THREE INVERSE EIGENVALUE PROBLEMS FOR SYMMETRIC DOUBLY ARROW MATRICES

BY

MISS WANWISA PENGUDOM

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS
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THAMMASAT UNIVERSITY
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ENTITED

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ABSTRACT

In this thesis, we consider the inverse eigenvalue problems for constructing real symmetric doubly arrow matrices from spectral information. Three types of spectral information are used in this work: the minimal and maximal eigenvalues of all leading principal submatrices, one of the eigenpairs and eigenvalues of all leading principal submatrices, and the two eigenpairs. The necessary and sufficient conditions for the existence of a solution of the problems are derived. Moreover, the special cases for some problems are also discussed, that is, the nonnegative case and the uniqueness case. Examples to illustrate our results are also presented.

Keywords: Inverse eigenvalue problem, Doubly arrow matices, Symmetric matrix.

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Miss Wanwisa Pengudom

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

An inverse eigenvalue problem involves the reconstruction of a matrix from prescribed spectral information. The spectral information concerned may consist of complete or only partial information on eigenvalues or eigenvectors. The objective of an inverse eigenvalue problem is to construct a matrix that maintains a certain specific structure as well as the given spectral properties. Inverse eigenvalue problems arise in many applications, for example, inverse Sturm-Liouville problem, vibration analysis, control theory, and particle physics (see [1] – [8]).

Chu and Golub [6] classified inverse eigenvalue problems into seven types as follows.

1. Multivariate Inverse Eigenvalue Problems
2. Least Squares Inverse Eigenvalue Problems
3. Parameterized Inverse Eigenvalue Problems
4. Structured Inverse Eigenvalue Problems
5. Partially Described Inverse Eigenvalue Problems
6. Additive Inverse Eigenvalue Problems
7. Multiplicative Inverse Eigenvalue Problems

We are interested in structured inverse eigenvalue problems in this work. The objective for this type is to construct a matrix of a certain form, see [9] – [16].

Peng et al. [12] introduced two inverse eigenvalue problems for constructing a real symmetric bordered diagonal matrix (also called a real symmetric arrow matrix) from the minimal and maximal eigenvalues of all its leading principal submatrices, and from one of the eigenpairs and eigenvalues of all its leading principal submatrices. Later in 2007, Pickmann et al. [13] made a correction in one of the problems stated in [12]. In 2006, Wang et al. [14] introduced an inverse eigenvalue problem for constructing a real symmetric five-diagonal matrix from its three eigenpairs.

In this thesis, we are interested in constructing real symmetric doubly arrow matrices from certain spectral information. To achieve our objective, we considered the following problems.

Problem 1. Given the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for all $j = 1, 2, \ldots, n$, find a real symmetric doubly arrow matrix such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalue of the $j \times j$ leading principal submatrix of such a matrix for $j = 1, 2, \ldots, n$, respectively.

Problem 2. Given the real numbers $\lambda^{(j)}$ for all $j = 1, 2, \ldots, n$ and the real vector $\mathbf{x} = [x_1, x_2 \ldots, x_n]^T$, find a real symmetric doubly arrow matrix such that $\lambda^{(j)}$ is an eigenvalue of the $j \times j$ leading principal submatrix of such a matrix for $j = 1, 2, \ldots, n-1$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair.

Problem 3. Given two real numbers λ, μ and two nonzero real vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, $\mathbf{y} = [y_1, y_2 \ldots, y_n]^T$, find a real symmetric doubly arrow matrix such that $(\lambda, \mathbf{x})$ and $(\mu, \mathbf{y})$ are the two eigenpairs.

In addition, two special cases for Problem 1 and Problem 2 are also discussed, these are the nonnegative case and the uniqueness case.

For all three problems, we consider an inverse eigenvalue problem for constructing a real symmetric doubly arrow matrix of the form:

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & b_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{s-1} & b_{s-1} & 0 & \cdots & 0 & 0 \\ b_1 & b_2 & \cdots & b_{s-1} & a_s & 0 & \cdots & 0 & b_s \\ 0 & 0 & \cdots & 0 & 0 & a_{s+1} & \cdots & 0 & b_{s+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & \cdots & 0 & b_s & b_{s+1} & \cdots & b_{n-1} & a_n \end{pmatrix}, \tag{1.1}$$

where $a_j, b_j \in \mathbb{R}$, and $1 \leq s \leq n$.

Furthermore, we are interested in constructing other real symmetric doubly arrow matrices in Problem 3. There are of the form

$$\mathbf{B} = \begin{pmatrix} a_1 & \cdots & 0 & b_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{s-1} & b_{s-1} & 0 & \cdots & 0 \\ b_1 & \cdots & b_{s-1} & a_s & b_s & \cdots & b_{n-1} \\ 0 & \cdots & 0 & b_s & a_{s+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & 0 & \cdots & a_n \end{pmatrix}, \tag{1.2}$$

where $a_j, b_j \in \mathbb{R}$, $1 \leq s \leq n$, and

$$\mathbf{C} = \begin{pmatrix} a_1 & b_1 & \cdots & b_{s-1} & 0 & \cdots & 0 \\ b_1 & a_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{s-1} & 0 & \cdots & a_s & b_s & \cdots & b_{n-1} \\ 0 & 0 & \cdots & b_s & a_{s+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 & \cdots & a_n \end{pmatrix}, \tag{1.3}$$

where $a_j, b_j \in \mathbb{R}$, and $1 \leq s \leq n$. The doubly arrow matrices B and C were described by Pickmann et al. in [15] and Liu et al. in [16], respectively.

In Chapter 2, we give some properties for solving our problems. The necessary and sufficient conditions for the existence of a solutions and some examples of Problem 1, Problem 2, and Problem 3 are illustrated in Chapter 3, Chapter 4, and Chapter 5, respectively. Finally, Chapter 6 gives the conclusions and possible extensions of the work.

CHAPTER 2

PRELIMINARIES

Let **A** be a matrix of the form (1.1). The $j \times j$ leading principal submatrix A_j of matrix $A = A_n$ is obtained by deleting the last p rows and p columns of matrix A, where $p = n - j$. Notice that if $j = 1, 2, \ldots, s-1$, then the leading principal submatrix A_j is a diagonal matrix.

Lemma 2.1. *Let* **A** *be a matrix of the form (1.1). Then the sequence of characteristic polynomials* $\{P_j(\lambda)\}_{j=1}^{n}$ *satisfies the recurrence relation:*

$$P_j(\lambda) = \prod_{i=1}^{j}(\lambda - a_i)\ ; \qquad j = 1, 2, \ldots, s-1, \tag{2.1}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - \sum_{k=1}^{j-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{j-1}(\lambda - a_i)\ ; \qquad j = s, \tag{2.2}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda)\ ; \qquad j = s+1, s+2, \ldots, n-1, \tag{2.3}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - \sum_{k=s}^{j-1} b_k^2 P_{k-1}(\lambda) \prod_{i=k+1}^{j-1}(\lambda - a_i)\ ; \qquad j = n, \tag{2.4}$$

where $P_0(\lambda) = 1$.

Proof. This is easy to verify by expanding the determinants $\det(\lambda \mathbf{I}_j - \mathbf{A}_j)$ for $j = 1, 2, \ldots, n$, where $\mathbf{I}_j$ is the identity matrix of identical dimensions. □

Lemma 2.2. *Let* $P(\lambda)$ *be a monic polynomial of degree n with all real zeros. If* λ_1 *and* λ_n *are, respectively, the minimal and maximal zeros of* $P(\lambda)$*, then*

1. *If* $\beta < \lambda_1$*, we have* $(-1)^n P(\beta) > 0$.
2. *If* $\beta > \lambda_n$*, we have* $P(\beta) > 0$.

Proof. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$. If $\beta < \lambda_1$ and n is even, then $P(\beta) > 0$. If $\beta < \lambda_1$ and n is odd, then $P(\beta) < 0$, and hence $(-1)^n P(\beta) > 0$. If $\beta > \lambda_n$, then clearly $P(\beta) > 0$. □

The following lemma explains the establishing relations between the eigenvalues of a symmetric matrix and the eigenvalues of its principal submatrices, that is, the Cauchy interlacing property.

Lemma 2.3. *Let* $\mathbf{K}_n$ *be a real symmetric matrix of order n with eigenvalues* $\beta_1^{(n)} \leq \beta_2^{(n)} \leq \ldots \leq \beta_n^{(n)}$. *Let* $\mathbf{K}_{n-1}$ *be the principal submatrix of* $\mathbf{K}_n$ *with eigenvalue* $\beta_1^{(n-1)} \leq \beta_2^{(n-1)} \leq \ldots \leq \beta_{n-1}^{(n-1)}$. *Then*

$$\beta_1^{(n)} \leq \beta_1^{(n-1)} \leq \beta_2^{(n)} \leq \cdots \leq \beta_{n-1}^{(n)} \leq \beta_{n-1}^{(n-1)} \leq \beta_n^{(n)}.$$

Proof. See [17]. □

Observe that, by Lemma 2.3, if $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and the maximal eigenvalues of the leading principal submatrix A_j of matrix A for $j = 1, 2, \ldots, n$, respectively, then

$$\lambda_1^{(n)} \leq \lambda_1^{(n-1)} \leq \cdots \leq \lambda_1^{(3)} \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \lambda_3^{(3)} \leq \cdots \leq \lambda_{n-1}^{(n-1)} \leq \lambda_n^{(n)}. \quad (2.5)$$

CHAPTER 3

SOLUTION TO PROBLEM 1

The main result of Problem 1 is the construction of a doubly arrow matrix A of the form (1.1) with $1 \leq s \leq n$. If a matrix A has the form (1.1), then we have the general form of the characteristic polynomials $P_j(\lambda)$ for $j = 1, 2, \ldots, n$ as in Lemma 2.1. All results are constructive in the sense that they generate algorithmic procedures to compute the elements of the leading principal submatrix A_j for $j = 1, 2, \ldots, n$ of matrix $A = A_n$. The following theorem gives a sufficient condition for solving Problem 1.

Theorem 3.1. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1, 2, \ldots, n$ be given. Then there exists an $n \times n$ matrix A of the form (1.1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of matrix A for $j = 1, 2, \ldots, n$, respectively, if the following conditions are satisfied:*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} = \cdots = \lambda_1^{(s)} < \lambda_1^{(s-1)} = \cdots = \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}, \tag{3.1}$$

there exist real solutions a_s and b_k, $k = 1, 2, \ldots, s-1$ for the system of equations

$$\left(\lambda_j^{(s)} - a_s\right) P_{s-1}\left(\lambda_j^{(s)}\right) - \sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_j^{(s)} - \lambda_i^{(i)}\right) = 0\,, \quad j = 1, s, \tag{3.2}$$

and there exist real solutions a_n and b_k, $k = s, s+1, \ldots, n-1$ for the system of equations

$$\left(\lambda_j^{(n)} - a_n\right) P_{n-1}\left(\lambda_j^{(n)}\right) - \sum_{k=s}^{n-1} b_k^2 P_{k-1}\left(\lambda_j^{(n)}\right) \prod_{i=k+1}^{n-1} \left(\lambda_j^{(n)} - \lambda_i^{(i)}\right) = 0\,, \quad j = 1, n. \tag{3.3}$$

Proof. Assume that the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for all $j = 1, 2, \ldots, n$ satisfy condition (3.1). Since the leading principal submatrix A_{s-1} is a diagonal matrix, $a_j = \lambda_j^{(j)}$ for $j = 1, 2, \ldots, s-1$. Therefore, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the leading principal submatrix $A_j = diag\left\{\lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}\right\}$ for $j = 1, 2, \ldots, s-1$, respectively.

To show the existence of a matrix A_j for $j = s, s+1, \ldots, n$ where $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues, respectively, is equivalent to showing that the system of equations

$$\left.\begin{aligned} P_j\left(\lambda_1^{(j)}\right) &= 0 \\ P_j\left(\lambda_j^{(j)}\right) &= 0 \end{aligned}\right\} \tag{3.4}$$

has real solutions a_j, $j = s, s+1, \ldots, n$ and b_{j-1}, $j = 1, 2, \ldots, n$.

For $j = s$, if condition (3.2) holds, the system of equations (3.4) has real solutions a_s and b_k, $k = 1, 2, \ldots, s-1$.

For $j = s+1, s+2, \ldots, n-1$, the system of equations (3.4) has the form:

$$\left.\begin{aligned} \left(\lambda_1^{(j)} - a_j\right) P_{j-1}\left(\lambda_1^{(j)}\right) &= 0 \\ \left(\lambda_j^{(j)} - a_j\right) P_{j-1}\left(\lambda_j^{(j)}\right) &= 0 \end{aligned}\right\}. \tag{3.5}$$

From condition (3.1) and Lemma 2.2(2), the system of equations (3.5) has the real solution

$$a_j = \lambda_j^{(j)}$$

for all $j = s+1, s+2, \ldots, n-1$.

For $j = n$, if condition (3.3) holds, then the system of equations (3.4) has real solutions a_n and b_k for $k = s, s+1, \ldots, n-1$. Then, there exists the matrix $A_n = A$.

The resulting matrix $A_n = A$ is now a real symmetric doubly arrow matrix with $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1, 2, \ldots, n$ satisfying condition (3.1). Hence, from the Cauchy interlacing property (Lemma 2.3) and relation (2.5) $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of matrix A for $j = 1, 2, \ldots, n$, respectively. $\square$

The following corollary is for the nonnegative case of Problem 1. That is, we give the sufficient condition for the existence of a matrix A of the form (1.1) with $a_j \geq 0$ and $b_{j-1} \geq 0$ for all $j = 1, 2, \ldots, n$.

Corollary 3.2. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1, 2, \ldots, n$ be given. Then there exists an $n \times n$ nonnegative matrix A of the form (1.1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of matrix A for $j = 1, 2, \ldots, n$, respectively, if the following conditions are satisfied:*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} = \cdots = \lambda_1^{(s)} < \lambda_1^{(s-1)} = \cdots = \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}, \tag{3.6}$$

$$\lambda_1^{(1)} \geq 0, \tag{3.7}$$

there exist nonnegative real solutions a_s and b_k, $k = 1, 2, \ldots, s-1$ for the system of equations

$$\left(\lambda_j^{(s)} - a_s\right) P_{s-1}\left(\lambda_j^{(s)}\right) - \sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_j^{(s)} - \lambda_i^{(i)}\right) = 0\,, \quad j = 1, s, \tag{3.8}$$

and there exist nonnegative real solutions a_n and b_k, $k = s, s+1, \ldots, n-1$ for the system of equations

$$\left(\lambda_j^{(n)} - a_n\right) P_{n-1}\left(\lambda_j^{(n)}\right) - \sum_{k=s}^{n-1} b_k^2 P_{k-1}\left(\lambda_j^{(n)}\right) \prod_{i=k+1}^{n-1} \left(\lambda_j^{(n)} - \lambda_i^{(i)}\right) = 0\,, \quad j = 1, n. \tag{3.9}$$

Proof. Suppose that the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for all $j = 1, 2, \ldots, n$, satisfy conditions (3.6) and (3.7). From conditions (3.6), (3.7), we have

$$a_j = \lambda_j^{(j)} \geq 0$$

for $j = 1, 2, \ldots, s-1$.

For $j = s$, if the condition (3.8) holds and the system of equations (3.4) in Theorem 3.1 has nonnegative real solutions a_s and b_k, $k = 1, 2, \ldots, s-1$, then there exists the nonnegative matrix A_s.

For $j = s+1, s+2, \ldots, n-1$, the system of equations (3.4) has the form:

$$\left.\begin{aligned} \left(\lambda_1^{(j)} - a_j\right) P_{j-1}\left(\lambda_1^{(j)}\right) &= 0 \\ \left(\lambda_j^{(j)} - a_j\right) P_{j-1}\left(\lambda_j^{(j)}\right) &= 0 \end{aligned}\right\} \tag{3.10}$$

From conditions (3.6) and (3.7), and Lemma 2.2(2), the system of equations (3.10) has the nonnegative real solution

$$a_j = \lambda_j^{(j)} \geq 0$$

for all $j = s+1, s+2, \ldots, n-1$.

For $j = n$, if condition (3.9) holds and the system of equations (3.4) in Therem 3.1 has nonnegative real solutions a_n and b_k for $k = s, s+1, \ldots, n-1$, then there exists the nonnegative matrix $A_n = A$.

Now matrix $A_n = A$ is a real symmetric doubly arrow matrix with $\lambda_1^{(n)}$ and $\lambda_n^{(n)}$ for $j = 1, 2, \ldots, n$ satifying condition (3.6). Hence, from the Cauchy interlacing property (Lemma 2.3) and relation (2.5) $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of matrix A for $j = 1, 2, \ldots, n$, respectively. $\square$

In order to obtain the unique solution to Problem 1, we consider a real symmetric doubly arrow matrix A of the form

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \cdots & 0 & b & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & b & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{s-1} & b & 0 & \cdots & 0 & 0 \\ b & b & \cdots & b & a_s & 0 & \cdots & 0 & c \\ 0 & 0 & \cdots & 0 & 0 & a_{s+1} & \cdots & 0 & c \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-1} & c \\ 0 & 0 & \cdots & 0 & c & c & \cdots & c & a_n \end{pmatrix}, \tag{3.11}$$

where $a_j \in \mathbb{R}$ and $b, c > 0$

In this case, the recurrence relation of characteristic polynomials in Lemma 2.1 becomes:

$$P_j(\lambda) = \prod_{i=1}^{j}(\lambda - a_i)\ ; \qquad j = 1,2,\ldots,s-1, \tag{3.12}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - b^2 \sum_{k=1}^{j-1} \prod_{\substack{i=1 \\ i\neq k}}^{j-1}(\lambda - a_i)\ ; \qquad j = s, \tag{3.13}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda)\ ; \qquad j = s+1, s+2, \ldots, n-1, \tag{3.14}$$

$$P_j(\lambda) = (\lambda - a_j)P_{j-1}(\lambda) - c^2 \sum_{k=s}^{j-1} P_{k-1}(\lambda) \prod_{i=k+1}^{j-1}(\lambda - a_i)\ ; \qquad j = n, \tag{3.15}$$

where $P_0(\lambda) = 1$.

The following corollary gives a sufficient condition for the existence of a unique solution to Problem 1.

Corollary 3.3. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1,2,\ldots,n$ be given. Then there exists the unique $n \times n$ matrix A of the form (3.11) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of matrix A for $j = 1,2,\ldots,n$, respectively, if the following conditions are satisfied:*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} = \cdots = \lambda_1^{(s)} < \lambda_1^{(s-1)} = \cdots = \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}. \tag{3.16}$$

Proof. Suppose that the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for all $j = 1,2,\ldots,n$ satisfy condition (3.16). It is obvious from (3.16) that for $j = 1,2,\ldots,s-1$ there exists the unique matrix $A_j = diag\left\{\lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}\right\}$ with the minimal and maximal eigenvalues $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively.

As in the proof of Theorem 3.1, it is enough to show that the system of equations

$$\left.\begin{aligned} P_j\left(\lambda_1^{(j)}\right) &= 0 \\ P_j\left(\lambda_j^{(j)}\right) &= 0 \end{aligned}\right\} \tag{3.17}$$

has real solution b, c, and a_j, $j = s, s+1, \ldots, n$.

For $j = s$, the system (3.17) has the form:

$$\left.\begin{aligned} a_s P_{s-1}\left(\lambda_1^{(s)}\right) + b^2 \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_1^{(s)} - a_i\right) &= \lambda_1^{(s)} P_{s-1}\left(\lambda_1^{(s)}\right) \\ a_s P_{s-1}\left(\lambda_s^{(s)}\right) + b^2 \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_s^{(s)} - a_i\right) &= \lambda_s^{(s)} P_{s-1}\left(\lambda_s^{(s)}\right) \end{aligned}\right\}. \tag{3.18}$$

So, the coefficient matrix for the system (3.18) is

$$\begin{bmatrix} P_{s-1}\left(\lambda_1^{(s)}\right) & \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_1^{(s)} - a_i\right) \\ P_{s-1}\left(\lambda_s^{(s)}\right) & \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_s^{(s)} - a_i\right) \end{bmatrix}.$$

Let d_s be the determinant of this coefficient matrix. Then,

$$d_s = P_{s-1}\left(\lambda_1^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_s^{(s)} - a_i\right) - P_{s-1}\left(\lambda_s^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_1^{(s)} - a_i\right).$$

We will show that $d_s \neq 0$ by showing that $(-1)^{s-1} d_s > 0$. From Lemma 2.2, we have $(-1)^{s-1} P_{s-1}\left(\lambda_1^{(s)}\right) > 0$ and $P_{s-1}\left(\lambda_s^{(s)}\right) > 0$. Since $a_j = \lambda_j^{(j)}$ for $j = 1, 2, \ldots, s-1$ and condition (3.16), $\left(\lambda_s^{(s)} - a_i\right) > 0$ and $\left(\lambda_1^{(s)} - a_i\right) < 0$. Therefore,

$$(-1)^{s-1} P_{s-1}\left(\lambda_1^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_s^{(s)} - a_i\right) > 0.$$

Since there are $s-2$ factors in each product $\prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_1^{(s)} - a_i\right)$,

$$\begin{aligned} -(-1)^{s-1} P_{s-1}\left(\lambda_s^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda_1^{(s)} - a_i\right) &= (-1)^{s}(-1)^{s-2} P_{s-1}\left(\lambda_s^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(a_i - \lambda_1^{(s)}\right) \\ &= (-1)^{2(s-1)} P_{s-1}\left(\lambda_s^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(a_i - \lambda_1^{(s)}\right) > 0. \end{aligned}$$

Thus, $(-1)^{s-1} d_s > 0$ and hence $d_s \neq 0$. Consequently, the system (3.18) has a unique solution. By using the Cramer's rule, we obtain

$$a_s = \frac{\lambda_1^{(s)} P_{s-1}\left(\lambda_1^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i\neq k}}^{s-1} \left(\lambda_s^{(s)} - a_i\right) - \lambda_s^{(s)} P_{s-1}\left(\lambda_s^{(s)}\right) \sum_{k=1}^{s-1} \prod_{\substack{i=1 \\ i\neq k}}^{s-1} \left(\lambda_1^{(s)} - a_i\right)}{d_s},$$

and

$$b^2 = \frac{\left(\lambda_s^{(s)} - \lambda_1^{(s)}\right) P_{s-1}\left(\lambda_1^{(s)}\right) P_{s-1}\left(\lambda_s^{(s)}\right)}{d_s}.$$

Since $(-1)^{s-1}\left(\lambda_s^{(s)} - \lambda_1^{(s)}\right) P_{s-1}\left(\lambda_1^{(s)}\right) P_{s-1}\left(\lambda_s^{(s)}\right) > 0$ by Lemma 2.2,

$$b^2 = \frac{(-1)^{s-1}\left(\lambda_s^{(s)} - \lambda_1^{(s)}\right) P_{s-1}\left(\lambda_1^{(s)}\right) P_{s-1}\left(\lambda_s^{(s)}\right)}{(-1)^{s-1} d_s} > 0.$$

Then b is a real number which can be made positive. Hence, there exists the unique matrix A_s.

For $j = s+1, s+2, \ldots, n-1$, as in the proof of Theorem 3.1, we obtain

$$a_j = \lambda_j^{(j)}.$$

For $j = n$, the system (3.16) has the form:

$$\left.\begin{aligned} a_n P_{n-1}\left(\lambda_1^{(n)}\right) + c^2 \sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_1^{(n)} - a_i\right) &= \lambda_1^{(n)} P_{n-1}\left(\lambda_1^{(n)}\right) \\ a_n P_{n-1}\left(\lambda_n^{(n)}\right) + c^2 \sum_{k=s}^{n-1} P_{k-1}(\lambda_n^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_n^{(n)} - a_i\right) &= \lambda_n^{(n)} P_{n-1}\left(\lambda_n^{(n)}\right) \end{aligned}\right\}. \tag{3.19}$$

Thus, the coefficient matrix for the system (3.19) is

$$\begin{bmatrix} P_{n-1}\left(\lambda_1^{(n)}\right) & \sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_1^{(n)} - a_i\right) \\ P_{n-1}\left(\lambda_n^{(n)}\right) & \sum_{k=s}^{n-1} P_{k-1}(\lambda_n^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_n^{(n)} - a_i\right) \end{bmatrix}.$$

Let d_n be the determinant of this coefficient matrix. Then,

$$d_n = P_{n-1}\left(\lambda_1^{(n)}\right) \sum_{k=s}^{n-1} P_{k-1}(\lambda_n^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_n^{(n)} - a_i\right) - P_{n-1}\left(\lambda_n^{(n)}\right) \sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_1^{(n)} - a_i\right).$$

To show $d_n \neq 0$, we exhibit $(-1)^{n-1}d_n > 0$. We now consider the first factor of $(-1)^{n-1}d_n$. From Lemma 2.2, we have $(-1)^{n-1}P_{n-1}\left(\lambda_1^{(n)}\right) > 0$, and $P_{k-1}\left(\lambda_n^{(n)}\right) > 0$ for all $k = s, s+1, \ldots, n-1$. Since $a_j = \lambda_j^{(j)}$ for $j = s+1, \ldots, n-1$, $\left(\lambda_n^{(n)} - a_i\right) > 0$. Therefore,

$$(-1)^{n-1}P_{n-1}\left(\lambda_1^{(n)}\right)\sum_{k=s}^{n-1} P_{k-1}(\lambda_n^{(n)}) \prod_{i=k+1}^{n-1}\left(\lambda_n^{(n)} - a_i\right) > 0.$$

Next, we consider the second term of $(-1)^{n-1}d_n$. From Lemma 2.2(2), we have

$$P_{n-1}\left(\lambda_n^{(n)}\right) > 0.$$

It only remains to show that

$$-(-1)^{n-1}\sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right) > 0.$$

Since

$$\begin{aligned}
-(-1)^{n-1}\sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right) = {} & (-1)^n P_{s-1}(\lambda_1^{(n)}) \prod_{i=s+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right) \\
& + (-1)^n P_s(\lambda_1^{(n)}) \prod_{i=s+2}^{n-1}\left(\lambda_1^{(n)} - a_i\right) + \ldots \\
& + (-1)^n P_{n-s-1}(\lambda_1^{(n)}) \prod_{i=n-s+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right) + \ldots \\
& + P_{n-2}\left(\lambda_1^{(n)}\right), \qquad (3.20)
\end{aligned}$$

we examine each term of the right hand side of the equation (3.20) as follows.

For the first term, $\left(\lambda_1^{(n)} - a_i\right) < 0$ for $i = s+1, s+2, \ldots, n-1$ and there are $n-s-1$ factors in the product $\prod_{i=s+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right)$ thus,

$$\prod_{i=s+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right) = (-1)^{n-s-1}\prod_{i=s+1}^{n-1}\left(a_i - \lambda_1^{(n)}\right).$$

Therefore,

$$\begin{aligned}
(-1)^n P_{s-1}(\lambda_1^{(n)}) \prod_{i=s+1}^{n-1}\left(\lambda_1^{(n)} - a_i\right) &= (-1)^n P_{s-1}(\lambda_1^{(n)})(-1)^{n-s-1}\prod_{i=s+1}^{n-1}\left(a_i - \lambda_1^{(n)}\right) \\
&= (-1)^{2(n-s)}(-1)^{s-1}P_{s-1}(\lambda_1^{(n)})\prod_{i=s+1}^{n-1}\left(a_i - \lambda_1^{(n)}\right).
\end{aligned}$$

From Lemma 2.2(1), we have $(-1)^{s-1}P_{s-1}(\lambda_1^{(n)}) > 0$. Thus,

$$(-1)^n P_{s-1}(\lambda_1^{(n)}) \prod_{i=s+1}^{n-1} \left(\lambda_1^{(n)} - a_i\right) > 0.$$

For the second term, $\left(\lambda_1^{(n)} - a_i\right) < 0$ for $i = s+2, s+3, \ldots, n-1$ and there are $n-s-2$ factors in the product $\prod_{i=s+2}^{n-1} \left(\lambda_1^{(n)} - a_i\right)$,

$$\prod_{i=s+2}^{n-1} \left(\lambda_1^{(n)} - a_i\right) = (-1)^{n-s-2} \prod_{i=s+2}^{n-1} \left(a_i - \lambda_1^{(n)}\right).$$

Thus,

$$\begin{aligned}(-1)^n P_s(\lambda_1^{(n)}) \prod_{i=s+2}^{n-1} \left(\lambda_1^{(n)} - a_i\right) &= (-1)^n P_s(\lambda_1^{(n)})(-1)^{n-s-2} \prod_{i=s+2}^{n-1} \left(a_i - \lambda_1^{(n)}\right) \\ &= (-1)^{2(n-s-1)}(-1)^s P_s(\lambda_1^{(n)}) \prod_{i=s+2}^{n-1} \left(a_i - \lambda_1^{(n)}\right).\end{aligned}$$

From Lemma 2.2(1), we have $(-1)^s P_s(\lambda_1^{(n)}) > 0$. Thus,

$$(-1)^n P_s(\lambda_1^{(n)}) \prod_{i=s+2}^{n-1} \left(\lambda_1^{(n)} - a_i\right) > 0.$$

By using the similar process, we can show that all terms on the right hand of equation (3.18) are positive. Consequently,

$$-(-1)^{n-1} \sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_1^{(n)} - a_i\right) > 0.$$

Therefore, $(-1)^{n-1} d_n > 0$ and hence $d_n \neq 0$. Then, by the Cramer's rule, we obtain the unique solution of the system (3.19) as follows:

$$a_n = \frac{\lambda_1^{(n)} P_{n-1}\left(\lambda_1^{(n)}\right) \sum_{k=s}^{n-1} P_{k-1}(\lambda_n^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_n^{(n)} - a_i\right) - \lambda_n^{(n)} P_{n-1}\left(\lambda_n^{(n)}\right) \sum_{k=s}^{n-1} P_{k-1}(\lambda_1^{(n)}) \prod_{i=k+1}^{n-1} \left(\lambda_1^{(n)} - a_i\right)}{d_n},$$

and

$$c^2 = \frac{\left(\lambda_n^{(n)} - \lambda_1^{(n)}\right) P_{n-1}\left(\lambda_1^{(n)}\right) P_{n-1}\left(\lambda_n^{(n)}\right)}{d_n}.$$

Since $(-1)^{n-1}\left(\lambda_n^{(n)} - \lambda_1^{(n)}\right)P_{n-1}\left(\lambda_1^{(n)}\right)P_{n-1}\left(\lambda_n^{(n)}\right) > 0$ by Lemma 2.2,

$$c^2 = \frac{(-1)^{n-1}\left(\lambda_n^{(n)} - \lambda_1^{(n)}\right)P_{n-1}\left(\lambda_1^{(n)}\right)P_{n-1}\left(\lambda_n^{(n)}\right)}{(-1)^{n-1}d_n} > 0.$$

Then, c is a real number which can be made positive. Hence, there exists the unique matrix $A_n = A$

The resulting matrix $A_n = A$ is now a real symmetric doubly arrow matrix with $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for $j = 1, 2, \ldots, n$ satifying condition (3.16). Consequently, from the Cauchy interlacing property (Lemma 2.3) and relation (2.5) $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal submatrix A_j of matrix A for $j = 1, 2, \ldots, n$, respectively. $\square$

Some examples for Problem 1 are shown below.

Example 3.1 Given the real numbers

$\lambda_1^{(6)} = -6 \quad \lambda_1^{(5)} = -3 \quad \lambda_1^{(4)} = -3 \quad \lambda_1^{(3)} = -3 \quad \lambda_1^{(2)} = -1 \quad \lambda_1^{(1)} = -1$

$\lambda_2^{(2)} = 3 \quad \lambda_3^{(3)} = 6 \quad \lambda_4^{(4)} = 7 \quad \lambda_5^{(5)} = 9 \quad \lambda_6^{(6)} = 10$

that satisfy conditions (3.1), (3.2), and (3.3) of Theorem 3.1 with $s = 3$, we can construct a real symmetric doubly arrow matrix A of the form (1.1) with the required properties as follows:

$$\mathbf{A} = \begin{pmatrix} -1.0000 & 0.0000 & 1.2472 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 3.0000 & 4.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.2472 & 4.0000 & 0.4444 & 0.0000 & 0.0000 & 4.6626 \\ 0.0000 & 0.0000 & 0.0000 & 7.0000 & 0.0000 & 3.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 9.0000 & 2.0000 \\ 0.0000 & 0.0000 & 4.6626 & 3.0000 & 2.0000 & -0.0497 \end{pmatrix}.$$

Example 3.2 Given the real numbers

$\lambda_1^{(6)} = -7 \quad \lambda_1^{(5)} = -4 \quad \lambda_1^{(4)} = -4 \quad \lambda_1^{(3)} = 2 \quad \lambda_1^{(2)} = 2 \quad \lambda_1^{(1)} = 2$

$\lambda_2^{(2)} = 5 \quad \lambda_3^{(3)} = 7 \quad \lambda_4^{(4)} = 9 \quad \lambda_5^{(5)} = 10 \quad \lambda_6^{(6)} = 13$

that satisfy conditions (3.6), (3.7), (3.8), and (3.9) of Corollary 3.2 with $s = 4$, we can construct a nonnegative real symmetric doubly arrow matrix A of the form (1.1) with the required properties as follows:

$$\mathbf{A} = \begin{pmatrix} 2.0000 & 0.0000 & 0.0000 & 3.0000 & 0.0000 & 0.0000 \\ 0.0000 & 5.0000 & 0.0000 & 5.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 7.0000 & 1.4170 & 0.0000 & 0.0000 \\ 3.0000 & 5.0000 & 1.4170 & 0.4603 & 0.0000 & 7.1828 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 10.0000 & 2.0000 \\ 0.0000 & 0.0000 & 0.0000 & 7.1828 & 2.0000 & 5.4219 \end{pmatrix}.$$

Example 3.3 Given the real numbers

$\lambda_1^{(6)} = -9 \quad \lambda_1^{(5)} = -8 \quad \lambda_1^{(4)} = -8 \quad \lambda_1^{(3)} = -8 \quad \lambda_1^{(2)} = -6 \quad \lambda_1^{(1)} = -6$

$\lambda_2^{(2)} = -4 \quad \lambda_3^{(3)} = -2 \quad \lambda_4^{(4)} = -1 \quad \lambda_5^{(5)} = 6 \quad \lambda_6^{(6)} = 8$

that satisfy condition (3.16) of Corollary 3.3 with $s = 3$, we can construct a unique real symmetric doubly arrow matrix A of the form (3.11) with the required properties as follows:

$$\mathbf{A} = \begin{pmatrix} -6.0000 & 0.0000 & 2.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -4.0000 & 2.0000 & 0.0000 & 0.0000 & 0.0000 \\ 2.0000 & 2.0000 & -5.0000 & 0.0000 & 0.0000 & 3.4609 \\ 0.0000 & 0.0000 & 0.0000 & -1.0000 & 0.0000 & 3.4609 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 6.0000 & 3.4609 \\ 0.0000 & 0.0000 & 3.4609 & 3.4609 & 3.4609 & -0.2874 \end{pmatrix}.$$

CHAPTER 4

SOLUTION TO PROBLEM 2

In this chapter, we solve Problem 2 and give the sufficient conditions for the existence of a unique doubly arrow matrix A of the form (1.1) from one of the eigenpairs and eigenvalues of all leading principal submatrices of matrix A. The nonnegative case for this inverse eigenvalue problem is also discussed.

Assume that $\lambda^{(j)}$ is an eigenvalue of the $j \times j$ leading principal submatrix A_j of matrix A, $j = 1, 2, \ldots, n$, and $(\lambda^{(n)}, \mathbf{x})$ is an eigenpair of matrix $A = A_n$. That is,

$$P_j\left(\lambda^{(j)}\right) = 0, \qquad j = 1, 2, \ldots, n-1, \tag{4.1}$$

$$\mathbf{Ax} = \lambda^{(n)}\mathbf{x}. \tag{4.2}$$

We can rewrite the relations (4.1) and (4.2) as (4.3) and (4.4), respectively.

$$\left.\begin{aligned} \prod_{i=1}^{j}\left(\lambda^{(j)} - a_i\right) &= 0\,, \quad j = 1, \ldots, s-1, \\ \left(\lambda^{(j)} - a_j\right)P_{j-1}\left(\lambda^{(j)}\right) - \sum_{k=1}^{j-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{j-1}\left(\lambda^{(j)} - a_i\right) &= 0\,, \quad j = s, \\ \left(\lambda^{(j)} - a_j\right)P_{j-1}\left(\lambda^{(j)}\right) &= 0\,, \quad j = s+1, \ldots, n-1, \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} a_j x_j + b_j x_s &= \lambda^{(n)} x_j, \quad j = 1, \ldots, s-1, \\ \sum_{k=1}^{s-1} b_k x_k + a_s x_s + b_s x_n &= \lambda^{(n)} x_s, \quad j = s, \\ a_j x_j + b_j x_n &= \lambda^{(n)} x_j, \quad j = s+1, \ldots, n-1, \\ \sum_{k=s}^{n-1} b_k x_k + a_n x_n &= \lambda^{(n)} x_n, \quad j = n. \end{aligned}\right\} \tag{4.4}$$

Observe that the solvability of the system of equations (4.3) and (4.4) is equivalent to that of Problem 2.

The following theorem gives the sufficient conditions for Problem 2 to have a unique solution.

Theorem 4.1. *Problem 2 has a unique solution if the following conditions are satisfied:*

(i) $x_i \neq 0, \quad i = s, n,$

(ii) $P_{j-1}\left(\lambda^{(j)}\right) \neq 0, \quad j = s, s+1, \ldots, n-1.$

Proof. By the above observation, Problem 2 has a unique solution if and only if the system of equations (4.3) and (4.4) has a unique solution. That is, the system of equations satisfies conditions (i) and (ii).

All elements of doubly arrow matrix A can be calculated as follows.

For $j = 1, 2, \ldots, s-1$, from the first equation of (4.3) we have

$$a_j = \lambda^{(j)}.$$

Substituting a_j into the first equation of (4.4), we obtain

$$\lambda^{(j)} x_j + b_j x_s = \lambda^{(n)} x_j.$$

Since $x_s \neq 0$ by condition (i), we get

$$b_j = \left(\lambda^{(n)} - \lambda^{(j)}\right) \frac{x_j}{x_s}.$$

For $j = s$, we rewrite the second equation of (4.3) as

$$\lambda^{(s)} P_{s-1}(\lambda^{(s)}) - a_s P_{s-1}(\lambda^{(s)}) - \sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} (\lambda^{(s)} - a_i) = 0.$$

Since $P_{s-1}(\lambda^{(s)}) \neq 0$ by condition (ii), we obtain

$$a_s = \frac{\lambda^{(s)} P_{s-1}\left(\lambda^{(s)}\right) - \sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda^{(s)} - a_i\right)}{P_{s-1}(\lambda^{(s)})}.$$

Next, from the second equation of (4.4) and $x_n \neq 0$ by condition (i) we have

$$b_s = \frac{\lambda^{(n)} x_s - a_s x_s - \sum_{k=1}^{s-1} b_k x_k}{x_n}.$$

For $j = s+1, s+2, \ldots, n-1$, from the third equation of (4.3) and condition (ii) it follows that

$$a_j = \lambda^{(j)}.$$

Substituting a_j into the third equation of (4.4), we obtain

$$\lambda^{(j)} x_j + b_j x_n = \lambda^{(n)} x_j$$

and by condition (i) we get

$$b_j = \left(\lambda^{(n)} - \lambda^{(j)}\right) \frac{x_j}{x_n}.$$

For $j = n$, from the last equation of (4.4) and condition (i) we have

$$a_n = \lambda^{(n)} - \frac{\sum_{k=s}^{n-1} b_k x_k}{x_n}. \qquad \square$$

The following corollary is related to a nonnegative solution to Problem 2.

Corollary 4.2. *Problem 2 has a unique nonnegative solution if the following conditions are satisfied:*

(i) $\lambda^{(n)} \geq \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, s-1, s+1, \ldots, n-1,$

(ii) $x_i > 0, \quad i = 1, 2, \ldots, n,$

(iii) $P_{j-1}\left(\lambda^{(j)}\right) > 0, \quad j = s, s+1, \ldots, n-1,$

(iv) $\lambda^{(s)} \geq \dfrac{\sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda^{(s)} - a_i\right)}{P_{s-1}\left(\lambda^{(s)}\right)},$

(v) $\lambda^{(n)} \geq a_s + \dfrac{\sum_{k=1}^{s-1} b_k x_k}{x_s},$

(vi) $\lambda^{(n)} \geq \dfrac{\sum_{k=s}^{n-1} b_k x_k}{x_n}$.

Proof. From the proof of Theorem 4.1, conditions (ii) and (iii) guarantee that there exists a unique doubly arrow matrix A of the form (1.1). It remains to show that all elements of matrix A are nonnegative.

For $j = 1, 2, \ldots, s-1$, from $\lambda^{(j)} \geq 0$ by condition (i) and $x_j, x_s > 0$ by condition (ii) we obtain

$$a_j = \lambda^{(j)} \geq 0$$

and

$$b_j = \left(\lambda^{(n)} - \lambda^{(j)}\right)\frac{x_j}{x_s} \geq 0.$$

For $j = s$, $P_{s-1}\left(\lambda^{(s)}\right) > 0$ by condition (iii) and rewriting condition (iv) as

$$\lambda^{(s)} P_{s-1}\left(\lambda^{(s)}\right) - \sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda^{(s)} - a_i\right) \geq 0,$$

we have

$$a_s = \frac{\lambda^{(s)} P_{s-1}\left(\lambda^{(s)}\right) - \sum_{k=1}^{s-1} b_k^2 \prod_{\substack{i=1 \\ i \neq k}}^{s-1} \left(\lambda^{(s)} - a_i\right)}{P_{s-1}(\lambda^{(s)})} \geq 0.$$

Since $x_s, x_n > 0$ by condition (ii) and we can express condition (v) as

$$\lambda^{(n)} - a_s - \frac{\sum_{k=1}^{s-1} b_k x_k}{x_s} \geq 0,$$

we get

$$b_s = \frac{\lambda^{(n)} x_s - a_s x_s - \sum_{k=1}^{s-1} b_k x_k}{x_n} \geq 0.$$

For $j = s+1, s+2, \ldots, n-1$, from condition (i) we obtain

$$a_j = \lambda^{(j)} \geq 0.$$

By conditions (i) and (ii) we have

$$b_j = \left(\lambda^{(n)} - \lambda^{(j)}\right)\frac{x_j}{x_n} \geq 0.$$

For $j = n$, from conditions (ii) and (vi) we obtain

$$a_n = \lambda^{(n)} - \frac{\sum_{k=s}^{n-1} b_k x_k}{x_n} \geq 0. \qquad \square$$

Now, we give some examples to illustrate our results.

Example 4.1 Given the real numbers

$$\lambda^{(1)} = -4 \quad \lambda^{(2)} = -1 \quad \lambda^{(3)} = -2 \quad \lambda^{(4)} = 3 \quad \lambda^{(5)} = 2 \quad \lambda^{(6)} = 1$$

and the real vector $\mathbf{x} = [-2, 3, -1, 2, -3, 1]^T$ that satisfy conditions (i) and (ii) of Theorem 4.1 with $s = 3$, we can construct a unique real symmetric doubly arrow matrix A of the form (1.1) with the required properties, as follows:

$$\mathbf{A} = \begin{pmatrix} -4 & 0 & 10 & 0 & 0 & 0 \\ 0 & -1 & -6 & 0 & 0 & 0 \\ 10 & -6 & -16 & 0 & 0 & 21 \\ 0 & 0 & 0 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 21 & -4 & 3 & 39 \end{pmatrix}.$$

Example 4.2 Given the real numbers

$$\lambda^{(1)} = 7 \quad \lambda^{(2)} = 3 \quad \lambda^{(3)} = -1 \quad \lambda^{(4)} = 3 \quad \lambda^{(5)} = 5 \quad \lambda^{(6)} = 9$$

and the real vector $\mathbf{x} = [5, 3, 5, 1, 2, 5]^T$ that satisfy conditions (i) – (vi) of Corollary 4.2 with $s = 3$, we can construct a unique nonnegative real symmetric doubly arrow matrix A of the form (1.1) with the required properties, as follows:

$$\mathbf{A} = \begin{pmatrix} 7.0000 & 0.0000 & 2.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 3.0000 & 3.6000 & 0.0000 & 0.0000 & 0.0000 \\ 2.0000 & 3.6000 & 2.7400 & 0.0000 & 0.0000 & 2.1000 \\ 0.0000 & 0.0000 & 0.0000 & 3.0000 & 0.0000 & 1.2000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 5.0000 & 1.6000 \\ 0.0000 & 0.0000 & 2.1000 & 1.2000 & 1.6000 & 6.0200 \end{pmatrix}.$$

CHAPTER 5

SOLUTION TO PROBLEM 3

In this chapter, we solve Problem 3 and give the necessary and sufficient conditions for the existence of unique real symmetric doubly arrow matrices from the two eigenpairs. In our work, we are interested in constructing three different forms of a real symmetric doubly arrow matrix. Those are the matrices A, B, and C of the form (1.1), (1.2), and (1.3), respectively.

5.1. Constructing Doubly Arrow Matrix A

In this subsection, we construct a doubly arrow matrix A of the form (1.1). Assume that $(\lambda, \mathbf{x})$ and $(\mu, \mathbf{y})$ are eigenpairs of matrix A. That is, $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{Ay} = \mu\mathbf{y}$. We can rewrite these as

$$\left.\begin{aligned} a_1x_1 + b_1x_s = \lambda x_1, \\ a_1y_1 + b_1y_s = \mu y_1, \end{aligned}\right\} \tag{5.1.1}$$

$$\left.\begin{aligned} a_2x_2 + b_2x_s = \lambda x_2 \\ a_2y_2 + b_2y_s = \mu y_2, \end{aligned}\right\} \tag{5.1.2}$$

$$\vdots$$

$$\left.\begin{aligned} a_{s-1}x_{s-1} + b_{s-1}x_s = \lambda x_{s-1}, \\ a_{s-1}y_{s-1} + b_{s-1}y_s = \mu y_{s-1}, \end{aligned}\right\} \tag{5.1.s – 1}$$

$$\left.\begin{aligned} b_1x_1 + b_2x_2 + \cdots + b_{s-1}x_{s-1} + a_sx_s + b_sx_n = \lambda x_s, \\ b_1y_1 + b_2y_2 + \cdots + b_{s-1}y_{s-1} + a_sy_s + b_sy_n = \mu y_s, \end{aligned}\right\} \tag{5.1.s}$$

$$\left.\begin{aligned} a_{s+1}x_{s+1} + b_{s+1}x_n = \lambda x_{s+1}, \\ a_{s+1}y_{s+1} + b_{s+1}y_n = \mu y_{s+1}, \end{aligned}\right\} \tag{5.1.s + 1}$$

$$\vdots$$

$$\left.\begin{aligned} a_{n-1}x_{n-1} + b_{n-1}x_n = \lambda x_{n-1}, \\ a_{n-1}y_{n-1} + b_{n-1}y_n = \mu y_{n-1}, \end{aligned}\right\} \tag{5.1.n – 1}$$

$$\left.\begin{aligned} b_s x_s + b_{s+1} x_{s+1} + \cdots + b_{n-1} x_{n-1} + a_n x_n = \lambda x_n, \\ b_s y_s + b_{s+1} y_{s+1} + \cdots + b_{n-1} y_{n-1} + a_n y_n = \mu y_n. \end{aligned}\right\} \tag{5.1.n}$$

To solve this system, we denote all variables as follows:

$$\alpha = [a_1, b_1, a_2, b_2, \cdots, a_s, b_s, \cdots, a_{n-1}, b_{n-1}, a_n]^T,$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{Z}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}; \quad i = 1, 2, \ldots, n,$$

$$\mathbf{C}_i = \begin{bmatrix} \lambda x_i \\ \mu y_i \end{bmatrix} = \begin{bmatrix} c_i^{(1)} \\ c_i^{(2)} \end{bmatrix}; \quad i = 1, 2, \ldots, n,$$

$$\gamma = \left[c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_2^{(2)}, \cdots, c_s^{(1)}, c_s^{(2)}, \cdots, c_n^{(1)}, c_n^{(2)}\right]^T,$$

$$\mathbf{B}_{i,s} = \begin{bmatrix} x_i & x_s \\ y_i & y_s \end{bmatrix}; \quad i = 1, 2, \ldots, s-1,$$

$$\mathbf{B}_{s,n} = \begin{bmatrix} x_s & x_n \\ y_s & y_n \end{bmatrix},$$

$$\mathbf{B}_{i,n} = \begin{bmatrix} x_i & x_n \\ y_i & y_n \end{bmatrix}; \quad i = s+1, s+2, \ldots, n-1,$$

$$D_{i,s} = \det(\mathbf{B}_{i,s}); \quad i = 1, 2, \ldots, s-1,$$

$$D_{s,n} = \det(\mathbf{B}_{s,n}),$$

$$D_{i,n} = \det(\mathbf{B}_{i,n}); \quad i = s+1, s+2, \ldots, n-1,$$

$$D_{i,s}^{(a)} = \begin{vmatrix} c_i^{(1)} & x_s \\ c_i^{(2)} & y_s \end{vmatrix}, \quad D_{i,s}^{(b)} = \begin{vmatrix} x_i & c_i^{(1)} \\ y_i & c_i^{(2)} \end{vmatrix}; \quad i = 1, 2, \ldots, s-1,$$

$$D_{s,n}^{(a)} = \begin{vmatrix} c_s^{(1)} - b_1x_1 - \ldots - b_{s-1}x_{s-1} & x_n \\ c_s^{(2)} - b_1y_1 - \ldots - b_{s-1}y_{s-1} & y_n \end{vmatrix}, \quad D_{s,n}^{(b)} = \begin{vmatrix} x_s & c_s^{(1)} - b_1x_1 - \ldots - b_{s-1}x_{s-1} \\ y_s & c_s^{(2)} - b_1y_1 - \ldots - b_{s-1}y_{s-1} \end{vmatrix},$$

$$D_{i,n}^{(a)} = \begin{vmatrix} c_i^{(1)} & x_n \\ c_i^{(2)} & y_n \end{vmatrix}, \quad D_{i,n}^{(b)} = \begin{vmatrix} x_i & c_i^{(1)} \\ y_i & c_i^{(2)} \end{vmatrix}; \quad i = s+1, s+2, \ldots, n-1,$$

and

$$\mathbf{H} = \begin{bmatrix} \mathbf{B}_{1,s} & & & & & & & & \\ & \mathbf{B}_{2,s} & & & & & & & \\ & & \ddots & & & & & & \\ & & & \mathbf{B}_{s-1,s} & & & & & \\ \mathbf{Z}_1\mathbf{e}_2^T & \mathbf{Z}_2\mathbf{e}_2^T & \ldots & \mathbf{Z}_{s-1}\mathbf{e}_2^T & \mathbf{B}_{s,n} & & & & \\ & & & & & \mathbf{B}_{s+1,n} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \mathbf{B}_{n-1,n} & \\ & & & & \mathbf{Z}_s\mathbf{e}_2^T & \mathbf{Z}_{s+1}\mathbf{e}_2^T & \ldots & \mathbf{Z}_{n-1}\mathbf{e}_2^T & \mathbf{B}_{n,n} \end{bmatrix}_{2n\times 2n-1},$$

where $\mathbf{B}_{n,n} = \mathbf{Z}_n$. We now simply rewrite the linear system of equations (5.1.1) – (5.1.n) as

$$\mathbf{H}\alpha = \gamma. \tag{5.A}$$

Observe that the solvability of the system (5.A) is equivalent to that of Problem 3.

The following theorem gives the necessary and sufficient conditions for Problem 3 to have a unique solution matrix A.

Theorem 5.1. *Problem 3 has a unique solution if and only if the following conditions are satisfied:*

(i) $D_{i,s} \neq 0; \quad i = 1, 2, \ldots, s-1,$

(ii) $D_{s,n} \neq 0,$

(iii) $D_{i,n} \neq 0; \quad i = s+1, s+2, \ldots, n-1,$

(iv) $Rank(\mathbf{Z}_n) = Rank(\mathbf{Z}_n|\mathbf{C}_n - b_s\mathbf{Z}_s - b_{s+1}\mathbf{Z}_{s+1} - \ldots - b_{n-1}\mathbf{Z}_{n-1}) = 1.$

Proof. Since the solvability of linear system (5.A) is equivalent to that of Problem 3, Problem 3 has a unique solution if and only if linear system (5.A) has a unique solution. Therefore, linear system (5.A) has a unique solution if and only if the conditions (i) – (iv) are satisfied. □

We can find all elements of doubly arrow matrix A from the linear system of equations (5.1.1) – (5.1.n) and all conditions (i) – (iv) in Thorem 5.1 as follows. Solving the equations (5.1.1) – (5.1.s – 1) and using condition (i), we have

$$a_i = \frac{D_{i,s}^{(a)}}{D_{i,s}}, \quad b_i = \frac{D_{i,s}^{(b)}}{D_{i,s}}.$$

Substituting $b_1, b_2, \ldots, b_{s-1}$ into equation (5.1.s) and using condition (ii), we obtain

$$a_s = \frac{D_{s,n}^{(a)}}{D_{s,n}}, \quad b_s = \frac{D_{s,n}^{(b)}}{D_{s,n}}.$$

Similarly, we can also find a_i and b_i for $i = s+1, s+2, \ldots, n-1$ by solving the equations (5.1.s + 1) – (5.1.n – 1) under condition (iii). That is,

$$a_i = \frac{D_{i,n}^{(a)}}{D_{i,n}}, \quad b_i = \frac{D_{i,n}^{(b)}}{D_{i,n}}.$$

Substituting $b_s, b_{s+1}, \ldots, b_{n-1}$ into equation (5.1.n), if x_n and y_n are not both 0, then we get the last entry a_n under condition (iv) as follows:

$$\text{if } x_n \neq 0, \text{ then } a_n = \lambda - b_s\frac{x_s}{x_n} - b_{s+1}\frac{x_{s+1}}{x_n} - \ldots - b_{n-1}\frac{x_{n-1}}{x_n} \text{ and}$$

$$\text{if } y_n \neq 0,\ x_n = 0, \text{ then } a_n = \mu - b_s\frac{y_s}{y_n} - b_{s+1}\frac{y_{s+1}}{y_n} - \ldots - b_{n-1}\frac{y_{n-1}}{y_n}.$$

5.2.Constructing Doubly Arrow Matrix B

Now, we construct a doubly arrow matrix B of the form (1.2). Assume that $(\lambda, \mathbf{x})$ and $(\mu, \mathbf{y})$ are eigenpairs of matrix B. That is, $\mathbf{Bx} = \lambda\mathbf{x}$ and $\mathbf{By} = \mu\mathbf{y}$. We can rewrite these as

$$\left.\begin{aligned} a_1x_1 + b_1x_s &= \lambda x_1, \\ a_1y_1 + b_1y_s &= \mu y_1, \end{aligned}\right\} \tag{5.2.1}$$

$$\left.\begin{aligned} a_2x_2 + b_2x_s &= \lambda x_2, \\ a_2y_2 + b_2y_s &= \mu y_2, \end{aligned}\right\} \tag{5.2.2}$$

$$\vdots$$

$$\left.\begin{aligned} a_{s-1}x_{s-1} + b_{s-1}x_s &= \lambda x_{s-1}, \\ a_{s-1}y_{s-1} + b_{s-1}y_s &= \mu y_{s-1}, \end{aligned}\right\} \tag{5.2.s – 1}$$

$$\left.\begin{aligned} b_1x_1 + \cdots + b_{s-1}x_{s-1} + a_sx_s + b_sx_{s+1} + b_{s+1}x_{s+2} + \cdots + b_{n-1}x_n &= \lambda x_s, \\ b_1y_1 + \cdots + b_{s-1}y_{s-1} + a_sy_s + b_sy_{s+1} + b_{s+1}y_{s+2} + \cdots + b_{n-1}y_n &= \mu y_s, \end{aligned}\right\} \tag{5.2.s}$$

$$\left.\begin{aligned} b_sx_s + a_{s+1}x_{s+1} &= \lambda x_{s+1}, \\ b_sy_s + a_{s+1}y_{s+1} &= \mu y_{s+1}, \end{aligned}\right\} \tag{5.2.s + 1}$$

$$\vdots$$

$$\left.\begin{aligned} b_{n-1}x_s + a_nx_n &= \lambda x_n, \\ b_{n-1}y_s + a_ny_n &= \mu y_n. \end{aligned}\right\} \tag{5.2.n}$$

To solve the system, we denote

$$\alpha = [a_1, b_1, a_2, b_2, \cdots, a_s, b_s, \cdots, a_{n-1}, b_{n-1}, a_n]^T,$$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{Z}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}; \quad i = 1, 2, \ldots, n,$$

$$\mathbf{C}_i = \begin{bmatrix} \lambda x_i \\ \mu y_i \end{bmatrix} = \begin{bmatrix} c_i^{(1)} \\ c_i^{(2)} \end{bmatrix}; \quad i = 1, 2, \ldots, n,$$

$$\gamma = \left[c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_2^{(2)}, \cdots, c_s^{(1)}, c_s^{(2)}, \cdots, c_n^{(1)}, c_n^{(2)}\right]^T,$$

$$\mathbf{B}_{i,s} = \begin{bmatrix} x_i & x_s \\ y_i & y_s \end{bmatrix}; \quad i = 1, 2, \ldots, s-1,$$

$$\mathbf{B}_{s,i} = \begin{bmatrix} x_s & x_i \\ y_s & y_i \end{bmatrix}; \quad i = s+1, s+2, \ldots, n,$$

$$D_{i,s} = \det(\mathbf{B}_{i,s}); \quad i = 1, 2, \ldots, s-1,$$

$$D_{s,i} = \det(\mathbf{B}_{s,i}); \quad i = s+1, s+2, \ldots, n,$$

$$D_{i,s}^{(a)} = \begin{vmatrix} c_i^{(1)} & x_s \\ c_i^{(2)} & y_s \end{vmatrix}, \quad D_{i,s}^{(b)} = \begin{vmatrix} x_i & c_i^{(1)} \\ y_i & c_i^{(2)} \end{vmatrix}; \quad i = 1, 2, \ldots, s-1,$$

$$D_{s,i}^{(a)} = \begin{vmatrix} x_s & c_i^{(1)} \\ y_s & c_i^{(2)} \end{vmatrix}, \quad D_{s,i}^{(b)} = \begin{vmatrix} c_i^{(1)} & x_i \\ c_i^{(2)} & y_i \end{vmatrix}; \quad i = s+1, s+2, \ldots, n-1,$$

and

$$\mathbf{L} = \begin{bmatrix} \mathbf{B}_{1,s} & & & & & & & \\ & \mathbf{B}_{2,s} & & & & & & \\ & & \ddots & & & & & \\ & & & \mathbf{B}_{s-1,s} & & & & \\ \mathbf{Z}_1\mathbf{e}_2^T & \mathbf{Z}_2\mathbf{e}_2^T & \ldots & \mathbf{Z}_{s-1}\mathbf{e}_2^T & \mathbf{B}_{s,s} & \mathbf{Z}_{s+1}\mathbf{e}_1^T & \ldots & \mathbf{Z}_{n-1}\mathbf{e}_1^T & \mathbf{Z}_n\mathbf{e}_1^T \\ & & & & & \mathbf{B}_{s,s+1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \mathbf{B}_{s,n-1} & \\ & & & & & & & & \mathbf{B}_{s,n} \end{bmatrix}_{2n \times 2n-1},$$

where $\mathbf{B}_{s,s} = \mathbf{Z}_s$. We can rewrite the linear system of equations (5.2.1) – (5.2.n) as

$$\mathbf{L}\alpha = \gamma. \tag{5.B}$$

Observe that the solvability of the system (5.B) is equivalent to that of Problem 3.

The following corollary gives the necessary and sufficient conditions for Problem 3 to have a unique real symmetric doubly arrow matrix B of the form (1.2).

Corollary 5.2. *Problem 3 has a unique solution if and only if the following conditions are satisfied:*

(i) $D_{i,s} \neq 0; \quad i = 1,2,\ldots,s-1,$

(ii) $D_{s,i} \neq 0; \quad i = s+1, s+2, \ldots, n,$

(iii) $Rank(\mathbf{Z}_s) = Rank(\mathbf{Z}_s|\mathbf{C}_s - b_1\mathbf{Z}_1 - b_2\mathbf{Z}_2 - \ldots - b_{s-1}\mathbf{Z}_{s-1} - b_s\mathbf{Z}_{s+1} - b_{s+1}\mathbf{Z}_{s+2} - \ldots - b_{n-1}\mathbf{Z}_n) = 1.$

Proof. From the above observation, Problem 3 has a unique solution if and only if linear system (5.B) has a unique solution. That is, linear system (5.B) satisfies conditions (i) – (iii) □

All elements of doubly arrow matrix B can be found from the linear system of equations (5.2.1) – (5.2.n) and all conditions (i) – (iii), in a similar fashion to that used in finding doubly arrow matrix A. We obtain

$$a_i = \frac{D_{i,s}^{(a)}}{D_{i,s}}, \qquad b_i = \frac{D_{i,s}^{(b)}}{D_{i,s}}; \qquad i = 1,2,\ldots,s-1,$$

$$a_i = \frac{D_{s,i}^{(a)}}{D_{s,i}}, \quad b_{i-1} = \frac{D_{s,i}^{(b)}}{D_{s,i}}; \qquad i = s+1, s+2, \ldots, n,$$

and if x_s and y_s are not both 0, then we obtain the last entry a_s as follows:

if $x_s \neq 0$, then $a_s = \lambda - b_1\dfrac{x_1}{x_s} - b_2\dfrac{x_2}{x_s} - \ldots - b_{s-1}\dfrac{x_{s-1}}{x_s} - b_s\dfrac{x_{s+1}}{x_s} - \ldots - b_{n-1}\dfrac{x_n}{x_s}$ and

if $y_s \neq 0,\ x_s = 0$, then $a_s = \mu - b_1\dfrac{y_1}{y_s} - b_2\dfrac{y_2}{y_s} - \ldots - b_{s-1}\dfrac{y_{s-1}}{y_s} - b_s\dfrac{y_{s+1}}{y_s} - \ldots - b_{n-1}\dfrac{y_n}{y_s}.$

5.3. Constructing Doubly Arrow Matrix C

In this part, we construct a doubly arrow matrix C of the form (1.3). Assume that $(\lambda, \mathbf{x})$ and $(\mu, \mathbf{y})$ are eigenpairs of matrix C. That is, $C\mathbf{x} = \lambda\mathbf{x}$ and $C\mathbf{y} = \mu\mathbf{y}$. We can rewrite these as

$$\left.\begin{aligned} a_1x_1+b_1x_2+b_2x_3+\ldots+b_{s-1}x_s &= \lambda x_1, \\ a_1y_1+b_1y_2+b_2y_3+\ldots+b_{s-1}y_s &= \mu y_1, \end{aligned}\right\} \tag{5.3.1}$$

$$\left.\begin{aligned} b_1x_1+a_2x_2 &= \lambda x_2, \\ b_1y_1+a_2x_2 &= \lambda y_2, \end{aligned}\right\} \tag{5.3.2}$$

$$\vdots$$

$$\left.\begin{aligned} b_{s-2}x_1+a_{s-1}x_{s-1} &= \lambda x_{s-1}, \\ b_{s-2}y_1+a_{s-1}y_{s-1} &= \mu y_{s-1}, \end{aligned}\right\} \tag{5.3.s – 1}$$

$$\left.\begin{aligned} b_{s-1}x_1+a_sx_s+b_sx_{s+1}+\ldots+b_{n-1}x_n &= \lambda x_s, \\ b_{s-1}y_1+a_sy_s+b_sx_{s+1}+\ldots+b_{n-1}y_n &= \mu y_s, \end{aligned}\right\} \tag{5.3.s}$$

$$\left.\begin{aligned} b_sx_s+a_{s+1}x_{s+1} &= \lambda x_{s+1}, \\ b_sy_s+a_{s+1}y_{s+1} &= \mu y_{s+1}, \end{aligned}\right\} \tag{5.3.s + 1}$$

$$\vdots$$

$$\left.\begin{aligned} b_{n-1}x_s+a_nx_n &= \lambda x_n, \\ b_{n-1}y_s+a_ny_n &= \mu y_n. \end{aligned}\right\} \tag{5.3.n}$$

To solve the system, we denote

$$\alpha = [a_1,b_1,a_2,b_2,\cdots,a_s,b_s,\cdots,a_{n-1},b_{n-1},a_n]^T,$$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{Z}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}; \quad i = 1,2,\ldots,n,$$

$$\mathbf{C}_i = \begin{bmatrix} \lambda x_i \\ \mu y_i \end{bmatrix} = \begin{bmatrix} c_i^{(1)} \\ c_i^{(2)} \end{bmatrix}; \quad i = 1,2,\ldots,n,$$

$$\gamma = \left[c_1^{(1)},c_1^{(2)},c_2^{(1)},c_2^{(2)},\cdots,c_s^{(1)},c_s^{(2)},\cdots,c_n^{(1)},c_n^{(2)}\right]^T,$$

$$\mathbf{B}_{1,i} = \begin{bmatrix} x_1 & x_i \\ y_1 & y_i \end{bmatrix}; \quad i = 2, 3, \ldots, s,$$

$$\mathbf{B}_{s,i} = \begin{bmatrix} x_s & x_i \\ y_s & y_i \end{bmatrix}; \quad i = s+1, s+2, \ldots, n,$$

$$D_{1,i} = \det(\mathbf{B}_{1,i}); \quad i = 2, 3, \ldots, s,$$

$$D_{s,i} = \det(\mathbf{B}_{s,i}); \quad i = s+1, s+2, \ldots, n,$$

$$D_{1,i}^{(a)} = \begin{vmatrix} x_1 & c_i^{(1)} \\ y_1 & c_i^{(2)} \end{vmatrix}, \quad D_{1,i}^{(b)} = \begin{vmatrix} c_i^{(1)} & x_i \\ c_i^{(2)} & y_i \end{vmatrix}; \quad i = 2, 3, \ldots, s-1,$$

$$D_{1,s}^{(a)} = \begin{vmatrix} x_1 & c_s^{(1)} - b_s x_{s+1} - \ldots - b_{n-1} x_n \\ y_1 & c_s^{(2)} - b_s y_{s+1} - \ldots - b_{n-1} y_n \end{vmatrix}, \quad D_{1,s}^{(b)} = \begin{vmatrix} c_s^{(1)} - b_s x_{s+1} - \ldots - b_{n-1} x_n & x_s \\ c_s^{(2)} - b_s x_{s+1} - \ldots - b_{n-1} x_n & y_s \end{vmatrix},$$

$$D_{s,i}^{(a)} = \begin{vmatrix} x_s & c_i^{(1)} \\ y_s & c_i^{(2)} \end{vmatrix}, \quad D_{s,i}^{(b)} = \begin{vmatrix} c_i^{(1)} & x_i \\ c_i^{(2)} & y_i \end{vmatrix}; \quad i = s+1, s+2, \ldots, n,$$

and

$$\mathbf{M} = \begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{Z}_2\mathbf{e}_1^T & \ldots & \mathbf{Z}_{s-1}\mathbf{e}_1^T & & & & & \\ & \mathbf{B}_{1,2} & & & & & & & \\ & & \ddots & & & & & & \\ & & & \mathbf{B}_{1,s-1} & & & & & \\ & & & & \mathbf{B}_{1,s} & \mathbf{Z}_{s+1}\mathbf{e}_1^T & \ldots & \mathbf{Z}_{n-1}\mathbf{e}_1^T & \mathbf{Z}_n\mathbf{e}_1^T \\ & & & & & \mathbf{B}_{s,s+1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \mathbf{B}_{s,n-1} & \\ & & & & & & & & \mathbf{B}_{s,n} \end{bmatrix}_{2n \times 2n-1},$$

where $\mathbf{B}_{1,1} = \mathbf{Z}_1$. We now simply rewrite the linear system of equations (5.3.1) – (5.3.n) as

$$\mathbf{M}\alpha = \gamma. \tag{5.C}$$

Observe that the solvability of the system (5.C) is equivalent to that of Problem 3.

The following corollary gives the necessary and sufficient conditions for problem 3 to have a unique real symmetric doubly arrow matrix C of the form (1.3).

Corollary 5.3. *Problem 3 has a unique solution if and only if the following conditions are satisfied:*

(i) $D_{1,i} \neq 0; \quad i = 2, 3, \ldots, s,$

(ii) $D_{s,i} \neq 0; \quad i = s+1, s+2, \ldots, n,$

(iii) $Rank(\mathbf{Z}_1) = Rank(\mathbf{Z}_1 | \mathbf{C}_1 - b_1\mathbf{Z}_2 - b_2\mathbf{Z}_3 - \ldots - b_{s-1}\mathbf{Z}_s) = 1.$

Proof. Since the solvability of linear system (5.C) is equivalent to that of Problem 3, Problem 3 has a unique solution if and only if linear system (5.C) has a unique solution. Therefore, linear system (5.C) has a unique solution if and only if the conditions (i) – (iii) are satisfied. □

All elements of doubly arrow matrix C can be found from the system of equations (5.3.1) – (5.3.n) and all conditions (i) – (iii), in a similar manner to that used in finding doubly arrow matrix A. We obtain

$$a_i = \frac{D_{1,i}^{(a)}}{D_{1,i}}, \qquad b_{i-1} = \frac{D_{1,i}^{(b)}}{D_{1,i}}; \qquad i = 2, 3, \ldots, s-1,$$

$$a_i = \frac{D_{s,i}^{(a)}}{D_{s,i}}, \qquad b_{i-1} = \frac{D_{s,i}^{(b)}}{D_{s,i}}; \qquad i = s+1, s+2, \ldots, n,$$

$$a_s = \frac{D_{1,s}^{(a)}}{D_{1,s}}, \qquad b_{s-1} = \frac{D_{1,s}^{(b)}}{D_{1,s}}.$$

If x_1 and y_1 are not both 0, then we obtain the last entry a_1 as follows:

if $x_1 \neq 0$, then $a_1 = \lambda - b_1\dfrac{x_2}{x_1} - b_2\dfrac{x_3}{x_1} - \ldots - b_{s-1}\dfrac{x_s}{x_1}$ and

if $y_1 \neq 0,\ x_1 = 0,$ then $a_1 = \mu - b_1\dfrac{y_2}{y_1} - b_2\dfrac{y_3}{y_1} - \ldots - b_{s-1}\dfrac{y_s}{y_1}.$

Next, we give some examples to illustrate our results.

Example 5.1. Given the two real numbers $\lambda = 1,\ \mu = -7$ and two nonzero real vectors $\mathbf{x} = [-1,-4,1,-2,-2]^T$, $\mathbf{y} = [1,-4,-3,6,0]^T$, using Theorem 5.1 with $s = 3$ we can construct a real symmetric doubly arrow matrix A of the form (1.1) as follows:

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 4 & 0 & 0 \\ 0 & -1 & -8 & 0 & 0 \\ 4 & -8 & 5 & 0 & 16 \\ 0 & 0 & 0 & -7 & 8 \\ 0 & 0 & 16 & 8 & 1 \end{pmatrix}.$$

Example 5.2. Given the two real numbers $\lambda = -1,\ \mu = 3$ and two nonzero real vectors $\mathbf{x} = [1,-3,-2,2,-1]^T$, $\mathbf{y} = [3,1,0,-1,-2]^T$, by using Corollary 5.2 with $s = 3$ we can construct a real symmetric doubly arrow matrix B of the form (1.2) as follows:

$$\mathbf{B} = \begin{pmatrix} 3 & 0 & 2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 2 & -6 & 14 & 4 & -2 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & -2 & 0 & 3 \end{pmatrix}.$$

Example 5.3. Given the two real numbers $\lambda = 1,\ \mu = -2$ and two nonzero real vectors $\mathbf{x} = [0,1,2,3,-2]^T$, $\mathbf{y} = [-1,1,-3,1,-1]^T$, using Corollary 5.3 with $s = 4$ we can construct a real symmetric doubly arrow matrix C of the form (1.3) as follows:

$$\mathbf{C} = \begin{pmatrix} 33 & 3 & -9 & 5 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ -9 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & -3 & -6 \\ 0 & 0 & 0 & -6 & -8 \end{pmatrix}.$$

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

This thesis focused on three inverse eigenvalue problems for constructing a real symmetric doubly arrow matrix of the form:

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & b_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{s-1} & b_{s-1} & 0 & \cdots & 0 & 0 \\ b_1 & b_2 & \cdots & b_{s-1} & a_s & 0 & \cdots & 0 & b_s \\ 0 & 0 & \cdots & 0 & 0 & a_{s+1} & \cdots & 0 & b_{s+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & \cdots & 0 & b_s & b_{s+1} & \cdots & b_{n-1} & a_n \end{pmatrix}$$

from three types of spectral information.

The first inverse eigenvalue problem was to construct the doubly arrow matrix from the minimal and maximal eigenvalues of all leading principal submatrices. We solved this problem by giving the sufficient condition for the existence of such a matrix. In addition, the sufficient conditions for the existence of the nonnegative doubly arrow matrix and the unique doubly arrow matrix were also given.

The second inverse eigenvalue problem was to construct the doubly arrow matrix from one of the eigenpairs and eigenvalues of all leading principal submatrices. The sufficient condition for the existence of such a unique matrix was given. We also gave the sufficient condition for the existence of the nonnegative doubly arrow matrix.

The third inverse eigenvalue problem was to construct the doubly arrow matrices from the two eigenpairs. For this problem, we gave the necessary and sufficient conditions for the existence of such a unique matrix. In particular, we were interested in constructing other real symmetric doubly arrow matrices that were introduced by other researchers. These are

$$\begin{pmatrix} a_1 & \cdots & 0 & b_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{s-1} & b_{s-1} & 0 & \cdots & 0 \\ b_1 & \cdots & b_{s-1} & a_s & b_s & \cdots & b_{n-1} \\ 0 & \cdots & 0 & b_s & a_{s+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n-1} & 0 & \cdots & a_n \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & b_1 & \cdots & b_{s-1} & 0 & \cdots & 0 \\ b_1 & a_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{s-1} & 0 & \cdots & a_s & b_s & \cdots & b_{n-1} \\ 0 & 0 & \cdots & b_s & a_{s+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 & \cdots & a_n \end{pmatrix}.$$

The necessary and sufficient conditions for the existence of such unique matrices have also been given.

For all inverse eigenvalue problems, we presented examples to illustrate our results.

Future Work: The following are possible extensions of this thesis.

1. We are interested in finding an explicit formula of eigenvectors for the real symmetric doubly arrow matrices.

2. As described in this thesis, the spectral information concerned may consist of complete or only partial information on eigenvalues or eigenvectors. We will investigate other spectral information that can be used to construct our matrix.

3. We may consider inverse eigenvalue problems for constructing other matrices that have a certain form.

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Oral Presentations

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