ON THE IMPROVEMENT OF THE BANACH CONTRACTION MAPPINGS PRINCIPLE IN VARIOUS DISTANCE SPACES AND APPLICATIONS

BY

MR. PATHAITHEP KUMROD

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
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ENTITIED

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was approved as partial fulfillment of the requirements for the degree of Master of Science (Mathematics)

on July 21, 2016

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Dissertation Title	ON THE IMPROVEMENT OF THE BANACH CONTRACTION MAPPINGS PRINCIPLE IN VARIOUS DISTANCE SPACES AND APPLICATIONS
Author	Mr. Pathaithep Kumrod
Degree	Master of Science (Mathematics)
Major Field/Faculty/University	Mathematics Faculty of Science and Technology Thammasat University
Thesis Advisor	Wutiphol Sintunavarat, Ph.D.
Academic Year	2015

ABSTRACT

The aim of this thesis is to improve and generalize the notion of Banach contraction mapping and to establish fixed point theorems for new mappings in spaces of various distances. The main structure of this thesis can be divided into two parts. In the first part, we extend and improve the condition of contraction of the results of Jleli *et al.* (2014) by applying the concept of control functions of type-K and type-S_1. We also present some applications to fixed point results in partial metric space and demonstrate the existence of solutions to integral equations. In the second part, we present a new generalization of the main results of Asadi *et al.* (2014) under more general contractive conditions in M-metric spaces by utilizing the control functions of type-S_1, type-S_2 and weakly α-admissible mapping. We also prove fixed point theorems for Chatterjea contraction mappings in the framework of M-metric spaces. This provides a partial answer to a question posed by Asadi *et al.* concerning a fixed point for Chatterjea contraction mappings. Our results improve, extend, and unify several results from the previous literature.

Keywords: Partial metric spaces, M-metric spaces, φ-fixed points, Control functions

ACKNOWLEDGEMENTS

I would like to thank my advisor Dr. Wutiphol Sintunavarat for giving me the opportunity to be a part of his group, and for all the education, supporting and encouragement. He was always there to meet and talk about new ideas and supporting me throughout my master degree study.

I would like to thank all of the thesis committee members, Asst. Prof. Dr. Anchalee Kaewcharoen, Asst. Prof. Dr. Chakkrid Klin-eam, and Assoc. Prof. Samran Muntup, for their attention to my work and advice on my thesis. I could not have finished this thesis without the guidance of my advisor and committee.

I would also like to thank the staff of the Department of Mathematics and Statistics at Thammasat University, especially Mrs. Malai Kaewklin for always being so helpful and friendly.

I thank to my parents for the patience, support and encouragement that they have given me throughout my study.

Finally, I would like to acknowledge the financial support from the Faculty of Science and Technology, Department of Mathematics and Statistics, Thammasat University.

Mr. Pathaithep Kumrod

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CHAPTER 1

INTRODUCTION

In 1922, Banach [3] proved a theorem which ensures the existence and uniqueness of the fixed point for self-mappings on metric spaces under appropriate conditions. His result is called the Banach contraction mapping principle or Banach fixed point theorem.

Theorem 1.0.1 ([3])**.** *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a contraction mapping, i.e., there is* $k \in [0,1)$ *such that*

$$d(Tx,Ty) \le kd(x,y) \tag{1.0.1}$$

for all $x,y \in X$*. Then* T *has a unique fixed point. Moreover, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = Tx_{n-1}$ *for all* $n \in \mathbb{N}$*, where* $x_0 \in X$*, converges to the fixed point of* T*.*

Example 1.0.2. Let $X = [0,1]$ be a usual metric space. Define $T : X \to X$ by $Tx = \frac{x}{2}$. Then it is easy to see that all the conditions of Theorem 1.0.1 are satisfied and so T has a unique fixed point in X.

Next, we will give some numerical examples for approximating a unique fixed point of T in Figure 1.1. The convergence behavior of these iterations is shown in Figure 1.2.

x_0	0.2000	0.4000	0.6000	0.8000
x_1	0.1000	0.2000	0.3000	0.4000
x_2	0.0500	0.1000	0.1500	0.2000
x_3	0.0250	0.0500	0.0750	0.1000
x_4	0.0125	0.0250	0.0375	0.0500
x_5	0.0063	0.0125	0.0188	0.0250
x_6	0.0031	0.0063	0.0094	0.0125
x_7	0.0016	0.0031	0.0047	0.0063
x_8	0.0008	0.0016	0.0023	0.0031
x_9	0.0004	0.0008	0.0012	0.0016
x_{10}	0.0002	0.0004	0.0006	0.0008
x_{11}	0.0001	0.0002	0.0003	0.0004
x_{12}	0.0000	0.0001	0.0001	0.0002

Figure 1.1: Iterates of Picard iterations in Example 1.0.2

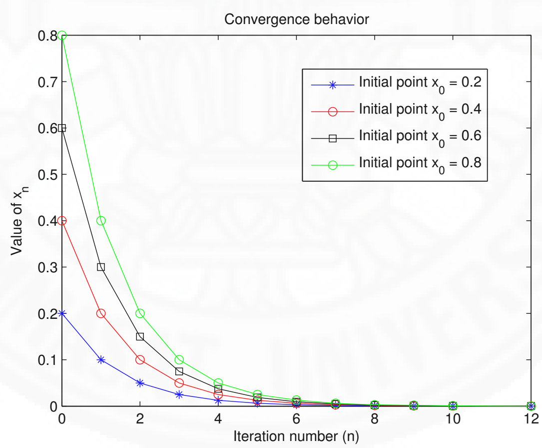

Figure 1.2: Convergence behavior for Example 1.0.2

The Banach contraction mapping principle guarantees the existence and uniqueness of the fixed points of nonlinear equations and provides a method for evaluating these fixed points. This principle is widely used in nonlinear analysis and has many useful applications and generalizations.

In fact, the origins of this principle are in the methods for solving nonlinear differential equations via successive approximations. However, since the Banach contraction principle is remarkable in its simplicity, a wide range of applications have been given in very different frameworks. In the last decades, a number of fixed point results have been obtained in attempts to generalize this principle. In parallel with the Banach contraction principle, Kannan [7] introduced a new contractive condition and proved the fixed point result for mappings satisfying this contraction.

Definition 1.0.3 ([7])**.** Let (X,d) be a metric space. A mapping $T : X \to X$ is called a *Kannan mapping* if there exists a constant $k \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx,Ty) \leq k[d(x,Tx)+d(y,Ty)] \text{ for all } x,y \in X. \tag{1.0.2}$$

Theorem 1.0.4 ([7])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a Kannan mapping. Then T has a unique fixed point. Moreover, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$, converges to the fixed point of T.*

Like Banach contraction Kannan mapping has a unique fixed points in a complete metric space. However, unlike the Banach condition, there exist discontinuous functions satisfying the definition of Kannan type mappings. Also, we note that Kannan's fixed point theorem is not an extension of the Banach contraction principle.

Example 1.0.5 ([7])**.** Let $X = [0,1]$ be a usual metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{4}, & x \in \left[0, \frac{1}{2}\right), \\ \frac{x}{5}, & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then it is easy to see that all the conditions of Theorem 1.0.4 are satisfied for $k = \frac{3}{8}$. So T has a unique fixed point.

Next, we will give some numerical examples for approximating the unique fixed point of T in Figure 1.3. The convergence behavior of these iterations is shown in Figure 1.4.

x_0	0.2000	0.4000	0.6000	0.8000
x_1	0.0500	0.1000	0.1200	0.1600
x_2	0.0125	0.0250	0.0240	0.0320
x_3	0.0031	0.0063	0.0048	0.0064
x_4	0.0008	0.0016	0.0010	0.0013
x_5	0.0002	0.0004	0.0002	0.0003
x_6	0.0000	0.0001	0.0000	0.0001
x_7	0.0000	0.0000	0.0000	0.0000
x_8	0.0000	0.0000	0.0000	0.0000
x_9	0.0000	0.0000	0.0000	0.0000
x_{10}	0.0000	0.0000	0.0000	0.0000

Figure 1.3: Iterates of Picard iterations in Example 1.0.2

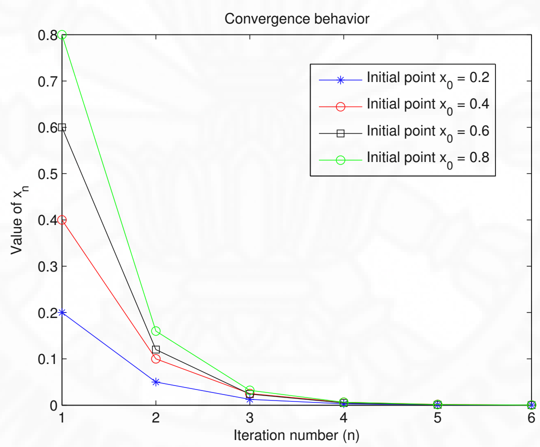

Figure 1.4: Convergence behavior for Example 1.0.5

In 1972, Chatterjea [4] introduced a new contractive condition and proved fixed point theorem for such mappings.

Definition 1.0.6 ([4]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *Chatterjea mapping* if there exists a constant $k \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{1.0.3}$$

Theorem 1.0.7 ([4]). *Let (X,d) be a complete metric space and let $T : X \to X$ be a Chatterjea mapping. Then T has a unique fixed point. Moreover, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X$, converges to the fixed point of T.*

In 2012, Samet [9] first introduced the concept known as α-admissible mapping. He also proved the fixed point theorems for nonlinear mappings in complete metric spaces by using the concept of α-admissibility.

Definition 1.0.8 ([9]). Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0,\infty)$ be a given mapping. We say that T is α-admissible if the following condition holds:

$$x,y \in X \text{ with } \alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1.$$

Example 1.0.9 ([9]). Let $X = [0,\infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \sqrt{x}$$

for all $x \in X$ and

$$\alpha(x,y) = \begin{cases} e^{x-y} & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then T is α-admissible.

Sintunavarat [10] weakened the concept of admissibility of mappings as follows:

Definition 1.0.10 ([10]). Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0,\infty)$ be a mapping. We say that T is weakly α-admissible if the following condition holds:

$$x \in X \text{ with } \alpha(x,Tx) \geq 1 \Longrightarrow \alpha(Tx,TTx) \geq 1.$$

Remark 1.0.11 ([10]). If T is an α-admissible mapping, then T is also a weakly α-admissible mapping. In general, the converse of the previous statement is not true.

Next, we give a example to show the real generalization of weakly α-admissibility of mappings. This example shows that T is a weakly α-admissible mapping, but not an α-admissible mapping.

Example 1.0.12. Let X=$\{1,2,3,...\}$. Define $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2xy^2 & \text{if } x,y \in \{1,2,3,4\}, \\ \dfrac{|x-y|}{xy^2} & \text{otherwise,} \end{cases}$$

and

$$Tx = \begin{cases} 7 & \text{if } x = 1, \\ 6 & \text{if } x = 2, \\ 5 & \text{if } x = 3, \\ x & \text{if } x = 4,5,6,\dots\,. \end{cases}$$

It is easy to see that T is not an α-admissible mapping. Indeed, for $x = 1, y = 3$, we see that

$$\alpha(x,y) = \alpha(1,3) = 18 \geq 1$$

but

$$\alpha(Tx,Ty) = \alpha(T1,T3) = \alpha(7,5) = \frac{2}{175} < 1.$$

Next, we show that T is a weakly α-admissible mapping. Suppose that $x \in X$ such that $\alpha(x,Tx) \geq 1$. Then we obtain that $x = 4$. Now we have

$$\alpha(Tx,TTx) = \alpha(T4,TT4) = \alpha(4,T4) = \alpha(4,4) = 128 \geq 1.$$

Therefore, T is a weakly α-admissible mapping.

Recently, Jleli *et al.* [6] introduced a new concept of φ-fixed points and contractive conditions. This is called the (F,φ)-contractive condition. Also, they proved the existentce of fixed point theorems for some (F,φ)-contraction mappings in metric space.

On the other hand, one of the extensions that has attracted attention is due to Matthews [8], who introduced in 1994 a Banach contractive mapping in a new space, which is called a partial metric space. This generalized the concept of a metric space in the sense that the distance from a point to itself need not be equal to zero. Several authors have obtained many useful fixed point results in these spaces.

Motivated by the notion of partial metrics spaces, Asadi *et al.* [2] proposed the concept of M-metric spaces, which is a generalization of partial metric space. They also established the fixed point theorem, which is a generalization of the Banach and Kannan fixed point theorems.

Theorem 1.0.13 ([2]). *Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in [0,1) \text{ such that } m(Tx,Ty) \leq km(x,y) \text{ for all } x,y \in X. \tag{1.0.4}$$

Then T has a unique fixed point.

Theorem 1.0.14 ([2]). *Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in \left[0,\frac{1}{2}\right) \text{ such that } m(Tx,Ty) \leq k[m(x,Tx)+m(y,Ty)] \text{ for all } x,y \in X. \tag{1.0.5}$$

Then T has a unique fixed point.

They also posed the following open problem.

Open problem 1.0.15 ([2]). *Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in \left[0,\frac{1}{2}\right) \text{ such that } m(Tx,Ty) \leq k[m(x,Ty)+m(y,Tx)] \text{ for all } x,y \in X. \tag{1.0.6}$$

Then T has a unique fixed point.

The main results presented in this thesis are divided into two parts, and the first part into two sections. The first section presents some new results on the existence of φ-fixed points for new nonlinear mappings in metric spaces, using control functions of type-K and type-S_1. The second section discusses some applications to fixed point results in partial metric spaces and confirms the existence of a solution for nonlinear integral equations, demonstrating the effectiveness of the main results.

The second part is also divided into two sections. The first section presents some fixed point results for new nonlinear mapping in M-metric spaces using control functions of type-S_1 and type-S_2 and the concept of weakly α-admissible mappings. Examples are provided to support our results. Our results extend and unify the main results of Asadi *et al.*. The second section presents a partial answer to a question posed by Asadi *et al.* concerning a fixed point for Chatterjea contraction mappings.

CHAPTER 2

PRELIMINARIES

In this thesis, the latter $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ denote the set of positive integers, non-negative real numbers and real numbers, respectively.

2.1 Metric spaces

Definition 2.1.1 ([5]). Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if it satisfies the following properties:

(d_1) $d(x,y) \geq 0$ for all $x,y \in X$ (non-negativity),

(d_2) $d(x,y) = 0$ if and only if $x = y$ (non-degenerated),

(d_3) $d(x,y) = d(y,x)$ for all $x,y \in X$ (symmetry),

(d_4) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$ (triangle inequality).

The ordered pair (X,d), where d is a metric on a nonempty set X, is called a *metric space.*

Example 2.1.2 ([5]). Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ be defined by

$$d(x,y) = |x-y|$$

for all $x,y \in \mathbb{R}$. Then d is a metric on $\mathbb{R}$ and it is called the *usual metric.*

Example 2.1.3 ([5]). Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then d is a metric on $\mathbb{R}^n$ and it is called the *Euclidean metric.*

Example 2.1.4 ([5])**.** Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$d(x,y) = \max_{1 \le i \le n} |x_i - y_i|$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then d is a metric on $\mathbb{R}^n$.

Example 2.1.5 ([5])**.** Let X be a nonempty set and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 0 \text{ if } & x = y, \\ 1 \text{ if } & x \neq y. \end{cases}$$

Then d is a metric on X and it is called the *discrete metric* on X.

Definition 2.1.6 ([5])**.** Let (X,d) be a metric space. For each $x \in X$ and each $\epsilon > 0$ the *open ball* with center x and radius ϵ is denoted by $B_\epsilon(x)$ and it is defined by

$$B_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}.$$

Definition 2.1.7 ([5])**.** Let (X,d) be a metric space. For each $x \in X$ and each $\epsilon > 0$ define the *closed ball* with center x and radius ϵ is denoted by $B_\epsilon[x]$ and it is defined by

$$B_\epsilon[x] = \{y \in X : d(x,y) \le \epsilon\}.$$

Definition 2.1.8 ([5])**.** Let (X,d_1) and (Y,d_2) be metric spaces. A mapping $T : X \to Y$ is called to be *continuous* at a point $x' \in X$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_1(x,x') < \delta \implies d_2(Tx,Tx') < \epsilon.$$

A mapping T is called to be continuous if it is continuous at every point of X.

Definition 2.1.9 ([5])**.** Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is called to be *convergent* to x if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon \text{ for all } n \ge N.$$

Theorem 2.1.10 ([5]). *A sequence $\{x_n\}$ in a metric space (X,d) converges to x if and only if*

$$d(x_n,x) \to 0, \text{ as } n \to \infty.$$

Definition 2.1.11 ([5]). Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_m,x_n) < \epsilon \text{ for all } m,n \geq N.$$

Theorem 2.1.12 ([5]). *Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if*

$$d(x_m,x_n) \to 0 \text{ as } m,n \to \infty.$$

Definition 2.1.13 ([5]). Let (X,d) be a metric space. A metric space X is called to be *complete* if every Cauchy sequence in X is convergent to an element of X.

2.2 Partial metric spaces

The notion of a partial metric space was introduced by Matthews [8] in 1994. In fact, a partial metric space is a generalization of metric spaces in which $d(x,x)$ are no longer necessarily zero.

Definition 2.2.1 ([8]). A mapping $p : X \times X \to [0,\infty)$, where X is a nonempty set, is said to be a partial metric (briefly, p-metric) if for any $x,y,z \in X$ satisfy the following conditions:

(p_1) $p(x,x) = p(y,y) = p(x,y) \Longleftrightarrow x = y$ (equality);

(p_2) $p(x,x) \leq p(x,y)$ (small self-distances);

(p_3) $p(x,y) = p(y,x)$ (symmetry);

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (triangularity).

The ordered pair (X,p) is also called a *partial metric space* (briefly, p-metric space).

Note that a metric is evidently a partial metric. However, a partial metric on X need not to be a metric on X. Here, we give some examples of a partial metric which is not a metric.

Example 2.2.2 ([8])**.** Let $X = [0,\infty)$ and the function $p : X \times X \to [0,\infty)$ be defined by

$$p(x,y) = \max\{x,y\}$$

for all $x,y \in X$. Then p is a partial metric, but if $x = y$, $p(x,y)$ may not be 0. Then it is not a metric on X.

Example 2.2.3 ([8])**.** Let $X = (-\infty,0]$ and the function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = -\min\{x,y\}.$$

Then p is a partial metric space on X, but if $x = y$, $p(x,y)$ may not be 0. Then it is not a metric on X.

Example 2.2.4 ([8])**.** Let $X = \{[a,b] : a,b \in \mathbb{R} \text{ and } a \leq b\}$ and the function $p : X \times X \to [0,\infty)$ defined by

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}.$$

Then p is a partial metric on X. But it is not metric on X. Indeed, if $x = y = [1,2]$, then $p([1,2],[1,2]) = 1$.

Each partial metric p on a nonempty set X generates a T_0-topology τ_p on X which has the family of open p-ball $\{B_p(x,\epsilon) : x \in X : \epsilon > 0\}$, where

$$B_p(x,\epsilon) := \{y \in X : p(x,y) < p(x,x) + \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$, forms a base of τ_p.

Next, we will give the concepts of convergence, Cauchy sequence and completeness in partial metric spaces.

Definition 2.2.5 ([8])**.** Let (X,p) be a partial metric space.

- A sequence $\{x_n\} \subset X$ converges to some $x \in X$ with respect to p if and only if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

- A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

- The partial metric space (X, p) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

Notice that the limit of a sequence in a partial metric space is not necessarily unique as the following example shows.

Example 2.2.6 ([8])**.** Let $X = [0, \infty)$ and the function $p : X \times X \to [0, \infty)$ be defined by

$$p(x, y) = \max\{x, y\}$$

for all $x, y \in X$. Consider the sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}$ in X.
Notice that

$$\lim_{n\to\infty} p(x_n, 1) = \lim_{n\to\infty} \max\left\{1 + \frac{1}{n}, 1\right\} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

and

$$\lim_{n\to\infty} p(x_n, 2) = \lim_{n\to\infty} \max\left\{1 + \frac{1}{n}, 2\right\} = \lim_{n\to\infty} 2 = 2.$$

Moreover, for any $a \geq 1$ we have $\lim_{n\to\infty} p(x_n, a) = a$.

Remark 2.2.7 ([8])**.** If p is a partial metric on X, then the function $d_p : X \times X \to [0, \infty)$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad x, y \in X \tag{2.2.1}$$

is a metric on X.

Lemma 2.2.8 ([8])**.** *Let (X, p) be a partial metric space. Then* *(i)* $\{x_n\}$ *is a Cauchy sequence in* (X,p) *if and only if* $\{x_n\}$ *is a Cauchy sequence in the metric space* (X,d_p).

(ii) *The partial metric space* (X,p) *is complete if and only if the metric space* (X,d_p) *is complete. Moreover, for each* $\{x_n\}$ *in* X *and* $x \in X$, *we have*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Longleftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

2.3 M-metric spaces

Motivated by the notion of partial metric spaces, Asadi *et al.* [2] proposed the concept of M-metric spaces which is a generalization of partial metric spaces. They also studied the topology on M-metric spaces.

For a nonempty set X and a function $m : X \times X \to \mathbb{R}_+$, the following notation are useful in the sequel:

1. $m_{x,y} := \min\{m(x,x), m(y,y)\}$;

2. $M_{x,y} := \max\{m(x,x), m(y,y)\}$.

Definition 2.3.1 ([2])**.** Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}_+$ is called an m-metric if the following conditions are satisfied for all $x,y,z \in X$:

(m_1) $m(x,x) = m(y,y) = m(x,y) \Leftrightarrow x = y$;

(m_2) $m_{x,y} \leq m(x,y)$;

(m_3) $m(x,y) = m(y,x)$;

(m_4) $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$.

Also, the ordered pair (X,m) is called an M-metric space.

Lemma 2.3.2 ([2])**.** *Every partial metric space is an* M*-metric space.*

From Lemma 2.3.2, we can write the following figure.

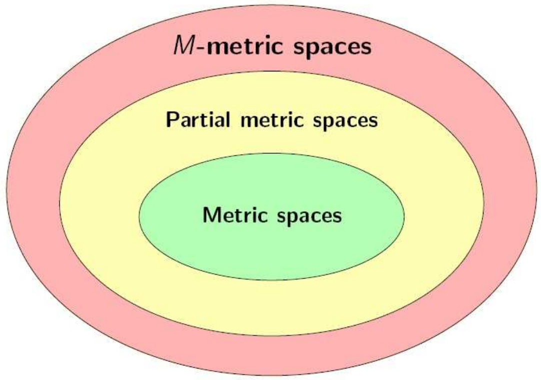

Figure 2.1: Relation between metric space, partial metric space and M-metric space

Next, we give some examples of m-metrics which are not partial metrics.

Example 2.3.3 ([2])**.** Let $X = [0,\infty)$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by

$$m(x,y) = \frac{x+y}{2}$$

for all $x,y \in X$. Then m is an m-metric on X, but it is not a partial metric.

Example 2.3.4. Let $X = \{1,2,3\}$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by

$$m(x,y) = \begin{cases} 1, & x = y = 1, \\ 9, & x = y = 2, \\ 5, & x = y = 3, \\ 10, & x,y \in \{1,2\} \text{ and } x \neq y, \\ 7, & x,y \in \{1,3\} \text{ and } x \neq y, \\ 8, & x,y \in \{2,3\} \text{ and } x \neq y. \end{cases}$$

Then m is an m-metric but it is not a partial metric.

Remark 2.3.5 ([2])**.** For every x,y in M-metric space (X,m), the following assertions hold:

1. $0 \leq M_{x,y} + m_{x,y} = m(x,x) + m(y,y)$;

2. $0 \le M_{x,y} - m_{x,y} = |m(x,x) - m(y,y)|$;

3. $M_{x,y} - m_{x,y} \le (M_{x,z} - m_{x,z}) + (M_{z,y} - m_{z,y})$.

If m is an m-metric on a nonempty set X, then the function $m^w, m^s : X \times X \to \mathbb{R}_+$ which are defined by

$$m^w(x,y) := m(x,y) - 2m_{x,y} + M_{x,y}$$

and

$$m^s(x,y) := \begin{cases} m(x,y) - m_{x,y}, & x \neq y, \\ 0, & x = y \end{cases}$$

are metrics on X.

Remark 2.3.6 ([2]). If m is an m-metric on a nonempty set X, then the following assertions hold for all $x, y \in X$:

1. $|m^w(x,y) - m(x,y)| \le M_{x,y}$;

2. $|m^s(x,y) - m(x,y)| \le M_{x,y}$.

Each m-metric m on nonempty set X generates a T_0-topology τ_m on X which has the family of open m-ball $\{B_m(x,\epsilon) : x \in X, \epsilon > 0\}$, where

$$B_m(x,\epsilon) := \{y \in X : m(x,y) < m_{x,y} + \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$, form a base of τ_m.

Next, we will give the concepts of convergence, m-Cauchy sequence and completeness in M-metric spaces.

Definition 2.3.7 ([2]). Let (X,m) be an M-metric space.

- A sequence $\{x_n\}$ in an M-metric space (X,m) converges to point $x \in X$ if and only if

$$\lim_{n\to\infty} (m(x_n,x) - m_{x_n,x}) = 0. \tag{2.3.1}$$

- A sequence $\{x_n\}$ in an M-metric space (X,m) is called an m-Cauchy sequence if

$$\lim_{n,m\to\infty}(m(x_n,x_m)-m_{x_n,x_m}) \text{ and } \lim_{n,m\to\infty}(M_{x_n,x_m}-m_{x_n,x_m}) \tag{2.3.2}$$

exist (and are finite).

- An M-metric space (X,m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x\in X$ such that

$$\lim_{n\to\infty}(m(x_n,x)-m_{x_n,x})=0 \text{ and } \lim_{n\to\infty}(M_{x_n,x}-m_{x_n,x})=0.$$

Lemma 2.3.8 ([2])**.** *Let (X,m) be an m-metric space. Then*

1. *$\{x_n\}$ is an m-Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X,m^w).*
2. *An M-metric space (X,m) is complete if and only if the metric space (X,m^w) is complete. Furthermore, for each $\{x_n\}$ in X and $x\in X$, we have*

$$\lim_{n\to\infty}m^w(x_n,x)=0 \Longleftrightarrow \left(\lim_{n\to\infty}(m(x_n,x)-m_{x_n,x})=0 \text{ and } \lim_{n\to\infty}(M_{x_n,x}-m_{x_n,x})=0\right).$$

Likewise the above definition holds also for m^s.

Example 2.3.9. Let $X=[0,\infty)$ and a function $m:X\times X\to\mathbb{R}_+$ be defined by

$$m(x,y)=\frac{x+y}{2}$$

for all $x,y\in X$. Then (X,m) is a complete M-metric space.

Proof. For $x,y\in X$, we have

$$\begin{aligned} m^w(x,y) &= m(x,y)-2m_{x,y}+M_{x,y} \\ &= \frac{x+y}{2}-2\min\{x,y\}+\max\{x,y\}. \end{aligned} \tag{2.3.3}$$

We consider the following two cases.

Case 1 : Suppose that $x\geq y$. From (2.3.3), we get

$$m^w(x,y)=\frac{x+y}{2}-2y+x=\frac{3}{2}|x-y|.$$

Case 2 : Suppose that $x < y$. From (2.3.3), we get

$$m^w(x,y) = \frac{x+y}{2} - 2x + y = \frac{3}{2}|y-x|.$$

Now we obtain that $m^w(x,y) = \frac{3}{2}|x-y|$ for all $x, y \in X$. Since we know that the closed interval $[0,\infty)$ with the usual metric is a complete metric space, thus (X, m^w) is also complete metric space. Using Lemma 2.3.8, we get (X,m) is a complete M-metric space. This completes the proof. □

Lemma 2.3.10 ([2])**.** *Let (X,m) be an M-metric space and $\{x_n\}, \{y_n\}$ be sequences in X. Assume that $x_n \to x \in X$ and $y_n \to y \in X$ as $n \to \infty$. Then*

$$\lim_{n\to\infty}(m(x_n,y_n) - m_{x_n,y_n}) = m(x,y) - m_{x,y}.$$

Lemma 2.3.11 ([2])**.** *Let (X,m) be an M-metric space and $\{x_n\}$ be a sequence in X. Assume that $x_n \to x \in X$ as $n \to \infty$. Then*

$$\lim_{n\to\infty}(m(x_n,y) - m_{x_n,y}) = m(x,y) - m_{x,y}$$

for all $y \in X$.

Lemma 2.3.12 ([2])**.** *Let (X,m) be an M-metric space and $\{x_n\}$ be a sequence in X. Assume that $x_n \to x \in X$ and $x_n \to y \in X$ as $n \to \infty$. Then $m(x,y) = m_{x,y}$. Furthermore, if $m(x,x) = m(y,y)$, then $x = y$.*

Lemma 2.3.13 ([2])**.** *Let $\{x_n\}$ be a sequence in an M-metric space (X,m) and there exists $r \in [0,1)$ such that*

$$m(x_{n+2},x_{n+1}) \leq r m(x_{n+1},x_n), \tag{2.3.4}$$

for all $n \in \mathbb{N}$. Then the following assertions hold:

(A) $\lim_{n\to\infty} m(x_{n+1},x_n) = 0$;

(B) $\lim_{n\to\infty} m(x_n,x_n) = 0$;

(C) $\lim_{n,m\to\infty} m_{x_m,x_n} = 0$;

(D) *$\{x_n\}$ is an m-Cauchy sequence.*

2.4 Fixed points and φ-fixed points

Definition 2.4.1. Let X be a nonempty set and let $T : X \to X$ be a mapping. A point $x \in X$ such that

$$Tx = x$$

is called a *fixed point* of T.

Example 2.4.2. Let $X = \mathbb{R}$ and $T : X \to X$ defined by

$$Tx = 2x - 2$$

for all $x \in X$. Then 2 is a fixed point of T, because $T(2) = 2$.

Example 2.4.3. Let $X = [0,1]$ and $T : X \to X$ defined by

$$Tx = \begin{cases} \frac{1}{2} + 2x, & x \in \left[0, \frac{1}{4}\right), \\ \frac{1}{2}, & x \in \left(\frac{1}{4}, 1\right] \end{cases}$$

for all $x \in X$. Then $\frac{1}{2}$ is a fixed point of T,

Example 2.4.4. Let $X = \mathbb{R}$ and $T : X \to X$ defined by

$$Tx = 2x^2 - x + 5$$

for all $x \in X$. Then T has no fixed points.

Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $T : X \to X$ be an mapping. We denote the set of all fixed points of T by

$$F_T := \{x \in X : Tx = x\}.$$

We denote the set of all zeros of the function φ by

$$Z_\varphi := \{x \in X : \varphi(x) = 0\}.$$

In 2014, Jleli *et al.* [6] introduced the concepts of φ-fixed points as follows:

Definition 2.4.5 ([6])**.** Let X be a nonempty set and $\varphi : X \to [0,\infty)$ be a given function. An element $z \in X$ is called a φ-fixed point of the mapping $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$.

Example 2.4.6. Let $X = [1,\infty)$, a mapping $T : X \to X$ be defined by

$$Tx = x^2 - x + 1,$$

and a function $\varphi : X \to [0,\infty)$ defined by

$$\varphi(x) = \ln x$$

for all $x \in X$. Then 1 is a φ-fixed point of T.

Example 2.4.7. Let $k \in [0,2)$, $X = [0,\infty)$, a mapping $T : X \to X$ be defined by

$$Tx = (4-k^2)x$$

for all $x \in X$ and a function $\varphi : X \to [0,\infty)$ defined by

$$\varphi(x) = \begin{cases} \frac{x}{3}, & x \leq 4, \\ \frac{x^2+2k+1}{2x^2}, & x \geq 4. \end{cases}$$

Then 0 is a φ-fixed point of T.

Example 2.4.8. Let $X = [0,\infty)$, a mapping $T : X \to X$ be defined by

$$Tx = 2x^2 - 6x + 6,$$

for all $x \in X$ and a function $\varphi : X \to [0,\infty)$ be defined by

$$\varphi(x) = \begin{cases} x, & x \in [0,3), \\ 0, & x \in [3,\infty). \end{cases}$$

Then $\frac{3}{2}$ and 2 are fixed points of T, but not φ-fixed points of T.

They also introduced the ideas of φ-Picard mappings and weakly φ-Picard mappings as follows:

Definition 2.4.9 ([6])**.** Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function. A mapping $T : X \to X$ is called a φ-Picard mapping if and only if

(i) $F_T \cap Z_\varphi = \{z\}$, where $z \in X$,

(ii) $T^n x \to z$ as $n \to \infty$, for each $x \in X$.

Definition 2.4.10 ([6])**.** Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function. We say that the mapping $T : X \to X$ is a weakly φ-Picard mapping if and only if

(i) T has at least one φ-fixed point,

(ii) the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is a φ-fixed point of T.

CHAPTER 3

FIXED POINT AND φ-FIXED POINT RESULTS IN METRIC SPACES

3.1 Overview

Recently, in 2014, Jleli *et al.* introduced the new concept of control function $F : [0,\infty)^3 \to [0,\infty)$ satisfies the following conditions:

(F1) $\max\{a,b\} \le F(a,b,c)$, for all $a,b,c \in [0,\infty)$;

(F2) $F(0,0,0) = 0$;

(F3) F is continuous.

Throughout this thesis unless otherwise specified, the class of all functions satisfying the conditions (F1)-(F3) is denoted by $\mathcal{F}$.

Example 3.1.1 ([6])**.** Let $F_1, F_2, F_3 : [0,\infty)^3 \to [0,\infty)$ be defined by

$$F_1(a,b,c) = a+b+c,$$

$$F_2(a,b,c) = \max\{a,b\}+c,$$

$$F_3(a,b,c) = a+a^2+b+c$$

for all $a,b,c \in [0,\infty)$. Then $F_1, F_2, F_3 \in \mathcal{F}$.

By using the control function in $\mathcal{F}$, Jleli *et al.* [6] defined the new contractive conditions and proved the φ-fixed point results as follows:

Definition 3.1.2 ([6])**.** Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ)-contraction with respect to the metric d if and only if there is $k \in (0,1)$ such that

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \le kF(d(x,y),\varphi(x),\varphi(y)) \tag{3.1.1}$$

for all $x,y \in X$.

Definition 3.1.3 ([6]). Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, and $F \in \mathcal{F}$. We say that the operator $T : X \to X$ is a graphic (F,φ)-contraction with respect to the metric d if and only if there is $k \in (0,1)$ such that

$$F(d(T^2x,Tx),\varphi(T^2x),\varphi(Tx)) \leq kF(d(Tx,x),\varphi(Tx),\varphi(x)) \tag{3.1.2}$$

for all $x \in X$.

Definition 3.1.4 ([6]). Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ)-weak contraction with respect to the metric d if and only if there are $k \in (0,1)$ and $L \geq 0$ such that

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \ \leq \ & kF(d(x,y),\varphi(x),\varphi(y)) \\ & +L[F(d(y,Tx),\varphi(y),\varphi(Tx)) \\ & -F(0,\varphi(y),\varphi(Tx))] \end{aligned} \tag{3.1.3}$$

for all $x,y \in X$.

Theorem 3.1.5 ([6]). *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. Suppose that the following conditions hold:*

(H1) φ is lower semi-continuous, i.e, if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ implies that $\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n)$;

(H2) $T : X \to X$ is an (F,φ)-contraction with respect to the metric d.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a φ-Picard mapping;

(iii) for all $x \in X$ and for all $n \in \mathbb{N}$, we have

$$d(T^n x, z) \leq \tfrac{k^n}{1-k} F(d(Tx,x),\varphi(Tx),\varphi(x)),$$

where $\{z\} \in F_T \cap Z_\varphi = F_T$.

Theorem 3.1.6 ([6]). *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. Suppose that the following conditions hold:*

(H1) φ is lower semi-continuous;

(H2) $T : X \to X$ is a graphic (F,φ)-contraction with respect to the metric d;

(H3) T is continuous.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a weakly φ-Picard mapping;

(iii) for all $x \in X$, if $T^n x \to z$ as $n \to \infty$, then

$$d(T^n x, z) \leq \tfrac{k^n}{1-k} F(d(Tx,x), \varphi(Tx), \varphi(x)),\ n \in \mathbb{N}.$$

Theorem 3.1.7 ([6]). *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, and $F \in \mathcal{F}$. Suppose that the following conditions hold:*

(H1) φ is lower semi-continuous;

(H2) $T : X \to X$ is an (F,φ)-weak contraction with respect to the metric d;

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a weakly φ-Picard mapping;

(iii) for all $x \in X$, if $T^n x \to z$ as $n \to \infty$, then

$$d(T^n x, z) \leq \tfrac{k^n}{1-k} F(d(x,Tx), \varphi(x), \varphi(Tx)).$$

In Chapter 3 we introduce the concept of generalized (F,φ)-contraction mapping in metric spaces and establish φ-fixed point results for such mappings. The presented theorems extend and generalize the φ-fixed point results of Jleli *et al.* [6]. We also show that the fixed point theorem on partial metric spaces can be applied to our main results on metric spaces and investigate the problem of the existence of solutions to integral and differential equations.

3.2 Fixed point and φ-fixed point results by using a control function type-K

Let J be the set of all functions $\theta : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(j_1) θ is a nondecreasing function, i.e., $t_1 < t_2$ implies $\theta(t_1) \leq \theta(t_2)$;

(j_2) θ is continuous;

(j_3) $\lim\limits_{n\to\infty} \theta^n(t) = 0$ for all $t \in (0,\infty)$;

(j_4) $\sum\limits_{n=0}^{\infty} \theta^n(t) < \infty$ for all $t > 0$.

Lemma 3.2.1. *If $\theta \in J$, then $\theta(t) < t$ for all $t > 0$.*

Proof. Assume that $\theta(t) \geq t$ for some $t > 0$. From (j_1), we get $\theta^n(t) \geq t$. Taking limit as $n \to \infty$, we get $t = 0$, which is a contradiction. □

Remark 3.2.2. From (j_1) and Lemma 3.2.1, we have $\theta(0) = 0$. Indeed, if we take $t_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, we obtain that $0 \leq \theta(t_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. It yields that $0 \leq \lim\limits_{n\to\infty} \theta(t_n) = \theta(0) \leq 0$ and hence $\theta(0) = 0$.

Example 3.2.3. The function $\theta : [0,\infty) \to [0,\infty)$ defined by

$$\theta(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ k\ln t, & t \geq 1, \end{cases}$$

where $k \in [0,1)$, belongs to J. The graph of θ for some case is given in Figure 3.1.

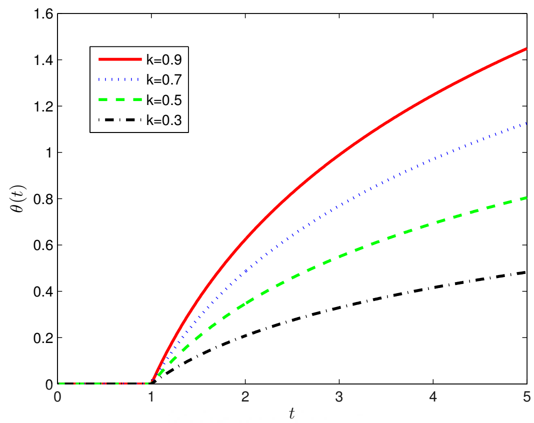

Figure 3.1: The graph of θ for $k = 0.3, 0.5, 0.7, 0.9$.

Here we define the new contractive condition in metric spaces as follows:

Definition 3.2.4. Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. The mapping $T : X \to X$ is said to be an (F,φ,θ)-contraction with respect to the metric d if and only if

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \theta(F(d(x,y),\varphi(x),\varphi(y))) \tag{3.2.1}$$

for all $x,y \in X$.

Now we give the existence of φ-fixed point results for (F,φ,θ)-contraction mappings, which is the first main result in this work.

Theorem 3.2.5. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. Assume that the following conditions are satisfied:*

(H1) φ is lower semi-continuous;

(H2) $T : X \to X$ is an (F,φ,θ)-contraction with respect to the metric d.

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) T is a φ-Picard mapping.

Proof. (i) Suppose that $\xi \in F_T$. Applying (3.2.1) with $x = y = \xi$, we get

$$F(0, \varphi(\xi), \varphi(\xi)) \leq \theta(F(0, \varphi(\xi), \varphi(\xi))).$$

By Lemma 3.2.1, we obtain that

$$F(0, \varphi(\xi), \varphi(\xi)) = 0. \tag{3.2.2}$$

From (F1), we have

$$\varphi(\xi) \leq F(0, \varphi(\xi), \varphi(\xi)). \tag{3.2.3}$$

Using (3.2.2) and (3.2.3), we get $\varphi(\xi) = 0$ and then $\xi \in Z_\varphi$. This means that $F_T \subseteq Z_\varphi$.

(ii) We assume that $x \in X$ is an arbitrary point. Using (3.2.1), we have

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \leq \theta(F(d(T^n x, T^{n-1}x), \varphi(T^n x), \varphi(T^{n-1}x)))$$

for all $n \in \mathbb{N}$. By induction, for each $n \in \mathbb{N}$, we get

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \leq \theta^n(F(d(Tx, x), \varphi(Tx), \varphi(x))),$$

which implies by property (F1) that

$$\begin{aligned} \max\{d(T^{n+1}x, T^n x), \varphi(T^{n+1}x)\} &\leq F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \\ &\leq \theta^n(F(d(Tx, x), \varphi(Tx), \varphi(x))) \end{aligned} \tag{3.2.4}$$

for all $n \in \mathbb{N}$. From (3.2.4), we have

$$d(T^{n+1}x, T^n x) \leq \theta^n(F(d(Tx, x), \varphi(Tx), \varphi(x)))$$

for all $n \in \mathbb{N}$.

Now, we claim that $\{T^n x\}$ is a Cauchy sequence. Suppose that $m,n \in \mathbb{N}$ such that $m > n$. By using the triangle inequality, we get

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{m-1} x, T^m x) \\ &= \theta^n(F(d(Tx,x),\varphi(Tx),\varphi(x))) + \theta^{n+1}(F(d(Tx,x),\varphi(Tx),\varphi(x))) \\ &\quad + \cdots + \theta^{m-1}(F(d(Tx,x),\varphi(Tx),\varphi(x))) \\ &= \sum_{i=1}^{m-1} \theta^i(F(d(Tx,x),\varphi(Tx),\varphi(x))) \\ &\quad - \sum_{k=1}^{n-1} \theta^k(F(d(Tx,x),\varphi(Tx),\varphi(x))). \end{aligned} \tag{3.2.5}$$

From the above inequality and (j_4), we get $\lim_{m,n\to\infty} d(T^n x, T^m x) = 0$, which implies that $\{T^n x\}$ is a Cauchy sequence. Since (X,d) is complete, there is some $z \in X$ such that

$$\lim_{n\to\infty} d(T^n x, z) = 0. \tag{3.2.6}$$

Next, we need to verify that z is a φ-fixed point of T. Observe that from (3.2.4), we get

$$\varphi(T^{n+1}x) \leq \theta^n(F(d(Tx,x),\varphi(Tx),\varphi(x))).$$

Letting $n \to \infty$ in the above inequality, by (j_3), we get

$$\lim_{n\to\infty} \varphi(T^{n+1}x) = 0. \tag{3.2.7}$$

From hypothesis (H_1), and conditions (3.2.6) and (3.2.7), we have

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(T^{n+1}x) = 0. \tag{3.2.8}$$

From (3.2.1), we have

$$F(d(T^{n+1}x, Tz), \varphi(T^{n+1}x), \varphi(Tz)) \leq \theta(F(d(T^n x, z), \varphi(T^n x), \varphi(z))).$$

Letting $n \to \infty$ in the above inequality, using (3.2.6), (3.2.7), (3.2.8), (F2), and the continuity of F, we get

$$F(d(z,Tz),0,\varphi(Tz)) \leq \theta(F(0,0,0)) = 0,$$

which implies from condition (F1) that

$$d(z,Tz) = 0. \tag{3.2.9}$$

It follows from (3.2.8) and (3.2.9) that z is a φ-fixed point of T.

To verify that z is a unique φ-fixed point we note that z' be another φ-fixed point of T. Applying (3.2.1) with $x = z$ and $y = z'$, we obtain

$$F(d(z,z'),0,0) \leq \theta(F(d(z,z'),0,0)).$$

By Lemma 3.2.1, we obtain that $F(d(z,z'),0,0) = 0$ and hence $d(z,z') = 0$, that is, $z = z'$. This implies that the φ-fixed point is unique. So T is a φ-Picard mapping.

□

Next, we give some examples to illustrate Theorem 3.2.5 and also give some numerical results.

Example 3.2.6. Let $X = [0,3]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete metric space. Assume that $T : X \to X$ is defined by

$$Tx = \begin{cases} 0, & 0 \leq x < 2.5, \\ k\ln(\frac{x}{2}), & 2.5 \leq x \leq 3, \end{cases}$$

where $k \in [0,1)$, the function $\varphi : X \to [0,\infty)$ is defined by $\varphi(x) = x$ for all $x \in X$, the function $F : [0,\infty)^3 \to [0,\infty)$ is defined by $F(a,b,c) = a+b+c$ and the function $\theta : [0,\infty) \to [0,\infty)$ is defined by

$$\theta(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ k\ln t, & t \geq 1. \end{cases}$$

It is easy to see that $F \in \mathcal{F}$, $\theta \in J$ and φ is lower semi-continuous.

Next we will show that T is an (F,φ,θ)-contraction mapping. Suppose that $x,y \in X$. We will divide the proof into three cases.

Case 1: Assume that $x, y \in [0, 2.5)$. Then the claim is obvious.

Case 2: Assume that $x, y \in [2.5, 3]$. Without loss of generality, we may suppose that $x \geq y$. Then we obtain that

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty) \\
&= \left|k\ln\left(\frac{x}{2}\right)-k\ln\left(\frac{y}{2}\right)\right|+k\ln\left(\frac{x}{2}\right)+k\ln\left(\frac{y}{2}\right) \\
&\leq 2k\ln\left(\frac{3}{2}\right) \\
&< k\ln(5) \\
&\leq k\ln(d(x,y)+\varphi(x)+\varphi(y)) \\
&= k\ln(F(d(x,y),\varphi(x),\varphi(y))) \\
&= \theta(F(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}$$

Case 3: Assume that $(x, y) \in [0, 2.5) \times [2.5, 3] \cup [2.5, 3] \times [0, 2.5)$. Without loss of generality, we may suppose that $x \in [2.5, 3]$ and $y \in [0, 2.5)$. Then we obtain that

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty) \\
&= \left|k\ln\left(\frac{x}{2}\right)-0\right|+k\ln\left(\frac{x}{2}\right)+0 \\
&\leq 2k\ln\left(\frac{3}{2}\right) \\
&\leq k\ln(2.5) \\
&\leq k\ln(d(x,y)+\varphi(x)+\varphi(y)) \\
&= k\ln(F(d(x,y),\varphi(x),\varphi(y))) \\
&\leq \theta(F(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}$$

This shows that all conditions of Theorem 3.2.5 are satisfied and so T has a φ-fixed point in X. In this case, a unique φ-fixed point of T is a point 0.

We can see some numerical experiments for approximate the φ-fixed point of T in Figure 3.2. Furthermore, the convergence behavior of these iteration is shown in Figure 3.3.

Iterate for $k = 0.5$	$x_0 = 2.4$	$x_0 = 2.6$	$x_0 = 2.8$	$x_0 = 3$
x_1	0.09116	0.13118	0.16823	0.20273
x_2	0.00000	0.00000	0.00000	0.00000
x_3	0.00000	0.00000	0.00000	0.00000
x_4	0.00000	0.00000	0.00000	0.00000
x_5	0.00000	0.00000	0.00000	0.00000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Iterate for $k = 0.9$	**$x_0 = 2.4$**	**$x_0 = 2.6$**	**$x_0 = 2.8$**	**$x_0 = 3$**
x_1	0.16409	0.23612	0.30282	0.36492
x_2	0.00000	0.00000	0.00000	0.00000
x_3	0.00000	0.00000	0.00000	0.00000
x_4	0.00000	0.00000	0.00000	0.00000
x_5	0.00000	0.00000	0.00000	0.00000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 3.2: Iterates of Picard iterations

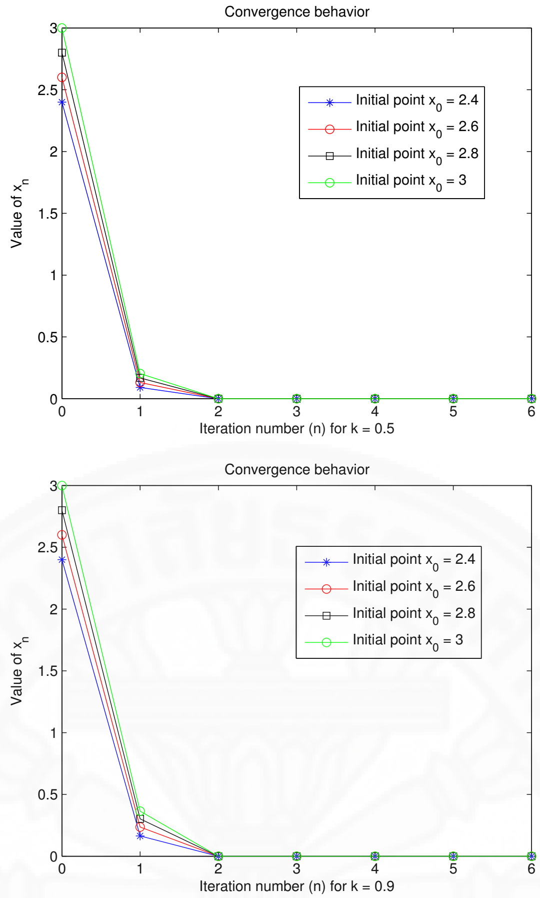

Figure 3.3: Convergence behavior for Example 3.2.6

Example 3.2.7. Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete metric space. Assume that $T : X \to X$ is defined by

$$Tx = \frac{kx^2}{2}$$

where $k \in [0,1)$, the function $\varphi : X \to [0,\infty)$ is defined by $\varphi(x) = x$ for all $x \in X$, the function $F : [0,\infty)^3 \to [0,\infty)$ is defined by $F(a,b,c) = a+b+c$ and the function $\theta : [0,\infty) \to [0,\infty)$ is defined by $\theta(t) = kt$ for all $t \in [0,\infty)$. It is easy to see that $F \in \mathcal{F}$, $\theta \in J$ and φ is lower semi-continuous.

Next we will show that T is (F,φ,θ)-contraction mapping. Suppose that $x,y \in X$, we get

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty) \\
&= \frac{k}{2}|x^2-y^2|+\frac{kx^2}{2}+\frac{ky^2}{2} \\
&= \frac{k}{2}|x+y||x-y|+\frac{kx^2}{2}+\frac{ky^2}{2} \\
&= k\left(\frac{|x+y||x-y|}{2}+\frac{x^2}{2}+\frac{y^2}{2}\right) \\
&\leq k(|x-y|+x+y) \\
&= k(d(x,y)+\varphi(x)+\varphi(y)) \\
&= k(F(d(x,y),\varphi(x),\varphi(y))) \\
&= \theta(F(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}$$

This shows that all conditions of Theorem 3.2.5 are satisfied and so T has a φ-fixed point in X. In this case, a unique φ-fixed point of T is a point 0.

We can see some numerical experiments for approximate the φ-fixed point of T in Figure 3.4. Furthermore, the convergence behavior of these iteration is shown in Figure 3.5.

Iterate for $k = 0.5$	$x_0 = 0.2$	$x_0 = 0.4$	$x_0 = 0.6$	$x_0 = 0.8$
x_1	0.010000	0.040000	0.090000	0.160000
x_2	0.000025	0.000400	0.002025	0.006400
x_3	0.000000	0.000000	0.000001	0.000010
x_4	0.000000	0.000000	0.000000	0.000000
x_5	0.000000	0.000000	0.000000	0.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Iterate for $k = 0.9$	$x_0 = 0.2$	$x_0 = 0.4$	$x_0 = 0.6$	$x_0 = 0.8$
x_1	0.018000	0.072000	0.162000	0.288000
x_2	0.000146	0.002333	0.011810	0.037325
x_3	0.000000	0.000002	0.000063	0.000627
x_4	0.000000	0.000000	0.000000	0.000000
x_5	0.000000	0.000000	0.000000	0.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 3.4: Iterates of Picard iterations

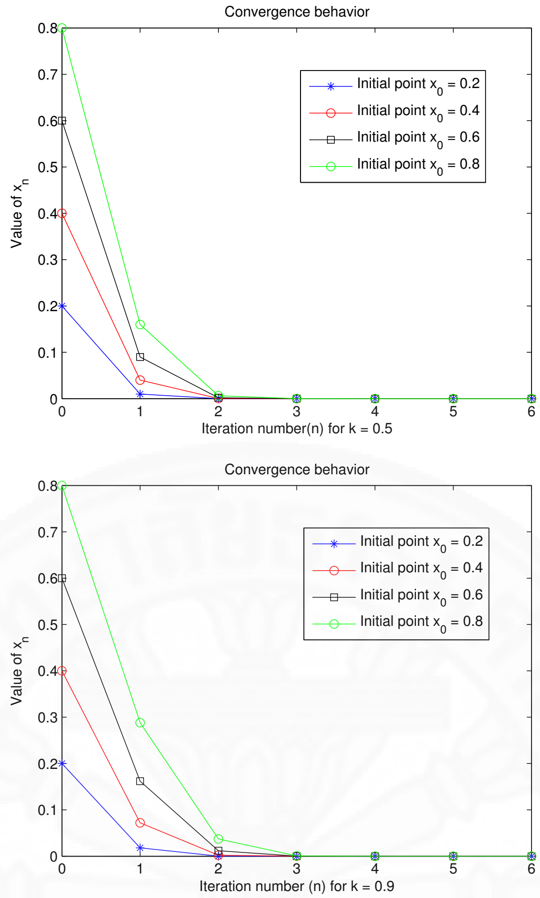

Figure 3.5: Convergence behavior for Example 3.2.7

Next we extend the contractive condition (3.1.3) and prove the second main result in this work.

Definition 3.2.8. Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. The mapping $T : X \to X$ is said to be an (F,φ,θ)-weak contraction with respect to the metric d if and only if

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \ \leq \ & \theta(F(d(x,y),\varphi(x),\varphi(y))) \\ & +L[F(N(x,y),\varphi(y),\varphi(Tx)) \\ & -F(0,\varphi(y),\varphi(Tx))] \end{aligned} \tag{3.2.10}$$

for all $x,y \in X$, where $N(x,y) := \min\{d(x,Tx),d(y,Ty),d(y,Tx)\}$ and $L \geq 0$.

Theorem 3.2.9. *Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. Assume that the following conditions are satisfied:*

(H1) φ *is lower semi-continuous;*

(H2) $T : X \to X$ *is an* (F,φ,θ)*-weak contraction with respect to the metric* d*.*

Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$*;*

(ii) T *is a weakly* φ*-Picard mapping.*

Proof. (i) Suppose that $\xi \in X$ is a fixed point of T. Applying (3.2.10) with $x = y = \xi$, we get

$$\begin{aligned} F(0,\varphi(\xi),\varphi(\xi)) &\leq \theta(F(0,\varphi(\xi),\varphi(\xi))) \\ &\quad +L[F(0,\varphi(\xi),\varphi(\xi)) - F(0,\varphi(\xi),\varphi(\xi))] \\ &= \theta(F(0,\varphi(\xi),\varphi(\xi))). \end{aligned}$$

From Lemma 3.2.1, which implies that

$$F(0,\varphi(\xi),\varphi(\xi)) = 0. \tag{3.2.11}$$

From $(F1)$, we have

$$\varphi(\xi) \leq F(0,\varphi(\xi),\varphi(\xi)). \tag{3.2.12}$$

Using (3.2.11) and (3.2.12), we obtain that $\varphi(\xi) = 0$, Then $F_T \subseteq Z_\varphi$.

(ii) We assume that $x \in X$ be an arbitrary point. Using (3.2.10), we have

$$\begin{aligned} F(d(T^{n+1}x,T^n x),\varphi(T^{n+1}x),\varphi(T^n x)) &\leq \theta(F(d(T^n x,T^{n-1}x),\varphi(T^n x),\varphi(T^{n-1}x))) \\ &\quad +L[F(0,\varphi(T^n x),\varphi(T^{n+1}x)) \\ &\quad -F(0,\varphi(T^n x),\varphi(T^{n+1}x))] \\ &= \theta(F(d(T^n x,T^{n-1}x),\varphi(T^n x),\varphi(T^{n-1}x))) \end{aligned}$$

for all $n \in \mathbb{N}$. Repeating this process, for each $n \in \mathbb{N}$, we get

$$F(d(T^{n+1}x,T^n x),\varphi(T^{n+1}x),\varphi(T^n x)) \leq \theta^n(F(d(Tx,x),\varphi(Tx),\varphi(x))).$$

The rest of the proof follows using similar argument to proof of Theorem 3.2.5.

□

Remark 3.2.10. If we take $\theta(t) := kt$ for all $t \in [0,\infty)$, where $k \in [0,1)$, then Theorem 3.2.5 and Theorem 3.2.9 reduce to Theorem 3.1.5 and Theorem 3.1.7, respectively. Based on the previous fact, our main results can be considered as a new contribution.

3.3 Fixed point and φ-fixed point results by using a control function type-S_1

Motivated and inspired by recently results of Jleli *et al.* and control function of Sintunavarat, we introduce new contractive condition for nonlinear mappings by using the ideas of φ-fixed point results and control function type-S_1.

Let X be a nonempty set and $T : X \to X$ be a given mapping. Throughout this thesis, we denote Λ_T^1 is the class of all functions of type-S_1, that is, $\lambda : X \to [0,1)$ satisfying $\lambda(Tx) \leq \lambda(x)$ for all $x \in X$.

Definition 3.3.1. Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ,λ)-contraction with respect to the metric d if and only if there exists a function $\lambda \in \Lambda_T^1$ such that

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \lambda(x)F(d(x,y),\varphi(x),\varphi(y)) \tag{3.3.1}$$

for all $x,y \in X$.

Theorem 3.3.2. *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, and $F \in \mathcal{F}$. Assume that $T : X \to X$ is an (F,φ,λ)-contraction with respect to the metric d. Then the following assertions hold:*

(i) $F_T \subseteq Z_\varphi$;

(ii) T *is a φ-Picard mapping;*

(iii) for all $x \in X$ and for all $n \in \mathbb{N}$, we have

$$d(T^n x, z) \leq \frac{[\lambda(x)]^n}{1-\lambda(x)} F(d(x,Tx), \varphi(x), \varphi(Tx)),$$

where $\{z\} \in F_T \cap Z_\varphi = F_T$.

Proof. First, we will prove that $F_T \subseteq Z_\varphi$. Let $\xi \in F_T$. Applying (3.3.1) with $x = y = \xi$, we get

$$F(0, \varphi(\xi), \varphi(\xi)) \leq \lambda(\xi) F(0, \varphi(\xi), \varphi(\xi)).$$

We obtain that

$$F(0, \varphi(\xi), \varphi(\xi)) = 0. \tag{3.3.2}$$

From (F1), we have

$$\varphi(\xi) \leq F(0, \varphi(\xi), \varphi(\xi)). \tag{3.3.3}$$

Using (3.3.2) and (3.3.3), we get $\varphi(\xi) = 0$ and then $\xi \in Z_\varphi$.
Next, we will show that T is a φ-Picard mapping. Let $x \in X$ be an arbitrary point. Using (3.3.1), we have

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \leq \lambda(T^n x) F(d(T^n x, T^{n-1}x), \varphi(T^n x), \varphi(T^{n-1}x)) \tag{3.3.4}$$

for all $n \in \mathbb{N}$. Since $\lambda \in \Lambda_T^1$, the relation (3.3.4) implies that

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \leq \lambda(x) F(d(T^n x, T^{n-1}x), \varphi(T^n x), \varphi(T^{n-1}x))$$

for all $n \in \mathbb{N}$. By induction, for each $n \in \mathbb{N}$, we get

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \leq [\lambda(x)]^n F(d(Tx, x), \varphi(Tx), \varphi(x)),$$

which implies by property (F1) that

$$\begin{aligned} \max\{d(T^{n+1}x, T^n x), \varphi(T^{n+1}x)\} &\leq F(d(T^n x, T^{n-1}x), \varphi(T^n x), \varphi(T^{n-1}x)) \\ &\leq [\lambda(x)]^n F(d(Tx, x), \varphi(Tx), \varphi(x)) \qquad (3.3.5) \end{aligned}$$

for all $n \in \mathbb{N}$. From (3.3.5), we have

$$d(T^{n+1}x, T^n x) \leq [\lambda(x)]^n F(d(Tx, x), \varphi(Tx), \varphi(x)).$$

Now, we show that $\{T^n x\}$ is a Cauchy sequence. Suppose that $m,n \in \mathbb{N}$ such that $m > n$. By using the triangle inequality, we get

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{m-1}x, T^m x) \\ &= [\lambda(x)]^n F(d(Tx,x),\varphi(Tx),\varphi(x)) \\ &\quad + [\lambda(x)]^{n+1} F(d(Tx,x),\varphi(Tx),\varphi(x)) \\ &\quad + \ldots + [\lambda(x)]^{m-1} F(d(Tx,x),\varphi(Tx),\varphi(x)) \\ &\leq \frac{[\lambda(x)]^n}{1-\lambda(x)} F(d(Tx,x),\varphi(Tx),\varphi(x)). \end{aligned} \tag{3.3.6}$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{m,n\to\infty} d(T^n x, T^m x) = 0$ since $\lambda(x) \in [0,1)$. This yields that $\{T^n x\}$ is a Cauchy sequence. Since (X,d) is complete, there is $z \in X$ such that

$$\lim_{n\to\infty} d(T^n x, z) = 0. \tag{3.3.7}$$

Next, we need to verify that z is a φ-fixed point of T. Observe that from (3.3.5), we get

$$\varphi(T^{n+1}x) \leq [\lambda(x)]^n F(d(Tx,x),\varphi(Tx),\varphi(x)).$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \varphi(T^{n+1}x) = 0. \tag{3.3.8}$$

From hypothesis (H_1), and conditions (3.3.7) and (3.3.8), we have

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(T^{n+1}x) = 0. \tag{3.3.9}$$

Using (3.3.1), we have

$$\begin{aligned} F(d(T^{n+1}x,Tz),\varphi(T^{n+1}x),\varphi(Tz)) &\leq \lambda(T^n x) F(d(T^n x,z),\varphi(T^n x),\varphi(z)) \\ &\leq \lambda(T^{n-1}x) F(d(T^n x,z),\varphi(T^n x),\varphi(z)) \\ &\ \ \vdots \\ &\leq \lambda(x) F(d(T^n x,z),\varphi(T^n x),\varphi(z)). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (3.3.7), (3.3.8), (3.3.9), (F2), and the continuity of F, we get

$$F(d(z,Tz),0,\varphi(Tz)) \leq \lambda(x)F(0,0,0) = 0,$$

which implies from condition (F1) that

$$d(z,Tz) = 0. \tag{3.3.10}$$

It follows from (3.3.9) and (3.3.10) that z is a φ-fixed point of T.

To verify that z is a unique φ-fixed point we note that z' be another φ-fixed point of T. Applying (3.3.1) with $x = z$ and $y = z'$, we obtain

$$F(d(z,z'),0,0) \leq \lambda(z)F(d(z,z'),0,0),$$

which implies that $F(d(z,z'),0,0) = 0$ and hence $d(z,z') = 0$, that is, $z = z'$. This implies that the φ-fixed point is unique. This completes the proof. $\square$

Now, we give some example to support validity of above Theorem.

Example 3.3.3. Let $X = [0,\infty)$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete metric space. Assume that $T : X \to X$, $\varphi : X \to [0,\infty)$, $F : [0,\infty)^3 \to [0,\infty)$ and $\lambda : X \to [0,1)$ are defined by

$$Tx = \begin{cases} 0, & 0 \leq x < 2, \\ \frac{1}{x}, & x \geq 2, \end{cases}$$

$$\varphi(x) = x \quad \text{for all } x \in X,$$

$$F(a,b,c) = a+b+c \quad \text{for all } a,b,c \in [0,\infty),$$

$$\lambda(x) = \max\left\{\frac{x}{1+x}, \frac{3}{4}\right\} \quad \text{for all } x \in X.$$

It is easy to see that $F \in \mathcal{F}$ and φ is lower semi-continuous. Here, we will show that $\lambda \in \Lambda_T^1$. For this, we distinguish the following cases:

Case 1: If $x \in [0,2)$, then we have $Tx = 0$. It follows that $\lambda(Tx) = \frac{3}{4} = \lambda(x)$.

Case 2: If $x \in [2,\infty)$, then we have $Tx = \frac{1}{x}$. It follows that

$$\begin{aligned}
\lambda(Tx) &= \max\left\{\frac{\frac{1}{x}}{1+\frac{1}{x}}, \frac{3}{4}\right\} \\
&= \max\left\{\frac{1}{1+x}, \frac{3}{4}\right\} \\
&\leq \max\left\{\frac{x}{1+x}, \frac{3}{4}\right\} \\
&= \lambda(x).
\end{aligned}$$

Next we will show that T is (F,φ,λ)-contraction mapping. Suppose that $x,y \in X$. We will divide the proof into four cases.

Case 1: If $x,y \in [0,2)$, the claim is obvious.

Case 2: If $x,y \in [2,\infty)$, we get

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty) \\
&= \left|\frac{1}{x}-\frac{1}{y}\right| + \frac{1}{x}+\frac{1}{y} \\
&= \left|\frac{x-y}{xy}\right| + \frac{1}{x}+\frac{1}{y} \\
&\leq \frac{|x-y|}{4} + \frac{1}{x}+\frac{1}{y} \\
&\leq \frac{3}{4}|x-y| + \frac{x}{2}+\frac{y}{2} \\
&\leq \frac{3}{4}(|x-y|+x+y) \\
&\leq \lambda(x)F(d(x,y),\varphi(x),\varphi(y)).
\end{aligned}$$

Case 3: If $x \in [0,2)$ and $y \in [2,\infty)$, we get

$$\begin{aligned}
F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty) \\
&= d\left(0,\frac{1}{y}\right)+\varphi(0)+\varphi\left(\frac{1}{y}\right) \\
&= \frac{2}{y} \\
&\leq \frac{y}{2} \\
&\leq \frac{3}{2}(2y-x) \\
&= \frac{3}{4}((y-x)+0+y) \\
&= \lambda(x)F(d(x,y),\varphi(x),\varphi(y)).
\end{aligned}$$

Case 4: If $x \in [2,\infty)$ and $y \in [0,2)$, we get

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty) \\ &= d\left(\frac{1}{x},0\right)+\varphi(Tx)+\varphi(0) \\ &\leq \frac{x}{2} \\ &\leq \frac{3}{4}(2x-y) \\ &= \frac{3}{4}((x-y)+x+0) \\ &= \lambda(x)F(d(x,y),\varphi(x),\varphi(y)). \end{aligned}$$

Then all conditions of Theorem 3.3.2 are satisfied and so T has a φ-fixed point in X.

Remark 3.3.4. It is well-known that the Banach contraction mapping is continuous. So the discontinuous mapping is not Banach contraction mapping. Note that T in above example is not continuous and thus the Banach contraction mapping principle can not be applied in this case.

Definition 3.3.5. Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is a graphic (F,φ,λ)-contraction with respect to the metric d if and only if there exists a function $\lambda \in \Lambda_T^1$ such that

$$F(d(T^2x,Tx),\varphi(T^2x),\varphi(Tx)) \leq \lambda(x)F(d(Tx,x),\varphi(Tx),\varphi(x)) \tag{3.3.11}$$

for all $x \in X$.

Theorem 3.3.6. *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, and $F \in \mathcal{F}$. Assume that $T : X \to X$ is a graphic (F,φ,λ)-contraction with respect to the metric d and T is continuous. Then the following assertions hold:*

(i) $F_T \subseteq Z_\varphi$;

(ii) $T : X \to X$ *is a weakly φ-Picard mapping;*

(iii) for all $x \in X$, if $T^n x \to z$ as $n \to \infty$, then

$$d(T^n x, z) \le \frac{[\lambda(x)]^n}{1-\lambda(x)} F(d(x,Tx), \varphi(x), \varphi(Tx)), \textit{ for all } n \in \mathbb{N}.$$

Proof. First, we will prove that $F_T \subseteq Z_\varphi$. Let $\xi \in F_T$. Applying (3.3.11) with $x = \xi$, we get

$$F(0, \varphi(\xi), \varphi(\xi)) \le \lambda(\xi) F(0, \varphi(\xi), \varphi(\xi)).$$

This implies that

$$F(0, \varphi(\xi), \varphi(\xi)) = 0. \tag{3.3.12}$$

From (F1), we have

$$\varphi(\xi) \le F(0, \varphi(\xi), \varphi(\xi)). \tag{3.3.13}$$

Using (3.3.12) and (3.3.13), we obtain $\varphi(\xi) = 0$, which prove (i).

Next, we will show that T is a weakly φ-Picard mapping. Let $x \in X$ be an arbitrary point. Using (3.3.11), we have

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \le \lambda(T^n x) F(d(T^n x, T^{n-1}x), \varphi(T^n x), \varphi(T^{n-1}x)) \tag{3.3.14}$$

for all $n \in \mathbb{N}$. Since $\lambda \in \Lambda_T^1$, the relation (3.3.14) implies that

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \le \lambda(x) F(d(T^n x, T^{n-1}x), \varphi(T^n x), \varphi(T^{n-1}x))$$

for all $n \in \mathbb{N}$. By induction, for each $n \in \mathbb{N}$, we get

$$F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \le [\lambda(x)]^n F(d(Tx, x), \varphi(Tx), \varphi(x)),$$

which implies by property (F1) that

$$\max\{d(T^{n+1}x, T^n x), \varphi(T^{n+1}x)\} \le [\lambda(x)]^n F(d(Tx, x), \varphi(Tx), \varphi(x))$$

for all $n \in \mathbb{N}$. This yields that

$$d(T^{n+1}x, T^n x) \le [\lambda(x)]^n F(d(Tx, x), \varphi(Tx), \varphi(x))$$

for all $n \in \mathbb{N}$. Now, we show that $\{T^n x\}$ is a Cauchy sequence. Suppose that

$m,n \in \mathbb{N}$ such that $m > n$. By using the triangle inequality, we get

$$\begin{aligned}
d(T^n x, T^m x) &\leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{m-1} x, T^m x) \\
&= [\lambda(x)]^n F(d(Tx,x), \varphi(Tx), \varphi(x)) \\
&\quad + [\lambda(x)]^{n+1} F(d(Tx,x), \varphi(Tx), \varphi(x)) \\
&\quad + \cdots + [\lambda(x)]^{m-1} F(d(Tx,x), \varphi(Tx), \varphi(x)) \\
&\leq \frac{[\lambda(x)]^n}{1-\lambda(x)} F(d(Tx,x), \varphi(Tx), \varphi(x)). \qquad (3.3.15)
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get $\lim\limits_{m,n\to\infty} d(T^n x, T^m x) = 0$ since $\lambda(x) \in [0,1)$. This yields that $\{T^n x\}$ is a Cauchy sequence. Since (X,d) is complete, there is $z \in X$ such that

$$\lim_{n\to\infty} d(T^n x, z) = 0. \qquad (3.3.16)$$

Next, we need to verify that z is a φ-fixed point of T. Observe that from (3.3.15), we get

$$\varphi(T^{n+1} x) \leq [\lambda(x)]^n F(d(Tx,x), \varphi(Tx), \varphi(x)).$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \varphi(T^{n+1} x) = 0. \qquad (3.3.17)$$

From hypothesis (H_1), and conditions (3.3.16) and (3.3.17), we have

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(T^{n+1} x) = 0. \qquad (3.3.18)$$

On the other hand, using the continuity of T and (3.3.16), we get

$$z = \lim_{n\to\infty} T^{n+1} x = \lim_{n\to\infty} T(T^n x) = T(\lim_{n\to\infty} T^n x) = Tz.$$

Then z is a φ-fixed point of T. So T is a weakly φ-Picard operator. This completes the proof. $\square$

Next, we introduce the generalized contractive condition (3.3.1).

Definition 3.3.7. Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is an (F,φ,λ)-weak contraction with respect to the metric d if and only if there exists $\lambda \in \Lambda_T^1$ and $L \geq 0$ such that

$$\begin{aligned} F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &\leq \lambda(x)F(d(x,y),\varphi(x),\varphi(y)) \\ &\quad +L[F(d(y,Tx),\varphi(y),\varphi(Tx)) \\ &\quad -F(0,\varphi(y),\varphi(Tx))] \qquad (3.3.19) \end{aligned}$$

for all $x,y \in X$.

Theorem 3.3.8. *Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, and $F \in \mathcal{F}$. Assume that $T : X \to X$ is an (F,φ,λ)-weak contraction with respect to the metric d. Then the following assertions hold:*

(i) $F_T \subseteq Z_\varphi$;

(ii) $T : X \to X$ *is a weakly φ-Picard mapping;*

(iii) for all $x \in X$, if $T^n x \to z$ as $n \to \infty$, then

$$d(T^n x, z) \leq \tfrac{[\lambda(x)]^n}{1-\lambda(x)} F(d(x,Tx),\varphi(x),\varphi(Tx)), \text{ for all } n \in \mathbb{N}.$$

Proof. Let $\xi \in X$ is a fixed point of T. Applying (3.3.19) with $x = y = \xi$, we get

$$\begin{aligned} F(0,\varphi(\xi),\varphi(\xi)) &\leq \lambda(\xi)F(0,\varphi(\xi),\varphi(\xi)) + L[F(0,\varphi(\xi),\varphi(\xi)) - F(0,\varphi(\xi),\varphi(\xi))] \\ &= \lambda(\xi)F(0,\varphi(\xi),\varphi(\xi)), \end{aligned}$$

which implies that

$$F(0,\varphi(\xi),\varphi(\xi)) = 0. \qquad (3.3.20)$$

From $(F1)$, we have

$$\varphi(\xi) \leq F(0,\varphi(\xi),\varphi(\xi)). \qquad (3.3.21)$$

Using (3.3.20) and (3.3.21), we obtain that $\varphi(\xi) = 0$, then $\xi \in Z_\varphi$.

Next, we will show that T is a weakly φ-Picard mapping. Assume that $x \in X$ be an arbitrary point. Using (3.3.19), we have

$$\begin{aligned} F(d(T^{n+1}x,T^n x),\varphi(T^{n+1}x),\varphi(T^n x)) &\leq \lambda(T^n x)F(d(T^n x,T^{n-1}x),\varphi(T^n x),\varphi(T^{n-1}x)) \\ &\quad +L[F(0,\varphi(T^n x),\varphi(T^{n+1}x)) - \\ &\quad F(0,\varphi(T^n x),\varphi(T^{n+1}x))] \\ &= \lambda(T^n x)F(d(T^n x,T^{n-1}x),\varphi(T^n x),\varphi(T^{n-1}x)) \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lambda \in \Lambda_T^1$, the previous relation implies that

$$F(d(T^{n+1}x,T^n x),\varphi(T^{n+1}x),\varphi(T^n x)) \leq \lambda(x)F(d(T^n x,T^{n-1}x),\varphi(T^n x),\varphi(T^{n-1}x))$$

for all $n \in \mathbb{N}$. Repeating this process, for each $n \in \mathbb{N}$, we get

$$F(d(T^{n+1}x,T^n x),\varphi(T^{n+1}x),\varphi(T^n x)) \leq [\lambda(x)]^n(x)F(d(Tx,x),\varphi(Tx),\varphi(x)).$$

The rest of the proof follows that in proof of Theorem 3.3.2. □

3.4 Applications

From the fixed point result in Sections 3.2 and 3.3, we show that we can deduce easily various fixed point theorems on partial metric spaces and the application to nonlinear integral equations.

3.4.1 Application to the fixed point results in partial metric spaces

In this subsection, as application in Sections 3.2 and 3.3, we derive a results in partial metric spaces.

From Theorem 3.2.5 we get the following theorem.

Theorem 3.4.1. *Let (X,p) be a complete partial metric space, and $\theta \in J$ and $T : X \to X$ be a given function. Assume that the following conditions are satisfied:*

(P1) $\theta(2t) = 2\theta(t)$ for all $t \in [0,\infty)$;

(P2) the mapping T satisfies

$$p(Tx,Ty) \leq \theta(p(x,y))$$

for all $x,y \in X$.

Then the following assertions hold:

(i) T *has a unique fixed point* $z \in X$*,*

(ii) $p(z,z) = 0$.

Proof. Let $x, y \in X$. Clearly, (P1) and (P2) implies

$$\begin{aligned}2p(Tx,Ty) - p(Tx,Tx) - p(Ty,Ty) + p(Tx,Tx) + p(Ty,Ty) \\ \leq \theta(2p(x,y) - p(x,x) - p(y,y) + p(x,x) + p(y,y)).\end{aligned}$$

Setting the metric d_p on X which is defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \; \forall x, y \in X$$

and the function $\varphi : X \to [0,\infty)$ which is defined by $\varphi(x) = p(x,x)$ for all $x \in X$, we get

$$d_p(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \leq \theta(d_p(x,y) + \varphi(x) + \varphi(y))$$

for all $x, y \in X$. It is easy to verify that $\varphi(x) = p(x,x)$ is lower semi-continuous. Assume that function $F : [0,\infty)^3 \to [0,\infty)$ defined by $F(a,b,c) = a+b+c$ and using Lemma 2.2.8. It suffices to verify that

$$\forall x, y \in X : F(d_p(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq \theta(F(d_p(x,y), \varphi(x), \varphi(y))).$$

Then the hypothesis of Theorem 3.2.5 is automatically satisfied and then T has a unique φ-fixed point (say z). This implies that T has a unique fixed point $z \in X$ such that $p(z,z) = 0$. This completes the proof. □

Similarly, from Theorem 3.2.9, we obtain the following result.

Theorem 3.4.2. *Let* (X,p) *be a complete partial metric space,* $\theta \in J$ *and* $T : X \to X$ *be a given mapping. Assume that the following conditions are satisfied:*

(P1) $\theta(2t) = 2\theta(t)$ *for all* $t \in [0,\infty)$*;*

(P2) the mapping T *satisfies*

$$p(Tx,Ty) \leq \theta(p(x,y)) + L\left[N(x,y) - \frac{p(y,y) + p(Tx,Tx)}{2}\right]$$

for all $x, y \in X$*, where* $N(x,y) := \min\{p(x,Tx), p(y,Ty), p(y,Tx)\}$ *and* $L \geq 0$.

Then the following assertion hold:

(i) T *has a fixed point* $z \in X$,

(ii) $p(z,z) = 0$.

By using Theorem 3.3.2 and 3.3.8 with the same technique in the proof of Theorem 3.4.1, we get the following results.

Theorem 3.4.3. *Let* (X,p) *be a complete partial metric space and* $T : X \to X$ *be a given mapping. Suppose that there exists a function* $\lambda \in \Lambda_T^1$ *such that*

$$p(Tx,Ty) \leq \lambda(x)p(x,y) \tag{3.4.1}$$

for all $x,y \in X$. *Then* T *has a unique fixed point* $z \in X$. *Moreover, we have* $p(z,z) = 0$.

Theorem 3.4.4. *Let* (X,p) *be a complete partial metric space and* $T : X \to X$ *be a given mapping. Suppose that there exists a function* $\lambda \in \Lambda_T^1$ *such that*

$$p(Tx,Ty) \leq \lambda(x)p(x,y) + L\left[N(x,y) - \frac{p(y,y)+p(Tx,Tx)}{2}\right] \tag{3.4.2}$$

for all $x,y \in X$, *where* $N(x,y) := \min\{p(x,Tx), p(y,Ty), p(y,Tx)\}$ *and* $L \geq 0$. *Then* T *has a unique fixed point* $z \in X$. *Moreover, we have* $p(z,z) = 0$.

Taking $\theta(t) = kt$, where $k \in [0,1)$ in Theorems 3.4.1 and 3.4.2 (or taking $\lambda(t) = k$, where $k \in [0,1)$ in Theorems 3.4.3 and 3.4.4), we obtain immediately Corollaries 3.4.5 and 3.4.6.

Corollary 3.4.5 ([6])**.** *Let* (X,p) *be a complete partial metric space and* $T : X \to X$ *be a mapping satisfying the following condition:*

$$\exists k \in [0,1) \text{ such that } p(Tx,Ty) \leq kp(x,y) \text{ for all } x,y \in X. \tag{3.4.3}$$

Then T *has a unique fixed point.*

Corollary 3.4.6 ([6]). *Let (X,p) be a complete partial metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in [0,1) \text{ such that } p(Tx,Ty) \leq kp(x,y) + L\left[N(x,y) - \frac{p(y,y)+p(Tx,Tx)}{2}\right] \tag{3.4.4}$$

for all $x,y \in X$, where $N(x,y) := \min\{p(x,Tx), p(y,Ty), p(y,Tx)\}$ and $L \geq 0$. Then T has a fixed point.

By using Theorem 3.3.6 with the same technique in the proof of Theorem 3.4.1, we get the following Theorem.

Theorem 3.4.7. *Let (X,p) be a complete partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\lambda \in \Lambda_T^1$ such that*

$$p(T^2x,Tx) \leq \lambda(Tx)p(Tx,x) \tag{3.4.5}$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$. Moreover, we have $p(z,z) = 0$.

Taking $\lambda(t) = k$, where $k \in [0,1)$ in Theorems 3.4.7, we obtain immediately Corollaries 3.4.8.

Corollary 3.4.8 ([6]). *Let (X,p) be a complete partial metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$\exists k \in [0,1) \text{ such that } p(T^2x,Tx) \leq kp(Tx,x) \text{ for all } x,y \in X.$$

Then T has a fixed point.

3.4.2 Application to the nonlinear integral equations

During the last three decades the theory of differential and integral equation has been widely used in the various fields of science and engineering. The main aim of this subsection is to investigate the existence and uniqueness of solution for the nonlinear integral equation:

$$x(t) = \phi(t) + \int_a^t K(t,s,x(s))ds, \tag{3.4.6}$$

where $a \in \mathbb{R}$, $x \in C[a,b]$ (the set of all continuous functions form $[a,b]$ into $\mathbb{R}$), $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are two given function.

Theorem 3.4.9. *Consider the nonlinear integral equation (3.4.6). Suppose that the following condition hold:*

(i) $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous,*

(ii) for each $x, y \in C[a,b]$ *and* $t, s \in [a,b]$

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq \frac{\theta(|x(s) - y(s)|)}{b-a},$$

where $\theta \in J$.

Then the nonlinear integral equation (3.4.6) has a unique solution.

Proof. Let $X = C[a,b]$ and let $T : X \to X$ be defined by

$$(Tx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds$$

for all $x \in X$. Clearly, X with the metric $d : X \times X \to \mathbb{R}_+$ given by

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

for all $x, y \in X$, is a complete metric space.

Next, we define functions $F : [0,\infty)^3 \to [0,\infty)$ and $\varphi : X \to [0,\infty)$ by

$$F(a,b,c) = a + b + c \text{ for all } a,b,c \in [0,\infty)$$

and $\varphi = 0$ for all $x \in X$.

Here, we will show that T is an (F,φ,θ)-contraction mapping.

Assume that $x,y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}
|(Tx)(t)-(Ty)(t)| &= \left| \int_a^t K(t,s,x(s))ds - \int_a^t K(t,s,y(s))ds \right| \\
&= \left| \int_a^t (K(t,s,x(s)) - K(t,s,y(s)))ds \right| \\
&\leq \int_a^t |(K(t,s,x(s)) - K(t,s,y(s)))|ds \\
&\leq \frac{1}{b-a} \int_a^t \theta(|x(s)-y(s)|)ds \\
&\leq \frac{1}{b-a} \int_a^t \theta(d(x,y))ds \\
&\leq \frac{1}{b-a} \theta(d(x,y))[b-a] \\
&= \theta(d(x,y)).
\end{aligned}$$

This implies that $\max_{t\in[a,b]} |(Tx)(t)-(Ty)(t)| \leq \theta(d(x,y))$ and hence

$$d(Tx,Ty) \leq \theta(d(x,y))$$

for all $x,y \in X$. It follows that

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \theta(F(d(x,y),\varphi(x),\varphi(y)))$$

and so T is an (F,φ,θ)-contraction mapping. Thus all the condition of Theorem 3.2.5 are satisfied and hence T has a unique fixed point in X. It follows that there exists a unique solution of the nonlinear equation (3.4.6). $\square$

Theorem 3.4.10. *Consider the nonlinear integral equation (3.4.6). Suppose that the following conditions hold:*

(i) $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous,*

(ii) for each $x,y \in C[a,b]$ *and* $t,s \in [a,b]$, *there exists a function* $\lambda : C[a,b] \to [0,1)$ *for which* $\lambda(Tx) \leq \lambda(x)$, *where* $T : C[a,b] \to C[a,b]$ *is defined by*

$$(Tx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds,$$

such that

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq \frac{\lambda(x)|x(s)-y(s)|}{b-a}.$$

Then the nonlinear integral equation (3.4.6) has a unique solution.

Proof. Let $X = C[a,b]$. Clearly, X with the metric $d : X \times X \to \mathbb{R}_+$ given by

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$$

for all $x,y \in X$, is a complete metric space. It is easy to see that $\lambda \in \Lambda_T$. Next, we define functions $F : [0,\infty)^3 \to [0,\infty)$ and $\varphi : X \to [0,\infty)$ by

$$F(a,b,c) = a+b+c \text{ for all } a,b,c \in [0,\infty)$$

and $\varphi(x) = 0$ for all $x \in X$. Here, we will show that T is an (F,λ)-contraction mapping. Assume that $x,y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &= \left|\int_a^t K(t,s,x(s))ds - \int_a^t K(t,s,y(s))ds\right| \\
&= \left|\int_a^t (K(t,s,x(s)) - K(t,s,y(s)))ds\right| \\
&\leq \int_a^t |(K(t,s,x(s)) - K(t,s,y(s)))|ds \\
&\leq \frac{1}{b-a}\int_a^t \lambda(x)|x(s) - y(s)|ds \\
&\leq \frac{1}{b-a}\int_a^t \lambda(x)d(x,y)ds \\
&\leq \frac{1}{b-a}\lambda(x)d(x,y)[b-a] \\
&= \lambda(x)d(x,y).
\end{aligned}$$

This implies that $\max_{t\in[a,b]}|(Tx)(t) - (Ty)(t)| \leq \lambda(x)d(x,y)$ and hence

$$d(Tx,Ty) \leq \lambda(x)d(x,y)$$

for all $x,y \in X$. It follows that

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \lambda(x)F(d(x,y),\varphi(x),\varphi(y))$$

and so T is an (F,φ,λ)-contraction mapping. Thus all the condition of Theorem 3.3.2 are satisfied and hence T has a unique fixed point in X. It follows that there exists a unique solution of the nonlinear equation (3.4.6). $\square$

CHAPTER 4

FIXED POINT FOR GENERALIZED CONTRACTIONS IN M-METRIC SPACES

In this chapter, we introduce a new generalization of Banach and Kannan contraction mappings in the setting of M-metric space. We also prove the existence of fixed point theorems for these mappings, which are generalization of the Banach and Kannan fixed point theorems on M-metric spaces (Theorems 1.0.13 and 1.0.14) and give some illustrative examples to support our results. Finally, we give a partial answer to Open Problem 1.0.15. This result confirms the existence of fixed point theorems for Chatterjea contraction mappings in the framework of M-metric spaces. We also give some examples to illustrate the usability of our results.

4.1 Fixed point results by using a control functions type-S_1 and type-S_2

In this section, we recall the class of control functions as appeared in Section 3.3. For a nonempty set X and a self mapping $T : X \to X$, we let Λ_T^1 be the class of all functions of type-S_1, that is, $\lambda : X \to [0,1)$ satisfying $\lambda(Tx) \leq \lambda(x)$ for all $x \in X$. Furthermore, for a nonempty set X and a self mapping $T : X \to X$, we denote Λ_T^2 as the class of all functions of type-S_2, that is, $\lambda : X \to \left[0, \frac{1}{2}\right)$ satisfying $\lambda(Tx) \leq \lambda(x)$ for all $x \in X$. Some examples and numerical results are given to support our main results. Finally we leave an open question for those who might be interested.

Theorem 4.1.1. *Let (X,m) be a complete M-metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$m(Tx,Ty) \leq \lambda(x)m(x,y) \tag{4.1.1}$$

for any $x,y \in X$, where $\lambda \in \Lambda_T^1$. Then T has a unique fixed point. Moreover, the

Picard iteration $\{x_n\}$*, which is defined by* $x_n = Tx_{n-1}$ *for all* $n \in \mathbb{N}$*, where* $x_0 \in X$*, converges to the fixed point of* T*.*

Proof. Let x_0 be an arbitrary point in X and the Picard iteration of T be given by

$$x_n = Tx_{n-1} \ \ \forall n \in \mathbb{N}.$$

According to (4.1.1), with $x = x_n$ and $y = x_{n-1}$, we have that

$$m(x_{n+1}, x_n) = m(Tx_n, Tx_{n-1}) \leq \lambda(x_n) m(x_n, x_{n-1}). \tag{4.1.2}$$

Since $\lambda \in \Lambda_T^1$, the relation (4.1.2) implies that

$$m(x_{n+1}, x_n) = m(Tx_n, Tx_{n-1}) \leq \lambda(x_0) m(x_n, x_{n-1}). \tag{4.1.3}$$

Using Lemma 2.3.13, we get (A), (B), (C) and (D) of Lemma 2.3.13 hold. From the completeness of X, we get $x_n \to x$ for some $x \in X$. From (2.3.1), we have

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0.$$

Since condition (B) of Lemma 2.3.13 and definition of $m_{x_n,x}$, we deduce that $m(x_n, x) \to 0$. By (4.1.1) and the fact that $\lambda(x) \in \Lambda_T^1$, we get

$$\begin{aligned}
m(Tx_n, Tx) &\leq \lambda(x_n) m(x_n, x) \\
&= \lambda(Tx_{n-1}) m(x_n, x) \\
&\leq \lambda(x_{n-1}) m(x_n, x) \\
&\ \vdots \\
&\leq \lambda(x_0) m(x_n, x) \to 0.
\end{aligned}$$

Then by (m_2), we have

$$m_{Tx_n,Tx} \leq m(Tx_n, Tx) \to 0.$$

By using equation (2.3.1), we get $Tx_n \to Tx$. Thus, form Lemma 2.3.10, $x_n \to x$ and $Tx_n \to Tx$, we have

$$\lim_{n\to\infty} (m(x_n, Tx_n) - m_{x_n,Tx_n}) = m(x, Tx) - m_{x,Tx}. \tag{4.1.4}$$

By using (A) and (C) of Lemma 2.3.13, we obtain that (4.1.4) implies $m(x,Tx) = m_{x,Tx}$. We observe that (4.1.1) implies

$$m(Tx,Tx) \leq \lambda(x)m(x,x) \leq m(x,x).$$

Then we get $m(x,Tx) = m_{x,Tx} = m(Tx,Tx)$. Next, we claim that $m(x,x) = m_{x,Tx}$. By Lemma 2.3.10, $x_n \to x$ and $Tx_n \to Tx$ imply that

$$0 = \lim_{n\to\infty}(m(x_n,x_{n+1}) - m_{x_n,Tx_n}) = m(x,x) - m_{x,Tx}.$$

Thus, $m(x,x) = m_{x,Tx}$. Now, we obtain that

$$m(x,x) = m(x,Tx) = m(Tx,Tx),$$

i.e. $x = Tx$. To prove the uniqueness of X. Assume that there exists another fixed point z such that $z \neq x$. From (4.1.1), we have

$$m(x,z) = m(Tx,Tz) \leq \lambda(x)m(x,z) < m(x,z),$$

which is a contradiction. Then T has a unique fixed point $x \in X$. This completes the proof. □

Example 4.1.2. Let $X = [0,3]$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by $m(x,y) = \frac{x+y}{2}$ for all $x,y \in X$. From Example 2.3.9, we get (X,m) is a complete M-metric space. Define $T : X \to X$ and $\lambda : X \to [0,1)$ by

$$Tx = \begin{cases} x^2, & 0 \leq x \leq \frac{3}{4}, \\ \frac{x^2}{1+x}, & \frac{3}{4} < x \leq 3, \end{cases}$$

and

$$\lambda(x) = \begin{cases} \frac{3}{4}, & 0 \leq x \leq \frac{3}{4}, \\ \frac{4x}{1+4x}, & \frac{3}{4} < x \leq 3, \end{cases}$$

for all $x \in X$. We will show that $\lambda \in \Lambda_T^1$. For this, we distinguish the following cases:

Case 1: If $x \in \left[0, \frac{3}{4}\right]$, then we have $Tx = x^2$. It follows that

$$\lambda(Tx) = \lambda\left(x^2\right) = \frac{3}{4} = \lambda(x).$$

Case 2: If $x \in \left(\frac{3}{4}, 3\right]$, then we have $Tx = \frac{x^2}{1+x}$. It follows that

$$\lambda(Tx) = \lambda\left(\frac{x^2}{1+x}\right) \leq \frac{4x}{1+4x} = \lambda(x).$$

Next, we will show that condition (4.1.1) holds for all $x, y \in X$. Suppose that $x, y \in X$. We will divide the proof into four cases.

Case 1: If $x, y \in \left[0, \frac{3}{4}\right]$, then we have

$$\begin{aligned} m(Tx, Ty) &= \frac{x^2}{2} + \frac{y^2}{2} \\ &\leq \frac{3}{4}\left(\frac{x+y}{2}\right) \\ &= \lambda(x)m(x,y). \end{aligned}$$

Case 2: If $x, y \in \left(\frac{3}{4}, 3\right]$, then we have

$$\begin{aligned} m(Tx, Ty) &= \frac{x^2}{2(1+x)} + \frac{y^2}{2(1+y)} \\ &\leq \frac{3}{4}\left(\frac{x+y}{2}\right) \\ &= \lambda(x)m(x,y). \end{aligned}$$

Case 3: If $x \in \left[0, \frac{3}{4}\right]$ and $y \in \left(\frac{3}{4}, 3\right]$, then we have

$$\begin{aligned} m(Tx, Ty) &= \frac{x^2}{2} + \frac{y^2}{2(1+y)} \\ &\leq \frac{3}{4}\left(\frac{x+y}{2}\right) \\ &= \lambda(x)m(x,y). \end{aligned}$$

Case 4: If $x \in \left(\frac{3}{4}, 3\right]$ and $y \in \left[0, \frac{3}{4}\right]$, then we have

$$\begin{aligned} m(Tx, Ty) &= \frac{x^2}{2(1+x)} + \frac{y^2}{2} \\ &\leq \frac{4x}{1+4x}\left(\frac{x+y}{2}\right) \\ &= \lambda(x)m(x,y). \end{aligned}$$

Therefore, all conditions in Theorem 4.1.1 are satisfied. So T has a unique fixed point in X. In this example, a point 0 is a unique fixed point of T.

Next, we will give some numerical experiments for approximate a unique fixed point of T in Figure 4.1. Moreover, the convergence behavior of these iteration is shown in Figure 4.2.

x_0	0.25000000	0.70000000	1.50000000	3.00000000
x_1	0.06250000	0.49000000	0.90000000	2.25000000
x_2	0.00390625	0.24010000	0.42631579	1.55769231
x_3	0.00001526	0.05764801	0.12742280	0.94866975
x_4	0.00000000	0.00332329	0.01440149	0.46184034
x_5	0.00000000	0.00001104	0.00020446	0.14590957
x_6	0.00000000	0.00000000	0.00000004	0.01857878
x_7	0.00000000	0.00000000	0.00000000	0.00033888
x_8	0.00000000	0.00000000	0.00000000	0.00000011
x_9	0.00000000	0.00000000	0.00000000	0.00000000
x_{10}	0.00000000	0.00000000	0.00000000	0.00000000

Figure 4.1: Iterates of Picard iterations in Example 4.1.2

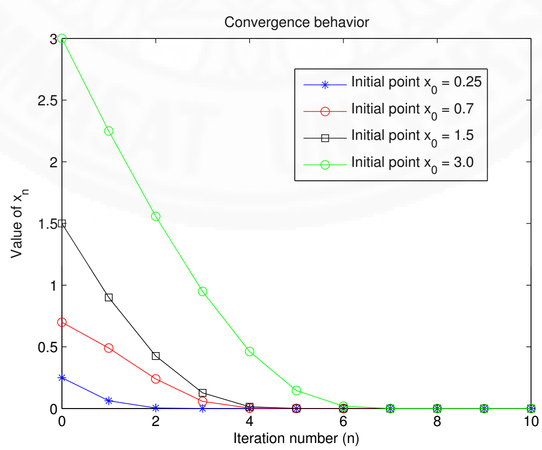

Figure 4.2: Convergence behavior for Example 4.1.2

Theorem 4.1.3. *Let (X,m) be a complete M-metric space and $T:X\to X$ be a mapping satisfying the following condition:*

$$m(Tx,Ty)\leq\lambda(x)[m(x,Tx)+m(y,Ty)] \tag{4.1.5}$$

for any $x,y\in X$, where $\lambda\in\Lambda_T^2$. Then T has a unique fixed point. Moreover, the Picard iteration $\{x_n\}$, which is defined by $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$, where $x_0\in X$, converges to the fixed point of T.

Proof. Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X such that $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$. From (4.1.5) and condition (m_4), we get

$$\begin{aligned}
m(x_{n+1},x_n) &= m(Tx_n,Tx_{n-1})\\
&\leq \lambda(x_n)\,(m(x_n,x_{n+1})+m(x_{n-1},x_n))\\
&= \lambda(Tx_{n-1})\,(m(x_n,x_{n+1})+m(x_{n-1},x_n))\\
&\leq \lambda(x_{n-1})\,(m(x_n,x_{n+1})+m(x_{n-1},x_n))\\
&\vdots\\
&\leq \lambda(x_0)\,(m(x_n,x_{n+1})+m(x_{n-1},x_n))
\end{aligned}$$

for all $n\in\mathbb{N}$. This implies that

$$m(x_{n+1},x_n)\leq rm(x_n,x_{n-1})$$

for all $n\in\mathbb{N}$, where $0\leq r:=\frac{\lambda(x_0)}{1-\lambda(x_0)}<1$. Using Lemma 2.3.13, we get (A), (B), (C) and (D) of Lemma 2.3.13 hold. By Lemma 2.3.13 and completeness of X, we get $x_n\to x$ for some $x\in X$. So

$$(m(x_n,x)-m_{x_n,x})\to 0 \text{ as } n\to\infty$$

and

$$(M_{x_nx}-m_{x_n,x})\to 0 \text{ as } n\to\infty.$$

Since $m_{x_n,x}\to 0$ as $n\to\infty$, we have $m(x_n,x)\to 0$ and $M_{x_n,x}\to 0$ as $n\to\infty$. Therefore by Remark 2.3.5 (1), we have

$$m(x,x)=0=m_{x,Tx}. \tag{4.1.6}$$

Next, we will show that $m(x,Tx)=0$. From (4.1.5) and $\lambda \in \Lambda_T^2$, we get

$$\begin{aligned} m(x_{n+1},Tx) &= m(Tx_n,Tx) \\ &\leq \lambda(x_n)\,(m(x_n,x_{n+1})+m(x,Tx)) \\ &= \lambda(Tx_{n-1})\,(m(x_n,x_{n+1})+m(x,Tx)) \\ &\leq \lambda(x_{n-1})\,(m(x_n,x_{n+1})+m(x,Tx)) \\ &\vdots \\ &\leq \lambda(x_0)\,(m(x_n,x_{n+1})+m(x,Tx)). \end{aligned}$$

Letting $n \to \infty$ and using condition (A) of Lemma 2.3.13, then we have

$$\limsup_{n\to\infty} m(x_{n+1},Tx) = \limsup_{n\to\infty} m(Tx_n,Tx) \leq \lambda(x_0)m(x,Tx). \tag{4.1.7}$$

On the other hand, from (m_4), we get

$$\begin{aligned} m(x,Tx)-m_{x,Tx} &\leq m(x,x_n)-m_{x,x_n}+m(x_n,Tx)-m_{x_n,Tx} \\ &\leq m(x,x_n)+m(x_n,Tx). \end{aligned}$$

By using (4.1.6), (4.1.7) and the fact that $m(x_n,x)\to 0$, we obtain that

$$\begin{aligned} m(x,Tx) &\leq \limsup_{n\to\infty} m(x_n,Tx) \\ &\leq \lambda(x_0)m(x,Tx). \end{aligned}$$

This yields that $m(x,Tx)=0$. By contractive condition (4.1.5), we have

$$0 \leq m(Tx,Tx) \leq 2\lambda(x)m(x,Tx) = 0,$$

which implies that $m(Tx,Tx)=0$. So $m(x,x)=m(x,Tx)=m(Tx,Tx)$. Follows from (m_1), we get $x=Tx$. Finally, we will show that T has a unique fixed point. Assume that z is another fixed point of T such that $x \neq z$. From inequality (4.1.5), we get

$$\begin{aligned} m(x,z) &= m(Tx,Tz) \\ &\leq \lambda(x)(m(x,Tz)+m(z,Tx)) \\ &= \lambda(x)(m(x,z)+m(z,x)) \\ &= 2\lambda(x)m(x,z) \\ &< m(x,z) \end{aligned}$$

having in this way a contradiction, completing therefore the proof. $\square$

Example 4.1.4. Let $X = [0,\infty)$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by $m(x,y) = \frac{x+y}{2}$ for all $x,y \in X$. From Example 2.3.9, we get (X,m) is a complete M-metric space. Define $T : X \to X$ and $\lambda : X \to [0,\frac{1}{2})$ by

$$Tx = \begin{cases} 0 & \text{if } x < 2, \\ \frac{x}{1+x} & \text{if } x \geq 2 \end{cases}$$

and

$$\lambda(x) = \begin{cases} \frac{1}{3} & \text{if } x < 1, \\ \frac{x}{1+2x} & \text{if } x \geq 1. \end{cases}$$

It is easy to see that $\lambda \in \Lambda_T^2$. Here, we will show that condition (4.1.5) holds for all $x,y \in X$. We will divide the proof into three cases.

Case 1: If $x,y \in [0,2)$, the claim is obvious.

Case 2: If $x,y \in [2,\infty)$, we get

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{2}\left(\frac{x}{1+x} + \frac{y}{1+y}\right) \\ &\leq \frac{1}{2}\left(\frac{x}{3} + \frac{y}{3}\right) \\ &\leq \frac{1}{3}\left(\frac{x + \frac{x}{1+x}}{2} + \frac{y + \frac{y}{1+y}}{2}\right) \\ &\leq \lambda(x)(m(x,Tx) + m(y,Ty)). \end{aligned}$$

Case 3: Assume that $(x,y) \in [2,\infty) \times [0,2) \cup [0,2) \times [2,\infty)$. Without loss of generality, we may assume that $x \in [0,2)$ and $y \in [2,\infty)$. Then we obtain that

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{2}\left(\frac{y}{1+y}\right) \\ &\leq \frac{1}{2}\left(\frac{y}{3}\right) \\ &\leq \frac{1}{3}\left(\frac{y + \frac{y}{1+y}}{2}\right) \\ &\leq \lambda(x)(m(x,Tx) + m(y,Ty)). \end{aligned}$$

Therefore, all conditions in Theorem 4.1.3 are satisfied. So T has a unique fixed point.

Now we give the following open problem which is challenging for other mathematicians.

Open problem 4.1.5. *Let (X,m) be a complete M-metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$m(Tx,Ty) \leq \lambda(x)[m(x,Ty)+m(y,Tx)] \tag{4.1.8}$$

for any $x,y \in X$, where $\lambda \in \Lambda_T^2$. Does T always have a unique fixed point?

4.2 Fixed point result of weakly α-admissible mappings

In this section, we introduce concepts of new type of mappings in the framework of M-metric spaces and prove fixed point results for these mappings along with weak α-admissibility.

Now we define an α-m-Banach contraction and an α-m-Kannan contraction in the setting of an M-metric space as follows:

Definition 4.2.1. Let (X,m) be an M-metric space and $\alpha : X \times X \to [0,\infty)$ be a given mapping. The mapping $T : X \to X$ is said to be an α-m-Banach contraction if

$$\exists k \in [0,1) \text{ such that } \alpha(x,y)m(Tx,Ty) \leq km(x,y) \text{ for all } x,y \in X. \tag{4.2.1}$$

Definition 4.2.2. Let (X,m) be an M-metric space and $\alpha : X \times X \to [0,\infty)$ be a given mapping. The mapping $T : X \to X$ is said to be an α-m-Kannan contraction if

$$\exists k \in \left[0,\frac{1}{2}\right) \text{ such that } \alpha(x,y)m(Tx,Ty) \leq k\left[m(x,Tx)+m(y,Ty)\right] \text{ for all } x,y \in X. \tag{4.2.2}$$

Theorem 4.2.3. *Let (X,m) be a complete M-metric space and $T : X \to X$ be an α-m-Banach contraction mapping satisfying conditions:*

(i) T is a weakly α-admissible mapping;

(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

(iii) If $\{x_n\}$ *is sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to \infty$*, then* $\alpha(x, x) \geq 1$ *and* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$*.*

Then the fixed point problem of T *has a solution, that is, there exists* $x^* \in X$ *such that* $Tx^* = x^*$*.*

Proof. Starting from x_0 in (ii) we have $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Since T is weakly α-admissible, we have

$$\alpha(x_{n-1}, x_n) \geq 1$$

for all $n \in \mathbb{N}$. From an α-m-Banach contractive condition, we get

$$\begin{aligned} m(x_n, x_{n+1}) &= m(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) m(Tx_{n-1}, Tx_n) \\ &\leq km(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Using Lemma 2.3.13, we get (A), (B), (C) and (D) of Lemma 2.3.13 hold. From (D) of Lemma 2.3.13, we get $\{x_n\}$ is an m-Cauchy sequence in X. By completeness of X we get $x_n \to x$ for some $x \in X$. From (B) of Lemma 2.3.13, we get $\lim_{n\to\infty} m(x_n, x_n) = 0$. This implies that

$$\lim_{n\to\infty} m_{x_n, x} = \lim_{n\to\infty} \min\{m(x_n, x_n), m(x, x)\} = \min\{0, m(x, x)\} = 0.$$

It follows that

$$\lim_{n\to\infty} m(x_n, x) = 0. \tag{4.2.3}$$

From condition (iii), we get

$$\alpha(x, x) \geq 1 \tag{4.2.4}$$

and

$$\alpha(x_n, x) \geq 1 \tag{4.2.5}$$

for all $n \in \mathbb{N}$. Again, by using the α-m-Banach contractive condition, we obtain that

$$m(Tx_n, Tx) \leq \alpha(x_n, x)m(Tx_n, Tx) \leq km(x_n, x) \tag{4.2.6}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in (4.2.6), we have $m(Tx_n, Tx) \to 0$ as $n \to \infty$. By (m_2), we have $m_{Tx_n,Tx} \to 0$ as $n \to \infty$. It follows from the definition of convergent sequence that $Tx_n \to Tx$ as $n \to \infty$. Using (A) of Lemma 2.3.13, we obtain that $m(x_n, Tx_n) = m(x_n, x_{n+1}) \to 0$. From (B) of Lemma 2.3.13, we obtain that

$$\lim_{n\to\infty} m_{x_n,Tx_n} = \lim_{n\to\infty} m_{x_n,x_{n+1}} = 0.$$

By Lemma 2.3.10, we get

$$m(x, Tx) - m_{x,Tx} = \lim_{n\to\infty}(m(x_n, Tx_n) - m_{x_n,Tx_n}) = 0,$$

that is,

$$m(x, Tx) = m_{x,Tx}$$

From (4.2.4) and the α-m-Banach contractive condition, we get $m(Tx, Tx) \leq m(x, x)$ and thus $m_{x,Tx} = m(Tx, Tx)$. Therefore, we get

$$m(x, Tx) = m_{x,Tx,} = m(Tx, Tx).$$

On the other hand, by Lemma 2.3.10 and $x_n = Tx_{n-1} \to x$ as $n \to \infty$

$$0 = \lim_{n\to\infty}(m(x_n, Tx_n) - m_{x_n,Tx_n}) = \lim_{n\to\infty}(m(x_n, x_{n+1}) - m_{x_n,Tx_n}) = m(x, x) - m_{x,Tx}.$$

This implies that $m(x, x) = m(x, Tx)$. Now we have

$$m(x, Tx) = m_{x,Tx} = m(Tx, Tx).$$

By (m_1), we get $x = Tx$. This completes the proof. □

Theorem 4.2.4. *Let (X, m) be a complete M-metric space and $T : X \to X$ be an α-m-Kannan contraction mapping satisfying conditions:*

(i) T is a weakly α-admissible mapping;

(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

(iii) If $\{x_n\}$ *is sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to \infty$*, then* $\alpha(x, x) \geq 1$ *and* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$*.*

Then the fixed point problem of T *has a solution, that is, there exists* $x^* \in X$ *such that* $Tx^* = x^*$*.*

Proof. Starting from x_0 in *(ii)* we have $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Since T is a weakly α-admissible, we have

$$\alpha(x_{n-1}, x_n) \geq 1$$

for all $n \in \mathbb{N}$. From the α-m-Kannan contractive condition, we get

$$\begin{aligned} m(x_n, x_{n+1}) &= m(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) m(Tx_{n-1}, Tx_n) \\ &\leq k\left(m(x_{n-1}, x_n) + m(x_n, x_{n+1})\right). \end{aligned}$$

This implies that

$$m(x_n, x_{n+1}) \leq r m(x_{n-1}, x_n),$$

where $0 \leq r = \frac{k}{1-k} < 1$ for all $n \in \mathbb{N}$. It follows that (A), (B), (C) and (D) of Lemma 2.3.13 hold. By Lemma 2.3.13 and completeness of X and $x_n \to x$ as $n \to \infty$ for some $x \in X$. Then we get

$$m(x_n, x) - m_{x_n, x} \to 0 \text{ as } n \to \infty$$

and

$$M_{x_n, x} - m_{x_n, x} \to 0 \text{ as } n \to \infty.$$

Since $m_{x_n, x} \to 0$ as $n \to \infty$ we have $m(x_n, x) \to 0$ as $n \to \infty$ and $M_{x_n, x} \to 0$ as $n \to \infty$. By Remark 2.3.5, we get

$$m(x, x) = 0 = m_{x, Tx}.$$

Next, we will show that $m(x,Tx)=0$. From condition (iii), we get

$$\alpha(x,x)\geq 1 \tag{4.2.7}$$

and

$$\alpha(x_n,x)\geq 1 \tag{4.2.8}$$

for all $n\in\mathbb{N}$. Using the α-m-Kannan contractive condition, we have

$$\begin{aligned} m(x_{n+1},Tx) &= m(Tx_n,Tx) \\ &\leq \alpha(x_n,x)m(Tx_n,Tx) \\ &\leq k[m(x_n,x_{n+1})+m(x,Tx)]. \end{aligned}$$

Since $m(x_n,x_{n+1})\to 0$ as $n\to\infty$, we get

$$\limsup_{n\to\infty} m(x_{n+1},Tx)\leq km(x,Tx).$$

From (m_4), we get

$$\begin{aligned} m(x,Tx) &= m(x,Tx)-m_{x,Tx} \\ &\leq (m(x,x_n)-m_{x,x_n})+(m(x_n,Tx)-m_{x_n,Tx}) \\ &\leq m(x,x_n)+m(x_n,Tx). \end{aligned}$$

Taking limit supremum in above relation, we obtain that

$$m(x,Tx)\leq \limsup_{n\to\infty}(m(x,x_n)+m(x_n,Tx))\leq km(x,Tx).$$

This implies that

$$m(x,Tx)=0.$$

From (4.2.7) and an α-m-Kannan contractive condition, we have

$$m(Tx,Tx)\leq \alpha(x,x)m(Tx,Tx)\leq 2km(x,Tx)=0$$

and so

$$m(Tx,Tx)=0.$$

Now we have,

$$m(x,Tx)=m_{x,Tx}=m(Tx,Tx).$$

Follows from (m_1), we get $x=Tx$. This completes the proof. □

By remark 1.0.11,Theorem 4.2.3, and Theorem 4.2.4, we immediately obtain the following two corollaries.

Corollary 4.2.5. *Let (X,m) be a complete M-metric space and $T:X\to X$ be an α-m-Banach contraction mapping satisfying conditions:*

(i) *T is an α-admissible mapping;*

(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*

(iii) *If $\{x_n\}$ is sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x,x)\geq 1$ and $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$.*

Then the fixed point problem of T has a solution, that is, there exists $x^*\in X$ such that $Tx^*=x^*$.

Corollary 4.2.6. *Let (X,m) be a complete M-metric space and $T:X\to X$ be an α-m-Kannan contraction mapping satisfying conditions:*

(i) *T is an α-admissible mapping;*

(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;*

(iii) *If $\{x_n\}$ is sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x,x)\geq 1$ and $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$;*

Then the fixed point problem of T has a solution, that is, there exists $x^*\in X$ such that $Tx^*=x^*$.

Fixed point results for α-m-Banach contraction mapping and α-m-Kannan contraction mapping in M-metric spaces along with weak α-admissibility and α-admissibility were investigated under some suitable conditions. The generalized Banach and Kannan contractive conditions for the existence of fixed point results in various spaces have been introduced and studied by many mathematicians. The fixed point results for nonlinear mappings satisfy other famous contractive conditions such as the Chatterjea contractive condition, Ciric contractive condition, Berinde contractive condition etc. have also been studied by several mathematicians. Therefore, these problems remain open for interested mathematicians.

4.3 Fixed point result for Chatterjea contraction mappings

In this section, we give a partial answer to open problem posed by Asadi *et al.* and also furnish some illustrative examples to demonstrate the validity of the hypotheses and degree of utility of our results. Finally we leave an open question for those who might be interested.

Theorem 4.3.1. *Let (X,m) be a complete M-metric space and let $T:X\to X$ be a mapping satisfying the following condition:*

$$\exists k\in\left[0,\frac{1}{2}\right) \text{ such that } m(Tx,Ty)\le k[m(x,Ty)+m(Tx,y)] \text{ for all } x,y\in X. \tag{4.3.1}$$

If there is $x_0\in X$ such that

$$m(T^n x_0,T^n x_0)\le m(T^{n-1}x_0,T^{n-1}x_0) \tag{4.3.2}$$

for all $n\in\mathbb{N}$, then T has a unique fixed point. Moreover, if the Picard sequence $\{x_n\}$ in X which is defined by $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$ such that x_0 is an initial point in condition (4.3.2), then $\{x_n\}$ converges to a fixed point of T.

Proof. Starting from $x_0\in X$ in hypothesis, we will construct the sequence $\{x_n\}$ in X such that

$$x_n=Tx_{n-1}$$

for all $n \in \mathbb{N}$. From (4.3.1) and condition (m_4), we get

$$\begin{aligned}
m(x_{n+1}, x_n) &= m(Tx_n, Tx_{n-1}) \\
&\leq k\,(m(x_n, x_n) + m(x_{n+1}, x_{n-1})) \\
&= k\Big(m(x_n, x_n) + m(x_{n+1}, x_{n-1}) - m_{x_{n+1}, x_{n-1}} + m_{x_{n+1}, x_{n-1}}\Big) \\
&\leq k(m(x_n, x_n) + m(x_{n+1}, x_n) - m_{x_{n+1}, x_n} + m(x_n, x_{n-1}) \\
&\quad -m_{x_n, x_{n-1}} + m_{x_{n+1}, x_{n-1}}) \\
&= k(m(x_n, x_n) + m(x_{n+1}, x_n) - m(x_{n+1}, m_{n+1}) \\
&\quad +m(x_n, x_{n-1}) - m(x_n, x_n) + m(x_{n+1}, x_{n+1})) \\
&= k\,(m(x_{n+1}, x_n) + m(x_n, x_{n-1}))
\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that

$$m(x_{n+1}, x_n) \leq rm(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$, where $0 \leq r := \frac{k}{1-k} < 1$. By using Lemma 2.3.13, we get (A), (B), (C) and (D) in Lemma 2.3.13 hold. From the completeness of X, we get $x_n \to x$ for some $x \in X$. So

$$m(x_n, x) - m_{x_n, x} \to 0 \text{ as } n \to \infty$$

and

$$M_{x_n, x} - m_{x_n, x} \to 0 \text{ as } n \to \infty.$$

Since $m_{x_n, x} \to 0$ as $n \to \infty$, we have $m(x_n, x) \to 0$ and $M_{x_n, x} \to 0$ as $n \to \infty$. By Remark 2.3.5, we have

$$m(x, x) = 0 = m_{x, Tx}.$$

Next, we will show that $m(x,Tx)=0$. From (m_4), we get

$$\begin{aligned}
m(x,Tx) &\leq \limsup_{n\to\infty} m(x,x_n)+\limsup_{n\to\infty} m(x_n,Tx)\\
&= \limsup_{n\to\infty} m(x_n,Tx)\\
&\leq \limsup_{n\to\infty}\ (k(m(x_{n-1},Tx)+m(Tx_{n-1},x)))\\
&\leq k\left(\limsup_{n\to\infty} m(x_{n-1},Tx)+\limsup_{n\to\infty} m(x_n,x)\right)\\
&\leq k\left(\limsup_{n\to\infty}\ (m(x_{n-1},x)-m_{x_{n-1},x}+m(x,Tx)-m_{x,Tx})\right)\\
&\leq k\left(\limsup_{n\to\infty}\ (m(x_{n-1},x)+m(Tx,x))\right)\\
&= km(x,Tx).
\end{aligned}$$

This implies that

$$m(x,Tx)=0.$$

By contractive condition (4.3.1), we have

$$0\leq m(Tx,Tx)\leq 2km(x,Tx)=0.$$

Now, we obtain that

$$m(x,x)=m(Tx,Tx)=m(x,Tx).$$

Follows from (m_1), we get $x=Tx$. Next, we will show that T has a unique fixed point. Assume that y is another fixed point of T. From (4.3.1) we get

$$\begin{aligned}
m(x,y) &= m(Tx,Ty)\\
&\leq k(m(x,Ty)+m(y,Tx))\\
&= k(m(x,y)+m(y,x))\\
&\leq 2km(x,y)\\
&< m(x,y),
\end{aligned}$$

which is a contraction. Then T has a unique fixed point. This completes the proof. $\square$

Next, we give some examples to illustrate Theorem 4.3.1 and also give some numerical results.

Example 4.3.2. Let $X = [0,\infty)$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by $m(x,y) = \frac{x+y}{2}$ for all $x,y \in X$. From Example 2.3.9, we get (X,m) is a complete M-metric space. Let $T : X \to X$ be given by

$$Tx = \begin{cases} 0, & 0 \leq x < 3, \\ \frac{x}{1+x}, & x \geq 3. \end{cases}$$

We shall show that also condition (4.3.1) is satisfied with $k = \frac{1}{4}$. Suppose that $x,y \in X$. Then there are three possibilities:

Case 1: If $x,y \in [0,3)$, the claim is obvious.

Case 2: If $x,y \in [3,\infty)$, we get

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{2}\left(\frac{x}{1+x} + \frac{y}{1+y}\right) \\ &\leq \frac{1}{2}\left(\frac{x}{4} + \frac{y}{4}\right) \\ &\leq \frac{1}{4}\left(\frac{x}{2} + \frac{y}{2(1+y)} + \frac{y}{2} + \frac{x}{2(1+x)}\right) \\ &= k\left(m(x,Ty) + m(y,Tx)\right). \end{aligned}$$

Case 3: Assume that $(x,y) \in [3,\infty) \times [0,3) \cup [0,3) \times [3,\infty)$. Without loss of generality, we may assume that $x \in [0,3)$ and $y \in [3,\infty)$. Then we obtain that

$$\begin{aligned} m(Tx,Ty) &= \frac{1}{2}\left(\frac{y}{1+y}\right) \\ &\leq \frac{1}{2}\left(\frac{y}{4}\right) \\ &\leq \frac{1}{4}\left(0 + \frac{y}{2(1+y)} + \frac{y}{2} + 0\right) \\ &= k\left(m(x,Ty) + m(y,Tx)\right). \end{aligned}$$

Clearly, T satisfies condition (4.3.2) for all $x,y \in X$. Thus, all conditions of Theorem 4.3.1 are satisfied and the existence of a fixed point of T follows. In this case, a unique fixed point of T is a point 0.

We can see some numerical experiments for approximate the a unique fixed point of T in Figure 4.3. Furthermore, the convergence behavior of these iteration is shown in Figure 4.4.

x_0	3.00000000	5.00000000	7.00000000	9.00000000
x_1	0.75000000	0.83333333	0.87500000	0.90000000
x_2	0.00000000	0.00000000	0.00000000	0.00000000
x_3	0.00000000	0.00000000	0.00000000	0.00000000
x_4	0.00000000	0.00000000	0.00000000	0.00000000
x_5	0.00000000	0.00000000	0.00000000	0.00000000
x_6	0.00000000	0.00000000	0.00000000	0.00000000
x_7	0.00000000	0.00000000	0.00000000	0.00000000
x_8	0.00000000	0.00000000	0.00000000	0.00000000
x_9	0.00000000	0.00000000	0.00000000	0.00000000
x_{10}	0.00000000	0.00000000	0.00000000	0.00000000

Figure 4.3: Iterates of Picard iterations in Example 4.3.2

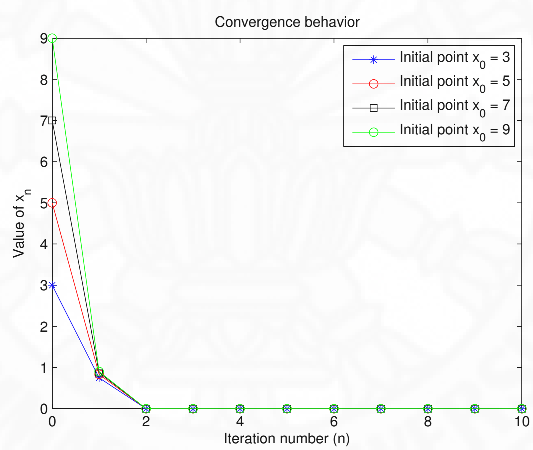

Figure 4.4: Convergence behavior for Example 4.3.2

Example 4.3.3. Let $X = [0,\infty)$ and a function $m : X \times X \to \mathbb{R}_+$ be defined by $m(x,y) = \frac{x+y}{2}$ for all $x,y \in X$. From Example 2.3.9, we get (X,m) is a complete M-metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} x^2, & 0 \le x < \frac{1}{2}, \\ \frac{1}{4}, & x \ge \frac{1}{2}. \end{cases}$$

Now we shall claim that T satisfies the condition (4.3.1) with $k = \frac{1}{3}$. Suppose that

$x, y \in X$. We will divide the proof into three cases.

Case 1: If $x, y \in \left[0, \frac{1}{2}\right)$, then we get

$$\begin{aligned} m(Tx, Ty) &= \frac{x^2}{2} + \frac{y^2}{2} \\ &\le \frac{x}{6} + \frac{y^2}{6} + \frac{y}{6} + \frac{x^2}{6} \\ &= \frac{1}{3}\left(\frac{x}{2} + \frac{y^2}{2} + \frac{y}{2} + \frac{x^2}{2}\right) \\ &= \frac{1}{3}(m(x, Ty) + m(y, Tx)). \end{aligned}$$

Case 2: If $x, y \in \left[\frac{1}{2}, \infty\right)$, then we get

$$\begin{aligned} m(Tx, Ty) &= \frac{1}{8} + \frac{1}{8} \\ &\le \frac{x}{6} + \frac{1}{24} + \frac{y}{6} + \frac{1}{24} \\ &= \frac{1}{3}\left(\frac{x}{2} + \frac{1}{8} + \frac{y}{2} + \frac{1}{8}\right) \\ &= \frac{1}{3}(m(x, Ty) + m(y, Tx)). \end{aligned}$$

Case 3: Assume that $(x, y) \in \left[0, \frac{1}{2}\right) \times \left[\frac{1}{2}, \infty\right) \cup \left[\frac{1}{2}, \infty\right) \times \left[0, \frac{1}{2}\right)$. Without loss of generality, we may assume that $x \in \left[0, \frac{1}{2}\right)$ and $y \in \left[\frac{1}{2}, \infty\right)$. Since $\frac{x^2}{2} \le \frac{x}{6} + \frac{x^2}{6}$ and $\frac{1}{8} \le \frac{1}{24} + \frac{y}{6}$. Then we obtain that

$$\begin{aligned} m(Tx, Ty) &= \frac{x^2}{2} + \frac{1}{8} \\ &\le \frac{x}{6} + \frac{1}{24} + \frac{y}{6} + \frac{x^2}{6} \\ &= \frac{1}{3}\left(\frac{x}{2} + \frac{1}{8} + \frac{y}{2} + \frac{x^2}{2}\right) \\ &= \frac{1}{3}(m(x, Ty) + m(y, Tx)). \end{aligned}$$

This implies that T satisfies the condition (4.3.1) with $k = \frac{1}{3}$. It easy to see that T satisfies condtition (4.3.2). Therefore, all conditions in Theorem 4.3.1 are satisfied. So T has a unique fixed point. In this example, a unique fixed point of T is a point 0.

We can see some numerical experiments for approximate the a unique fixed point of T in Figure 4.5. Furthermore, the convergence behavior of these iteration is shown in Figure 4.6.

x_0	0.20000000	0.30000000	0.40000000	0.50000000
x_1	0.04000000	0.09000000	0.16000000	0.25000000
x_2	0.00160000	0.00810000	0.02560000	0.06250000
x_3	0.00000256	0.00006561	0.00065536	0.00390625
x_4	0.00000000	0.00000000	0.00000043	0.00001526
x_5	0.00000000	0.00000000	0.00000000	0.00000000
x_6	0.00000000	0.00000000	0.00000000	0.00000000
x_7	0.00000000	0.00000000	0.00000000	0.00000000
x_8	0.00000000	0.00000000	0.00000000	0.00000000
x_9	0.00000000	0.00000000	0.00000000	0.00000000
x_{10}	0.00000000	0.00000000	0.00000000	0.00000000

Figure 4.5: Iterates of Picard iterations in Example 4.3.3

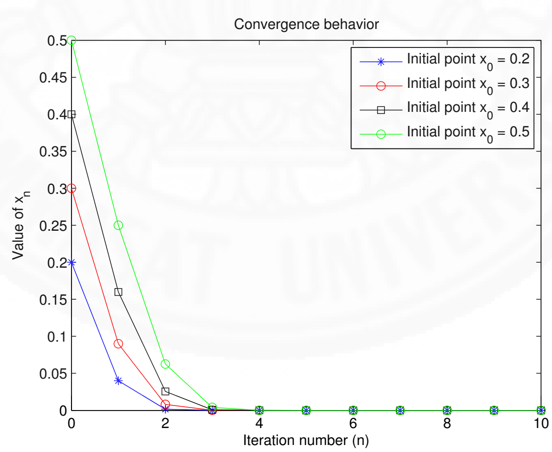

Figure 4.6: Convergence behavior for Example 4.3.3

There are many challenging and open questions in this field. There are:

- **Problem 1.**: Can Theorem 4.3.1 be proved without the condition (4.3.2)?

- **Problem 2.**: Does there exist any other condition (4.3.2) to prove Theorem 4.3.1?

CHAPTER 5

CONCLUSION

In this chapter, we summarize all fixed point results and applications in this thesis. In Chapter 3, we prove new φ-fixed points results for new nonlinear mappings by using control functions type-K and type-S_1 as follows:

(1) Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. Assume that the following conditions are satisfied:

  (i) φ is lower semi-continuous;

  (ii) $T : X \to X$ is an (F,φ,θ)-contraction with respect to the metric d.

  Then the following assertions hold:

  (i) $F_T \subseteq Z_\varphi$;

  (ii) T is a φ-Picard mapping.

(2) Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. Assume that the following conditions are satisfied:

  (H1) φ is lower semi-continuous;

  (H2) $T : X \to X$ is an (F,φ,θ)-weak contraction with respect to the metric d.

  Then the following assertions hold:

  (i) $F_T \subseteq Z_\varphi$;

  (ii) T is a weakly φ-Picard mapping.

(3) Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, and $F \in \mathcal{F}$. Assume that $T : X \to X$ is an (F,φ,λ)-contraction with respect to the metric d. Then the following assertions hold:

  (i) $F_T \subseteq Z_\varphi$;

(ii) T is a φ-Picard mapping;

(iii) for all $x \in X$ and for all $n \in \mathbb{N}$, we have

$$d(T^n x, z) \leq \tfrac{[\lambda(x)]^n}{1-\lambda(x)} F(d(x,Tx), \varphi(x), \varphi(Tx)),$$

where $\{z\} \in F_T \cap Z_\varphi = F_T$.

(4) Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, and $F \in \mathcal{F}$. Assume that $T : X \to X$ is a graphic (F,φ,λ)-contraction with respect to the metric d and T is continuous. Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) $T : X \to X$ is a weakly φ-Picard mapping;

(iii) for all $x \in X$, if $T^n x \to z$ as $n \to \infty$, then

$$d(T^n x, z) \leq \tfrac{[\lambda(x)]^n}{1-\lambda(x)} F(d(x,Tx), \varphi(x), \varphi(Tx)), \text{ for all } n \in \mathbb{N}.$$

(5) Let (X,d) be a complete metric space, $\varphi : X \to [0,\infty)$ be a lower semi-continuous function, and $F \in \mathcal{F}$. Assume that $T : X \to X$ is an (F,φ,λ)-weak contraction with respect to the metric d. Then the following assertions hold:

(i) $F_T \subseteq Z_\varphi$;

(ii) $T : X \to X$ is a weakly φ-Picard mapping;

(iii) for all $x \in X$, if $T^n x \to z$ as $n \to \infty$, then

$$d(T^n x, z) \leq \tfrac{[\lambda(x)]^n}{1-\lambda(x)} F(d(x,Tx), \varphi(x), \varphi(Tx)), \text{ for all } n \in \mathbb{N}.$$

(6) Let (X,p) be a complete partial metric space, and $\theta \in J$ and $T : X \to X$ be a given function. Assume that the following conditions are satisfied:

(P1) $\theta(2t) = 2\theta(t)$ for all $t \in [0,\infty)$;

(P2) the mapping T satisfies

$$p(Tx,Ty) \leq \theta(p(x,y))$$

for all $x,y \in X$.

Then the following assertions hold:

(i) T has a unique fixed point $z \in X$;

(ii) $p(z,z) = 0$.

(7) Let (X,p) be a complete partial metric space, $\theta \in J$ and $T : X \to X$ be a given mapping. Assume that the following conditions are satisfied:

(P1) $\theta(2t) = 2\theta(t)$ for all $t \in [0,\infty)$;

(P2) the mapping T satisfies

$$p(Tx,Ty) \leq \theta(p(x,y)) + L\left[N(x,y) - \frac{p(y,y) + p(Tx,Tx)}{2}\right]$$

for all $x,y \in X$, where $N(x,y) := \min\{p(x,Tx), p(y,Ty), p(y,Tx)\}$ and $L \geq 0$.

Then the following assertions hold:

(i) T has a fixed point $z \in X$;

(ii) $p(z,z) = 0$.

(8) Let (X,p) be a complete partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\lambda \in \Lambda_T^1$ such that

$$p(Tx,Ty) \leq \lambda(x)p(x,y) \tag{5.0.1}$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$. Moreover, we have $p(z,z) = 0$.

(9) Let (X,p) be a complete partial metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\lambda \in \Lambda_T$ such that

$$p(T^2x,Tx) \leq \lambda(Tx)p(Tx,x) \tag{5.0.2}$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$. Moreover, we have $p(z,z) = 0$.

(10) Let (X,p) be a complete partial metric space and $T:X\to X$ be a given mapping. Suppose that there exists a function $\lambda\in\Lambda_T$ such that

$$p(Tx,Ty)\le\lambda(x)p(x,y)+L\left[N(x,y)-\frac{p(y,y)+p(Tx,Tx)}{2}\right] \tag{5.0.3}$$

for all $x,y\in X$, where $N(x,y):=\min\{p(x,Tx),p(y,Ty),p(y,Tx)\}$ and $L\ge 0$. Then T has a unique fixed point $z\in X$. Moreover, we have $p(z,z)=0$.

In Chapter 4, we prove some results of generalization of the Banach and Kannan type contraction in the setting of M-metric space and also give a partial answer to Open problem 1.0.15 concerning a fixed point for Chatterjea contraction mappings as follows:

(1) Let (X,m) be a complete M-metric space and $T:X\to X$ be an α-m-Banach contraction mapping satisfying conditions:

(i) T is a weakly α-admissible mapping;

(ii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;

(iii) If $\{x_n\}$ is sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x,x)\ge 1$ and $\alpha(x_n,x)\ge 1$ for all $n\in\mathbb{N}$.

Then the fixed point problem of T has a solution, that is, there exists $x^*\in X$ such that $Tx^*=x^*$.

(2) Let (X,m) be a complete M-metric space and $T:X\to X$ be an α-m-Kannan contraction mapping satisfying conditions:

(i) T is a weakly α-admissible mapping;

(ii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;

(iii) If $\{x_n\}$ is sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x,x)\ge 1$ and $\alpha(x_n,x)\ge 1$ for all $n\in\mathbb{N}$.

Then the fixed point problem of T has a solution, that is, there exists $x^*\in X$ such that $Tx^*=x^*$.

(3) Let (X,m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition:

$$\exists k \in \left[0, \frac{1}{2}\right) \text{ such that } m(Tx,Ty) \leq k[m(x,Ty)+m(Tx,y)] \text{ for all } x,y \in X. \tag{5.0.4}$$

If there is $x_0 \in X$ such that

$$m(T^n x_0, T^n x_0) \leq m(T^{n-1}x_0, T^{n-1}x_0) \tag{5.0.5}$$

for all $n \in \mathbb{N}$, then T has a unique fixed point. Moreover, if the Picard sequence $\{x_n\}$ in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ such that x_0 is an initial point in condition (4.3.2), then $\{x_n\}$ converges to a fixed point of T.

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BIOGRAPHY

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Publications

1. Kumrod, P., & Sintunavarat, W. (2015). A new generalization of the Banach and Kannan contraction principle in the sense of M-metric spaces, Proceedings of the 20th Annual Meeting in Mathematics (AMM2015), (pp. 147-155), Nakhon Pathom, http://www.amm2015.com/images/documents/AMM2015-Procs-Final.pdf.

2. Kumrod, P., & Sintunavarat, W. (2016). A new contractive condition approach to φ-fixed point results in metric spaces and its applications, Journal of Computational and Applied Mathematics (Accept).

Poster Presentation

1. Kumrod, P., & Sintunavarat, W. (2016). A new contractive condition approach to φ-fixed point results in metric spaces and its applications, Science Research Conference 8th, 30-31 May 2016, University of Phayao, Thailand.

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Awards

1. Poster Presentation Award (1st place), A new contractive condition approach to φ-fixed point results in metric spaces and its applications, Science Research Conference 8th, 30-31 May 2016, University of Phayao, Thailand.