \). It is used to measure skewness of the sample.

- Kurtosis of the sample

The standardized 4\textsuperscript{th} sample central moment is

\[
b_2 = \frac{m_4}{m_2^2} = \frac{n}{\sum_{j=1}^{n} (X_j - \bar{X})^4} \left[ \sum_{j=1}^{n} (X_j - \bar{X})^2 \right]^2.
\]

where \( m_4 = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X})^4 \). Moreover, \( \sqrt{b_1} \) and \( b_2 \) have figured in tests for departure from normality (Bartosznski & Niewiadomska-Bugaj, 2008; Bowman & Shenton, 1986; Krishnamoorthy, 2006 and Lefebvre, 2006).

2.1.6 Method of Moment Estimation

The method of moment is, sometimes, the oldest method of finding point estimators. It is quite simple to use and almost always yields some sort of estimate
Let $X_1, \ldots, X_n$ be a sample from a population with p.d.f. or p.m.f. $f(x|\theta_1, \ldots, \theta_k)$. Method of moment estimators are found by equating the first $k$ moments to the corresponding $k$ population moments, and solving the resulting system of simultaneous equations. More precisely, the system of equations is defined as follows:

$$
m'_1 = \frac{1}{n} \sum_{j=1}^{n} X_j^1, \quad \mu_1 = E(X); \\
m'_2 = \frac{1}{n} \sum_{j=1}^{n} X_j^2, \quad \mu_2 = E(X^2); \\
\vdots \\
m'_k = \frac{1}{n} \sum_{j=1}^{n} X_j^k, \quad \mu_k = E(X^k).$$

The population moment $\mu_j$ is a function of $\theta_1, \ldots, \theta_k$, say $\mu_j(\theta_1, \ldots, \theta_k)$. Therefore, the method of moments estimators $(\hat{\theta}_1, \ldots, \hat{\theta}_k)$ of $(\theta_1, \ldots, \theta_k)$ is obtained by solving the following system of equations for $(\theta_1, \ldots, \theta_k)$ in terms of $(m'_1, \ldots, m'_k)$:

$$
m'_1 = \mu_1(\theta_1, \ldots, \theta_k), \\
m'_2 = \mu_2(\theta_1, \ldots, \theta_k), \\
\vdots \\
m'_k = \mu_k(\theta_1, \ldots, \theta_k).$$

(Casella and Berger, 1990)

### 2.1.7 Cumulants (Semi-invariants)

**Definition 2.** The cumulant generating function of a random variable $X$ is

$$
\psi_X(t) = \ln \varphi_X(t);
$$

where $\varphi_X(t)$ is the characteristic function of $X$. 

Theorem 1. Let \( X \) be a random variable with the cumulant generating function \( \psi \). If \( E[|X|^n] < \infty \) for some \( n = 1, 2, \ldots \), then

\[
\psi_X(t) = \sum_{j=1}^{m} \frac{k_j}{j!} (t)^j + o(|t|^m) \quad \text{as} \quad n \to 0;
\]

and the coefficient \( k_j \) are called the cumulants or semi-invariants of the distribution. The meaning of important cumulants are shown as follows:

1\(^{st}\) cumulant, \( k_1 = \mu_X \)
2\(^{nd}\) cumulant, \( k_2 = \sigma_X^2 \)
3\(^{rd}\) cumulant, \( k_3 = \mu_3 \)
4\(^{th}\) cumulant, \( k_4 = \mu_4 - 3(\sigma_X^2)^2 \).

Additionally, the raw moment are related to the cumulants by the following formula (Cramér, 1999 and Gut, 2005):

1\(^{st}\) raw moment, \( \mu'_1 = k_1 \)
2\(^{nd}\) raw moment, \( \mu'_2 = k_1^2 + k_2 \)
3\(^{rd}\) raw moment, \( \mu'_3 = k_1^3 + 3k_1k_2 + k_3 \)
4\(^{th}\) raw moment, \( \mu'_4 = k_1^4 + 6k_1^2k_2 + 4k_1k_3 + k_4 \).

2.1.8 Taylor Series Expansion

Definition 3. Let a function \( f(x) \) has derivatives of all orders on an open interval containing the point \( t \) and \( x \) then the Taylor series of function \( f(x) \) is

\[
f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(t)}{j!} (x - t)^j
\]

\[
= f(t) + \sum_{j=1}^{m} \frac{f^{(j)}(t)}{j!} (x - t)^j + R_m(x);
\]

where \( f^{(j)}(t) \) is the \( j^{th} \) derivative evaluated at \( t \) and \( R_m(x) \) is a remainder term. The Taylor series converges to the function \( f(x) \) if the remainder \( R_m(x) \to 0 \) as \( m \to \infty \). The remainder is defined by:

\[
R_m(x) = \frac{f^{(m+1)}(\xi)(x - t)^{m+1}}{(m + 1)!};
\]
for $\xi$ between $x$ and $t$.

If $t = 0$, the Taylor series expansion is called Maclaurin series expansion. A function $f(x)$ in some neighbourhoods of $x = 0$ has $m$ continuous derivatives. Then, the Maclaurin expansion is

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} (x)^j$$

$$= f(0) + \sum_{j=1}^{m} \frac{f^{(j)}(0)}{j!} (x)^j + R_m(x) ;$$

where

$$R_m(x) = \frac{f^{(m)}(\varpi x) - f^{(m)}(0)}{m!} x^m ;$$

$0 < \varpi < 1$, and $f^{(m)}(\varpi x) - f^{(m)}(0)$ tends to zero with $x$ by hypothesis. Furthermore, as $x$ tends to zero and $f(x)$ is also complex, then

$$f(x) = f(0) + \sum_{j=1}^{m} \frac{f^{(j)}(0)}{j!} (x)^j + O(x^m) .$$

(Cramer, 1999; Gradshteyn & Ryzhik, 2007 and Ver Hoef, 2012 )

2.1.9 The Delta Method

Delta method is generally used by statisticians and other scientists in three distinct meanings: as an approximation for the variance of a function of a random variable, as a bias correction for the expectation of a function of a random variable, and as the limiting distribution of a function of a random variable (Ver Hoef, 2012).

An Approximation Variance

The Delta method is extensively used to find approximations based on the Taylor series expansion to the variance of functions of random variables. Also, the Delta method is known as Taylor series approximations (Casella and Berger, 1990). Especially, an approximation for the variances of the estimators will be
considered. The Taylor series expansion of a function \( f(\cdot) \) about a value \( a \) is given as
\[
Y = f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + \cdots,
\]
where \( x \) is a value of a random variable \( X \), and \( f'(a) \) and \( f''(a) \) are the first and second derivatives of \( f \) evaluated at \( a \), respectively. The Taylor expansion can be expressed up to any order with a remainder term.

\[
Y = f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x - a)^j + R,
\]
(2.33)
where \( f^{(j)}(a) \) is the \( j \)th derivative evaluated at \( a \) and \( R \) is a remainder term. Particularly, the first order of the Taylor expansion is considered in the following way:

\[
f(x) - f(a) \approx f'(a)(x - a) .
\]
Letting \( a = \mu \), the mean of \( X \), then the Taylor series expansion of \( f(x) \) about \( \mu \) gives the approximation:

\[
f(x) - f(\mu) \approx f'(\mu)(x - \mu) ,
\]
and after squaring and taking expectation,

\[
Var(Y) = Var(f(x)) \approx [f'(\mu)]^2\sigma^2 ,
\]
(2.34)
where \( \sigma^2 \) is the variance of \( X \).

Similarly, a two-variable Taylor series expansion can be determined in the following way. Suppose we have two random variables \( X \) and \( Y \). The Taylor expansion of \( f(x, y) \) about the values \( (x_0, y_0) \) is given by:

\[
f(x, y) = f(x_0, y_0) + \frac{\partial f(x, y)}{\partial x}_{(x_0, y_0)} (x - x_0) + \frac{\partial f(x, y)}{\partial y}_{(x_0, y_0)} (y - y_0) + R \quad (2.35)
\]
where \( R \) is the remainder term.

By dropping the remainder term, we obtain the following approximation:

\[
f(x, y) \approx f(x_0, y_0) + \frac{\partial f(x, y)}{\partial x}_{(x_0, y_0)} (x - x_0) + \frac{\partial f(x, y)}{\partial y}_{(x_0, y_0)} (y - y_0) \quad (2.36)
\]
In addition, this first-order approximation to the variance of functions of random variables is usually referred to as the Delta method.
Bias Correction

The Delta method for bias correction also starts with the Taylor series expansion (2.33). The second-order term is particularly considered, the remainder term is dropped, and expectations are taken. Finally, we obtain

\[ E(Y) = E(f(X)) \approx f(\mu) + \frac{1}{2} f''(\mu) \sigma^2. \] (2.37)

A Limiting Distribution

The Delta method for a limiting distribution is defined as an approximate probability distribution for a function of an asymptotically normal statistical estimator. If there is a sequence of random variable \( X_n \) depending on \( n \) and satisfying

\[ \sqrt{n}(X_n - \mu) \xrightarrow{D} N(0, \sigma^2), \]

where \( \mu, \sigma^2 \) are finite valued constants, and \( \xrightarrow{D} \) implies convergence in distribution, then

\[ \sqrt{n}(f(X_n) - f(\mu)) \xrightarrow{D} N(0, [f'(\mu)]^2 \sigma^2), \] (2.38)

for any function \( f \) satisfying the property that \( f'(\mu) \) exists and is non-zero valued. The proof was shown in Agresti (1990, p. 429).

Moreover, while many people use the delta method, few people know who invented it. This method is often used without citation or it is cited to a secondhand authors. The monograph of Ver Hoef (2012) tell us that Robert Dorfman invent the delta method. In 1938, Robert Dorfman originally proposed a "\( \delta \)-method" as an approximate variance for a nonlinear function of one or more random variables. He calls it the "\( \delta \)-method" in the title of his article and uses the \( \delta \) instead of "Delta". The name has persisted but the association to has not.
2.2 Review of Literature

In this part, reviews of literature are given. The researches involving this dissertation are presented as the followings.

Khattree (1989) presented a characterization of inverse Gaussian and gamma distributions through their length biased distributions. A length biased random variable $Y$ associated with $X$ having an inverse Gaussian or gamma distribution can be written as a linear combination of $X$ and a ch-square random variable. Conversely, a random variable $X$ can be characterized through this relation. Finally, the Wald distribution, a special case of inverse Gaussian distribution when the mean parameter is equal to one, was characterized.

Jörgensen et al. (1991) provided a generalizations of the inverse Gaussian distribution. It is the mixture of the inverse Gaussian distribution and its complementary reciprocal, and it is called the \textit{inverse Gaussian mixture distribution}. The new three-parameter generalized inverse Gaussian distribution was formed by combining two independent probability density functions: the pdf. of inverse Gaussian distribution with the weight $1 - p$ and the pdf. of length biased inverse Gaussian distribution with the weight $p$ where $0 \leq p \leq 1$, and it was denoted as $\text{M-IG}(\mu, \sigma^2, p)$. They derived two representations of the new distribution, one as the mixture of the inverse Gaussian distribution, and the second as a sum of an inverse Gaussian variable and an independent compound Bernoulli variable. Additionally, they also provided statistical inference for the new family of distribution and it is shown that there were many applications for right-skewed unimodal data and, especially, duration or failure time data.

Akman and Gupta (1992) had studied the length biased inverse Gaussian distribution. They provided comparative simulation studies of various estimators for the mean of the inverse Gaussian distribution. The variance of inverse Gaussian distribution was assumed to be proportional to the mean, the data were postulated to be available from the inverse Gaussian distribution and its length biased version, and the MVUE and MLE were compared in terms of their variances and mean square errors from both kinds of data.
Gupta and Akman (1998) proposed statistical properties of the arithmetic and harmonic means of length biased distribution utilized for inverse Gaussian distribution. They developed confident intervals and asymptotic tests regarding the mean and the coefficient of variation of inverse Gaussian distribution based on the corresponding length biased data. An estimator the reliability function based on the length biased data was derived and its efficiency was also investigated.

Ahmed et al. (2008) introduced the new parametrization of Birnbaum-Saunders distribution. Importantly, the physical phenomena under this study is fitted by this re-parametrization since the suggested parameters correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively. The original shape and scale parameters of the distribution do not give this physical interpretation. They also provided the relationship between the usual parameters and the proposed parameters. Moreover, they proposed the standard methods of point parameter estimation including maximum likelihood estimation, method of moment estimation, and regression-quantile estimation.

Lisawadi (2009) presented two new classes of distributions based on the proposed parameters provided by Ahmed et al. (2008) and they were called the two-sided Birnbaum-Saunders and inverse Gaussian lifetime distributions. The two lifetime distributions were studied in reliability aspects. These distributions are considered in the case when a crack develops from two sides. For example, on a metallic object, which has a rectangular form with fixed on two sides, a pressure is applied to both upper and lower sides of the object that leads to a crack development from two sides. Based upon the non-classical parametrization, the method of moments of parameter estimation of the new distributions is adapted. Also, asymptotic statistical properties of the suggested estimators are developed.
CHAPTER 3

RESEARCH METHODOLOGY

In this chapter, research methodology will be presented in two parts corresponding to the research objectives.

3.1 Research Methodology for Theoretical Part

In this part, the goals are to establish a new lifetime distribution called the two-sided length biased inverse Gaussian distribution and to investigate its statistical properties. In parameter estimation scheme, we develop the method of moment to estimate the parameters of the new distribution.

Referring to the monograph of Ahmed, Budsaba, Lisawadi, and Volodin (2008), they introduced the new parametrization of Birnbaum-Saunders distribution. They also provided the relationship between the usual parameters and the proposed parameters. Moreover, they proposed the standard methods of point parameters estimation including maximum likelihood estimation, method of moment estimation, and regression-quantile estimation.

Lisawadi (2009) presented two new classes of distributions called the two-sided Birnbaum-Saunders and inverse Gaussian lifetime distributions. These distributions are considered in the case when a crack develops from two sides. For example, on a metallic sample, which has a rectangular form with fixed on two sides, a crack development is applied to both upper and lower sides of the plate with the same speed or distribution. Based upon the non-classical parametrization given in Ahmed et al. (2008), the method of moment of parameter estimation of the new distributions is investigated. Also, asymptotic statistical properties of the
suggested estimators are developed.

In this dissertation, we study in situation when a crack develops from two sides. The physical phenomena in this study is fitted by the re-parametrization of Ahmed et al. (2008) since the suggested estimators, $\lambda$ and $\theta$, correspond to the thickness of the object under consideration and the nominal treatment loading on the object, respectively. The classical scale and shape parameters do not give these characteristics. Probability model of the two-sided length biased inverse Gaussian distribution will be formed by applying the approach of Lisawadi (2009). Also, the traditional parameter estimation, method of moment, will be developed in the new distribution and the asymptotic variances and covariance of the proposed estimators will be investigated.

A brief summary of the important functions and statistical properties concerning the two-sided length biased inverse Gaussian distribution is provided as follows: probability model (cumulative distribution function and probability density function), the reciprocal properties, the first four cumulants or semi-invariants, the first four moments, the first four raw moments, the method of moment estimation, and asymptotic property of the estimates by the method of moment. The following relevant topics explaining what will be done are presented.

3.1.1 Probability Model of the Two-Sided Length Biased Inverse Gaussian Distribution

Probability model of the two-sided length biased inverse Gaussian distribution will be established based on the re-parametrization model. We explain the physics of the phenomena under the case when a crack development is applied to both upper and lower sides of the specified product or object with the same distribution function of the time reaching the critical value. For example, on a metallic sample, which has a rectangular form with fixed on two sides, a pressure is applied to both upper and lower sides of the block. Additionally, the two new random variables will be defined as a speed of the crack evolution and assumed to be independent, identically distributed. Finally, we will combine the two random
variables together. Let $F(t)$, $t > 0$, be the distribution function of the moment of the object break down $\tau$ for one-sided loading. The case under consideration is $F(t) = F_{\text{LBIG}}(t; \lambda, \theta)$. Let $Y = k/\tau$ be the random variable which can be interpreted as a speed of the crack evolution. The physical phenomena of this situation can be explained in the following way. At the bottom side of the metallic block, a crack is developing with the distribution function of the time to reach the length $k$. Simultaneously, at the top side of the block, a crack is developing with the same distribution function as the bottom side. Then, we have two random variables $\tau_1$ and $\tau_2$, and they are assumed to be independent and identically distributed. The speed of the crack growth for this two-sided case thus equals $Y_1 + Y_2 = k\tau_1^{-1} + k\tau_2^{-1}$, and the random variable

$$\tau = \frac{k}{Y_1 + Y_2} = \frac{k}{k\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)} = \frac{1}{\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)} = \left[\tau_1^{-1} + \tau_2^{-1}\right]^{-1}$$

corresponds to a moment of the block break down.

**Cumulative Distribution Function**

The cumulative distribution function (c.d.f.) also called distribution function is used in the term of complementary cumulative distribution function (c.c.d.f.) or simply the tail distribution. It is defined by

$$\bar{F}(x) = P(X > x) = 1 - F(x).$$

The function $\bar{F}(x)$ is also called the survival function in survival analysis and it is denoted as $S(x)$. Furthermore, it is commonly used as reliability function in engineering and is denoted by $R(x)$. According to the probability model stated above, the distribution function of the two-sided length biased inverse Gaussian distribution will be determined.

**Probability Density Function**

According to the previous topic, the probability density function of the two-sided length biased inverse Gaussian distribution will be investigated by dif-
ferentiating its distribution function:
\[ f(x) = \bar{F}'(x) = \frac{d}{dx} \bar{F}(x). \]

### 3.1.2 Reciprocal Properties

The reciprocal properties are important for derivation of the main statistical properties such as the first four moments of the two-sided length biased inverse Gaussian distribution. The reciprocal random variable will be solved by the definition of c.d.f. and the chain rule.

### 3.1.3 The First Four Cumulants

We firstly calculate the natural logarithm of the characteristic function. We have to check that it satisfies the condition \( t \in \mathbb{R} \). Because of the difficulty of direct derivation, the Maclaurin series expansion is applied and the first four terms are considered to obtain the first four cumulants or semi-invariants.

### 3.1.4 The First Four Moments

To examine the first four moment of the two-sided length biased inverse Gaussian distribution we use the following steps. Initially, we find the characteristic function of the related distribution, i.e., the inverse Gaussian distribution. Secondly, the Maclaurin series expansion is adapted to get the first four cumulants. Thirdly, applying the reciprocal properties the original cumulants will be modified by substituting the parameters obtained from the reciprocal results. Finally, we combine them to obtain the first four moment of the two-sided random variable \( X = \tau_1^{-1} + \tau_2^{-1} \).

### 3.1.5 The First Four Raw Moments

The relation of cumulants and population raw moment is used to find the first four raw moments by algebraic computation.
3.1.6 Parameter Estimation by the Method of Moment

In order to investigate the method of moment of parameter point estimation; firstly, we have to find the first four moments of the two-sided length biased inverse Gaussian distribution as the population moments. Secondly, we determine the corresponding sample moments. Finally, we equate the population moments and the sample moments, then the equations will be solved to obtain the desired estimators.

3.1.7 General Approach to Asymptotic Analysis of the Estimates by the Method of Moment

We provide general approach to asymptotic analysis of the moment estimators by using the Delta method for variance approximations.

3.1.8 Asymptotic Properties of the Estimates by the Method of Moment

Asymptotic properties of the estimates by the method of moment of the two-sided length biased inverse Gaussian distribution will be offered based upon the variances and covariance of the estimators $\hat{\lambda}$ and $\hat{\theta}$ by the strategy given above.

3.2 Research Methodology for Computational Part

In this part, the purpose is to appraise the performance of the proposed estimators by considering the bias, variance, coefficient of variation, and mean square error. Monte-Carlo simulation study will be performed for small, moderate, and large sample sizes. The R program version 3.1.2 will be used for the simulation of data and data analyses. In order to achieve our goal, we provide two random number generators for generating the two-sided length biased inverse Gaussian distribution. The first generator is called "Procedure". We propose the two-sided length biased inverse Gaussian random number generation procedure using the connection between common distributions. The second generator is called "Package". We apply the package called "statmod" in R. It is commonly
used to generate IG random number for usual parametrization. Recall that the length biased inversed Gaussian distribution is the same as that of the reciprocal of IG variate \((\mu^2/X)\). We use this relation to generate LBIG random number for non-classical parametrization. For two-sided length biased inversed Gaussian distribution, after we obtain a pair of one-sided LBIG random numbers \(\tau_1\) and \(\tau_2\), we compute \(X = \tau_1^{-1} + \tau_2^{-1}\). The problem is which generator is better than each other? Then, we will compare the performance of this two generators via an absolute difference-percentage of population and sample moments. After we obtain the conclusion Monte-Carlo simulation will be conducted to assess the performance of the proposed estimators.

In computer simulations, we consider all combinations of the followings values of \(\lambda, \theta, \) and sample size \(n\).

\[
\begin{align*}
\lambda &= 2, 5, 10, 20, 50, \\
\theta &= 1, 5, 10, 50, \\
n &= 10, 50, 100, 200, 500.
\end{align*}
\]

The number of iterations will be fixed at 5,000 for each combination of \(\lambda, \theta, \) and \(n\). Additionally, real life data sets based on published data are used to illustrate the proposed estimation method. The following topics explaining what can be done to achieve the objective of this part.

### 3.2.1 TS-LBIG-Random Number Generation Procedure

In this section, we develop the two-sided length biased inverse Gaussian random number generation using the connection between common distributions, e.g., connection between LBIG and IG distributions and connection between IG and Chi-squared distributions. It is an extension of the monograph of Michael, Achucany, and Haas (1976) for different parametrization.

### 3.2.2 TS-LBIG-Random Number Generator Selection

As mentioned above, the first three population and sample raw moments \((m'_1, m'_2, m'_3)\) obtained from the results of theoretical part are computed from both
generators. An absolute difference-percentage of population and sample moments (ADPM) is defined as

\[ ADPM = \frac{|\mu'_k - m'_k|}{\mu_k} \times 100, \quad k = 1, 2, 3, \]

where \( \mu'_k \) and \( m'_k \) are population and sample raw moments, respectively. We compare the ADPM of "Procedure" and "Package" for all combinations of \( \lambda, \theta, \) sample size \( n, \) and iterations as follows.

\[
\lambda = 2, 10, 50, \\
\theta = 1, 5, 50, \\
n = 10, 50, 100, 500 \\
\text{iterations} = 1,000, 5,000.
\]

3.2.3 Research Simulation Procedure

After we obtain the LBIG-random number generator and for each of the choices of \( \lambda, \theta, \) and sample size \( n \) stated above, we will perform the following steps.

1. Set the sample sizes \( n \) and the parameters \( \lambda \) and \( \theta. \)
2. Generate \( n \) random number of a one-sided \( LBIG(\lambda, \theta) \) distribution.
3. For two-sided \( LBIG(\lambda, \theta) \) distribution, after we obtain a pair of one-sided \( LBIG \) random numbers \( \tau_1 \) and \( \tau_2, \) we compute \( X = \tau_1^{-1} + \tau_2^{-1}. \)
4. Calculate the proposed parameter estimates.
5. Repeat steps 1 to 4 for 5,000 times.
6. Compute the average estimates, bias, variance, coefficient of variation, and mean square error of the point estimators.

The flowchart for the simulation study is shown in figure 3.1.
Figure 3.1: Flowchart for the simulation study
3.2.4 The Estimates by the Method of Moment

We compute the suggested parameter estimates for all combinations of \( \lambda, \theta \) and \( n \) previously mentioned, then observe their values.

3.2.5 The Desirable Properties of the Estimators

We consider the following properties of the method of moment estimates obtained from the simulation study.

- Minimum variances and coefficient of variations
- Unbiased estimators
- Minimum mean square errors

3.2.6 Histograms of Errors

We construct histograms of the error of the estimators. The error is the difference between the true and estimated values.

3.2.7 Asymptotic Analysis of the Estimates by the Method of Moment

Asymptotic analysis of the estimates by the method of moment is obtained by comparing of the simulated and asymptotic variances of the proposed estimators, \( \hat{\lambda} \) and \( \hat{\theta} \).

3.2.8 Illustrative Examples

The practical applications of the suggested estimators are illustrated in this section. Two real data sets are considered as the followings.

**Example 1.** The following data provided by Lieblein and Zelen (1956) on the fatigue life of the 23 deep groove ball bearings:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.
The above data had been analyzed by Gupta and Akman (1998) and the result indicated that the data set comes from the length biased inverse Gaussian distribution. For two-sided case, we divide all data in pairs, we have only 11 pairs of observations, and the last one is dropped. We obtain 11 observations: \((y_1, y_2), (y_3, y_4), \cdots, (y_{21}, y_{22})\). Let \(u_1 = (1/y_1) + (1/y_2), u_2 = (1/y_3) + (1/y_4), \cdots, u_{11} = (1/y_{21}) + (1/y_{22})\) be the observations drawn from the TS-LBIG distribution. In this example, the point estimates are computed. Moreover, using the relationship given in equation (2.11), the point estimates for original parameters, \(\mu\) and \(\beta\), are also computed.

**Example 2.** This example was taken from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibers (in GPa) as follows:

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<th>0.81</th>
<th>0.85</th>
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<td>4.70</td>
<td>4.90</td>
<td>4.91</td>
<td>5.08</td>
<td>5.56</td>
</tr>
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</table>

These data had been analyzed by assuming the Weibull distribution. In our case, the data was ascendingly ordered. Dealing with the analogous manner of the Example 1, finally, we obtain 50 observations drawn from the TS-LBIG distribution. The point estimates for the proposed and original parameters are calculated.

Besides, we compare our parameter estimation strategy with Lisawadi’s method for both examples.
CHAPTER 4

RESULTS OF THEORETICAL PART

In this part, the main functions and statistical properties of the two-sided length biased inverse Gaussian distribution are given.

4.1 Probability Model of the Two-Sided Length Biased Inverse Gaussian Distribution

According to section 3.1.1 we describe the definition of the two-sided random variable. In this study, we consider $F(t) = F_{LBIG}(t; \lambda, \theta)$. Let $\tau$ be the break down time moment (of the block) for one-sided loading and $k$ is a critical length of the crack. The random variable $Y = k/\tau$ represents a speed of the crack evolution. The distribution function of the random variable $Y$ followed by the reliability function is defined as

$$
\bar{F}_Y(t) = 1 - F(k/t) = 1 - F(kt^{-1}),
$$

and the probability density function of $Y$ is

$$
f_Y(t) = \bar{F}_Y'(t) = \frac{d}{dt} \bar{F}_Y(t) = kt^{-2}f(kt^{-1}).
$$

The cumulative distribution function (c.d.f.) and probability density function (p.d.f.) of the two-sided length biased inverse Gaussian distribution are presented in the following theorems.
Theorem 2. A random variable $\tau$ has a two-sided length biased inverse Gaussian distribution with parameters $\lambda$ and $\theta$ denoted as $\text{TS-LBIG}(\lambda, \theta)$, if it has a distribution function in the form:

$$F_\tau(u) = 1 - \int_0^{u^{-1}} f\left(\frac{1}{t}\right) \frac{dt}{t^2} \int_0^{u^{-1}-t} f\left(\frac{1}{s}\right) \frac{ds}{s^2},$$

and a density function in the following form:

$$f_\tau(u) = u^{-2} \int_0^{u^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{u^{-1}-t}\right) \frac{dt}{t^2 (u^{-1}-t)^2},$$

where $u > 0$.

Proof.

Let $\tau$ be a random variable defined above, $T = 1/\tau_1$ and $S = 1/\tau_2$. Then

$$F_T(t) = P(T \leq t) = P\left(\frac{1}{\tau_1} \leq t\right) = P\left(\tau_1 > \frac{1}{t}\right) = 1 - F_{\tau_1}\left(\frac{1}{t}\right),$$

$$f_T(t) = F_T'(t) = f_{\tau_1}\left(\frac{1}{t}\right) \cdot \frac{1}{t^2} = f\left(\frac{1}{t}\right) \cdot \frac{1}{t^2}.$$  

Similarly,

$$F_S(s) = P(S \leq s) = P\left(\frac{1}{\tau_2} \leq s\right) = P\left(\tau_2 > \frac{1}{s}\right) = 1 - F_{\tau_2}\left(\frac{1}{s}\right),$$

$$f_S(s) = F_S'(s) = f_{\tau_2}\left(\frac{1}{s}\right) \cdot \frac{1}{s^2} = f\left(\frac{1}{s}\right) \cdot \frac{1}{s^2}.$$
The distribution function of the random variable $\tau$ is defined by

$$F_\tau(z) = P(\tau \leq z) = P\left( [\tau_1^{-1} + \tau_2^{-1}]^{-1} \leq z \right)$$

$$= P\left( \tau_1^{-1} + \tau_2^{-1} > z^{-1} \right) = P\left( T + S > z^{-1} \right)$$

$$= \int \int_{t+s > z^{-1}} f_{TS}(t,s) \, dt \, ds = \int \int_{t+s > z^{-1}} f_T(t) f_S(s) \, dt \, ds$$

$$= \int \int_{t+s > z^{-1}} f\left( \frac{1}{t} \right) f\left( \frac{1}{s} \right) \frac{dt \, ds}{t^2 s^2}$$

$$= 1 - \int \int_{t+s \leq z^{-1}} f\left( \frac{1}{t} \right) f\left( \frac{1}{s} \right) \frac{dt \, ds}{t^2 s^2}$$

$$= 1 - \int_0^{z^{-1}} f\left( \frac{1}{t} \right) \frac{dt}{t^2} \int_0^{z^{-1} - t} f\left( \frac{1}{s} \right) \frac{ds}{s^2},$$

and the probability density function is given by

$$f_\tau(z) = F'_\tau(z) = \frac{d}{dz} \left[ 1 - \int_0^{z^{-1}} f\left( \frac{1}{t} \right) \frac{dt}{t^2} \int_0^{z^{-1} - t} f\left( \frac{1}{s} \right) \frac{ds}{s^2} \right]$$

$$= \frac{d}{dz} \left[ - \int_0^{z^{-1}} \left\{ \int_0^{z^{-1} - t} f\left( \frac{1}{t} \right) \frac{1}{t^2} f\left( \frac{1}{s} \right) \frac{1}{s^2} ds \right\} \, dt \right]$$

$$= \frac{d}{dz} \left[ - \int_{\psi_1(z)}^{\psi_2(z)} g(z,t) \, dt \right]$$

$$= - \int_{\psi_1(z)}^{\psi_2(z)} g'(z,t) \, dt + g(z, \psi_2(z)) \psi_2'(z) - g(z, \psi_1(z)) \psi_1'(z)$$

$$= - \int_0^{z^{-1}} g'_z(z,t) \, dt + g(z, z^{-1}) \cdot \left( - \frac{1}{z^2} \right) - g(z, 0) \cdot (0)$$

$$= - \int_0^{z^{-1}} g'_z(z,t) \, dt.$$ (4.1)
Consider $g'_z(z, t)$.

\[
g'_z(z, t) = \frac{d}{dz} \left[ \int_0^{z^{-1}-t} f \left( \frac{1}{t} \right) \frac{1}{t^2} f \left( \frac{1}{s} \right) \frac{1}{s^2} ds \right]
\]

\[
= \frac{d}{dz} \left[ \int_0^{z^{-1}-t} h(z, s) ds \right]
\]

\[
= \int_0^{z^{-1}-t} h'_z(z, s) ds + h(z, z^{-1} - t) \cdot \left( -\frac{1}{z^2} \right) - h(z, 0) \cdot \frac{1}{z^2}
\]

\[
= -\frac{1}{z^2} h(z, z^{-1} - t)
\]

\[
= -\frac{1}{z^2} f \left( \frac{1}{t} \right) \frac{1}{t^2} f \left( \frac{1}{z^{-1} - t} \right) \frac{1}{(z^{-1} - t)^2}.
\]  \hspace{1cm} (4.2)

By substitution (4.2) into (4.1), finally, the p.d.f. of the random variable $\tau$ is

\[
f_\tau(z) = z^{-2} \int_0^{z^{-1}} f \left( \frac{1}{t} \right) f \left( \frac{1}{z^{-1} - t} \right) \frac{1}{t^2(z^{-1} - t)^2} dt, \text{ where } u > 0
\]  \hspace{1cm} (4.3)

We say that the random variable $\tau$ has a two-sided length biased inverse Gaussian distribution abbreviated as $\tau \sim TS-LBIG(\lambda, \theta)$. However, the p.d.f. of the $TS-LBIG(\lambda, \theta)$ has no explicit form or closed form. Its form involves the integral sign and this may be difficult to directly find important functions such as a characteristic function and a moment generating function. Figures 4.1 and 4.2 show variety of the probability density functions of TS-LBIG distribution. It is indicated that the TS-LBIG is positively skewed distribution. Interestingly, it gives families of asymmetric distributions which are useful for skewed data analysis.
Figure 4.1: Density functions of $TS-LBIG(\lambda, \theta)$ for fixed $\lambda = 2$ and increasing $\theta$

Figure 4.2: Density functions of $TS-LBIG(\lambda, \theta)$ for fixed $\theta = 5$ and increasing $\lambda$
4.2 Reciprocal Properties

**Proposition 1.** If random variable \( \tau > 0 \) has the probability density function \( f_\tau(x) \), then the reciprocal random variable \( 1/\tau \) has the probability density function \( f_{1/\tau}(x) = x^{-2}f_\tau(1/x) \).

**Proof.**
For the reciprocal random variable \( 1/\tau \), the distribution function is given by

\[
F_{1/\tau}(x) = P\left(\frac{1}{\tau} \leq x\right) = P\left(\tau \geq \frac{1}{x}\right) = 1 - P\left(\tau \leq \frac{1}{x}\right) = 1 - F_\tau\left(\frac{1}{x}\right),
\]

and applying the chain rule the density function is

\[
f_{1/\tau}(x) = F'_{1/\tau}(x) = -F'_\tau\left(\frac{1}{x}\right) = -\left[f_\tau\left(\frac{1}{x}\right)\right] \cdot \left(-\frac{1}{x^2}\right) = x^{-2}f_\tau\left(\frac{1}{x}\right) \quad \square
\]

**Proposition 2.** If random variable \( \tau > 0 \) has \( LBIG(\lambda, \theta) \) distribution, then the reciprocal random variable \( 1/\tau \) is \( IG[\lambda, 1/(\lambda^2 \theta)] \) distributed.

**Proof.**
By Proposition 1,

\[
f_{1/\tau}(x) = x^{-2}f_{LB}\left(\frac{1}{x}; \lambda, \theta\right)
\]

\[
= \frac{x^{-2}\theta^{-1/2}x^{1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\lambda \theta^{1/2}x^{1/2} - \theta^{-1/2}x^{-1/2}\right)^2\right\}
\]

\[
= \frac{x^{-3/2}}{\sqrt{2\pi} \theta} \exp\left\{-\frac{1}{2} \left(\lambda \sqrt{\theta}x - \frac{1}{\sqrt{\theta}x}\right)^2\right\}
\]

\[
= \frac{x^{-3/2}}{\sqrt{2\pi} \theta} \exp\left\{-\frac{1}{2} \left(\lambda \sqrt{\frac{1}{(\lambda^2 \theta)}} - \sqrt{\frac{x}{1/(\lambda^2 \theta)}}\right)^2\right\}
\]
$$f_{1/2}(x) = \frac{\lambda(1/\lambda^2\theta)^{1/2}}{\sqrt{2\pi}} x^{-3/2} e^{\exp\left\{ -\frac{1}{2} \left( \lambda \sqrt{\frac{1/(\lambda^2\theta)}{x}} - \sqrt{\frac{x}{1/(\lambda^2\theta)}} \right)^2 \right\}$$

$$= \text{IG}[x; \lambda, 1/(\lambda^2\theta)] \Box$$

4.3 The First Four Cumulants

Finding the first four cumulants of the two-sided length biased inverse Gaussian distribution, the reciprocal property is necessarily needed (Proposition 2). That is, we have to start with the inverse Gaussian distribution. Its characteristic function is required. Because of the difficulty of direct derivation, Maclaurin expansion (the Taylor series expansion evaluated at zero) is applied. The first four terms are considered to be the corresponding cumulants.

**Theorem 3.** If a random variable $X$ has two-sided length biased inverse Gaussian distribution with parameters $\lambda$ and $\theta$ denoted as $TS-LBIG(\lambda, \theta)$, the correspondingly first four cumulants are

$$k_1(X) = \frac{2}{\lambda\theta}, \quad k_2(X) = \frac{2}{\lambda^3\theta^2}, \quad k_3(X) = \frac{6}{\lambda^5\theta^3}, \quad \text{and} \quad k_4(X) = \frac{30}{\lambda^7\theta^4}.$$

**Proof.**

If $\varphi(t)$ is a characteristic function, then its cumulants (semi-invariants) $k_j, \ j = 1, 2, ..., m$, are defined from the Maclaurin expansion of the logarithm of the characteristic function:

$$\ln \varphi(t) = \sum_{j=1}^{m} \frac{k_j}{j!} (it)^j + o(|t|^m). \quad (4.3)$$

Firstly, we fine the first four cumulants of the $IG(\lambda, \theta)$ distribution. From equation (2.16), the characteristic function of the $IG(\lambda, \theta)$ is

$$\varphi_{IG}(t; \lambda, \theta) = \exp\left\{ \lambda \left[ 1 - (1 - 2\theta tt)^{-1/2} \right] \right\},$$
and its logarithm of the characteristic function is

\[ \ln \varphi(t) = \lambda \left[ 1 - (1 - 2\theta it)^{1/2} \right] = \lambda - \lambda (1 - 2\theta it)^{1/2}. \]  

(4.4)

For the Maclaurin expansion, we use

\[ (1 - x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - O(x^5), \]

(4.5)

where \(|x| < 1\).

In this case, \(x = 2\theta it\). Applying the equations (4.4) and (4.5) , we obtain

\[ \ln \varphi(t) = \lambda - \lambda \left[ 1 - \frac{2(\theta it)}{2} - \frac{4(\theta it)^2}{8} - \frac{8(\theta it)^3}{16} - \frac{5 \cdot 16(\theta it)^4}{128} - O(t^5) \right], |2\theta it| < 1, \]

\[ = \lambda (\theta it) + \frac{\lambda}{2} (\theta it)^2 + \frac{\lambda}{2} (\theta it)^3 + \frac{5\lambda}{8} (\theta it)^4 + O(t^5), \left| t \right| < \frac{1}{2\theta}, \]

\[ = \frac{(it)}{1!} \lambda \theta + \frac{(it)^2}{2!} \lambda \theta^2 + \frac{(it)^3}{3!} 3\lambda \theta^3 + \frac{(it)^4}{4!} 15\lambda \theta^4 + O(t^5), \left| t \right| < \frac{1}{2\theta}. \]  

(4.6)

This expansion gives us the following cumulants of \(IG(\lambda, \theta)\) distribution.

\[ k_1 = \mu = \lambda \theta, \quad k_2 = \sigma^2 = \lambda \theta^2, \quad k_3 = \mu_3 = 3\lambda \theta^3 \quad \text{and} \quad k_4 = \mu_4 - 3\sigma^4 = 15\lambda \theta^4. \]

According to Proposition 2, if \(\tau\) has \(LBIG(\lambda, \theta)\) distribution, then the reciprocal \(1/\tau\) has \(IG(\lambda, 1/(\lambda^2 \theta))\) distribution. Substituting \(\theta\) by \(1/(\lambda^2 \theta)\), \(1/\tau\) has the cumulants for the \(LBIG(\lambda, \theta)\):

\[ k_1(1/\tau) = \frac{1}{\lambda \theta}, \quad k_2(1/\tau) = \frac{1}{\lambda^3 \theta^2}, \]

\[ k_3(1/\tau) = \frac{3}{\lambda^5 \theta^3} \quad \text{and} \quad k_4(1/\tau) = \frac{15}{\lambda^7 \theta^4}. \]

(4.7)

Because of the i.i.d. property, the first four cumulants for the random variable \(X = \tau_1^{-1} + \tau_2^{-1}\) will be obtained by combining (doubling) the cumulants for \(1/\tau\).

Finally, the first four cumulants for the two-sided length biased inverse Gaussian distribution are:
\[ k_1(X) = \frac{2}{\lambda \theta}, \quad k_2(X) = \frac{2}{\lambda^3 \theta^2}, \quad k_3(X) = \frac{6}{\lambda^5 \theta^3}, \quad \text{and} \quad k_4(X) = \frac{30}{\lambda^7 \theta^4} \]

It is noticed that the two important points we should concern: the remainder of the Macluarin expansion and the condition of the possible values of \( t \) which satisfies (4.3). In the first point, our goal is the first four cumulants, then the fifth and higher powers is considered to be trivial. That is, we sum the fifth and higher powers in the term \( O(x^5) \). In the second point, from (4.6) the condition satisfying (4.3) is \( |t| < \frac{1}{2\theta} \).

### 4.4 The First Four Moments

**Theorem 4.** If a random variable \( X \) has two-sided length biased inverse Gaussian distribution with parameters \( \lambda \) and \( \theta \) denoted as \( TS-LBIG(\lambda, \theta) \), the correspondingly first four moments are

\[
\mu(X) = k_1(X) = \frac{2}{\lambda \theta}, \quad \sigma^2(X) = k_2(X) = \frac{2}{\lambda^3 \theta^2},
\]

\[
\mu_3(X) = k_3(X) = \frac{6}{\lambda^5 \theta^3},
\]

and \( \mu_4(X) = k_4(X) + 3\sigma^4(X) = \frac{30}{\lambda^7 \theta^4} + \frac{12}{\lambda^6 \theta^4} = \frac{3(10 + 4\lambda)}{\lambda^7 \theta^4} = \frac{30 + 12\lambda}{\lambda^7 \theta^4} \).

**Proof.**

In order to find the first four moments for the two-sided length biased inverse Gaussian distribution, we deal with the analogous derivation of the cumulants. Using the information in (4.7) and solving the corresponding cumulants for \( 1/\tau \), and the different point between the first four cumulants and the corresponding moments is the last term, i.e., \( \mu_4(X) = k_4(X) + 3\sigma^4(X) \). Finally, the first four moments for the two-sided length biased inverse Gaussian distribution are:

\[
\mu(X) = k_1(X) = \frac{2}{\lambda \theta}, \quad \sigma^2(X) = k_2(X) = \frac{2}{\lambda^3 \theta^2},
\]

\[
\mu_3(X) = k_3(X) = \frac{6}{\lambda^5 \theta^3},
\]
and \( \mu_4(X) = k_4(X) + 3\sigma^4(X) = \frac{30}{\lambda^7\theta^4} + \frac{12}{\lambda^6\theta^4} = \frac{3(10 + 4\lambda)}{\lambda^7\theta^4} = \frac{30 + 12\lambda}{\lambda^7\theta^4} \). 

\[ \Box \]

4.5 The First Four Raw Moments

**Theorem 5.** If a random variable \( X \) has two-sided length biased inverse Gaussian distribution with parameters \( \lambda \) and \( \theta \) denoted as \( TS-LBIG(\lambda, \theta) \). Then

- 1\(^{st}\) raw moment, \( \mu_1' = \frac{2}{\lambda\theta} \),
- 2\(^{nd}\) raw moment, \( \mu_2' = \frac{2(1 + 2\lambda)}{\lambda^3\theta^2} \),
- 3\(^{rd}\) raw moment, \( \mu_3' = \frac{2(4\lambda^2 + 6\lambda + 3)}{\lambda^5\theta^3} \),
- 4\(^{th}\) raw moment, \( \mu_4' = \frac{2(8\lambda^3 + 54\lambda + 15)}{\lambda^7\theta^4} \).

**Proof.**

From the previous section, the first four cumulants for \( TS-LBIG(\lambda, \theta) \) distribution are given by

\[ k_1(X) = \frac{2}{\lambda\theta}, \quad k_2(X) = \frac{2}{\lambda^3\theta^2}, \quad k_3(X) = \frac{6}{\lambda^5\theta^3}, \quad \text{and} \quad k_4(X) = \frac{30}{\lambda^7\theta^4}, \]

and it is known that the relation between the raw moments and the cumulants corresponds to the following formulas:

- 1\(^{st}\) raw moment, \( \mu_1' = k_1 \)
- 2\(^{nd}\) raw moment, \( \mu_2' = k_1^2 + k_2 \)
- 3\(^{rd}\) raw moment, \( \mu_3' = k_1^3 + 3k_1k_2 + k_3 \)
- 4\(^{th}\) raw moment, \( \mu_4' = k_1^4 + 6k_1^2k_2 + 4k_1k_3 + k_4 \).
Hence, by substituting the cumulants into these equations above and using algebraic calculation, we obtain

\[ \mu'_1 = \mu(X) = \frac{2}{\lambda \theta}, \]

\[ \mu'_2 = \frac{2(1 + 2\lambda)}{\lambda^3 \theta^2}, \]

\[ \mu'_3 = \frac{2(4\lambda^2 + 6\lambda + 3)}{\lambda^5 \theta^3}, \]

\[ \mu'_4 = \frac{2(8\lambda^3 + 54\lambda + 15)}{\lambda^7 \theta^4}. \]

\[ \Box \]

4.6 Parameter Estimation by the Method of Moment

In this section, we provide a strategy of parameter estimation by method of moment for estimating the parameters of the two-sided length biased inverse Gaussian distribution. The required theorem is given as follows.

**Theorem 6.** Given \( X \sim TS-LBIG(\lambda, \theta) \). Let \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \),

\[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2, \text{ and } T = S^2 / \bar{X}^2. \]

Then the method of moment estimators for \( \lambda \) and \( \theta \) are:

\[ \hat{\lambda} = \frac{1}{2T} \text{ and } \hat{\theta} = \frac{2}{\lambda \bar{X}}. \]

**Proof.**

Estimation of the parameters \( \lambda \) and \( \theta \) by the method of moments for the two-sided length biased inverse Gaussian distribution can be derived in the following way. In the previous section, the moments are investigated, and thus we obtain following formulas for the expectation and variance:

\[ E(X) = \mu(X) = \frac{2}{\lambda \theta}, \quad Var(X) = \sigma^2(X) = \frac{2}{\lambda^3 \theta^2}. \]
Hence, the method of moment estimations are

\[ \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{2}{\lambda \theta}, \]  

(4.8)

\[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{2}{\lambda^3 \theta^2}. \]  

(4.9)

Solving these two equations for \( \lambda \) and \( \theta \) we will obtain the estimates which we are interested.

Let

\[ T = \frac{S^2}{\bar{X}^2}. \]

Then

\[ T = \frac{2}{\lambda^3 \theta^2} \cdot \frac{\lambda^2 \theta^2}{4} = \frac{1}{2\lambda} \]

or

\[ 2T\lambda - 1 = 0. \]

Finally, we obtain the method of moment estimators, namely

\[ \hat{\lambda} = \frac{1}{2T} \quad \text{and} \quad \hat{\theta} = \frac{2}{\lambda \bar{X}}. \]

\[ \square \]

4.7 General Approach to Asymptotic Analysis of the Estimates by the Method of Moment

In this section, an approximation for the variances of the suggested estimators will be determined by using the \textit{Delta method}. Referring to a two-variable Taylor series expansion suppose we have two random variables \( X \) and \( Y \). The Taylor expansion of \( f(x, y) \) about the values \((x_0, y_0)\) is given by:

\[ f(x, y) = f(x_0, y_0) + \frac{\partial f(x, y)}{\partial x} \bigg|_{(x_0, y_0)} (x - x_0) + \frac{\partial f(x, y)}{\partial y} \bigg|_{(x_0, y_0)} (y - y_0) + R \]  

(4.10)

where \( R \) is the remainder term.
The asymptotic distribution of an estimate which smoothly depends on
sample moments is commonly obtained by their decomposition into the Taylor
series expansion in the neighborhood of the true values of the moments. The
leading term of the asymptotic distribution is given by the linear part of this
decomposition. For our case, $\lambda = \hat{\lambda}(\bar{X}, S^2)$, $\theta = \hat{\theta}(\bar{X}, S^2)$, $\mu(X)$ is the true
value of $\bar{X}$, and $\sigma^2(X)$ is the true value of $S^2$.

Let $a_1$ and $a_2$ be the values of partial derivatives of $\hat{\lambda}$ by $\bar{X}$ and $S^2$,
correspondingly, at the point $(\mu(X), \sigma^2(X))$. $b_1$ and $b_2$ are denoted as the values
of analogous derivatives of $\hat{\theta}$.

For the differentiation, it is better to use the chain rule since $\hat{\lambda}$ depends on
the arguments that we differentiate only through the statistics $T = S^2/\bar{X}^2$. For
the $\hat{\theta}$, we mention that it depends on $\hat{\lambda}$ and $\bar{X}$. Note also that $\hat{\lambda}(\mu(X), \sigma^2(X)) = \lambda$ and $\hat{\theta}(\mu(X), \sigma^2(X)) = \theta$.

Therefore, applying the equation (2.35) the Taylor series expansion can
be written as

$$
\hat{\lambda} = \lambda + a_1(\bar{X} - \mu(X)) + a_2(S^2 - \sigma^2(X)) + O_p(1/n),
$$

$$
\hat{\theta} = \theta + b_1(\bar{X} - \mu(X)) + b_2(S^2 - \sigma^2(X)) + O_p(1/n).
$$

The notation $X_n = O_p(1/n)$, where $\{X_n, n \geq 1\}$ is a sequence of random
variables, means that there exists a constant $C > 0$ such that

$$
\lim_{n \to \infty} \sup_{n \geq 1} P|X_n - C/n| > t = 0.
$$

Additionally, these expressions are understood in terms of convergence in
probability. Thus, the vector $(\sqrt{n}(\hat{\lambda} - \lambda), \sqrt{n}(\hat{\theta} - \theta))$ has in limit in distribution,
the two-dimensional normal distribution with zero means, variances

$$
Var(\hat{\lambda}) = a_1^2 Var(\bar{X}) + a_2^2 Var(S^2) + 2a_1a_2 Cov(\bar{X}, S^2),
$$

$$
Var(\hat{\theta}) = b_1^2 Var(\bar{X}) + b_2^2 Var(S^2) + 2b_1b_2 Cov(\bar{X}, S^2),
$$

and covariance

$$
Cov(\hat{\lambda}, \hat{\theta}) = a_1b_1 Var(\bar{X}) + a_2b_2 Var(S^2) + 2(a_1b_2 + a_2b_1) Cov(\bar{X}, S^2).
$$
For derivation of the variance and covariance of the sample moments, we apply the formulas from Sections 27.2 - 27.4 provided by Cramér (1999) and the result of Cho and Cho (2008) as follows:

\[
Var(\bar{X}) = \frac{\sigma^2(X)}{n},
\]

\[
Var(S^2) = \frac{1}{n} \left( \mu_4(X) - \frac{n-3}{n-1} \mu_2^2(X) \right),
\]

(4.14)

\[
Cov(\bar{X}, S^2) = \frac{n-1}{n^2} \mu_3(X) \cdot
\]

4.8 Asymptotic Properties of the Estimates by the Method of Moment

In this section, we give formulas of asymptotic variances and covariance of the suggested estimators of the two-sided length biased inverse Gaussian distribution by applying the delta method for variance approximation. The required theorem is provided as follows.

**Theorem 7.** Given \(X \sim TS-LBIG(\lambda, \theta)\). Let \(\hat{\lambda}\) and \(\hat{\theta}\) be the method of moment estimators for \(\lambda\) and \(\theta\). As \(n \to \infty\), the \(Var(\hat{\lambda}), Var(\hat{\theta})\) and \(Cov(\hat{\lambda}, \hat{\theta})\) are given by

\[
Var(\hat{\lambda}) = \frac{4\lambda^2 n^2 + 7\lambda n^2 + 5\lambda n - 12\lambda}{2n^2(n-1)},
\]

\[
Var(\hat{\theta}) = \frac{(2\lambda + 3)\theta^2 n^2 + 6\theta^2 n - 9\theta^2}{\lambda n^2(n-1)},
\]

\[
Cov(\hat{\lambda}, \hat{\theta}) = -\frac{(4\lambda \theta + 18\theta)n^2 - 15\theta n - 3\theta}{2n^2(n-1)}.
\]
Proof.
In previous section, the first four moments of the two-sided length biased inverse Gaussian distribution are presented. Applying the equation (4.14) to the corresponding moments given in the theorem 4 we obtain:

\[
Var(\bar{X}) = \frac{2}{\lambda^3 \theta^2 n},
\]

\[
Var(S^2) = \frac{8\lambda n + 30n - 30}{n(n-1)\lambda^3 \theta^4},
\]

\[
Cov(\bar{X}, S^2) = \frac{6(n-1)}{n^2 \lambda^5 \lambda^3}.
\]

Initially, we collect the formulas for the TS-LBIG(\lambda, \theta) distribution obtained in previous section:

\[
\hat{\lambda}(\bar{X}, S^2) = \frac{1}{2T(\bar{X}, S^2)},
\]

\[
\hat{\theta}(\bar{X}, S^2) = \frac{2}{\lambda(\bar{X}, S^2)\bar{X}},
\]

\[
T(\bar{X}, S^2) = \frac{S^2}{\bar{X}^2},
\]

\[
\mu = \frac{2}{\lambda \theta},
\]

\[
\sigma^2 = \frac{2}{\lambda^3 \theta^2}.
\]

According to our asymptotic analysis procedure we assume that

\[
X(\mu, \sigma^2) = \mu = \frac{2}{\lambda \theta} \quad \text{and} \quad S^2(\mu, \sigma^2) = \sigma^2 = \frac{2}{\lambda^3 \theta^2}.
\]
These formulas lead to:

\[ T(\mu, \sigma^2) = \frac{2}{\lambda^3 \theta^2} \cdot \frac{\lambda^2 \theta^2}{4} = \frac{1}{2\lambda}, \]

\[ \hat{\lambda}(\mu, \sigma^2) = \lambda, \]

\[ \hat{\theta}(\mu, \sigma^2) = \theta, \]

\[ \frac{\partial T}{\partial X}(\mu, \sigma^2) = -\frac{4}{\lambda^3 \theta^2} \left( \frac{\lambda \theta}{2} \right)^3 = -\frac{\theta}{2}, \]

\[ \frac{\partial T}{\partial S^2}(\mu, \sigma^2) = \left( \frac{\lambda \theta}{2} \right)^2 = \frac{\lambda^2 \theta^2}{4}. \]

The next step, the partial derivatives of the estimates \( \hat{\lambda} \) and \( \hat{\theta} \) by \( \bar{X} \) and \( S^2 \), respectively will be calculated, the chain rule is necessarily needed, then the true values of the expectation and variance will be substituted into these derivatives, i.e., at the point \((\mu(X), \sigma^2(X))\).

We firstly consider the derivatives of the function \( \hat{\lambda} = \frac{1}{2T} \) and we have

\[ \frac{\partial \hat{\lambda}}{\partial \bar{X}} = \frac{d\hat{\lambda}}{dT} \cdot \frac{\partial T}{\partial \bar{X}} = -\frac{1}{2T^2} \cdot \frac{\partial T}{\partial \bar{X}}. \]

Exchanging \( \bar{X} \) and \( S^2 \) on \( \mu \) and \( \sigma^2 \), we get

\[ a_1 = \frac{\partial \hat{\lambda}}{\partial \bar{X}} = -\frac{1}{2T^2} \cdot \frac{\partial T}{\partial \bar{X}} = -\frac{1}{2T^2} \cdot \left( -\frac{\theta}{2} \right) = \left( \frac{4\lambda^2}{2} \right) \cdot \left( \frac{\theta}{2} \right) = \lambda^2 \theta. \]

Next, we have

\[ \frac{\partial \hat{\lambda}}{\partial S^2} = \frac{d\hat{\lambda}}{dT} \cdot \frac{\partial T}{\partial S^2} = -\frac{1}{2T^2} \cdot \frac{\partial T}{\partial S^2}. \]

Based on the appropriate substitution, we obtain

\[ a_2 = \frac{\partial \hat{\lambda}}{\partial S^2} = -\frac{1}{2T^2} \cdot \frac{\partial T}{\partial S^2} = -\frac{1}{2T^2} \cdot \left( \frac{\lambda^2 \theta^2}{4} \right) = \left( -\frac{4\lambda^2}{2} \right) \cdot \left( \frac{\lambda^2 \theta^2}{4} \right) = -\frac{\lambda^4 \theta^2}{2}. \]

We secondly provide the analogous derivations of the derivatives of the estimate \( \hat{\theta} \) and we have
\[
\frac{\partial \hat{\theta}}{\partial \bar{X}} = \frac{d\hat{\theta}}{d\lambda} \cdot \frac{\partial \lambda}{\partial \bar{X}} = \left( \frac{d}{d\lambda} \left( \frac{2}{\lambda \bar{X}} \right) \right) \cdot \frac{\partial \lambda}{\partial \bar{X}} \\
= \left( \frac{d}{d\lambda} \left( \frac{2}{\lambda} \right) \right) \cdot \frac{\partial \lambda}{\partial \bar{X}} \cdot \bar{X}^{-1} + \left( \frac{2}{\lambda} \right) (-1) \bar{X}^{-2} \\
= -\frac{2}{\lambda^2} \cdot \frac{\partial \lambda}{\partial \bar{X}} \cdot \frac{1}{\bar{X}} - \frac{2}{\lambda \bar{X}^2} \\
= - \left[ \frac{2}{\lambda^2 \bar{X}} \cdot \frac{\partial \lambda}{\partial \bar{X}} + \frac{2}{\lambda \bar{X}^2} \right].
\]

Note that \( \frac{\partial \lambda}{\partial \bar{X}} (\mu, \sigma^2) = a_1 \), and therefore

\[
b_1 = \frac{\partial \hat{\theta}}{\partial \bar{X}} (\mu, \sigma^2) \\
= - \left[ \frac{2}{\lambda^2 \left( \frac{2}{\lambda \theta} \right)} \cdot \lambda^2 \theta + \frac{2}{\lambda \left( \frac{2}{\lambda \theta} \right)^2} \right] \\
= - \left[ \lambda \theta^2 + \lambda \theta^2 \right] \\
= - \frac{3}{2} \lambda \theta^2.
\]

In order to investigate \( b_2 \) we consider

\[
\frac{\partial \hat{\theta}}{\partial S^2} = \frac{d\hat{\theta}}{d\lambda} \cdot \frac{\partial \lambda}{\partial S^2} = \left( \frac{d}{d\lambda} \left( \frac{2}{\lambda \bar{X}} \right) \right) \cdot \frac{\partial \lambda}{\partial S^2} \\
= \left( \frac{d}{d\lambda} \left( \frac{2}{\lambda} \right) \right) \cdot \frac{\partial \lambda}{\partial S^2} \cdot \bar{X}^{-1} \\
= -\frac{2}{\lambda^2 \bar{X}} \cdot \frac{\partial \lambda}{\partial S^2}.
\]
Note that \( \frac{\partial \hat{\lambda}}{\partial S^2}(\mu, \sigma^2) = a_2 \), and hence

\[
b_2 = \frac{\partial \hat{\theta}}{\partial S^2}(\mu, \sigma^2)
\]

\[
= -\frac{2}{\lambda^2 \left( \frac{n}{\lambda} \right)} \cdot \left( -\frac{\lambda^4 \theta^2}{2} \right)
\]

\[
= \left( \frac{2}{\lambda^2} \right) \cdot \left( \frac{\lambda \theta}{2} \right) \cdot \left( \frac{\lambda^4 \theta^2}{2} \right)
\]

\[
= \frac{\lambda^3 \theta^3}{2}.
\]

Then, we substitute \( a_1, a_2, b_1, b_2, \text{Var}(\bar{X}), \text{Var}(S^2) \) and \( \text{Cov}(\bar{X}, S^2) \) into the equations (4.11), (4.12) and (4.13) to obtain \( \text{Var}(\hat{\lambda}), \text{Var}(\hat{\theta}) \) and \( \text{Cov}(\hat{\lambda}, \hat{\theta}) \).

Hence,

\[
\text{Var}(\hat{\lambda}) = (\lambda^2 \theta)^2 \left( \frac{2}{\lambda^3 \theta^2 n} \right) + \left( -\frac{\lambda^4 \theta^2}{2} \right)^2 \left( \frac{8 \lambda n + 30n - 30}{n(n-1)\lambda^7 \theta^4} \right) + 2 \left( \frac{\lambda^2 \theta}{2} \right) \left( \frac{n-1}{n^2} \right) \left( \frac{6}{\lambda^5 \theta^3} \right)
\]

\[
= \frac{2\lambda}{n} + \frac{4\lambda^2 n + 15\lambda n - 15\lambda}{2n(n-1)} - \frac{(n-1)6\lambda}{n^2}
\]

\[
= \frac{4\lambda^2 n^2 + 7\lambda n^2 + 5\lambda n - 12\lambda}{2n^2(n-1)}.
\]
Thus,

\[
Var(\hat{\theta}) = \left( -\frac{3\lambda \theta^2}{2} \right)^2 \left( \frac{2}{\lambda^3 \theta^2 n} \right) + \left( \frac{\lambda^3 \theta^3}{2} \right)^2 \left( \frac{8\lambda n + 30n - 30}{n(n-1)\lambda^7 \theta^4} \right)
\]

\[
+ 2 \left( -\frac{3\lambda \theta^2}{2} \right) \left( \frac{\lambda^3 \theta^3}{2} \right) \left( \frac{6(n-1)}{n^2 \lambda^5 \theta^4} \right)
\]

\[
= \frac{9\theta^2}{2\lambda n} + \frac{4\lambda \theta^2 n + 15\theta^2 n - 15\theta^2}{2\lambda n(n-1)} + \frac{18\theta^2 (n-1)}{2\lambda n^2}
\]

\[
= \frac{(2\lambda + 3)\theta^2 n^2 + 6\theta^2 n - 9\theta^2}{\lambda n^2(n-1)}.
\]

Therefore,

\[
Cov(\hat{\lambda}, \hat{\theta}) = (\lambda^2 \theta) \left( -\frac{3\lambda \theta^2}{2} \right) \left( \frac{2}{\lambda^3 \theta^2 n} \right) + \left( -\frac{\lambda^4 \theta^2}{2} \right) \left( \frac{\lambda^3 \theta^3}{2} \right) \left( \frac{8\lambda n + 30n - 30}{n(n-1)\lambda^7 \theta^4} \right)
\]

\[
+ 2 \left\{ (\lambda^2 \theta) \left( \frac{\lambda^3 \theta^3}{2} \right) + \left( -\frac{\lambda^4 \theta^2}{2} \right) \left( -\frac{3\lambda \theta^2}{2} \right) \right\} \left( \frac{6(n-1)}{n^2 \lambda^5 \theta^4} \right)
\]

\[
= -\frac{3\theta}{n} - \frac{4\lambda \theta n^2 + 15\theta n - 15\theta}{2n(n-1)} + \frac{15\theta(n-1)}{n^2}
\]

\[
= -\frac{(4\lambda \theta + 18\theta)n^2 - 15\theta n - 3\theta}{2n^2(n-1)}
\]
CHAPTER 5

RESULTS OF COMPUTATIONAL PART

5.1 TS-LBIG-Random Number Generation Procedure

In this section, we develop the two-sided length biased inverse Gaussian random number generation procedure using the connection between common distributions. The related topics are presented as follows.

5.1.1 Connection Between LBIG and IG Distributions

**Proposition 3.** If a random variable $Y$ has $IG(\lambda, \theta)$ distribution, a random variable $Z$ has $\chi^2(1)$ distribution (Chi-squared distribution with one degree of freedom), the random variables $Y$ and $Z$ are independent, then the random variable $X = Y + \theta Z$ has $LBIG(\lambda, \theta)$ distribution.

**Proof.**

It is well known that the characteristic function of $\chi^2(n)$ is $(1 - 2it)^{-n/2}$. Hence, the characteristic function of the random variable $\theta Z$ is $\varphi_{\theta Z}(t) = (1 - 2\theta it)^{-1/2}$.

By equation (2.16), we know that the characteristic function of $IG(\lambda, \theta)$ distribution is given by $\varphi_Y(t) = \exp\left\{\lambda \left[1 - (1 - 2\theta it)^{1/2}\right]\right\}$. Because of independent property, the characteristic function of $X = Y + \theta Z$ is defined by

$$\varphi_X(t) = \varphi_{Y+\theta Z}(t) = \varphi_Y(t) \cdot \varphi_{\theta Z}(t)$$

$$= \exp\left\{\lambda \left[1 - (1 - 2\theta it)^{1/2}\right]\right\} (1 - 2\theta it)^{-1/2}$$

$$= \varphi_{LBIG}(t; \lambda, \theta)$$

\[\square\]
Consequently, in order to generate the LBIG-random number, the IG-random number generation procedure is necessarily needed.

5.1.2 Connection Between IG and Chi-squared Distributions

The following proposition is related to the results of Chhikara and Folks (1989) and Shuster (1968), where it is provided with different parametrizations and without proof.

**Proposition 4.** If a random variable $\tau$ is inverse Gaussian distributed with parameters $\lambda$ and $\theta$ or $\tau \sim IG(\lambda, \theta)$, then $\frac{(\tau - \lambda \theta)^2}{\theta \tau}$ has a chi-squared distribution with one degree of freedom.

**Proof.**

Let a random variable $Y = \frac{(\tau - \lambda \theta)^2}{\theta \tau}$. The distribution function of $Y$ is

$$F_Y(t) = P(Y \leq t) = P \left( \frac{(\tau - \lambda \theta)^2}{\theta \tau} \leq t \right) = P \left\{ \tau^2 - \theta (2\lambda + t) \tau + \lambda^2 \theta^2 \leq 0 \right\} = P(u_1(t) \leq \tau \leq u_2(t)) ,$$

where

$$u_1(t) = \lambda \theta + \frac{\theta}{2} \left[ t - \sqrt{t^2 + 4\lambda t} \right] , \quad u_2(t) = \lambda \theta + \frac{\theta}{2} \left[ t + \sqrt{t^2 + 4\lambda t} \right]$$

are the solutions of the quadratic equation $u^2 - \theta (2\lambda + t) u + \lambda^2 \theta^2 = 0$. Therefore, $F_Y(t) = F_\tau(u_2) - F_\tau(u_1)$ and the density function is $f_Y(t) = f_\tau(u_2) \cdot u_2(t) - f_\tau(u_1) \cdot u_1(t)$. Before we substitute $u_1$ and $u_2$ into the IG density function, note that $u_1$ and $u_2$ are the solutions of the quadratic equation $u^2 - \theta (2\lambda + t) u + \lambda^2 \theta^2 = 0$, we obtain $u_1 + u_2 = \theta (2\lambda + t)$ and $u_1 \cdot u_2 = \lambda^2 \theta^2$. Then,

$$\left( u_1^{1/2} - u_2^{1/2} \right)^2 = u_1 + u_2 - 2 (u_1 u_2)^{1/2} = \theta t.$$
Also note that \( \frac{u_2 - u_1}{\theta} = \sqrt{t^2 + 4\lambda t} \) and \( \frac{u_2 + u_1}{\theta} = 2\lambda + t \). By using these informations, we obtain

\[
u_1'(t) = \frac{\theta}{2} \left( 1 - \frac{2\lambda + t}{\sqrt{t^2 + 4\lambda t}} \right) = \frac{\theta}{2} \left( 1 - \frac{u_2 + u_1}{u_2 - u_1} \right) = \frac{\theta u_1}{u_2 - u_1},
\]

and

\[
u_2'(t) = \frac{\theta}{2} \left( 1 - \frac{2\lambda + t}{\sqrt{t^2 + 4\lambda t}} \right) = \frac{\theta}{2} \left( 1 + \frac{u_2 + u_1}{u_2 - u_1} \right) = \frac{\theta u_2}{u_2 - u_1}.
\]

Furthermore,

\[
\frac{(u_1 - \lambda \theta)^2}{u_1 \theta} = \frac{(u_2 - \lambda \theta)^2}{u_2 \theta} = t.
\]

Recall that

\[
\frac{(u_1 - \lambda \theta)^2}{u_1 \theta} = t
\]

is equivalent to

\[
u_1^2 - \theta (2\lambda + t) u_1 + \lambda^2 \theta^2 = 0,
\]

which is obviously true. Therefore,

\[
\exp \left\{-\frac{(u_1 - \lambda \theta)^2}{2u_1 \theta}\right\} = \exp \left\{-\frac{(u_2 - \lambda \theta)^2}{2u_2 \theta}\right\} = \exp \left\{-\frac{t}{2}\right\}.
\]

Collecting all these facts, we get

\[
f_Y(t) = f_r(u_2)u_2' - f_r(u_1)u_1'
\]

\[
= \frac{\lambda \theta^{1/2}}{\sqrt{2\pi}} u_2^{-3/2} \exp \left\{-\frac{(u_2 - \lambda \theta)^2}{2u_2 \theta}\right\} \cdot \frac{\theta u_2}{u_2 - u_1}
\]

\[
+ \frac{\lambda \theta^{1/2}}{\sqrt{2\pi}} u_1^{-3/2} \exp \left\{-\frac{(u_1 - \lambda \theta)^2}{2u_1 \theta}\right\} \cdot \frac{\theta u_1}{u_2 - u_1}
\]

\[
= \frac{\lambda \theta^{3/2} \exp \left\{-t/2\right\}}{\sqrt{2\pi}} \cdot \frac{u_2^{-1/2} + u_1^{-1/2}}{u_2 - u_1}
\]
\[ f_Y(t) = \frac{\lambda^{3/2} \exp\{-t/2\}}{\sqrt{2\pi}} \cdot \frac{u_2^{1/2} + u_1^{1/2}}{(u_2 u_1)^{1/2} \left(u_2^{1/2} + u_1^{1/2}\right) \left(u_2^{1/2} - u_1^{1/2}\right)} \]

\[ = \frac{\lambda^{3/2} \exp\{-t/2\}}{\lambda \theta \sqrt{2\pi}} \cdot \frac{1}{(\theta t)^{1/2}} \]

\[ = \frac{1}{\sqrt{2\pi}} t^{-1/2} \exp\{-t/2\}, \]

which is a density function of the chi-squared distribution with one degree of freedom.

\[ \square \]

### 5.1.3 IG-Random Number Generation Procedure

Observations drawn from the chi-squared distribution are easily generated by the squares of standard normal random variables. For each chi-squared variate \( t \), we have to solve the equation \( g(x) = \frac{(x - \lambda \theta)^2}{\theta x} = t \) for \( x \) to obtain a corresponding observation from the inverse Gaussian distribution. For any \( t > 0 \), there are exactly two roots, i.e., \( u_1 = \lambda \theta + \frac{\theta}{2} \left[t - \sqrt{t^2 + 4\lambda t}\right] \) and \( u_2 = \frac{\lambda^2 \theta^2}{u_1} \) (see the proof of Proposition 4) of the associated quadratic equation. The difficulty for generating observations with \( IG(\lambda, \theta) \) distribution lies in choosing between the two roots. We now follow an argument given in Michael, Achucany, and Haas (1976) for our different parametrization.

We consider the interval \((t - h, t + h)\), where \( h > 0 \). Referring to the inverse function theorem, for \( h \) sufficiently small, the inverse image \( g^{-1} \) of the interval \((t - h, t + h)\) is comprised of two disjoint intervals about the roots \( u_1 \) and \( u_2 \). Let the intervals containing the roots, \( u_1 \) and \( u_2 \), be denoted by \((\nu_{11}, \nu_{12})\) and \((\nu_{21}, \nu_{22})\), respectively. If \( p_1(t) \) represents the probability with which an observation should be chosen from the first interval \((\nu_{11}, \nu_{12})\) given that \( Y \) is in the interval \((t - h, t + h)\), then the conditional probability is defined by
\[ p_1^h(t) = P(\nu_{11} < X < \nu_{12} | t - h < Y < t + h) \]

\[ = \frac{P(\nu_{11} < X < \nu_{12} \text{ and } t - h < Y < t + h)}{P(t - h < Y < t + h)} \]

\[ = \frac{P(\nu_{11} < X < \nu_{12})}{P(\nu_{11} < X < \nu_{12}) + P(\nu_{21} < X < \nu_{22})} \]

\[ = \frac{F_{IG}(\nu_{12}) - F_{IG}(\nu_{11})}{F_{IG}(\nu_{12}) - F_{IG}(\nu_{11}) + F_{IG}(\nu_{22}) - F_{IG}(\nu_{21})} \]

\[ = \left( 1 + \frac{F_{IG}(\nu_{22}) - F_{IG}(\nu_{21})}{F_{IG}(\nu_{12}) - F_{IG}(\nu_{11})} \right)^{-1}. \]

Note that \( \lim_{h \to 0} (t - h, t + h) = t, \lim_{h \to 0} (\nu_{11}, \nu_{12}) = u_1, \) and \( p_1(t) = \lim_{h \to 0} p_1^h(t), \) we will yield the conditional probability with which the first root \( u_1 \) should be selected.

Therefore,

\[ p_1(t) = \lim_{h \to 0} p_1^h(t) \]

\[ = \left( 1 + \lim_{h \to 0} \frac{F_{IG}(\nu_{22}) - F_{IG}(\nu_{21})}{F_{IG}(\nu_{12}) - F_{IG}(\nu_{11})} \right)^{-1} \]

\[ = \left( 1 + \lim_{h \to 0} \frac{(\nu_{22} - \nu_{21})/h}{(\nu_{12} - \nu_{11})/h} \cdot \left( \frac{F_{IG}(\nu_{22}) - F_{IG}(\nu_{21})}{F_{IG}(\nu_{12}) - F_{IG}(\nu_{11})} \right) \right)^{-1} \]

\[ = \left( 1 + \frac{|g'(u_1)|}{g'(u_2)} \cdot \frac{f_{IG}(u_2)}{f_{IG}(u_1)} \right)^{-1}. \]

The absolute value in the expression \( \frac{|g'(u_1)|}{g'(u_2)} \) exists since \( \frac{\nu_{22} - \nu_{21}}{\nu_{12} - \nu_{11}} > 0 \) always.
Note that $g'(x) = \frac{x^2 - \lambda^2 \theta^2}{\theta x^2}$, and using the relations $u_1 u_2 = \lambda^2 \theta^2$ and
\[ \exp \left\{ -\frac{(u_2 - \lambda \theta)^2}{2u_2 \theta} \right\} = \exp \left\{ -\frac{(u_1 - \lambda \theta)^2}{2u_1 \theta} \right\} \]
provided in the proof of Proposition 4, we obtain
\[ \frac{g'(u_1)}{g'(u_2)} = \frac{(u_2^2 - \lambda^2 \theta^2) u_1^2}{(u_1^2 - \lambda^2 \theta^2) u_2^2} = \frac{u_1 (u_1 - u_2) u_2^2}{u_2 (u_2 - u_1) u_1^2} = -\frac{u_2}{u_1} = -\frac{\lambda^2 \theta^2}{u_1^2} \]
and
\[ \frac{f_{IG}(u_2)}{f_{IG}(u_1)} = \frac{u_2^{-3/2}}{u_1^{-3/2}} = \frac{u_1^3}{\lambda^3 \theta^3}. \]
Hence, the smaller root $u_1$ should be selected with probability $p_1(t) = \frac{\lambda \theta}{\lambda \theta + u_1}$ and the larger root $u_2 = \lambda^2 \theta^2 / u_1$ should be selected with probability $p_2(t) = 1 - p_1(t) = \frac{u_1}{\lambda \theta + u_1}$. Based on the previous results, finally, we obtain the procedure to generate an $IG(\lambda, \theta)$ random numbers. The following steps of an $IG(\lambda, \theta)$ random number generator procedure are given below.

1. Generate a uniform $[0, 1]$ random number $v$ and an independently standard normal number $z$.
2. Calculate $u = \lambda \theta + \frac{\theta}{2} \left[ z^2 - \sqrt{z^4 + 4 \lambda z^2} \right]$.
3. If $v < \frac{\lambda \theta}{(\lambda \theta + u)}$, then take $IG = u$, otherwise $IG = \frac{\lambda^2 \theta^2}{u}$.

### 5.1.4 Random One-Sided and Two-Sided LBIG-Numbers Generation Procedure

Using the connection between LBIG and IG distributions, we yield $LBIG(\lambda, \theta)$ random numbers generator procedure as follows.

1. Generate a uniform $[0, 1]$ random number $v$ and independently two standard normal numbers $z_1$ and $z_2$.
2. Calculate $u = \lambda \theta + \frac{\theta}{2} \left[ z_1^2 - \sqrt{z_1^4 + 4 \lambda z_1^2} \right]$.
3. If $v < \frac{\lambda \theta}{(\lambda \theta + u)}$, then take $IG = u$, otherwise $IG = \frac{\lambda^2 \theta^2}{u}$.
4. Take $LBIG = IG + \theta \cdot z^2$. 

For two-sided LBIG distribution, after we obtain a pair of one-sided LBIG random numbers $\tau_1$ and $\tau_2$, we compute $X = \tau_1^{-1} + \tau_2^{-1}$, then we get the Two-Sided LBIG random numbers.

5.2 TS-LBIG-Random Number Generator Selection

Two random number generators are compared: “Procedure” and “Package”. The “Procedure” generator is the proposed procedure provided in section 5.1 for generating the TS-LBIG random variable. The “Package” generator is the procedure created by the package called “statmod” in R. It is commonly used to generate IG random numbers for usual parameters ($\mu$ and $\beta$). We used the relation between IG and LBIG distributions to obtain $TS-LBIG(\lambda, \theta)$ random number (see section 3.2). Tables 5.1-5.12 show the comparison of the two generators via considering the absolute difference-percentage between the first three population and sample raw moments (ADPM). The ADPM formula is given as follows:

$$ADPM = \left| \frac{\mu'_k - m'_k}{\mu'_k} \right| \times 100, \ k = 1, 2, 3,$$

where $\mu'_k$ and $m'_k$ are population and sample raw moments, accordingly. The ADPM for the 1st - 3rd raw moments are presented in Tables 5.1-5.12 when $n = 10, 50, 100, 500$, respectively. The underlined numbers indicates that the generator performs better than another one when we consider in each row or each situation. In other words, if the value of the ADPM of the specified generator is smaller than that of another generator in considered row, it is implied that the specified generator performs better than another one.

The simulation results suggest that our proposed generator (Procedure) is generally better than the “Package” generator. It is noticed that the “Package” generator performs better when sample sizes $n$ increases and the exponent of parameter increases. Hence, we choose the “Procedure” generator to be the generator for evaluating the suggested estimators.
Table 5.1: Absolute Difference-Percentage of the 1st Raw Moment for $n = 10$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\theta$</th>
<th>Number of Iterations</th>
<th>$\mu_1'$</th>
<th>ADPM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Package</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1,000</td>
<td>1.0000</td>
<td>26.5188</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5,000</td>
<td>1.0000</td>
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Table 5.6: Absolute Difference-Percentage of the 2\textsuperscript{nd} Raw Moment for \( n = 50 \).

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Table 5.12: Absolute Difference-Percentage of the 3rd Raw Moment for $n = 500$.

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5.3 The Desirable Properties of the Estimators

We consider the properties of the method of moment estimators ($\hat{\lambda}$ and $\hat{\theta}$) based on values of variance, coefficient of variation (C.V.), bias and mean square error (MSE) from the simulation studies. The simulation results are summarized as follows.

5.3.1 Minimum Variances and Coefficient of Variations

Tables 5.13-5.17 present the variances and coefficient of variations of the proposed estimators $\hat{\lambda}$ and $\hat{\theta}$ for $n = 10, 50, 100, 200, 500$, respectively. Importantly, the numerical study reveals that the method of moment estimators for estimating the parameters of the TS-LBIG distribution are asymptotically consistent since variance and coefficient of variation are decreasing functions of sample sizes $n$. In other words, when sample sizes are sufficiently large, variance and coefficient of variation are very small. Additionally, the more increasing values of the parameters, the more growing are the variance and coefficient of variation. In particular where $n = 10$, the magnitude of variance and coefficient of variation is quite big. However, the variance and coefficient of variation slightly decrease when $n = 50$. They rapidly decrease when $n = 500$. Obviously, when sample sizes are small, the variance and coefficient of variation for $\lambda$ are larger than those for $\theta$. However, the variance and coefficient of variation of both $\theta$ and $\lambda$ look similar when sample sizes increase. In some situations, the variance of $\hat{\theta}$ are very larger than that of $\hat{\lambda}$. For example, when $n = 50$, the values of $\lambda = 2$ and $\theta = 50$; the values of $\lambda = 5$ and $\theta = 50$; and the values of $\lambda = 10$ and $\theta = 50$, the corresponding variances are 157.2717, 125.8491 and 114.6885, respectively. On the other hand, when we considered the corresponding coefficient of variation of $\hat{\theta}$ comparing to $\hat{\lambda}$, we found that both estimators gave similar performance. This is the reason why we use coefficient of variation together with variance for appraisal the proposed estimators? The coefficient of variation gives additional information a part from the variance. Most importantly, as sample sizes increase, the magnitude of the variance and coefficient of variation of the two estimators decreases and approach
to zero as \( n \to \infty \). This result corresponds to the theoretical background since the variance and coefficient of variation are decreasing functions of sample sizes.

Table 5.13: Simulated Variances and C.V. of the Estimators for \( n = 10 \).

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Table 5.14: Simulated Variances and C.V. of the Estimators for \( n = 50 \).

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Table 5.16: Simulated Variances and C.V. of the Estimators for \( n = 200 \).

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Table 5.17: Simulated Variances and C.V. of the Estimators for \( n = 500 \).

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5.3.2 Asymptotically Unbiased Estimators

Tables 5.18-5.22 show the biases of the proposed estimators $\hat{\lambda}$ and $\hat{\theta}$. It is observed that the simulated biases of $\hat{\lambda}$ are negative for all situations, so it is an overestimate, i.e., the $\hat{\lambda}$ gives the estimated value greater than the true value. In contrast, those of $\hat{\theta}$ are systematically positive, then it is an underestimate, i.e., the $\hat{\theta}$ offers the estimated value less than the true value. When sample sizes are relatively small, the amount of bias is large particularly when the true value of at least one parameter is sufficiently big. For instance for $\hat{\lambda}$, when $n = 10$, the values of $\lambda = 50$ and $\theta = 1$; and the values of $\lambda = 50$ and $\theta = 50$, the biases are $-14.0534$ and $-14.9923$, respectively. Nevertheless, when sample sizes increase, the bias systematically decreases. Obviously, when sample sizes are small, the biases of $\lambda$ are larger than those of $\theta$. However, the biases for both $\theta$ and $\lambda$ have similar behaviors when sample sizes are sufficiently large. Most importantly, as sample sizes increase, the magnitude of the bias for the two estimators decreases and tends to zero as $n \to \infty$. This result corresponds to the theoretical background since the bias is a decreasing function of sample sizes $n$. It is implied that both estimators ($\hat{\lambda}$ and $\hat{\theta}$) are consistent. Therefore, the method of moment estimators for estimating the parameters of the two-sided length biased inverse Gaussian distribution are asymptotically unbiased.
Table 5.18: Simulated Biases of the Estimators for $n = 10$.

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Table 5.21: Simulated Biases of the Estimators for $n = 200$.  

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Table 5.22: Simulated Biases of the Estimators for $n = 500$.

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Tables 5.23-5.27 report the mean square error of the proposed estimators $\hat{\lambda}$ and $\hat{\theta}$. As seen in the numerical results, it indicates that the method of moment estimators for estimating the parameters of the two-sided length biased inverse Gaussian distribution are asymptotically consistent since the mean square error is a decreasing function of sample sizes $n$. In other words, when sample sizes are sufficiently big especially where $n = 500$, the mean square error is very small. Furthermore, the magnitude of mean square error increases as values of the parameters increase, particularly when $n = 10$. When sample sizes decrease, the MSE seriously increase. However, the mean square error slightly decreases when $n = 50$. It rapidly decrease when $n = 500$. It is noticed that when sample sizes are small, the mean square error of $\hat{\lambda}$ are bigger than that of $\hat{\theta}$. However, the mean square errors for both estimators are comparable when sample sizes are adequately large. Most importantly, as sample sizes increase, the magnitude of the mean square error for the two estimators decreases and tends to zero as $n \to \infty$. Correspondingly, it is implied that the proposed estimators are desirable.
Table 5.23: Simulated Mean Square Errors of the Estimators for $n = 10$.

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Table 5.24: Simulated Mean Square Errors of the Estimators for $n = 50$.

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Table 5.25: Simulated Mean Square Errors of the Estimators for $n = 100$.

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Table 5.26: Simulated Mean Square Errors of the Estimators for $n = 200$.

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Table 5.27: Simulated Mean Square Errors of the Estimators for \( n = 500 \).

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5.4 Histograms of Errors

The selected histograms of errors (difference between true and estimated values of the parameter) are illustrated for each combination of $\lambda$, $\theta$ and $n$.

Figure 5.1: Histogram of the Error for $\lambda$ when $n = 500$, $\lambda = 2$ and $\theta = 1$.

Figure 5.1 shows the histogram of error $(\lambda - \hat{\lambda})$ when $n = 500$, $\lambda = 2$ and $\theta = 1$. It is noticed that the histogram is symmetric and is under the bell curve. That is, the error for $\lambda$ follows a normal distribution. It is implied that $\hat{\lambda}$ is suitable for estimating the true value of parameter.
Figure 5.2: Histogram of the Error for $\theta$ when $n = 500$, $\lambda = 2$ and $\theta = 1$.

Figure 5.2 presents the histogram of error $(\theta - \hat{\theta})$ when $n = 500$, $\lambda = 2$ and $\theta = 1$. It is observed that the histogram is symmetrical and is under the bell curve. It is implied that the error for $\theta$ follows a normal distribution. There are no more outliers or extreme values of the error. It is indicated that $\hat{\theta}$ is suitable for estimating the true value of parameter.
Figure 5.3: Histogram of the Error for $\lambda$ when $n = 500$, $\lambda = 2$ and $\theta = 50$.

Figure 5.3 displays the histogram of error ($\lambda - \hat{\lambda}$) when $n = 500$, $\lambda = 2$ and $\theta = 50$. The result indicates that the histogram is symmetric and is under the bell curve. Therefore, the error for $\lambda$ follows a normal distribution. There are no more outliers or extreme values of the error. It is suggested that $\hat{\lambda}$ is suitable for estimating the true value of parameter.
Figure 5.4 illustrates the histogram of error ($\theta - \hat{\theta}$) when $n = 500$, $\lambda = 2$ and $\theta = 50$. It is noticed that the histogram is symmetric and is under the bell curve. Accordingly, the error for $\theta$ follows a normal distribution. The error for $\theta$ gives no more outliers or extreme values. It is implied that $\hat{\theta}$ is suitable for estimating the true value of parameter.
Figure 5.5 reports the histogram of error $(\lambda - \hat{\lambda})$ when $n = 500$, $\lambda = 10$ and $\theta = 5$. Obviously, the histogram is symmetrical and is under the bell curve. The error for $\lambda$ follows a normal distribution. The error for $\lambda$ gives no more outliers or extreme values. Correspondingly, it is implied that $\hat{\lambda}$ is suitable for estimating the true value of parameter.
Figure 5.6: Histogram of the Error for $\theta$ when $n = 500$, $\lambda = 10$ and $\theta = 5$.

Figure 5.6 shows the histogram of error $(\theta - \hat{\theta})$ when $n = 500$, $\lambda = 10$ and $\theta = 5$. It is noticed that the histogram is symmetric and is under the bell curve. The error for $\theta$ follows a normal distribution. There are no more outliers or extreme values of the error. It is indicated that $\hat{\theta}$ is suitable for estimating the true value of parameter.
Figure 5.7: Histogram of the Error for $\lambda$ when $n = 500$, $\lambda = 20$ and $\theta = 1$. It is noticed that the histogram is symmetric and is under the bell curve. The error for $\lambda$ follows a normal distribution. There are no more outliers or extreme values of the error. Hence, $\hat{\lambda}$ is suitable for estimating the true value of parameter.
Figure 5.8 displays the histogram of error $(\theta - \hat{\theta})$ when $n = 500$, $\lambda = 20$ and $\theta = 1$. It is observed that the histogram is symmetric and is under the bell curve. That is, the error for $\theta$ follows a normal distribution. The error for $\theta$ gives no more outliers or extreme values. Consequently, $\hat{\theta}$ is suitable for estimating the true value of parameter.
Figure 5.9: Histogram of the Error for $\lambda$ when $n = 500$, $\lambda = 50$ and $\theta = 50$.

Figure 5.9 illustrates the histogram of error $(\lambda - \hat{\lambda})$ when $n = 500$, $\lambda = 50$ and $\theta = 50$. It is observed that the histogram is symmetrical and is under the bell curve, i.e., the error for $\lambda$ follows a normal distribution. The error for $\lambda$ offers no more outliers or extreme values. Thus, $\hat{\theta}$ is suitable for estimating the true value of parameter.
Figure 5.10 shows the histogram of error $(\theta - \hat{\theta})$ when $n = 500$, $\lambda = 50$ and $\theta = 50$. It is noticed that the histogram is symmetric and is under the bell curve. The error for $\theta$ follows a normal distribution. There are no more outliers or extreme values of the error. Hence, $\hat{\theta}$ is suitable for estimating the true value of parameter.
5.5 Asymptotic Analysis of the Estimates by the Method of Moment

The results of asymptotic analysis of the estimates by the method of moment for the two-sided length biased inverse Gaussian distribution are reported in the Tables 5.28-5.32. The numerical results show that the asymptotic variances for $\hat{\lambda}$ and $\hat{\theta}$ are decreasing functions as sample sizes increase. Generally, we see that the asymptotic variances are bigger than the simulated variances. When sample sizes are small, the asymptotic variances are such larger than the simulated variances especially when $n = 10$. However, the asymptotic and simulated variances are comparable when sample sizes $n$ increase in particular when $n = 500$. Although the simulated and asymptotic variances of $\hat{\lambda}$ still large, those of $\hat{\theta}$ are desirable. More precisely, it is necessarily needed the sample sizes increasing to $n = 1,000$. 
Table 5.28: Asymptotic Analysis Results for $n = 10$.

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<td>$\lambda$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.0760</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.0737</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.0724</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.0747</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.3412</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.3344</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.3512</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.3366</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1.1887</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.2533</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.1825</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>1.2051</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>4.5012</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4.4879</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>4.6000</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>26.5713</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>26.6369</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>27.1059</td>
</tr>
</tbody>
</table>
Table 5.32: Asymptotic Analysis Results for $n = 500$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Simulated</th>
<th>Asymptotic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\theta$</td>
<td>$\text{Var}(\hat{\lambda})$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.0299</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.0290</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.0292</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.0295</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.1370</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.1356</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.1374</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.1350</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0.4792</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.4796</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.4833</td>
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<tr>
<td></td>
<td>50</td>
<td>0.4791</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>1.8315</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.8183</td>
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<tr>
<td></td>
<td>10</td>
<td>1.7991</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>1.7843</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>10.4708</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>10.5334</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>10.4651</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>10.7711</td>
</tr>
</tbody>
</table>
5.6 Illustrative Examples

The practical applications of the suggested estimation strategy are illustrated in this section. The method of moment point estimates were computed. Moreover, using the relations given in equation (2.11), the point estimates for original parameters, \( \mu \) and \( \beta \), were also computed. The two real examples based on published data are considered and the results are reported as the followings tables.

Table 5.33: Point Estimates for Example 1.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Proposed Parameters</th>
<th>Usual Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \lambda )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Proposed MME</td>
<td>1.5648</td>
<td>34.1267</td>
</tr>
<tr>
<td>Lisawadi’s MME</td>
<td>1.2579</td>
<td>76.2042</td>
</tr>
</tbody>
</table>

Table 5.33 shows point estimates by method of moment of proposed strategy and Lisawadi’s method, used for the two-sided IG distribution. It is noticed that our point estimate for \( \lambda \) is slightly different from the estimate by Lisawadi (2009). Conversely, our estimate for \( \theta \) is far from that of Lisawadi’s method. It seems to be approximately two times of the estimate by Lisawadi (2009). Similarly, our point estimates for usual parameters are far from the Lisawadi’s estimates. They are around 1.5 times of Lisawadi’s strategy. However, we now consider the proposed parameters \( \lambda \) and \( \theta \).
Table 5.34: Point Estimates for Example 2.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Proposed Parameters</th>
<th>Usual Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>Proposed MME</td>
<td>1.3736</td>
<td>1.5579</td>
</tr>
<tr>
<td>Lisawadi’s MME</td>
<td>1.0452</td>
<td>4.0061</td>
</tr>
</tbody>
</table>

Table 5.34 illustrates point estimates by method of moment of proposed strategy and Lisawadi’s method, used for the two-sided IG distribution. It is observed that our point estimate for $\lambda$ is slightly different from the estimate by Lisawadi (2009). In contrast, our estimate for $\theta$ is far from that of Lisawadi’s method. It seems to be approximately 2.5 times of the estimate by Lisawadi (2009). Similarly, our point estimates for usual parameters are far from the Lisawadi’s estimates. They are nearly two times of Lisawadi’s strategy. However, we now consider the proposed parameters $\lambda$ and $\theta$.

Interestingly, the proposed point estimate of $\lambda$ is comparable with that of Lisawadi (2009). Nevertheless, the estimate for $\theta$ seems to be approximately two times of Lisawadi’s estimate. More importantly, the estimators $\hat{\lambda}$ and $\hat{\theta}$ stand for the thickness of a specified object and the nominal treatment pressure on the object, respectively. On the other hand, the estimators $\hat{\mu}$ and $\hat{\beta}$ do not offer this physical explanation.
CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

In this chapter, conclusions of this dissertation will be presented in two parts corresponding to the research objectives. Finally, recommendations for further research are given.

6.1 Theoretical Part

The purposes of this research in this part are to introduce the new lifetime distribution based on non-classical parametrization under the situation when a crack develops from two sides, to investigate some statistical properties of the proposed distribution, and to apply the traditional point estimation, method of moment, for estimating the parameters of the distribution. Concluding remark and some general comments are summarized as follows.

The new distribution called the two-sided length biased inverse Gaussian distribution with two parameters $\lambda$ and $\theta$ is proposed. It is shortly denoted as $TS-LBIG(\lambda, \theta)$ distribution. The parameter $\lambda$ stands for a thickness of an object under consideration and $\theta$ corresponds to a nominal treatment pressure on the object. Importantly, the original shape and scale parameters ($\mu$ and $\beta$) do not give these physical characteristics. The probabilistic model of the $TS-LBIG(\lambda, \theta)$ distribution consisting of its probability density function (p.d.f.) and cumulative distribution function (c.d.f.) is given. However, the density function has no explicit form or closed form, i.e., it involves an integral sign. Consequently, it may be difficult to directly find important functions such as a characteristic function and a moment generating function. Reciprocal properties involving with derivation
of statistical properties and parameter estimation method for the $TS-LBIG(\lambda, \theta)$ distribution are presented. The first four cumulants or semi-invariants of the distribution are investigated by applying the Maclaurin series expansion (the Taylor series expansion evaluated at zero) since the direct derivation is difficult. The characteristic function of $IG(\lambda, \theta)$ is necessarily needed. The first four cumulants are defined from the Maclaurin expansion of the logarithm of the characteristic function. There are two important points we should concern: the remainder of the Maclaurin expansion and the existing condition of the Maclaurin extended terms. In the first point, our goal is investigation of the first four cumulants since the first four terms involve important characteristics, i.e., mean, variance, skewness, and kurtosis of the distribution, then the fifth and higher exponents are considered to be trivial. That is, we sum the fifth and higher powers in the term $O(x^5)$. In the second point, the first four cumulants for the $TS-LBIG(\lambda, \theta)$ distribution exist when the condition $|t| < \frac{1}{2\theta}$ is satisfied. It only depends on the parameter $\theta$. Subsequently, the corresponding moments of the new distribution are also provided using similar manner for finding the cumulants. The first three terms of moments are the same as those of cumulants and the existing condition is also the same, i.e., $|t| < \frac{1}{2\theta}$. The first four raw moments are obtained from the relation between the raw moments and the cumulants.

In parameter estimation scheme, we only focus on a point estimation. The conventional point estimation, method of moment, is developed to estimate the parameters of the proposed distribution. We equate the first two moments previously obtained and the corresponding sample moments. Eventually, we gain the desired estimators $\hat{\lambda}$ and $\hat{\theta}$. General approach to asymptotic analysis of the estimate by the method of moment is provided by applying the delta method also called the Taylor series approximation. Then, the asymptotic variances and covariance of the suggested estimators are given.
6.2 Computational Part

The goal of this part is to appraise the performance of the proposed estimators of the two-sided length biased inverse Gaussian distribution. In order to obtain numerical information about the properties of the point estimators of interest, Monte-Carlo simulations were performed. The conclusions of this study are summarized as follows.

In order to achieve our goal, two-sided length biased inverse Gaussian random numbers were generated from two generators. The first generator is called “Procedure”. We propose the two-sided length biased inverse Gaussian random number generation procedure using the connections between common distributions. The connection between LBIG and IG distributions and the connection between Chi-squared and IG distribution are reported. The second generator is called “Package”. We apply the package called `statmod` in R statistical package. It is commonly used to generate IG random number for usual parametrization. Recall that the length biased inverted Gaussian distribution is the same as that of the reciprocal of IG variate ($\mu^2/X$). We use this relation to generate LBIG random number for non-classical parametrization. The criterion of the generator selection is an absolute difference-percentage between the first three population and sample raw moments (ADPM). The simulation results suggest that our proposed generator generally performs better than the “Package” generator. It is noticed that the “Package” generator performs better when sample sizes $n$ increases and the exponent of functions of parameters increases. Hence, we choose the “Procedure” generator to be the generator for evaluating the suggested estimators. The results are shown in Table 6.1.

To assess the performance of the presented estimators Monte-Carlo simulations were operated for small, moderate and large sample sizes. A finite sampling experiment is used to handle our goal. We have considered the sets of parameters and sample sizes as follows: $\lambda = 2, 5, 10, 20, 50$, $\theta = 1, 5, 10, 50$, and $n = 10, 50, 100, 200, 500$. The statistical R package is used for calculation and data analyses. The number of simulations was fixed at 5,000 for each combination of
\(\lambda, \theta,\) and \(n\). The criteria for evaluating the performance of the method of moment estimates obtained from simulation studies are minimum variance, coefficient of variation, bias and mean square error.

Table 6.1: TS-LBIG Random Number Generator Selection

<table>
<thead>
<tr>
<th>Moment</th>
<th>(n)</th>
<th>Generator Selection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Package</td>
</tr>
<tr>
<td>1\textsuperscript{st}</td>
<td>10</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>✓</td>
</tr>
<tr>
<td>2\textsuperscript{nd}</td>
<td>10</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>✓</td>
</tr>
<tr>
<td>3\textsuperscript{rd}</td>
<td>10</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>✓</td>
</tr>
</tbody>
</table>

✓ indicates that it is better.

As the results starting from section 5.3, the numerical study reveals that the method of moment estimators for estimating the parameters of the two-sided length biased inverse Gaussian distribution are asymptotically consistent and unbiased since variance, coefficient of variation, bias and mean square error are decreasing function of sample sizes \(n\). In other words, when sample sizes increase, variance, coefficient of variation, bias and mean square error decrease. Correspondingly, the result of asymptotic analysis of the estimates by the method of moment indicates that the asymptotic variances decrease as \(n\) increases. These results correspond to the theoretical background. Importantly, we estimate the
parameters which reflect the physical nature of an object analyzed by a statistical point of view of the distribution.

6.3 Recommendations

According to the previous results, because the density function of the $TS-LBIG(\lambda, \theta)$ distribution has no closed form, the characteristics and moment generating functions can not be solved directly. We may use the reciprocal property to accomplish this problem. One can continue investigations for other point estimation schemes such as maximum likelihood estimator, although we expect to encounter mathematical derivation difficulties. As previously stated, the density function of the $TS-LBIG(\lambda, \theta)$ distribution involves an integral sign and finding a maximum of their products is not an easy task. Furthermore, we may consider new estimators such as a regression-quantile (least square). This method is dealt with the regression analysis of sample quantiles. However, the method of moment estimation can be examined as a preliminary topic of studying the $TS-LBIG(\lambda, \theta)$ distribution since it has a satisfying property, e.g. simple computation. Interval estimation and hypothesis testing issues remain to be interesting for further investigation. As presented results of the asymptotic analysis, tests and confident interval estimation procedures will be explored regarding on power of the test and coverage probabilities. Nevertheless, they are above the scope of this research.
REFERENCES


APPENDICES
Appendix A1

R Source Code for Calculating an Absolute Difference-Percentage of the 1st Moment by Using Package

#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)
library ( statmod )
para.est <- function (n,lambda,theta,M)
{
  mu <- lambda*theta
  beta <- (lambda^2)*theta
  mu1 <- 2/(lambda*theta)
  x1 <- rep(0,n)
  x2 <- rep(0,n)
  tau1 <- rep(0,n)
  tau2 <- rep(0,n)
  z <- rep(0,n)
  z.bar <- rep(0,M)
  z.s.sq <- rep(0,M)
  T <- rep(0,M)
  lambda.hat <- rep(0,M)
  theta.hat <- rep(0,M)

  for(p in 1:M)
  {
    for (j in 1:n) {
      x1[j] <- rinvgauss(1,mu,beta)
      tau1[j] <- mu^2/x1[j]
      }
  }
for (j in 1:n) {
  x2[j] <- rinvgauss(1,mu,beta)
  tau2[j] <- mu^2/x2[j]
}
for (j in 1:n) {
  z[j] <- 1/tau[j] + 1/tau2[j]
}

z.bar[p] <- mean(z)
z.s.sq[p] <- var(z)
T[p] <- z.s.sq[p]/(z.bar[p]^2)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])

#End loop M
}

lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)
m1 <- 2/(lambda.n.hat*theta.n.hat)
ADPM <- abs((mu1 - m1)/mu1)*100

cat("\t-------------------------------------------------------------",'
')
cat("\t For sample sizes(n) =",n," , lambda =",lambda," ,
')
cat("\t The 1st population moment =",mu1," ,",'
')
cat("\t The 1st sample moment =", m1," ,",'
')
cat("\t The ADPM for 1st moment =",ADPM," ,",'
')
cat("\t-------------------------------------------------------------",'
')
}
Appendix A2

R Source Code for Calculating an Absolute Difference-Percentage of the 2nd Moment by Using Package

#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)
library (statmod)
para.est <- function (n,lambda,theta,M) {
  mu <- lambda*theta
  beta <- (lambda^2)*theta
  mu2 <- 2*(1 + 2*lambda)/(lambda^3)*(theta^2)
  x1 <- rep(0,n)
  x2 <- rep(0,n)
  tau1 <- rep(0,n)
  tau2 <- rep(0,n)
  z <- rep(0,n)
  z.bar <- rep(0,M)
  z.s.sq <- rep(0,M)
  T <- rep(0,M)
  lambda.hat <- rep(0,M)
  theta.hat <- rep(0,M)

  for(p in 1: M) {
    for (j in 1:n) {
      x1[j] <- rinvgauss(1,mu,beta)
      tau1[j] <- mu^2/x1[j]
for (j in 1:n) {
  x2[j] <- rinvgauss(1,mu,beta)
  tau2[j] <- mu^2/x2[j]
}

for (j in 1:n) {
  z[j] <- 1/tau1[j] + 1/tau2[j]
}

z.bar[p] <- mean(z)
z.s.sq[p] <- var(z)
T[p] <- z.s.sq[p]/(z.bar[p]^2)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])

#End loop M

lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)
m2 <- 2*(1 + 2*lambda.n.hat)/(lambda.n.hat^3)*(theta.n.hat^2)
ADPM <- abs((mu2 - m2)/mu2)*100

cat("\t------------------------------------------------------\t\n")
cat("\t For sample sizes(n) =",n,"\t\n")
  theta =",theta, "and M =",M, "\t\n")
cat("\t The 2nd population moment =",mu2,"\t\n")
cat("\t The 2nd sample moment =", m2,"\t\n")
cat("\t The ADPM for 2nd moment =",ADPM,"\t\n")
cat("\t------------------------------------------------------\t\n")}
Appendix A3

R Source Code for Calculating an Absolute Difference-Percentage of the 3rd Moment by Using Package

```r
#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)
library ( statmod )
para.est <- function (n,lambda,theta,M)
{
  mu <- lambda*theta
  beta <- (lambda^2)*theta
  mu3 <- 2*(4*lambda^2 + 6*lambda + 3)/(lambda^5)*(theta^3)
  x1 <- rep(0,n)
  x2 <- rep(0,n)
  tau1 <- rep(0,n)
  tau2 <- rep(0,n)
  z <- rep(0,n)
  z.bar <- rep(0,M)
  z.s.sq <- rep(0,M)
  T <- rep(0,M)
  lambda.hat <- rep(0,M)
  theta.hat <- rep(0,M)

  for(p in 1: M)
  {

    for (j in 1:n) {
      x1[j] <- rinvgauss(1,mu,beta)
      tau1[j] <- mu^2/x1[j]
    }

    for (j in 1:n) {
      x2[j] <- rinvgauss(1,mu,beta)
      tau2[j] <- mu^2/x2[j]
    }

  }
}
```
for (j in 1:n) {
    x2[j] <- rinvgauss(1,mu,beta)
    tau2[j] <- mu^2/x2[j]
}
for (j in 1:n) {
    z[j] <- 1/tau1[j] + 1/tau2[j]
}
z.bar[p] <- mean(z)
z.s.sq[p] <- var(z)
T[p] <- z.s.sq[p]/(z.bar[p]^2)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])

#End loop M
}
lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)
m3 <- 2*(4*lambda.n.hat^2 +6*lambda.n.hat +3)/
    (lambda.n.hat^5)*(theta.n.hat^3)
ADPM <- abs((mu3 - m3)/mu3)*100

cat("\t------------------------------------------------------",'\n')
cat("\t For sample sizes(n) =",n",lambda =",lambda,",
    theta =",theta, "and M =",M, ":","\n")
cat("\t The 3rd population moment =",mu3,"",'\n")
cat("\t The 3rd sample moment =", m3,"",'\n")
cat("\t The ADPM for 3rd moment =",ADPM,"",'\n")
cat("\t------------------------------------------------------",'\n")
}
Appendix B1

R Source Code for Calculating an Absolute Difference-Percentage of the 1st Moment by Using Proposed Procedure

#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)
para.est <- function (n,lambda,theta,M)
{
  mu1 <- 2/(lambda*theta)
  v1 <- rep(0,n)
  v2 <- rep(0,n)
  z11 <- rep(0,n)
  z12 <- rep(0,n)
  z21 <- rep(0,n)
  z22 <- rep(0,n)
  u1 <- rep(0,n)
  u2 <- rep(0,n)
  c1 <- rep(0,n)
  c2 <- rep(0,n)
  IG1 <- rep(0,n)
  IG2 <- rep(0,n)
  tau1 <- rep(0,n)
  tau2 <- rep(0,n)
  z <- rep(0,n)
  z.bar <- rep(0,M)
  z.s.sq <- rep(0,M)
  T <- rep(0,M)
  lambda.hat <- rep(0,M)
  theta.hat <- rep(0,M)
bias.lambda <- rep(0,M)
bias.theta <- rep(0,M)

for(p in 1: M)
{
    #create the LBIG random variable tua1
    for (j in 1:n) {
        v1[j] <- runif(1,0,1)
z11[j] <- rnorm(1,0,1)
z12[j] <- rnorm(1,0,1)
u1[j] <- (lambda*theta) + (theta/2)*(z11[j]^2 -
            sqrt(z11[j]^4 + (4*lambda*z11[j]^2)))
c1[j] <- (lambda*theta)/(lambda*theta + u1[j])
if(v1[j] < c1[j]) {IG1[j] <- u1[j]}
if(v1[j] >= c1[j]) {IG1[j] <- (lambda^2)*(theta^2)/u1[j]}
tau1[j] <- IG1[j] + theta*(z12[j]^2)
}

#create the LBIG random variable tua2
for (j in 1:n) {
    v2[j] <- runif(1,0,1)
z21[j] <- rnorm(1,0,1)
z22[j] <- rnorm(1,0,1)
u2[j] <- (lambda*theta) + (theta/2)*(z21[j]^2 -
            sqrt(z21[j]^4 + (4*lambda*z21[j]^2)))
c2[j] <- (lambda*theta)/(lambda*theta + u2[j])
if(v2[j] < c2[j]) {IG2[j] <- u2[j]}
if(v2[j] >= c2[j]) {IG2[j] <- (lambda^2)*(theta^2)/u2[j]}
tau2[j] <- IG2[j] + theta*(z22[j]^2)
}
# create the two-sided LBIG random variable Z
for (j in 1:n) {
  z[j] <- (1/tau1[j]) + (1/tau2[j])
}

z.bar[p] <- mean(z)
zs.sq[p] <- var(z)
T[p] <- z.s.sq[p]/(z.bar[p]^2)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])

#End loop M
}

lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)
m1 <- 2/(lambda.n.hat*theta.n.hat)
ADPM <- abs((mu1 - m1)/mu1)*100

cat("\t------------------------------------------------------",'
')
cat("\t For sample sizes(n) =",n"," ,lambda =",lambda," ,
cat("\t The 1st population moment =",mu1," ,",',\n')
cat("\t The ADPM for 1st moment =",ADPM," ,",',\n')
cat("\t------------------------------------------------------",',\n')
}
Appendix B2

R Source Code for Calculating an Absolute Difference-Percentage of the 2nd Moment by Using Proposed Procedure

#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)

para.est <- function (n,lambda,theta,M)
{
  mu2 <- 2*(1 + 2*lambda)/(lambda^3)*(theta^2)
  v1 <- rep(0,n)
  v2 <- rep(0,n)
  z11 <- rep(0,n)
  z12 <- rep(0,n)
  z21 <- rep(0,n)
  z22 <- rep(0,n)
  u1 <- rep(0,n)
  u2 <- rep(0,n)
  c1 <- rep(0,n)
  c2 <- rep(0,n)
  IG1 <- rep(0,n)
  IG2 <- rep(0,n)
  tau1 <- rep(0,n)
  tau2 <- rep(0,n)
  z <- rep(0,n)
  z.bar <- rep(0,M)
  z.s.sq <- rep(0,M)
  T <- rep(0,M)
  lambda.hat <- rep(0,M)
  theta.hat <- rep(0,M)
bias.lambda <- rep(0,M)
bias.theta <- rep(0,M)

for(p in 1: M) {
  #create the LBIG random variable tua1
  for (j in 1:n) {
    v1[j] <- runif(1,0,1)
z11[j] <- rnorm(1,0,1)
z12[j] <- rnorm(1,0,1)
u1[j] <- (lambda*theta) + (theta/2)*(z11[j]^2 -
    sqrt(z11[j]^4 + (4*lambda*z11[j]^2)))
c1[j] <- (lambda*theta)/(lambda*theta + u1[j])
    if(v1[j] < c1[j]) {IG1[j] <- u1[j]}
    if(v1[j] >= c1[j]) {IG1[j] <- (lambda^2)*(theta^2)/u1[j]}
tau1[j] <- IG1[j] + theta*(z12[j]^2)
  }

  #create the LBIG random variable tua2
  for (j in 1:n) {
    v2[j] <- runif(1,0,1)
z21[j] <- rnorm(1,0,1)
z22[j] <- rnorm(1,0,1)
u2[j] <- (lambda*theta) + (theta/2)*(z21[j]^2 -
    sqrt(z21[j]^4 + (4*lambda*z21[j]^2)))
c2[j] <- (lambda*theta)/(lambda*theta + u2[j])
    if(v2[j] < c2[j]) {IG2[j] <- u2[j]}
    if(v2[j] >= c2[j]) {IG2[j] <- (lambda^2)*(theta^2)/u2[j]}
tau2[j] <- IG2[j] + theta*(z22[j]^2)}
#create the two-sided LBIG random variable Z
for (j in 1:n) {
  z[j] <- (1/tau1[j]) + (1/tau2[j])
}

z.bar[p] <- mean(z)
zs.sq[p] <- var(z)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])

#End loop M

lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)
m2 <- 2*(1 + 2*lambda.n.hat) / (lambda.n.hat^3)*(theta.n.hat^2)
ADPM <- abs((mu2 - m2)/mu2)*100

cat("\t------------------------------------------------------",'\n')
cat("\t For sample sizes(n) =",n,"\n"
    theta =",theta, "and M =",M, ":",',\n"')
cat("\t The 1st population moment =",mu2,"\n")
cat("\t The ADPM for 2nd moment =",ADPM,"\n")
cat("\t------------------------------------------------------",',\n")}
Appendix B3

R Source Code for Calculating an Absolute Difference-Percentage of the 3rd Moment by Using Proposed Procedure

#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)
para.est <- function (n,lambda,theta,M)
    para.est <- function (n,lambda,theta,M)
    {
        mu3 <- 2*(4*lambda^2 +6*lambda +3)/(lambda^5)*(theta^3)
        v1 <- rep(0,n)
        v2 <- rep(0,n)
        z11 <- rep(0,n)
        z12 <- rep(0,n)
        z21 <- rep(0,n)
        z22 <- rep(0,n)
        u1 <- rep(0,n)
        u2 <- rep(0,n)
        c1 <- rep(0,n)
        c2 <- rep(0,n)
        IG1 <- rep(0,n)
        IG2 <- rep(0,n)
        tau1 <- rep(0,n)
        tau2 <- rep(0,n)
        z <- rep(0,n)
        z.bar <- rep(0,M)
        z.s.sq <- rep(0,M)
        T <- rep(0,M)
        lambda.hat <- rep(0,M)
\[
\theta_{\hat{\cdot}} \leftarrow \text{rep}(0, M)
\]
\[
\text{bias.lambda} \leftarrow \text{rep}(0, M)
\]
\[
\text{bias.theta} \leftarrow \text{rep}(0, M)
\]

\[
\text{for}(p \text{ in } 1: M) \{
\]
\hspace{1em} \# create the LBIG random variable tua1
\hspace{1em} \text{for } (j \text{ in } 1:n) \{
\hspace{2em} v1[j] \leftarrow \text{runif}(1, 0, 1)
\hspace{2em} z11[j] \leftarrow \text{rnorm}(1, 0, 1)
\hspace{2em} z12[j] \leftarrow \text{rnorm}(1, 0, 1)
\hspace{2em} u1[j] \leftarrow (\lambda \theta) + (\theta/2)(z11[j]^2 - \\
\hspace{6em} \text{sqrt}(z11[j]^4 + (4\lambda z11[j]^2)))
\hspace{2em} c1[j] \leftarrow (\lambda \theta)/(\lambda \theta + u1[j])
\hspace{2em} \text{if}(v1[j] < c1[j]) \{ IG1[j] \leftarrow u1[j] \}
\hspace{2em} \text{if}(v1[j] >= c1[j]) \{ IG1[j] \leftarrow (\lambda^2)(\theta^2)/u1[j] \}
\hspace{2em} \tau1[j] \leftarrow IG1[j] + \theta(z12[j]^2)
\hspace{1em} \}
\]
\hspace{1em} \# create the LBIG random variable tua2
\hspace{1em} \text{for } (j \text{ in } 1:n) \{
\hspace{2em} v2[j] \leftarrow \text{runif}(1, 0, 1)
\hspace{2em} z21[j] \leftarrow \text{rnorm}(1, 0, 1)
\hspace{2em} z22[j] \leftarrow \text{rnorm}(1, 0, 1)
\hspace{2em} u2[j] \leftarrow (\lambda \theta) + (\theta/2)(z21[j]^2 - \\
\hspace{6em} \text{sqrt}(z21[j]^4 + (4\lambda z21[j]^2)))
\hspace{2em} c2[j] \leftarrow (\lambda \theta)/(\lambda \theta + u2[j])
\hspace{2em} \text{if}(v2[j] < c2[j]) \{ IG2[j] \leftarrow u2[j] \}
\hspace{2em} \text{if}(v2[j] >= c2[j]) \{ IG2[j] \leftarrow (\lambda^2)(\theta^2)/u2[j] \}
\hspace{2em} \tau2[j] \leftarrow IG2[j] + \theta(z22[j]^2)
\hspace{1em} \}
\]
# create the two-sided LBIG random variable Z
for (j in 1:n) {
  z[j] <- (1/tau1[j]) + (1/tau2[j])
}

z.bar[p] <- mean(z)
z.s.sq[p] <- var(z)
T[p] <- z.s.sq[p]/(z.bar[p]^2)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])

#End loop M
}

lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)
m3 <- 2*(4*lambda.n.hat^2 +6*lambda.n.hat +3)/
     (lambda.n.hat^5)*(theta.n.hat^3)
ADPM <- abs((mu3 - m3)/mu3)*100

cat("\t------------------------------------------------------","\n")
cat("\t For sample sizes(n) =" ,n," ,lambda =" ,lambda," ,
       theta =" ,theta," and M =" ,M," ,":","\n")
cat("\t The 1st population moment =" ,mu3," ," ,\n")
cat("\t The ADPM for 3rd moment =" ,ADPM," ," ,\n")
cat("\t------------------------------------------------------","\n")
}
Appendix C

R Source Code for Evaluating the Performance of the Proposed Estimators

#n represents sample size
#lambda represents parameter
#theta represents parameter
#M represents the number of simulations (M = 5,000)
para.est <- function(n,lambda,theta,M)
{
  v1 <- rep(0,n)
  v2 <- rep(0,n)
  z11 <- rep(0,n)
  z12 <- rep(0,n)
  z21 <- rep(0,n)
  z22 <- rep(0,n)
  u1 <- rep(0,n)
  u2 <- rep(0,n)
  c1 <- rep(0,n)
  c2 <- rep(0,n)
  IG1 <- rep(0,n)
  IG2 <- rep(0,n)
  ta1 <- rep(0,n)
  ta2 <- rep(0,n)
  z <- rep(0,n)
  z.bar <- rep(0,M)
  z.s.sq <- rep(0,M)
  T <- rep(0,M)
  lambda.hat <- rep(0,M)
  theta.hat <- rep(0,M)
  bias.lambda <- rep(0,M)
}
bias.theta <- rep(0,M)

for(p in 1: M)
{
    #create the LBIG random variable tua1
    for (j in 1:n) {
        v1[j] <- runif(1,0,1)
        z11[j] <- rnorm(1,0,1)
        z12[j] <- rnorm(1,0,1)
        u1[j] <- (lambda*theta) + (theta/2)*(z11[j]^2 - sqrt(z11[j]^4 + (4*lambda*z11[j]^2)))
        c1[j] <- (lambda*theta)/(lambda*theta + u1[j])
        if(v1[j] < c1[j]) {IG1[j] <- u1[j]}
        if(v1[j] >= c1[j]) {IG1[j] <- (lambda^2)*(theta^2)/u1[j]}
        tau1[j] <- IG1[j] + theta*(z12[j]^2)
    }
    #create the LBIG random variable tua2
    for (j in 1:n) {
        v2[j] <- runif(1,0,1)
        z21[j] <- rnorm(1,0,1)
        z22[j] <- rnorm(1,0,1)
        u2[j] <- (lambda*theta) + (theta/2)*(z21[j]^2 - sqrt(z21[j]^4 + (4*lambda*z21[j]^2)))
        c2[j] <- (lambda*theta)/(lambda*theta + u2[j])
        if(v2[j] < c2[j]) {IG2[j] <- u2[j]}
        if(v2[j] >= c2[j]) {IG2[j] <- (lambda^2)*(theta^2)/u2[j]}
        tau2[j] <- IG2[j] + theta*(z22[j]^2)
    }
}
#create the two-sided LBIG random variable Z
for (j in 1:n) {
  z[j] <- (1/tau1[j]) + (1/tau2[j])
}

z.bar[p] <- mean(z)
z.s.sq[p] <- var(z)
T[p] <- z.s.sq[p]/(z.bar[p]^2)
lambda.hat[p] <- 1/(2*T[p])
theta.hat[p] <- 2/(lambda.hat[p]*z.bar[p])
bias.lambda[p] <- lambda - lambda.hat[p]
bias.theta[p] <- theta - theta.hat[p]

#End loop M
}

lambda.n.hat <- mean(lambda.hat)
theta.n.hat <- mean(theta.hat)

aver.bias.lambda <- lambda - lambda.n.hat
aver.bias.theta <- theta - theta.n.hat

var.bias.lambda <- var(lambda.hat)
var.bias.theta <- var(theta.hat)

aver.cv.lambda <- sd(lambda.hat)*100/lambda.n.hat
aver.cv.theta <- sd(theta.hat)*100/theta.n.hat

mse.lambda <- var.bias.lambda + (aver.bias.lambda)^2
mse.theta <- var.bias.theta + (aver.bias.theta)^2
cat("t-------------------------------------------------------",'\n')
cat("tFor sample sizes(n) ="n",lambda =",lambda,"and 
    theta =",theta, ":",'\n ')
cat("t\tlambda hat =",lambda.n.hat,"",'\n ') 
cat("t\ttheta hat =",theta.n.hat,"",'\n ') 
cat("t\tbias for lambda hat =",aver.bias.lambda,"",'\n') 
cat("t\tbias for theta hat =",aver.bias.theta, ",",'\n') 
cat("t\tvvariance for lambda hat =",var.bias.lambda,"",'\n') 
cat("t\tvvariance for theta hat =",var.bias.theta,"",'\n') 
cat("t\tC.V. for lambda hat =",aver.cv.lambda,"",'\n') 
cat("t\tC.V. for theta hat =",aver.cv.theta,"",'\n') 
cat("t\tMSE for lambda hat =",mse.lambda,"",'\n') 
cat("t\tMSE for theta hat =",mse.theta,"",'\n') 
cat("t-------------------------------------------------------",'\n')

hist(bias.lambda)
hist(bias.theta)

library(xlsx)
mydata <- data.frame(lambda, theta, lambda.n.hat, theta.n.hat, 
    aver.bias.lambda, aver.bias.theta, var.bias.lambda, 
    var.bias.theta, aver.cv.lambda, aver.cv.theta, 
    mse.lambda, mse.theta, lambda.hat, theta.hat, 
    bias.lambda, bias.theta)
write.xlsx(mydata, "d:/2pp.data.n10.12.t1.xlsx", row.names = FALSE)
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Publications
