# ON GENERALIZED STABILITY AND HYPERSTABILITY RESULTS FOR SOME FUNCTIONAL EQUATIONS BY USING FIXED POINT METHOD

BY

MISS LADDAWAN AIEMSOMBOON

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
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| Thesis Title | ON GENERALIZED STABILITY AND HYPERSTABILITY RESULTS FOR SOME FUNCTIONAL EQUATIONS BY USING FIXED POINT METHOD |
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## ABSTRACT

In this thesis, we investigate stability and hyperstability of various functional equations by using fixed point theorem. In case of stability results, we focus on radical quadratic functional equation, generalized logarithmic Cauchy functional equation and generalized Cauchy functional equation. In case of hyperstability results, we focus on general linear functional equation and Drygas functional equation. Stability and hyperstability results in this thesis generalize and extend several results concerning classical researchs in this field.

**Keywords:** Stability, Hyperstability, Fixed point, Functional equation

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Miss Laddawan Aiemsomboon

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# INTRODUCTION

Throughout this thesis, $\mathbb{N}_0$, $\mathbb{Z}^+$ (or $\mathbb{N}$), $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}_+$,$\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}_{n_0}$ denote the set of nonnegative integers, positive integers, integers, rational numbers, nonnegative real numbers, positive real numbers, real numbers, complex numbers, and the set of all integers greater than or equal to $n_0$, respectively. Also, $B^A$ denotes the family of all functions map from a set $A \neq \emptyset$ into a set $B \neq \emptyset$.

## 1.1 Literature review

The functional equations have been studied by d'Alembert more than 260 years ago, but many papers concerning functional equations were published during the last 60 years. In 1812, Cauchy first found general continuous solution of the following additive Cauchy functional equation

$$f(x+y) = f(x) + f(y),$$

where $f$ is mapping from $\mathbb{R}$ into $\mathbb{R}$. He also studied three other types of functional equations as follows:

$$f(x+y) = f(x)f(y),$$

$$f(xy) = f(x) + f(y),$$

$$f(xy) = f(x)f(y).$$

These equations are called the exponential Cauchy functional equation, the logarithmic Cauchy functional equation, and the multiplicative Cauchy functional equation, respectively.

The problem of stability of functional equations is one of the essential results in the theory of functional equations. Also, it is connected with perturbation theory and the notions of shadowing in dynamical systems and controlled chaos (see in [8] and [15]). The starting point of the stability theory of functional equations was the problem of Ulam concerning the stability of group homomorphism in 1940.

**Question 1.1.1** ([18])**.** Let $(G_1, *_1)$, $(G_2, *_2)$ be two groups and $d : G_2 \times G_2 \to [0, \infty)$ be a metric group. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x *_1 y), h(x) *_2 h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In 1941, Hyers [9] gave partial answer of Ulam's question and established the stability result as follows:

**Theorem 1.1.2** ([9])**.** *Let $E_1, E_2$ be two Banach spaces and $f : E_1 \to E_2$ be a function such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{1.1.1}$$

*for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit*

$$A(x) := \lim_{n \to \infty} 2^{-n} f(2^n x)$$

*exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive Cauchy function such that*

$$\|f(x) - A(x)\| \leq \delta$$

*for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the function $A$ is linear.*

In view of this result, the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is said to have the Ulam-Hyers stability on $(E_1, E_2)$ if for each function $f : E_1 \to E_2$ satisfies the inequality (1.1.1) for some $\delta > 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on $E_1$.

In 1950, Aoki [1] proved the following stability result for functions that do not have bounded Cauchy difference.

**Theorem 1.1.3** ([1]). *Let $E_1$ and $E_2$ be two Banach spaces and $f : E_1 \to E_2$ be a function. If $f$ satisfies the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1.2}$$

*for some $\theta \ge 0$, $p$ is a real number with $0 \le p < 1$ and for all $x, y \in E_1$, then there exists a unique additive Cauchy function $A : E_1 \to E_2$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p \tag{1.1.3}$$

*for each $x \in E_1$.*

In 1978, Rassias [13] gave the complement of Theorem 1.1.3 as follows.

**Theorem 1.1.4** ([13]). *Let $E_1$ and $E_2$ be two Banach spaces and $f : E_1 \to E_2$ be a function. If $f$ satisfies the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1.4}$$

*for some $\theta \ge 0$, $p$ is a real number with $0 \le p < 1$ and for all $x, y \in E_1$, then there exists a unique additive Cauchy function $A : E_1 \to E_2$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p \tag{1.1.5}$$

*for each $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the function $A$ is linear.*

Note that Theorem 1.1.4 reduces to the first result of stability due to Hyers [9] if $p = 0$. In [14], Rassias note that the proof of Theorem 1.1.4 can be applied to the proof of the following result.

**Theorem 1.1.5** ( [14]). *Let $E_1$ and $E_2$ be two Banach spaces and $f : E_1 \to E_2$ be a function. If $f$ satisfies the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{1.1.6}$$

*for some $\theta \geq 0$, $p$ is a real number with $p < 0$ and for all $x, y \in E_1 \backslash \{0\}$, then there exists a unique additive Cauchy function $A : E_1 \to E_2$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \tag{1.1.7}$$

*for each $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1 \backslash \{0\}$, then the function $A$ is linear.*

Afterward, Gajda [7] obtained this result for $p > 1$ and gave an example to show that Theorem 1.1.5 fails whenever $p = 1$.

**Theorem 1.1.6** ([7])**.** *Let $E_1$ be a normed space, $E_2$ be a Banach space and $f : E_1 \to E_2$ be a function. If $f$ satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.1.8}$$

*for some $\theta \geq 0$, $p$ is a real number with $p \neq 1$ and for all $x, y \in E_1$, then there exists a unique additive Cauchy function $A : E_1 \to E_2$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.1.9}$$

*for each $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the function $A$ is linear.*

Subsequently, several mathematicians studied and extended Hyers-Ulam stability in many functional equations such as the exponential Cauchy functional equation, the logarithmic Cauchy functional equation, the multiplicative functional equation, the cosine functional equation and the sine functional equation, and they also proved general solutions. In addition, many mathematicians studied the hyperstability in many type of functional equations such as additive Cauchy functional equation , general linear functional equation, and Jensen functional equation.

In 2011, Brzdęk et al. [3] proved the existence of fixed point theorem for nonlinear operator. Also, they used this result to study the stability of function equation in non-Archimedean metric spaces and obtained the fixed point result in arbitrary metric spaces (see Theorem 1.1.7).

**Theorem 1.1.7** ([3]). *Let $X$ be a nonempty set, $(Y,d)$ be a complete metric space and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be non-decreasing operator satisfying the following hypothesis:*

$$\lim_{n\to\infty} \Lambda\delta_n = 0 \text{ for every sequence } \{\delta_n\}_{n\in\mathbb{N}} \text{ in } \mathbb{R}_+^X \text{ with } \lim_{n\to\infty} \delta_n = 0.$$

*Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality*

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \le \Lambda(\Delta(\xi,\mu))(x), \quad \xi,\mu \in Y^X, x \in X, \tag{1.1.10}$$

*where $\Delta : (Y^X)^2 \to \mathbb{R}_+^X$ is a mapping which is defined by*

$$\Delta(\xi,\mu)(x) := d(\xi(x),\mu(x)), \quad \xi,\mu \in Y^X, x \in X \tag{1.1.11}$$

*and function $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ are such that*

$$d((\mathcal{T}\varphi)(x), \varphi(x)) \le \varepsilon(x), \quad x \in X \tag{1.1.12}$$

*and*

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty, \quad x \in X. \tag{1.1.13}$$

*Then for every $x \in X$, the limit*

$$\lim_{n\to\infty} (\mathcal{T}^n \varphi)(x) := \psi(x), \quad x \in X \tag{1.1.14}$$

*exists and the function $\psi \in Y^X$, defined in this way, is a fixed point of $\mathcal{T}$ with*

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x), \quad x \in X \tag{1.1.15}$$

In the same year, Brzdęk et al. [2] gave the stability results by using the new fixed point theorem which is proved bythemselves as follows:

**Theorem 1.1.8** ([2]). *Let $U$ be a nonempty set, $(Y,d)$ be a complete metric space, $f_1,...,f_k : U \to U$ and $L_1,...,L_k : U \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^U \to Y^U$ is an operator satisfying the inequality*

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \le \sum_{i=1}^{k} L_i(x) d(\xi(f_i(x)), \mu(f_i(x))) \tag{1.1.16}$$

*for all* $\xi, \mu \in Y^U$ *and* $x \in U$. *Assume that there are functions* $\varepsilon : U \to \mathbb{R}_+$ *and* $\varphi : U \to Y$ *fulfil the following condition for each* $x \in U$ :

$$d((\mathcal{T}\varphi)(x), \varphi(x)) \leq \varepsilon(x)$$

*and*

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty,$$

*where* $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ *is defined by*

$$(\Lambda\delta)(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x))$$

*for all* $\delta \in \mathbb{R}_+^U$ *and* $x \in U$. *Then there exists a unique fixed point* $\psi$ *of* $\mathcal{T}$ *with*

$$d(\varphi(x), \psi(x)) \leq \varepsilon^*(x)$$

*for all* $x \in U$. *Moreover,* $\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n\varphi)(x)$ *for all* $x \in U$.

In 2013, Brzdęk [4] proved the stability results of the additive Cauchy functional equation by using the special case of Theorem 1.1.8 in the following from:

**Theorem 1.1.9.** *Let* $U$ *be a nonempty set,* $(Y, d)$ *be a complete metric space and* $f_1, f_2$ *be a self mapping on* $U$. *Assume that* $\mathcal{T} : Y^U \to Y^U$ *is an operator satisfying the inequality*

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \leq d(\xi(f_1(x)), \mu(f_1(x))) + d(\xi(f_2(x)), \mu(f_2(x))) \qquad (1.1.17)$$

*for all* $\xi, \mu \in Y^U$ *and* $x \in U$. *Suppose that there exist functions* $\varepsilon : U \to \mathbb{R}_+$ *and* $\varphi : U \to Y$ *such that for each* $x \in U$,

$$d((\mathcal{T}\varphi)(x), \varphi(x)) \leq \varepsilon(x)$$

*and*

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty,$$

*where* $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ *is an operator defined by*

$$(\Lambda\delta)(x) := \delta(f_1(x)) + \delta(f_2(x)) \qquad (1.1.18)$$

*for all* $\delta \in \mathbb{R}_+^U$ *and* $x \in U$. *Then there exists a unique fixed point* $\psi$ *of* $\mathcal{T}$ *with*

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x)$$

*for all* $x \in U$. *Moreover,* $\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x)$ *for* $x \in U$.

On the other hand, Piszczek [10] proved hyperstability results by using the following fixed point result which is a special case of Theorem 1.1.8

**Theorem 1.1.10.** *Let* $U$ *be a nonempty set,* $Y$ *be a Banach space,* $f_1, ..., f_k : U \to U$ *and* $L_1, ..., L_k : U \to \mathbb{R}_+$ *be given mappings. Suppose that* $\mathcal{T} : Y^U \to Y^U$ *is an operator satisfying the inequality*

$$\|(\mathcal{T}\xi)(x) - (\mathcal{T}\mu)(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\| \tag{1.1.19}$$

*for all* $\xi, \mu \in Y^U$ *and* $x \in U$. *Assume that there are functions* $\varepsilon : U \to \mathbb{R}_+$ *and* $\varphi : U \to Y$ *fulfil the following condition for each* $x \in U$ :

$$\|(\mathcal{T}\varphi)(x) - \varphi(x)\| \le \varepsilon(x)$$

*and*

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty,$$

*where* $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ *is defined by*

$$(\Lambda\delta)(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)) \tag{1.1.20}$$

*for all* $\delta \in \mathbb{R}_+^U$ *and* $x \in U$. *Then there exists a unique fixed point* $\psi$ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x)$$

*for all* $x \in U$. *Moreover,* $\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x)$ *for all* $x \in U$.

### 1.2 Overview

In this thesis, we investigate stability and hyperstability of various functional equations by using fixed point theorem of Brzdęk in the forms like Theorem 1.1.9 and 1.1.10. First topic, we use the fixed point method of Brzdęk (Theorem 1.1.9) to prove the stability results for the following functional equations:

- radical quadratic functional equation of the form

$$f(\sqrt{x^2+y^2}) = f(x)+f(y), \tag{1.2.1}$$

where $f$ is is mapping from $\mathbb{R}$ into $\mathbb{R}$ and $a,b \in \mathbb{N}$,

- generalized logarithmic Cauchy functional equation of the form

$$f(|x|^a \cdot |y|^b) = af(x)+bf(y),$$

where $f$ is mapping from $\mathbb{R}$ into $\mathbb{R}$ and $a,b \in \mathbb{N}$,

- generalized additive Cauchy functional equation of the form

$$f_1(ax+by) = af_1(x)+bf_1(y),$$

where $f_1$ is mapping from $\mathbb{R}$ into $\mathbb{R}$ and generalized Cauchy functional equation of the form

$$f_2(ax*by) = af_2(x) \diamond bf_2(y),$$

where $f_2$ is a mapping from a commutative semigroup $(G_1,*)$ to a commutative group $(G_2,\diamond)$ and $a,b \in \mathbb{N}$.

Second topic, we use the fixed point method of Brzdęk (Theorem 1.1.10) to prove the hyperstability results for the following functional equations:

- general linear functional equation of the form

$$g(ax+by) = Ag(x)+Bg(y), \tag{1.2.2}$$

where $g: X \to Y$ is a mapping and $a,b \in \mathbb{F}\backslash\{0\}$, $A,B \in \mathbb{K}$,

- Drygas functional equation of the form

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y),$$

where $f$ maps from $X$ into $Y$ and $x,y \in X$ with $x+y, x-y \in X$.

Stability and hyperstability results in this thesis generalize and extend several results concerning classical researches in this field.

# CHAPTER 2

# PRELIMINARIES

## 2.1 Binary operation

**Definition 2.1.1.** A **binary operation** on a set $S$ is a mapping $*$ whose domain is a subset of $S \times S$ and whose range is a subset of $S$. We will denote this: $* : S \times S \to S$, and the image $*(a,b)$ will be denoted by $a * b$.

**Example 2.1.2.** Addition and multiplication are binary operations on $\mathbb{Z}$.

**Example 2.1.3.** Let $n \in \mathbb{N}$. Define $\mathbb{Z}_n = \{0,1,2,...,n-1\}$ and binary operation on $\mathbb{Z}_n$ are following:

1. addition modulo $n$: $a+b=c$ if and only if $a+b \equiv c(mod\ n)$ for all $a,b,c \in \mathbb{Z}_n$.
2. multiplication modulo $n$: $a \cdot b = c$ if and only if $a \cdot b \equiv c(mod\ n)$ for all $a,b,c \in \mathbb{Z}_n$.

For instance in $\mathbb{Z}_3 := \{0,1,2\}$, we get

$$0+0=0, \quad 0+1=1, \quad 0+2=2, \quad 1+1=2, \quad 1+2=0, \quad 2+2=1$$

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 0 \cdot 2 = 0, \quad 1 \cdot 1 = 1, \quad 1 \cdot 2 = 2 \quad 2 \cdot 2 = 1.$$

## 2.2 Fields

**Definition 2.2.1.** The set $\mathbb{F}$ is called a **field** when two binary operations $+$ and $\cdot$ on $\mathbb{F}$, which we call addition and multiplication, satisfy the following conditions for all $a,b,c \in \mathbb{F}$:

1. $a+b=b+a$ and $a \cdot b = b \cdot a$;
2. $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

3. there exists an element $0_{\mathbb{F}} \in \mathbb{F}$ such that $a + 0_{\mathbb{F}} = a$;

4. there exists an element $1_{\mathbb{F}} \in \mathbb{F}$ such that $a \cdot 1_{\mathbb{F}} = a$;

5. for each $a \in \mathbb{F}$, there exists an element $-a \in \mathbb{F}$ such that $a + (-a) = 0_{\mathbb{F}}$;

6. if $a \neq 0$, then there exists an element $a^{-1} \in \mathbb{F}$, such that $a \cdot (a^{-1}) = 1_{\mathbb{F}}$;

7. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

**Example 2.2.2.** $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are fields under the usual addition and multiplication. Note that $(\mathbb{Z}, +, \cdot)$ is not field because 3 has no multiplicative inverse.

**Example 2.2.3.** $(\mathbb{Z}_p, +, \cdot)$ is a field, where $p$ is a prime, $+_p$ is addition modulo $p$ and $\cdot$ is multiplication modulo $p$.

## 2.3 Vector spaces

**Definition 2.3.1.** The set $V$ is called a **vector space** over a field $\mathbb{F}$ when the vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ satisfy the following properties for every $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$:

1. $u + v = v + u$;

2. $(u + v) + w = u + (v + w)$;

3. there exists an element $\mathbf{0} \in V$ such that $u + \mathbf{0} = u$;

4. for each $u \in V$, there exists an element $-u \in V$ such that $u + (-u) = \mathbf{0}$;

5. $(\alpha\beta)u = \alpha(\beta u)$;

6. $\alpha(u + v) = \alpha u + \alpha v$;

7. $(\alpha + \beta)u = \alpha u + \beta u$;

8. $1_{\mathbb{F}} u = u$.

The vector space $X$ is called a real vector space when $\mathbb{F} = \mathbb{R}$ and a complex vector space when $\mathbb{F} = \mathbb{C}$.

**Example 2.3.2.** The set $M_{m\times n}(\mathbb{F})$ of all $m \times n$ matrices with entries from a field $\mathbb{F}$ is a vector space under two algebraic operations defined by

$$A + B = (a_{ij} + b_{ij})_{m\times n},$$

$$\alpha \cdot A = (\alpha \cdot a_{ij})_{m\times n}$$

for $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{F})$ and each scalar $\alpha$.
For instance in $M_{2\times 2}(\mathbb{R})$

$$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 4 & 7 \end{pmatrix},$$

$$3\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 12 \\ 3 & 6 \end{pmatrix}.$$

**Example 2.3.3.** The set $\mathbb{R}^n$ is a real vector space over a field $\mathbb{R}$ with the two algebraic operations defined by

$$u + v = (u_1 + v_1, u_2 + v_2, + \cdots, u_n + v_n),$$

$$\alpha u = (\alpha u_1, \alpha u_2 +, + \cdots, \alpha u_n)$$

for each $u = (u_1, u_2, \cdots, u_n), v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and each $\alpha \in \mathbb{R}$.

**Example 2.3.4.** The set of all real valued continuous functions on $[0,1]$ is vector space over a field $\mathbb{R}$ with the two algebraic operations defined by

$$(f + g)(x) = f(x) + g(x),$$

$$(\alpha f)(x) = \alpha f(x)$$

for all continuous functions $f, g : [0,1] \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and for all $x \in [0,1]$.

## 2.4 Normed spaces

**Definition 2.4.1.** Let $V$ be a real or complex vector spaces. A function $\|\cdot\|: V \to \mathbb{R}$ is called a **norm** on $V$ if the following conditions hold for every $u, v \in V$:

1. $\|u\| = 0$ if and only if $u = \mathbf{0}$;
2. $\|\alpha u\| = |\alpha|\|u\|$ for all scalars $\alpha$;
3. $\|u+v\| \leq \|u\| + \|v\|$.

The ordered pair $(V, \|\cdot\|)$ is called **normed space**.

**Remark 2.4.2.** In a normed space $(V, \|\cdot\|)$, we obtain that $\|u\| \geq 0$ for all $u \in V$.

**Definition 2.4.3.** A sequence $\{x_n\}$ of element in a normed space $(V, \|\cdot\|)$ is said to be **converge** to a point $x \in V$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$ for all $n > N$, denoted by $\lim_{n\to\infty} x_n = x$ or, simply, $x_n \to x$. If the sequence $\{x_n\}$ converges, it is said to be convergent.

**Remark 2.4.4.** A sequence $\{x_n\}$ of element in a normed space $(V, \|\cdot\|)$ is convergent sequence if and only if $\lim_{n\to\infty} \|x_n - x\| = 0$.

**Definition 2.4.5.** A sequence $\{x_n\}$ of element in a normed space $(V, \|\cdot\|)$ is called **Cauchy sequence** if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$ for all $n, m > N$.

**Remark 2.4.6.** A sequence $\{x_n\}$ of element in a normed space $(V, \|\cdot\|)$ is Cauchy sequence if and only if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$.

**Definition 2.4.7.** A normed space $(V, \|\cdot\|)$ is called **complete** if for every Cauchy sequence in $V$ converges.

**Definition 2.4.8.** A complete normed vector space over field $\mathbb{F}$ is called a **Banach space**.

**Example 2.4.9.** The set $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a real (or complex) Banach space with normed defined by

$$\|u\| := \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2}$$

where $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$).

**Example 2.4.10.** The set $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a real (or complex) Banach space with normed defined by

$$\|u\|_p := \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}}$$

and

$$\|u\|_\infty := \max\{|u_1|, |u_2|, ..., |u_n|\},$$

where $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and $1 < p < \infty$.

In this case, $\|\cdot\|_p$ is called $p$**-norm** and $\|\cdot\|_\infty$ is called **infinity norm** or **maximum norm**.

**Example 2.4.11.** Let $p \geq 1$ be a fixed real numbers and $V$ be the set of all sequence of real (or complex) numbers, that is, each element of $V$ is a real (or complex) sequence

$$u = \{u_1, u_2, ...\} \ \text{ briefly } \ u = \{u_i\}, \quad i = 1, 2, 3, ...$$

such that $\sum_{i=1}^{\infty} |u_i|^p < \infty$. Define a function $\|\cdot\| : V \times V \to \mathbb{R}$ by

$$\|u\|_p := \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}}$$

where $u = \{u_i\} \in V$. Then $(V, \|\cdot\|)$ is a Banach space with is denoted $\ell^p$.

**Example 2.4.12.** Let $V$ be the set of all bounded sequences of real (or complex) numbers, that is,

$$|u_i| \leq c_u \quad \forall i \in \{1, 2, ...\}$$

for all $u = \{u_i\} \in V$, where $c_u$ is a real numbers which depend on $u$, but not depend on $i$. Define a function $\|\cdot\| : V \times V \to \mathbb{R}$ by

$$\|u\|_\infty := \sup_{i \in \mathbb{N}} |u_i|,$$

where $u = \{u_i\} \in V$ and sup denotes the supremum (least upper bound). Then $(V, \|\cdot\|)$ is a Banach space with is denoted $\ell^\infty$.

## 2.5 Metric spaces

**Definition 2.5.1.** Let $X$ be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** on $X$ if the following conditions hold for all $x, y, z \in X$:

1. $d(x,y) = 0 \Leftrightarrow x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,z) \leq d(x,y) + d(y,z)$.

The ordered pair $(X,d)$ is called **metric space**.

**Remark 2.5.2.** In a metric space $(X,d)$, we get $d(x,y) \geq 0$ for all $x, y \in X$.

**Example 2.5.3.** Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) = |x-y|$$

for all $x, y \in \mathbb{R}$. Then $d$ is a metric on $\mathbb{R}$ and it is called **usual** or **standard metric**.

**Example 2.5.4.** Let $r > 0$ and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) = r|x-y|$$

for all $x, y \in \mathbb{R}$. Then $d$ is a metric on $\mathbb{R}$.

**Example 2.5.5.** Let $X$ be a nonempty set and $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

Then $d$ is a metric on $X$ and it is called the **discrete metric** or the **trivial metric**.

**Example 2.5.6.** Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) = |x-y|^2$$

for all $x,y \in \mathbb{R}$. Then $d$ is not a metric on $\mathbb{R}$.

**Example 2.5.7.** Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = |x_1| + |x_2| + |y_1| + |y_2|,$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then $d$ is not a metric on $\mathbb{R}^2$.

**Example 2.5.8.** Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2},$$

where $x = (x_1, x_2, \cdots x_n), y = (y_1, y_2, \cdots y_n) \in \mathbb{R}^n$. Then $d$ is a metric on $\mathbb{R}^n$ and it is called **Euclidian metric on** $\mathbb{R}^n$.

**Example 2.5.9.** Let $d : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ be defined by

$$d(x,y) = \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2 + \cdots + |x_n-y_n|^2},$$

where $x = (x_1, x_2, \cdots x_n), y = (y_1, y_2, \cdots y_n) \in \mathbb{C}^n$. Then $d$ is a metric on $\mathbb{C}^n$ and it is called **Euclidian metric on** $\mathbb{C}^n$.

**Definition 2.5.10.** Let $(X,d)$ be a metric space. The **open ball** of radius $r > 0$ and center $x \in X$ is the set $B(x,r) \subset X$ defined by

$$B(x,r) := \{y \in X : d(x,y) < r\}$$

(see in Figure 2.1).

**Definition 2.5.11.** Let $(X,d)$ and $(Y,d)$ be two metric spaces. A function $f : X \to Y$ is called **continuous** at $x \in X$ if for every $r > 0$, there exist $s > 0$ such that

$$f(B(x,s)) \subset B(f(x),r).$$

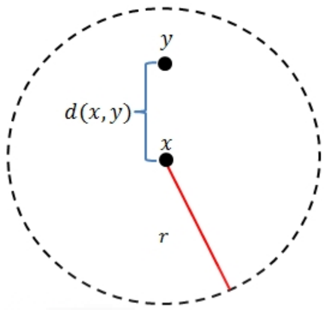


Figure 2.1: Open ball of radius $r$ and center $x$ in Euclidean space $\mathbb{R}^2$

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**Definition 2.5.12.** A sequence $\{x_n\}$ of element in a metric space $(X,d)$ is said to be **converge** to a point $x \in X$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n,x) < \epsilon$ for all $n > N$, denoted by $\lim_{n\to\infty} x_n = x$ or, simply, $x_n \to x$. If the sequence $\{x_n\}$ converges, it is said to be convergent.

**Remark 2.5.13.** A sequence $\{x_n\}$ of element in a metric space $(X,d)$ is convergent sequence if and only if

$$\lim_{n\to\infty} d(x_n,x) = 0.$$

**Definition 2.5.14.** A sequence $\{x_n\}$ of element in a metric space $(X,d)$ is called **Cauchy sequence** if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n,x_m) < \epsilon$ for all $n,m > N$.

**Remark 2.5.15.** A sequence $\{x_n\}$ of element in a metric space $(X,d)$ is Cauchy sequence if and only if

$$\lim_{m,n\to\infty} d(x_m,x_n) = 0.$$

**Definition 2.5.16.** A metric space $(X,d)$ is called **complete** if for every Cauchy sequence in $X$ converges.

## 2.6 Groups

**Definition 2.6.1.** The nonempty set $S$ is called **semigroup** in which $*$ is a binary associative operation on $S$, i.e., the equation

$$(a*b)*c = a*(b*c)$$

holds for all $a,b,c \in S$.

**Example 2.6.2.** $(\mathbb{Z}^+,+)$ and $(\mathbb{Z}^+,\cdot)$ are semigroups under the usual addition and multiplication.

**Example 2.6.3.** The set $M_{n\times n}(\mathbb{R})$ of all real matrices $n \times n$ with multiplication of matrix is a semigroup.

**Definition 2.6.4.** The nonempty set $G$ together with a binary operation $*$ on $G$ is called a **group** if the following conditions hold:

1. $a*(b*c) = (a*b)*c$ for all $a,b,c \in G$;
2. there exists an identity element $e \in G$ such that $e*a = a$ and $a*e = a$ for all $a \in G$;
3. for each $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a*a^{-1} = e$ and $a^{-1}*a = e$.

A notation for group $G$ with $*$ is denoted by $(G,*)$.

**Example 2.6.5.** $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are groups under the usual addition.

**Example 2.6.6.** $(\mathbb{Q}-\{0\},\cdot)$, $(\mathbb{R}-\{0\},\cdot)$ and $(\mathbb{C}-\{0\},\cdot)$ are groups under the usual multiplication.

**Example 2.6.7.** $(\mathbb{Z}_n,+)$ is a group, where $+$ is addition modulo $n$ .

**Example 2.6.8.** $(\mathbb{Z}_p-\{0\},\cdot)$ is a group, where $p$ is a prime and $\cdot$ is multiplication modulo $n$.

**Example 2.6.9.** The set $M_{n\times n}(\mathbb{R})$ of all real matrices $n\times n$ with $+$ as defined in Example 2.3.2 is a group.

**Example 2.6.10.** $(\mathbb{Z},\cdot)$ is not group because 3 has no inverse element.

**Definition 2.6.11.** A group $(G,*)$ is said to be an **abelian** (or **commutative**) if $a*b=b*a$ for all $a,b\in G$.

Groups in Examples 2.6.5, 2.6.6, 2.6.7, 2.6.8, 2.6.9 are also an abelian group.

Next, we give example of group but not abelian group.

**Example 2.6.12.** Let

$$M := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d\in\mathbb{R} \text{ and } ad-bc\neq 0 \right\}.$$

Note that binary operation $\cdot$ has associative property and the identity element is $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, for $N=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M$ there exists

$$N^{-1}=\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M$$

such that

$$NN^{-1}=I=N^{-1}N.$$

Then $M$ with multiplication of matrix is a group.

Next, we will show that $M$ with multiplication of matrix dose not have commutative property.
If $A=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $A,B\in M$ such that $AB=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $BA=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, that is, $AB\neq BA$. Then $M$ is a group but not abelain group.

**Definition 2.6.13.** Let $(G_1,\cdot),(G_2,*)$ be groups. The function $h:G_1\to G_2$ is said to be **homomorphism** if $h(a\cdot b)=h(a)*h(b)$ for every $a,b\in G_1$.

**Example 2.6.14.** Let $(G_1,\cdot),(G_2,*)$ be groups and the function $h: G_1 \to G_2$ defined by

$$h(a) = e_2 \text{ for all } a \in G_1,$$

when $e_2$ is identity of $G_2$. Then for every $a,b \in G_1$ we have

$$h(a \cdot b) = e_2 = e_2 * e_2 = h(a) \cdot h(b).$$

Therefore, $h$ is homomorphism and it is called the **zero function**.

**Example 2.6.15.** The function $h : (\mathbb{Z},+) \to (\mathbb{Z},+)$ defined by

$$h(a) = 9a \text{ for all } a \in \mathbb{Z}.$$

For each $a,b \in \mathbb{Z}$, we obtain that

$$h(a+b) = 9(a+b) = 9a+9b = h(a)+h(b).$$

Therefore, $h$ is homomorphism.

**Example 2.6.16.** The function $h : (\mathbb{R},+) \to (\mathbb{R}^+,\cdot)$ defined by

$$h(a) = 3^a \text{ for all } a \in \mathbb{R}.$$

For every $a,b \in \mathbb{R}$, we obtain that

$$h(a+b) = 3^{(a+b)} = 3^a \cdot 3^b = h(a) \cdot h(b).$$

Therefore, $h$ is homomorphism.

## 2.7 Metric groups

**Definition 2.7.1.** Let $(G,*)$ be a group. A metric $d$ on $G$ is said to be **left invariant** if for every $x,y,z \in G$,

$$d(y,z) = d(x*y, x*z).$$

**Definition 2.7.2.** A group $G$ with a left invariant metric such that inversion function $x \mapsto x^{-1}$ is continuous is called a **metric group**.

**Example 2.7.3.** The Euclidean space $\mathbb{R}^n$ is a metric group.

**Example 2.7.4.** Any group with the trivial metric is a metric group.

## 2.8 Fixed point basics

**Definition 2.8.1.** Let $X$ be a nonempty set. A point $x \in X$ is called **fixed point** of $f : X \to X$ if and only if $fx = x$ (see in Figure 2.2).

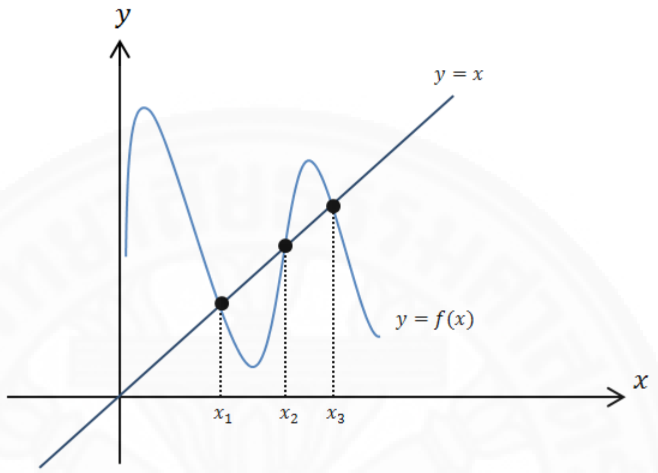


Figure 2.2: $x_1, x_2, x_3$ are fixed points of $f : \mathbb{R} \to \mathbb{R}$.

**Example 2.8.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$fx = x^2 - 4x + 6$$

for all $x \in \mathbb{R}$. Then $x = 2$ and 3 are fixed point of $f$.

**Example 2.8.3.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$fx = x^2 + x + 1$$

for all $x \in \mathbb{R}$. Then $f$ has no a fixed point.

## 2.9 Some functional equations

In this section, we give definitions and results of some classical functional equations.

**Definition 2.9.1.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be an **additive Cauchy function** if it satisfies the additive Cauchy functional equation

$$f(x+y)=f(x)+f(y) \tag{2.9.1}$$

for all $x,y\in\mathbb{R}$.

**Definition 2.9.2.** A function $f:\mathbb{R}\to\mathbb{R}$ is called a **linear function** if and only if it is of the form

$$f(x)=cx$$

for all $x\in\mathbb{R}$ and $c$ is an arbitrary constant.

**Definition 2.9.3.** A function $f:\mathbb{R}\to\mathbb{R}$ is said to be **rational homogeneous** if and only if

$$f(rx)=rf(x)$$

for all $x\in\mathbb{R}$ and $r\in\mathbb{Q}$.

**Example 2.9.4.** Let $f:\mathbb{R}\to\mathbb{R}$ be a linear function. Then $f$ satisfies the additive Cauchy functional equation. Moreover, $f$ satisfies the rational homogeneous.

**Theorem 2.9.5** ([17])**.** *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the additive Cauchy functional equation (2.9.1). Then $f$ is linear, that is, $f(x)=cx$ where $c$ is an arbitrary constant.*

**Remark 2.9.6.** Let $f:\mathbb{R}\to\mathbb{R}$ be an additive Cauchy function. Then

1. $f(0)=0$.
2. $f$ is an odd function.
3. $f$ is rational homogeneous
4. $f$ is linear on the set of rational numbers $\mathbb{Q}$.

**Definition 2.9.7.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be an **exponential Cauchy function** if it satisfies the exponential Cauchy functional equation

$$f(x+y)=f(x)f(y) \tag{2.9.2}$$

for all $x,y\in\mathbb{R}$.

**Example 2.9.8.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=n^{cx}$ for all $x\in\mathbb{R}$, where $n\in\mathbb{R}\backslash\{0\}$ and $c$ is an arbitrary constant. Then $f$ satisfies the exponential Cauchy functional equation.

**Theorem 2.9.9** ([17])**.** *If the functional equation (2.9.2), that is,*

$$f(x+y)=f(x)f(y),$$

*holds for all real numbers $x$ and $y$, then the general solution of (2.9.2) is given by*

$$f(x)=e^{A(x)} \quad \textit{and} \quad f(x)=0 \quad \textit{for all } x\in\mathbb{R},$$

*where $A:\mathbb{R}\to\mathbb{R}$ is an additive Cauchy function and $e$ is the Napierian base of logarithm.*

**Definition 2.9.10.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be a **logarithmic Cauchy function** if it satisfies the logarithmic Cauchy functional equation

$$f(xy)=f(x)+f(y) \tag{2.9.3}$$

for all $x,y\in\mathbb{R}$.

**Example 2.9.11.** Let $f:\mathbb{R}^{+}\to\mathbb{R}$ be defined by $f(x)=\log(x)$ for all $x\in\mathbb{R}^{+}$. Then $f$ satisfies the logarithmic Cauchy functional equation.

**Example 2.9.12.** Let $f:\mathbb{R}^{+}\to\mathbb{R}$ be defined by $f(x)=\ln(x)$ for all $x\in\mathbb{R}^{+}$. Then $f$ satisfies the logarithmic Cauchy functional equation.

**Theorem 2.9.13** ([17])**.** *If the functional equation (2.9.3), that is,*

$$f(xy)=f(x)+f(y),$$

*holds for all $x,y\in\mathbb{R}\backslash\{0\}$, then the general solution of (2.9.3) is given by*

$$f(x)=A(\ln|x|) \quad \textit{for all } x\in\mathbb{R}\backslash\{0\},$$

*where $A:\mathbb{R}\to\mathbb{R}$ is an additive Cauchy function.*

**Definition 2.9.14.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be a **multiplicative Cauchy function** if it satisfies the multiplicative Cauchy functional equation

$$f(xy) = f(x)f(y) \tag{2.9.4}$$

for all $x,y\in\mathbb{R}$.

**Example 2.9.15.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^n$ for all $x\in\mathbb{R}$, where $n$ is an arbitrary constant. Then $f$ satisfies the multiplicative Cauchy functional equation.

In the following theorem, we need the notion of the signum function. The signum function is denoted by $sgn(x)$ and defined as

$$sgn(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0. \end{cases}$$

**Theorem 2.9.16** ([17])**.** *The general solution of the multiplicative Cauchy functional equation (2.9.4), that is,*

$$f(xy) = f(x)f(y),$$

*holding for all $x,y\in\mathbb{R}$ is given by*

$$f(x)=0,$$

$$f(x)=1,$$

$$f(x)=e^{A(\ln|x|)}sgn(x),$$

*and*

$$f(x)=e^{A(\ln|x|)}|sgn(x)| \quad \text{for all } x,\in\mathbb{R},$$

*where $A:\mathbb{R}\to\mathbb{R}$ is an additive Cauchy function and $e$ is the Napierian base of logarithm.*

**Definition 2.9.17.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be a **radical quadratic equation** if it satisfies the radical quadratic functional equation

$$f(\sqrt{x^2+y^2}) = f(x)+f(y) \tag{2.9.5}$$

for all $x,y\in\mathbb{R}$.

**Example 2.9.18.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2$ for all $x\in\mathbb{R}$ Then $f$ satisfies the radical quadratic functional equation.

**Definition 2.9.19.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be a **general linear equation** if it satisfies the general linear functional equation

$$f(ax+by) = af(x)+bf(y) \tag{2.9.6}$$

for all $a,b\in\mathbb{R}$ with $a,b\neq 0$ and for all $x,y\in\mathbb{R}$.

**Example 2.9.20.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x$ for all $x\in\mathbb{R}$ Then $f$ satisfies the general linear functional equation.

The equation (2.9.7) was first considered in 1987 by Drygas [16]

**Definition 2.9.21.** The function $f:\mathbb{R}\to\mathbb{R}$ is said to be a **Drygas** if it satisfies Drygas functional equation

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y) \tag{2.9.7}$$

for all $x,y\in\mathbb{R}$.

**Example 2.9.22.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=cx$ for all $x\in\mathbb{R}$, where $c$ is an arbitrary constant. Then $f$ satisfies the Drygas functional equation.

**Example 2.9.23.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=cx^2$ for all $x\in\mathbb{R}$, where $c$ is an arbitrary constant. Then $f$ satisfies the Drygas functional equation.

**Remark 2.9.24.** If $f,g:\mathbb{R}\to\mathbb{R}$ satisfy Drygas functional equation, then $f\pm g$ is also satisfies the Drygas functional equation.

The general solution of (2.9.7) was derived by Ebanks *et. al* [6] as

$$f(x) = A(x) + Q(x),$$

where $A:\mathbb{R}\to\mathbb{R}$ is an additive function and $Q:\mathbb{R}\to\mathbb{R}$ is a quadratic function, that is, $A$ satisfies the additive functional equation

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in \mathbb{R}$ and $Q$ satisfies the quadratic functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in \mathbb{R}$.

# CHAPTER 3

# STABILITY OF FUNCTIONAL EQUATIONS

In 2013, Brzdęk [4] gave the fixed point results for some nonlinear mappings in metric spaces. By using this result, Brzdęk improved Hyers-Ulam-Rassias stability of Rassias. The aim of this chapter is to prove new stability results of several functional equations via fixed point result due to Brzdęk (Theorem 1.1.9).

## 3.1 Stability of radical quadratic functional equation

In this section, we show that the Brzdęk's fixed point result (Theorem 1.1.9) allows to investigate new type of stability for radical quadratic functional equation

$$f(\sqrt{x^2+y^2}) = f(x)+f(y), \tag{3.1.1}$$

where $f$ is self mapping on the set of real numbers. Our results generalize, extend and complement some earlier classical results concerning the Hyers-Ulam-Rassias stability for radical quadratic functional equation.

**Theorem 3.1.1.** *Let $d$ be a complete metric in $\mathbb{R}$ which is invariant (i.e., $d(x+z,y+z)=d(x,y)$ for $x,y,z\in\mathbb{R}$), and $h:\mathbb{R}\to\mathbb{R}_+$ be a function such that*

$$M_0 := \{n\in\mathbb{N} : s(n^2)+s(n^2+1) < 1\} \neq \emptyset,$$

*where $s(n) := \inf\{t\in\mathbb{R}_+ : h(nx^2)\leq th(x^2) \text{ for all } x\in\mathbb{R}\}$ for $n\in\mathbb{N}$. Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies the following inequality*

$$d\left(f\left(\sqrt{x^2+y^2}\right), f(x)+f(y)\right) \leq h(x^2)+h(y^2), \tag{3.1.2}$$

*for all $x,y\in\mathbb{R}$. Then there exists a unique radical quadratic function $T:\mathbb{R}\to\mathbb{R}$ such that*

$$d(f(x),T(x)) \leq s_0h(x^2), \quad x\in\mathbb{R}, \tag{3.1.3}$$

*with* $s_0 := \inf\left\{\dfrac{1+s(n^2)}{1-s(n^2)-s(n^2+1)} : n \in M_0\right\}.$

*Proof.* Replacing $y$ by $mx$, where $x, \in \mathbb{R}$ and $m \in \mathbb{N}$, in inequality (3.1.2) we get

$$d\left(f\left(\sqrt{(1+m^2)x^2}\right), f(x)+f(mx)\right) \leq [1+s(m^2)]h(x^2). \tag{3.1.4}$$

For each $m \in \mathbb{N}$, we define operators $\mathcal{T}_m : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}}$ by

$$\begin{aligned}
\mathcal{T}_m\xi(x) &:= \xi\left(\sqrt{(1+m^2)x^2}\right) - \xi(mx), \quad x \in \mathbb{R}, \xi \in \mathbb{R}^{\mathbb{R}}, \\
\Lambda_m\delta(x) &:= \delta\left(\sqrt{(1+m^2)x^2}\right) + \delta(mx), \quad x \in \mathbb{R}, \delta \in \mathbb{R}_+^{\mathbb{R}}.
\end{aligned} \tag{3.1.5}$$

Then it is easily seen that, for each $m \in \mathbb{N}$, $\Lambda := \Lambda_m$ has the form described in (1.1.18) with $f_1(x) = \sqrt{(1+m^2)x^2}$ and $f_2(x) = mx$. Moreover, since $d$ is invariant, (3.1.4) can be written in the form

$$d(\mathcal{T}_m f(x), f(x)) \leq [1+s(m^2)]h(x^2) =: \varepsilon_m(x) \tag{3.1.6}$$

for $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Also, for each $\xi, \mu \in \mathbb{R}^{\mathbb{R}}$, $x \in \mathbb{R}$ and $m \in \mathbb{N}$, we have

$$\begin{aligned}
d(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x)) &= d\Big(\xi\Big(\sqrt{(1+m^2)x^2}\Big) - \xi(mx), \mu\Big(\sqrt{(1+m^2)x^2}\Big) - \mu(mx)\Big) \\
&\leq d\Big(\xi\Big(\sqrt{(1+m^2)x^2}\Big) - \xi(mx), -\xi(mx) + \mu\Big(\sqrt{(1+m^2)x^2}\Big)\Big) \\
&\quad + d\Big(-\xi(mx) + \mu\Big(\sqrt{(1+m^2)x^2}\Big), \mu\Big(\sqrt{(1+m^2)x^2}\Big) - \mu(mx)\Big) \\
&= d\Big(\xi\Big(\sqrt{(1+m^2)x^2}\Big), \mu\Big(\sqrt{(1+m^2)x^2}\Big)\Big) \\
&\quad + d\Big(\xi(mx), \mu(mx)\Big).
\end{aligned} \tag{3.1.7}$$

Consequently, for each $m \in \mathbb{N}$, (1.1.17) is valid with $\mathcal{T} := \mathcal{T}_m$.

Next, we show that

$$\Lambda_m^n \varepsilon_m(x) \leq [1+s(m^2)]h(x^2)[s(m^2)+s(1+m^2)]^n \tag{3.1.8}$$

for $x \in \mathbb{R}, n \in \mathbb{N}_0$ and $m \in M_0$. It is easy to see that the inequality (3.1.8) holds for $n = 0$. By the definition of $\Lambda_m$ and $\varepsilon_m$, we can see that

$$\begin{aligned}
\Lambda_m\varepsilon_m(x) &= \varepsilon_m\left(\sqrt{(1+m^2)x^2}\right) + \varepsilon_m(mx) \\
&= [1+s(m^2)]h((1+m^2)x^2) + [1+s(m^2)]h(m^2x^2) \\
&\leq [1+s(m^2)][s(1+m^2)+s(m^2)]h(x^2),
\end{aligned} \tag{3.1.9}$$

for $x \in \mathbb{R}$. From above relation, we obtain that

$$\begin{aligned}
\Lambda_m^2 \varepsilon_m(x) &= \Lambda_m(\Lambda_m \varepsilon_m(x)) \\
&= \Lambda_m \varepsilon_m\left(\sqrt{(1+m^2)x^2}\right) + \Lambda_m \varepsilon_m(mx) \\
&\leq [1+s(m^2)][s(1+m^2)+s(m^2)]h((1+m^2)x^2) \\
&\quad +[1+s(m^2)][s(1+m^2)+s(m^2)]h(m^2x^2) \\
&\leq [1+s(m^2)][s(1+m^2)+s(m^2)]s(1+m^2)h(x^2) \\
&\quad +[1+s(m^2)][s(1+m^2)+s(m^2)]s(m^2)h(x^2) \\
&= [1+s(m^2)][s(1+m^2)+s(m^2)]^2 h(x^2).
\end{aligned}$$

By similar method, we get

$$\Lambda_m^n \varepsilon_m(x) \leq [1+s(m^2)]h(x^2)[s(m^2)+s(1+m^2)]^n$$

for $x \in \mathbb{R}, n \in \mathbb{N}_0$ and $m \in M_0$. Here, we obtain that

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} (\Lambda_m^n \varepsilon_m)(x) \\
&\leq (1+s(m^2))h(x^2) \sum_{n=0}^{\infty} (s(m^2)+s(1+m^2))^n \\
&= \frac{(1+s(m^2))h(x^2)}{(1-s(m^2)-s(1+m^2))},
\end{aligned}$$

for $x \in \mathbb{R}, m \in M_0$. By using Theorem 1.1.9 with $X = Y = \mathbb{R}$ and $\varphi = f$, we have the limit

$$T_m(x) := \lim_{n\to\infty} (\mathcal{T}_m^n f)(x)$$

exists for each $x \in \mathbb{R}$ and $m \in M_0$, and

$$d(f(x), T_m(x)) \leq \frac{(1+s(m^2))h(x^2)}{(1-s(m^2)-s(1+m^2))}, \quad x \in \mathbb{R}, m \in M_0. \tag{3.1.10}$$

Now we show that

$$d(\mathcal{T}_m^n f(\sqrt{x^2+y^2}), \mathcal{T}_m^n f(x) + \mathcal{T}_m^n f(y))$$

$$\leq [s(m^2)+s(1+m^2)]^n (h(x^2)+h(y^2)) \tag{3.1.11}$$

for every $x, y \in \mathbb{R}, n \in \mathbb{N}_0$ and $m \in M_0$. Since the case $n = 0$ is just (3.1.2), take $k \in \mathbb{N}_0$ and assume that (3.1.11) holds for $n = k$ and every $x, y \in \mathbb{R}, m \in M_0$. Then for each $x,y \in \mathbb{R}, m \in M_0$, we get

$$
\begin{aligned}
& d\Big(\mathcal{T}_m^{k+1} f(\sqrt{x^2+y^2}), \mathcal{T}_m^{k+1} f(x)+\mathcal{T}_m^{k+1} f(y)\Big) \\
= \;& d\bigg(\mathcal{T}_m^k f\left(\sqrt{(1+m^2)(x^2+y^2)}\right)-\mathcal{T}_m^k f\left(\sqrt{m^2(x^2+y^2)}\right), \\
& \mathcal{T}_m^k f\left(\sqrt{(1+m^2)x^2}\right)-\mathcal{T}_m^k f(mx)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)y^2}\right)-\mathcal{T}_m^k f(my)\bigg) \\
\leq \;& d\bigg(\mathcal{T}_m^k f\left(\sqrt{(1+m^2)(x^2+y^2)}\right)-\mathcal{T}_m^k f\left(\sqrt{m^2(x^2+y^2)}\right), \\
& -\mathcal{T}_m^k f\left(\sqrt{m^2(x^2+y^2)}\right)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)x^2}\right)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)y^2}\right)\bigg) \\
& +d\bigg(-\mathcal{T}_m^k f\left(\sqrt{m^2(x^2+y^2)}\right)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)x^2}\right)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)y^2}\right), \\
& \mathcal{T}_m^k f\left(\sqrt{(1+m^2)x^2}\right)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)y^2}\right)-\mathcal{T}_m^k f(mx)-\mathcal{T}_m^k f(my)\bigg) \\
= \;& d\left(\mathcal{T}_m^k f\left(\sqrt{(1+m^2)(x^2+y^2)}\right), \mathcal{T}_m^k f\left(\sqrt{(1+m^2)x^2}\right)+\mathcal{T}_m^k f\left(\sqrt{(1+m^2)y^2}\right)\right) \\
& +d\left(\mathcal{T}_m^k f\left(\sqrt{m^2(x^2+y^2)}\right), \mathcal{T}_m^k f(mx)+\mathcal{T}_m^k f(my)\right) \\
\leq \;& [s(m^2)+s(1+m^2)]^k[h((1+m^2)x^2)+h((1+m^2)y^2)] \\
& +[s(m^2)+s(1+m^2)]^k[h(m^2x^2)+h(m^2y^2)] \\
\leq \;& [s(m^2)+s(1+m^2)]^k[s(1+m^2)h(x^2)+s(1+m^2)h(y^2)] \\
& +[s(m^2)+s(1+m^2)]^k[s(m^2)h(x^2)+s(m^2)h(y^2)] \\
= \;& [s(m^2)+s(1+m^2)]^k[s(m^2)+s(1+m^2)][h(x^2)+h(y^2)] \\
= \;& [s(m^2)+s(1+m^2)]^{k+1}[h(x^2)+h(y^2)].
\end{aligned}
$$

Thus by induction we have shown that (3.1.11) holds for every $x,y \in \mathbb{R}, n \in \mathbb{N}_0$ and $m \in M_0$. Letting $n \to \infty$ in (3.1.11), we obtain the equality

$$T_m(\sqrt{x^2+y^2}) = T_m(x)+T_m(y), \quad x,y \in \mathbb{R}, m \in M_0. \tag{3.1.12}$$

This implies that $T_m : \mathbb{R} \to \mathbb{R}$, defined in this way, is a solution of the equation

$$T(x) = T\left(\sqrt{(1+m^2)x^2}\right) - T(mx). \tag{3.1.13}$$

Next, we prove that each function $T : \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$d(f(x), T(x)) \leq Lh(x^2), \quad x \in \mathbb{R} \tag{3.1.14}$$

with some $L>0$, is equal to $T_m$ for each $m\in M_0$. To this end, fix $m_0\in M_0$ and $T:\mathbb{R}\to\mathbb{R}$ satisfying (3.1.14). From (3.1.10), for each $x\in\mathbb{R}$, we get

$$\begin{aligned}
d(T(x),T_{m_0}(x)) &\leq d(T(x),f(x))+d(f(x),T_{m_0}(x)) \\
&\leq Lh(x^2)+\varepsilon^*_{m_0}(x) \\
&= Lh(x^2)+[1+s(m_0^2)]h(x^2)\sum_{n=0}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n \\
&= h(x^2)(L+[1+s(m_0^2)]\sum_{n=0}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n) \\
&= L_0h(x^2)\sum_{n=0}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n, \qquad (3.1.15)
\end{aligned}$$

where $L_0=L[1-s(m_0^2)-s(1+m_0^2)]+[1+s(m_0^2)]>0$ (the case $h(x)\equiv 0$ is trivial, so we exclude it here). Observe yet that $T$ and $T_{m_0}$ are solutions to equation (3.1.13) for all $m\in M_0$.

Here, we show that for each $j\in\mathbb{N}_0$, we have

$$d(T(x),T_{m_0}(x))\leq L_0h(x^2)\sum_{n=j}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n, \quad x\in\mathbb{R}. \qquad (3.1.16)$$

The case $j=0$ is exactly (3.1.15). So, fix $l\in\mathbb{N}_0$ and assume that (3.1.16) hold for $j=l$. Then, in view of (3.1.15), for each $x\in\mathbb{R}$, we get

$$\begin{aligned}
d\left(T(x),T_{m_0}(x)\right) &= d\left(T\left(\sqrt{(1+m_0^2)x^2}\right)-T(m_0x),T_{m_0}\left(\sqrt{(1+m_0^2)x^2}\right)-T_{m_0}(m_0x)\right) \\
&\leq d\left(T\left(\sqrt{(1+m_0^2)x^2}\right)-T(m_0x),T_{m_0}\left(\sqrt{(1+m_0^2)x^2}\right)-T(m_0x)\right) \\
&\quad +d\left(T_{m_0}\left(\sqrt{(1+m_0^2)x^2}\right)-T(m_0x),T_{m_0}\left(\sqrt{(1+m_0^2)x^2}\right)-T_{m_0}(m_0x)\right) \\
&= d\left(T\left(\sqrt{(1+m_0^2)x^2}\right),T_{m_0}\left(\sqrt{(1+m_0^2)x^2}\right)\right)+d(T(m_0x),T_{m_0}(m_0x)) \\
&\leq L_0\left[h\left(\sqrt{(1+m_0^2)x^2}\right)\right]\sum_{n=l}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n \\
&\quad +L_0h(m_0x)\sum_{n=l}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n \\
&= L_0\left[h\left(\sqrt{(1+m_0^2)x^2}\right)+h(m_0x)\right]\sum_{n=l}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n \\
&\leq L_0[s(m_0^2)+s(1+m_0^2)]h(x)\sum_{n=l}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n \\
&= L_0h(x)\sum_{n=l+1}^{\infty}[s(m_0^2)+s(1+m_0^2)]^n.
\end{aligned}$$

Thus we have shown (3.1.16). Now, letting $j \to \infty$ in (3.1.16), we get

$$T = T_{m_0}. \tag{3.1.17}$$

Thus we have also proved that $T_m = T_{m_0}$ for each $m \in M_0$, which (in view of (3.1.10)) yields

$$d(f(x), T_{m_0}(x)) \le \frac{(1+s(m^2))h(x^2)}{(1-s(m^2)-s(1+m^2))}, \quad x \in \mathbb{R}, m \in M_0. \tag{3.1.18}$$

This implies (3.1.3) with $T := T_{m_0}$; clearly, equality (3.1.17) means the uniqueness of $T$ as well. This completes the proof. □

Next, we give the subsequent corollary yielded by Theorem 3.1.1.

**Corollary 3.1.2.** *Let $d$ be as in Theorem 3.1.1 and $h : \mathbb{R} \to (0,\infty)$ be such that*

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{h(n^2x^2)+h((1+n)^2x^2)}{h(x^2)} = 0. \tag{3.1.19}$$

*Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies(3.1.2). Then there exists a unique radical quadratic function $T : \mathbb{R} \to \mathbb{R}$ such that*

$$d(f(x), T(x)) \le h(x^2), \quad x \in \mathbb{R}. \tag{3.1.20}$$

*Proof.* By the definition of $s(n)$, we can see that

$$s(n^2) = \sup_{x\in\mathbb{R}} \frac{h(n^2x^2)}{h(x^2)} \le \sup_{x\in\mathbb{R}} \frac{h(n^2x^2)+h((1+n^2)x^2)}{h(x^2)} \tag{3.1.21}$$

and

$$s(n^2+1) = \sup_{x\in\mathbb{R}} \frac{h((1+n^2)x^2)}{h(x^2)} \le \sup_{x\in\mathbb{R}} \frac{h(n^2x^2)+h((1+n^2)x^2)}{h(x^2)}. \tag{3.1.22}$$

Combining inequalities (3.1.21) and (3.1.22), we get

$$s(n^2)+s(n^2+1) \le 2\sup_{x\in\mathbb{R}} \frac{h(n^2x^2)+h((1+n^2)x^2)}{h(x^2)}$$

From, (3.1.19), there is a sequence $\{n_k\}$ of positive integer such that

$$\lim_{k\to\infty} \sup_{x\in\mathbb{R}} \frac{h(n_k^2x^2)+h((1+n_k^2)x^2)}{h(x^2)} = 0$$

and then we have

$$\lim_{k\to\infty}(s(n_k^2)+s(n_k^2+1))=0.$$

This implies that

$$\lim_{k\to\infty} s(n_k^2)=0,$$

and consequently

$$\lim_{k\to\infty}\frac{1+s(n_k^2)}{1-s(n_k^2)-s(n_k^2+1)}=1.$$

This mean that $s_0=1$. This finishes to the proof. $\square$

## 3.2 Stability of generalized logarithmic Cauchy functional equation

In this section, we consider the following generalized logarithmic Cauchy functional equation

$$f(x^a\cdot y^b)=af(x)+bf(y) \tag{3.2.1}$$

where $f$ is mapping from $\mathbb{R}$ into $\mathbb{R}$ and $a$ and $b$ are two fixed positive integer numbers. We investigate the new generalized Hyers-Ulam of generalized logarithmic Cauchy functional equation by using fixed point result of Brzdęk's (Theorem 1.1.9). Our results generalize some known outcomes.

**Theorem 3.2.1.** *Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant (i.e., $d(x+z,y+z)=d(x,y)$ for $x,y,z\in\mathbb{R}$) and $a,b$ be two given positive integer numbers and $h:\mathbb{R}\to\mathbb{R}_+$ be a function such that*

$$M_0:=\{n\in\mathbb{N}:s(n)+s(a+bn)<1\}\neq\emptyset,$$

*where $s(n):=\inf\{t\in\mathbb{R}_+:h(x^n)\le th(x) \text{ for all } x\in\mathbb{R}\}$ for $n\in\mathbb{N}$. Suppose that*

$$d(kx,ky)=d(x,y) \tag{3.2.2}$$

*for all $x,y\in\mathbb{R}$ and for all $k\in\{a,b\}$. If $f:\mathbb{R}\to\mathbb{R}$ satisfies the following inequality*

$$d(f(x^a\cdot y^b),af(x)+bf(y))\le ah(x)+bh(y), \tag{3.2.3}$$

*for all* $x, y \in \mathbb{R}\backslash\{0\}$*, then there exists a unique function* $T : \mathbb{R} \to \mathbb{R}$ *such that it satisfies the generalized logarithmic Cauchy functional equation (3.2.1) with respect to* $a$ *and* $b$ *for all* $x, y \in \mathbb{R}\backslash\{0\}$ *and*

$$d(f(x), T(x)) \leq s_0 h(x), \quad x \in \mathbb{R}\backslash\{0\}, \tag{3.2.4}$$

*with* $s_0 := \inf\left\{\dfrac{a+bs(n)}{1-s(n)-s(a+bn)} : n \in M_0\right\}$.

*Proof.* For each $m \in \mathbb{N}$ and $x \in \mathbb{R}\backslash\{0\}$ note that (3.2.3) with $y = x^m$ gives

$$d(f(x^{a+bm}), af(x) + bf(x^m)) \leq (a + bs(m))h(x). \tag{3.2.5}$$

Define operators $\mathcal{T}_m : \mathbb{R}^{\mathbb{R}\backslash\{0\}} \to \mathbb{R}^{\mathbb{R}\backslash\{0\}}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}\backslash\{0\}} \to \mathbb{R}_+^{\mathbb{R}\backslash\{0\}}$ by

$$\begin{aligned} \mathcal{T}_m\xi(x) &:= \frac{1}{a}\Big[\xi(x^{a+bm}) - b\xi(x^m)\Big], \quad m \in \mathbb{N}, x \in \mathbb{R}\backslash\{0\}, \xi \in \mathbb{R}^{\mathbb{R}\backslash\{0\}}, \\ \Lambda_m\delta(x) &:= \delta(x^{a+bm}) + \delta(x^m), \quad x \in \mathbb{R}\backslash\{0\}, \delta \in \mathbb{R}_+^{\mathbb{R}\backslash\{0\}}. \end{aligned} \tag{3.2.6}$$

Now we can see that, for each $m \in \mathbb{N}$, $\Lambda := \Lambda_m$ has the form described in (1.1.18) with $f_1(x) = x^{a+bm}$ and $f_2(x) = x^m$. Since $d$ is invariant and $d$ satisfies condition (3.2.2), inequality (3.2.5) follows that

$$\begin{aligned} d(\mathcal{T}_m f(x), f(x)) &= d(a\mathcal{T}_m f(x), af(x)) \\ &= d(f(x^{a+bm}) - bf(x^m), af(x)) \\ &= d(f(x^{a+bm}), af(x) + bf(x^m)) \\ &\leq (a + bs(m))h(x) =: \varepsilon_m(x), \quad x \in \mathbb{R}\backslash\{0\}, \end{aligned} \tag{3.2.7}$$

and

$$\begin{aligned} d(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x)) &= d\left(\frac{1}{a}\Big[\xi((x^{a+bm}) - b\xi(x^m)\Big], \frac{1}{a}\Big[\mu((x^{a+bm}) - b\mu(x^m)\Big]\right) \\ &= d(\xi(x^{a+bm}) - b\xi(x^m), \mu(x^{a+bm}) - b\mu(x^m)) \\ &\leq d(\xi(x^{a+bm}) - b\xi(x^m), -b\xi(x^m) + \mu(x^{a+bm}) \\ &\quad + d(-b\xi(x^m) + \mu(x^{a+bm}), \mu(x^{a+bm}) - b\mu(x^m)) \\ &= d(\xi(x^{a+bm}), \mu(x^{a+bm})) + d(b\xi(x^m), b\mu(x^m)) \\ &= d(\xi(x^{a+bm}), \mu(x^{a+bm})) + d(\xi(x^m), \mu(x^m)) \end{aligned} \tag{3.2.8}$$

for all $m \in \mathbb{N}, \xi, \mu \in \mathbb{R}^{\mathbb{R}\backslash\{0\}}$, $x \in \mathbb{R}\backslash\{0\}$. This implies that for each $m \in \mathbb{N}$, (1.1.17) is valid with $X := \mathbb{R}\backslash\{0\}$, $Y := \mathbb{R}$ and $\mathcal{T} := \mathcal{T}_m$.

Here, we will prove that

$$\Lambda_m^n \varepsilon_m(x) \leq [a+bs(m)][s(m)+s(a+bm)]^n h(x) \tag{3.2.9}$$

for all $n \in \mathbb{N}_0$, $m \in M_0$ and $x \in \mathbb{R}\backslash\{0\}$. Now, we have

$$\begin{aligned} \Lambda_m \varepsilon_m(x) &= \varepsilon_m(x^{a+bm}) + \varepsilon_m(x^m) \\ &= [a+bs(m)]h(x^{a+bm}) + [a+bs(m)]h(x^m) \\ &\leq [a+bs(m)]s(a+bm)h(x) + [a+bs(m)]s(m)h(x) \\ &\leq [a+bs(m)][s(m)+s(a+bm)]h(x). \end{aligned} \tag{3.2.10}$$

It follows that

$$\begin{aligned} \Lambda_m^2 \varepsilon_m(x) &= \Lambda_m(\Lambda_m \varepsilon_m(x)) \\ &= \Lambda_m \varepsilon_m(x^{a+bm}) + \Lambda_m \varepsilon_m(x^m) \\ &\leq [a+bs(m)][s(m)+s(a+bm)]h(x^{a+bm}) \\ &\quad +[a+bs(m)][s(m)+s(a+bm)]h(x^m) \\ &\leq [a+bs(m)][s(m)+s(a+bm)]s(a+bm)h(x) \\ &\quad +[a+bs(m)][s(m)+s(a+bm)]s(m)h(x) \\ &\leq [a+bs(m)][s(m)+s(a+bm)]^2 h(x). \end{aligned}$$

By similar method, we obtain that

$$\Lambda_m^n \varepsilon_m(x) \leq [a+bs(m)][s(m)+s(a+bm)]^n h(x)$$

for all $x \in \mathbb{R}\backslash\{0\}, n \in \mathbb{N}_0$ and $m \in M_0$. Here, we obtain that

$$\begin{aligned} \varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\ &\leq [a+bs(m)]h(x) \sum_{n=0}^{\infty} [s(m)+s(a+bm)]^n \\ &= \frac{(a+bs(m))h(x)}{1-s(m)-s(a+bm)}, \end{aligned}$$

for all $x \in \mathbb{R}\backslash\{0\}, m \in M_0$. By using Theorem 1.1.9 with $X = \mathbb{R}\backslash\{0\}$, $Y = \mathbb{R}$ and $\varphi = f$. We have the limit

$$T_m(x) := \lim_{n\to\infty} (\mathcal{T}_m^n f)(x)$$

exists for each $x \in \mathbb{R}\backslash\{0\}$ and $m \in M_0$, and

$$d(f(x), T_m(x)) \leq \frac{(a+bs(m))h(x)}{1-s(m)-s(a+bm)} \tag{3.2.11}$$

for all $x \in \mathbb{R}\backslash\{0\}$ and $m \in M_0$. Next, we will claim that

$$d(\mathcal{T}_m^n f(x^a \cdot y^b), a\mathcal{T}_m^n f(x) + b\mathcal{T}_m^n f(y)) \leq [s(m)+s(a+bm)]^n (ah(x)+bh(y)) \tag{3.2.12}$$

for all $x, y \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}_0$ and $m \in M_0$. From (3.2.3), we get inequality (3.2.12) holds for the case $n = 0$. Assume that (3.2.12) holds for $n = k$ and every $x, y \in \mathbb{R}\backslash\{0\}, m \in M_0$. Then we obtain that

$$
\begin{aligned}
& d\Big(\mathcal{T}_m^{k+1} f(x^a\cdot y^b), a\mathcal{T}_m^{k+1} f(x)+b\mathcal{T}_m^{k+1} f(y)\Big) \\
& \quad = d\bigg(\frac{1}{a}\Big[\mathcal{T}_m^k f((x^a\cdot y^b)^{a+bm}) - b\mathcal{T}_m^k f((x^a\cdot y^b)^m)\Big], \\
& \qquad \frac{1}{a}\Big[a\mathcal{T}_m^k f(x^{a+bm}) - ab\mathcal{T}_m^k f(x^m) + b\mathcal{T}_m^k f(y^{a+bm}) - b^2\mathcal{T}_m^k f(y^m)\Big]\bigg) \\
& \quad = d\bigg(\Big[\mathcal{T}_m^k f((x^a\cdot y^b)^{a+bm}) - b\mathcal{T}_m^k f((x^a\cdot y^b)^m)\Big], \\
& \qquad \Big[a\mathcal{T}_m^k f(x^{a+bm}) - ab\mathcal{T}_m^k f(x^m) + b\mathcal{T}_m^k f(y^{a+bm}) - b^2\mathcal{T}_m^k f(y^m)\Big]\bigg) \\
& \quad \leq d\bigg(\mathcal{T}_m^k f((x^a\cdot y^b)^{a+bm}) - b\mathcal{T}_m^k f((x^a\cdot y^b)^m), \\
& \qquad -b\mathcal{T}_m^k f((x^a\cdot y^b)^m) + a\mathcal{T}_m^k f(x^{a+bm}) + b\mathcal{T}_m^k f(y^{a+bm})\bigg) \\
& \qquad +d\bigg(-b\mathcal{T}_m^k f((x^a\cdot y^b)^m) + a\mathcal{T}_m^k f(x^{a+bm}) + b\mathcal{T}_m^k f(y^{a+bm}), \\
& \qquad a\mathcal{T}_m^k f(x^{a+bm}) - ab\mathcal{T}_m^k f(x^m) + b\mathcal{T}_m^k f(y^{a+bm}) - b^2\mathcal{T}_m^k f(y^m)\bigg) \\
& \quad = d\bigg(\mathcal{T}_m^k f((x^a\cdot y^b)^{a+bm}), a\mathcal{T}_m^k f(x^{a+bm}) + b\mathcal{T}_m^k f(y^{a+bm})\bigg) \\
& \qquad +d\bigg(b\mathcal{T}_m^k f((x^a\cdot y^b)^m), ab\mathcal{T}_m^k f(x^m) + b^2\mathcal{T}_m^k f(y^m)\bigg) \\
& \quad = d\bigg(\mathcal{T}_m^k f(x^{a(a+bm)}\cdot y^{b(a+bm)}), a\mathcal{T}_m^k f(x^{a+bm}) + b\mathcal{T}_m^k f(y^{a+bm})\bigg) \\
& \qquad +d\bigg(b\mathcal{T}_m^k f(x^{am}\cdot y^{bm}), ab\mathcal{T}_m^k f(x^m) + b^2\mathcal{T}_m^k f(y^m)\bigg) \\
& \quad = d\bigg(\mathcal{T}_m^k f(x^{a(a+bm)}\cdot y^{b(a+bm)}), a\mathcal{T}_m^k f(x^{a+bm}) + b\mathcal{T}_m^k f(y^{a+bm})\bigg) \\
& \qquad +d\bigg(\mathcal{T}_m^k f(x^{am}\cdot y^{bm}), a\mathcal{T}_m^k f(x^m) + b\mathcal{T}_m^k f(y^m)\bigg) \\
& \quad \leq [s(m)+s(a+bm)]^k[ah(x^{a+bm})+bh(y^{a+bm})] \\
& \qquad +[s(m)+s(a+bm)]^k[ah(x^m)+bh(y^m)] \\
& \quad \leq [s(m)+s(a+bm)]^k[as(a+bm)h(x)+bs(a+bm)h(y)] \\
& \qquad +[s(m)+s(a+bm)]^k[as(m)h(x)+bs(m)h(y)] \\
& \quad = [s(m)+s(a+bm)]^k[s(m)+s(a+bm)](ah(x)+bh(y)) \\
& \quad = [s(m)+s(a+bm)]^{k+1}[ah(x)+bh(y)].
\end{aligned}
$$

By mathematical induction, we have shown that (3.2.12) holds for every $x,y \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}_0$ and $m \in M_0$. Letting $n \to \infty$ in inequality (3.2.12), we obtain that

$$T_m(x^a\cdot y^b) = aT_m(x) + bT_m(y) \tag{3.2.13}$$

for all $x,y\in\mathbb{R}\backslash\{0\}$, $m\in M_0$ and the function $T_m:\mathbb{R}\backslash\{0\}\to\mathbb{R}$, defined in this way, is a solution of the equation

$$aT(x)=T(x^{a+bm})-bT(x^m). \tag{3.2.14}$$

Next, we prove that each generalized logarithmic Cauchy function $T:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ satisfying the inequality

$$d(f(x),T(x))\leq Lh(x),\quad x\in\mathbb{R} \tag{3.2.15}$$

with some $L>0$, is equal to $T_m$ for each $m\in M_0$. To this end, fix $m_0\in M_0$ and $T:\mathbb{R}\to\mathbb{R}$ satisfying (3.2.15). Then we observe that

$$\begin{aligned}
d(T(x),T_{m_0}(x)) &\leq d(T(x),f(x))+d(f(x),T_{m_0}(x))\\
&\leq Lh(x)+(a+bs(m_0))h(x)\sum_{n=0}^{\infty}(s(m_0)+s(a+bm_0))^n\\
&= h(x)[L+(a+bs(m_0))\sum_{n=0}^{\infty}(s(m_0)+s(a+bm_0))^n]\\
&= h(x)L_0\sum_{n=0}^{\infty}(s(m_0)+s(a+bm_0))^n
\end{aligned} \tag{3.2.16}$$

where $L_0=L(1-s(m_0)-s(a+bm_0))+(a+bs(m_0))$ (the case $h(x)\equiv 0$ is trivial, so we exclude it here). Note that $T$ and $T_{m_0}$ are solutions to equation (3.2.14) for all $m\in M_0$. Here we will show that for each $j\in\mathbb{N}_0$, we have

$$d(T(x),T_{m_0}(x))\leq h(x)L_0\sum_{n=j}^{\infty}(bs(m_0)+s(a+bm_0))^n,\quad x\in\mathbb{R}\backslash\{0\}. \tag{3.2.17}$$

The case $j=0$ is just (3.2.16). So, fix $l\in\mathbb{N}_0$ and assume that (3.2.17) hold for $j=l$. For $x\in\mathbb{R}\backslash\{0\}$, we get

$$\begin{aligned}
d(T(x),T_{m_0}(x)) &= d(aT(x),aT_{m_0}(x))\\
&= d(T(x^{a+bm_0})-bT(x^{m_0}),T_{m_0}(x^{a+bm_0})-bT_{m_0}(x^{m_0}))\\
&\leq d(T(x^{a+bm_0})-bT(x^{m_0}),-bT(x^{m_0})+T_{m_0}(x^{a+bm_0}))\\
&\quad +d(-bT(x^{m_0})+T_{m_0}(x^{a+bm_0}),bT_{m_0}(x^{m_0})-bT_{m_0}(x^{m_0}))\\
&= d(T(x^{a+bm_0}),T_{m_0}(x^{a+bm_0}))+d(bT(x^{m_0}),bT_{m_0}(x^{m_0}))\\
&= d(T(x^{a+bm_0}),T_{m_0}(x^{a+bm_0}))+d(T(x^{m_0}),T_{m_0}(x^{m_0}))\\
&\leq h((a+bm_0)x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n\\
&\quad +h(x^{m_0})L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n\\
&\leq s((a+bm_0))h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n\\
&\quad +s(m_0)h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n\\
&= [s(a+bm_0)+s(m_0)]h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n\\
&< h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n\\
&< h(x)L_0\sum_{n=l+1}^{\infty}[bs(m_0)+s(a+bm_0)]^n.
\end{aligned}$$

Therefore, (3.2.17) holds for all $n\in\mathbb{N}_0$. Letting $j\to\infty$ in (3.2.17), we get

$$T=T_{m_0}. \tag{3.2.18}$$

This implies that $T_m=T_{m_0}$ for each $m\in M_0$. So inequality (3.2.11) yields that

$$d(f(x),T_{m_0}(x))\leq\frac{(a+bs(m))h(x)}{(1-s(m)-s(a+bm))},\quad x\in\mathbb{R}\backslash\{0\},m\in M_0. \tag{3.2.19}$$

This implies (3.2.4) with $T:=T_{m_0}$; clearly, equality (3.2.18) means the uniqueness of $T$ as well. This completes the proof. □

**Corollary 3.2.2.** *Let $d$ be as in Theorem 3.2.1 and $h:\mathbb{R}\to(0,\infty)$ be such that*

$$\liminf_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{h(x^n)+h(x^{a+bn})}{h(x)}=0. \tag{3.2.20}$$

*Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies (3.2.3) for all $x,y\in\mathbb{R}\backslash\{0\}$. Then then there exists a unique function $T:\mathbb{R}\to\mathbb{R}$ such that it satisfies the generalized logarithmic Cauchy functional equation (3.2.1) for all $x,y\in\mathbb{R}\backslash\{0\}$ and*

$$d(f(x),T(x))\le ah(x) \tag{3.2.21}$$

*for all $x\in\mathbb{R}\backslash\{0\}$.*

*Proof.* It follows from (3.2.20) that there is a subsequence $\{n_k\}$ of positive integer such that

$$\lim_{k\to\infty}\sup_{x\in\mathbb{R}}\frac{h(x^{n_k})+h(x^{a+bn_k})}{h(x)}=0. \tag{3.2.22}$$

Using the property of infimum for $s(n_k)$ and $s(a+bn_k)$, we get

$$s(n_k)=\sup_{x\in\mathbb{R}}\frac{h(x^{n_k})}{h(x)}\le\sup_{x\in\mathbb{R}\backslash\{0\}}\frac{h(x^{n_k})+h(x^{a+bn_k})}{h(x)} \tag{3.2.23}$$

and

$$s(a+bn_k)=\sup_{x\in\mathbb{R}}\frac{h(x^{a+bn_k})}{h(x)}\le\sup_{x\in\mathbb{R}\backslash\{0\}}\frac{h(x^{n_k})+h(x^{a+bn_k})}{h(x)}. \tag{3.2.24}$$

Letting $k\to\infty$ in (3.2.23) and (3.2.24), we obtain that

$$\lim_{k\to\infty}s(n_k)=0 \tag{3.2.25}$$

and

$$\lim_{k\to\infty}s(a+bn_k)=0. \tag{3.2.26}$$

It follows that

$$\lim_{k\to\infty}\frac{a+bs(n_k)}{1-s(n_k)-s(a+bn_k)}=a.$$

This implies that $s_0=a$. □

Note that Theorem 3.2.1 and Corollary 3.2.2 yield the following results concerning the stability of logarithmic Cauchy functional equation.

**Corollary 3.2.3.** *Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant and $h:\mathbb{R}\to\mathbb{R}_+$ be a function such that*

$$M_0:=\{n\in\mathbb{N}:s(n)+s(1+n)<1\}\neq\emptyset,$$

*where* $s(n) := \inf\{t \in \mathbb{R}_+ : h(x^n) \leq th(x)$ *for all* $x \in \mathbb{R}\}$ *for* $n \in \mathbb{N}$. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the following inequality*

$$d(f(x \cdot y), f(x) + f(y)) \leq h(x) + h(y) \tag{3.2.27}$$

*for all* $x, y \in \mathbb{R}\backslash\{0\}$. *Then there exists a unique function* $T : \mathbb{R} \to \mathbb{R}$ *such that it satisfies the logarithmic Cauchy functional equation (2.9.3) for all* $x, y \in \mathbb{R}\backslash\{0\}$ *and*

$$d(f(x), T(x)) \leq s_0 h(x) \tag{3.2.28}$$

*for all* $x \in \mathbb{R}\backslash\{0\}$, *where* $s_0 := \inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)} : n \in M_0\right\}$.

**Corollary 3.2.4.** *Let* $d$ *be as in Corollary 3.2.3 and* $h : \mathbb{R} \to (0, \infty)$ *be such that*

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{h(x^n) + h(x^{1+n})}{h(x)} = 0. \tag{3.2.29}$$

*Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies (3.2.27). Then then there exists a unique function* $T : \mathbb{R} \to \mathbb{R}$ *such that it satisfies the logarithmic Cauchy functional equation (2.9.3) for all* $x, y \in \mathbb{R}\backslash\{0\}$ *and*

$$d(f(x), T(x)) \leq h(x) \tag{3.2.30}$$

*for all* $x \in \mathbb{R}\backslash\{0\}$.

## 3.3 Stability of generalized additive Cauchy functional equation

In this section, we investigate new type of stability results for generalized Cauchy functional equation of the form

$$f_1(ax + by) = af_1(x) + bf_1(y),$$

where $f_1$ is mapping from $\mathbb{R}$ into $\mathbb{R}$ and generalized Cauchy functional equation of the form

$$f_2(ax * by) = af_2(x) \diamond bf_2(y),$$

$f_2$ is a mapping from a commutative semigroup $(G_1, *)$ into a commutative group $(G_2, \diamond)$ and $a, b \in \mathbb{N}$ by using Brzdęk's fixed point theorem (Theorem 1.1.9).

### 3.3.1 Stability of generalized additive Cauchy functional equation by Brzdęk's technique on $\mathbb{R}$

In this subsection, we prove the new type of stability for the generalized additive Cauchy functional equation of the form

$$f(ax+by) = af(x)+bf(y),$$

where $f$ is mapping from $\mathbb{R}$ into $\mathbb{R}$. Here, we give the main result in this subsection.

**Theorem 3.3.1.** *Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant (i.e., $d(x+z,y+z) = d(x,y)$ for $x,y,z \in \mathbb{R}$) and $a,b$ be two given positive integer numbers and $h : \mathbb{R} \to \mathbb{R}_+$ be a function such that*

$$M_0 := \{n \in \mathbb{N} : s(n)+s(a+bn) < 1\} \neq \emptyset,$$

*where $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in \mathbb{R}\}$ for $n \in \mathbb{N}$. Suppose that*

$$d(kx,ky) = d(x,y) \tag{3.3.1}$$

*for all $x,y \in \mathbb{R}$ and for all $k \in \{a,b\}$. If $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality*

$$d(f(ax+by),af(x)+bf(y)) \leq ah(x)+bh(y) \tag{3.3.2}$$

*for all $x,y \in \mathbb{R}$, then there exists a unique generalized additive Cauchy function $T : \mathbb{R} \to \mathbb{R}$ such that*

$$d(f(x),T(x)) \leq s_0h(x), \quad x \in \mathbb{R}, \tag{3.3.3}$$

*with* $s_0 := \inf\left\{\dfrac{a+bs(n)}{1-s(n)-s(a+bn)} : n \in M_0\right\}.$

*Proof.* For each $m \in \mathbb{N}$ and $x \in \mathbb{R}$ note that (3.3.2) with $y = mx$ gives

$$d(f((a+bm)x),af(x)+bf(mx)) \leq (a+bs(m))h(x). \tag{3.3.4}$$

Define operators $\mathcal{T}_m : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}} \to \mathbb{R}_+^{\mathbb{R}}$ by

$$\begin{aligned} \mathcal{T}_m\xi(x) &:= \frac{1}{a}\left[\xi((a+bm)x)-b\xi(mx)\right], \quad m \in \mathbb{N}, x \in \mathbb{R}, \xi \in \mathbb{R}^{\mathbb{R}}, \\ \Lambda_m\delta(x) &:= \delta((a+bm)x)+\delta(mx), \quad x \in \mathbb{R}, \delta \in \mathbb{R}_+^{\mathbb{R}}. \end{aligned} \tag{3.3.5}$$

Then it is easily seen that, for each $m \in \mathbb{N}$, $\Lambda := \Lambda_m$ has the form described in (1.1.18) with $f_1(x) = (a+bm)x$ and $f_2(x) = mx$. Since $d$ is invariant and $d$ satisfies condition (3.3.1), inequality (3.3.4) follows that

$$\begin{aligned} d(\mathcal{T}_m f(x), f(x)) &= d(a\mathcal{T}_m f(x), af(x)) \\ &= d(f((a+bm)x) - bf(mx), af(x)) \\ &\leq (a+bs(m))h(x) =: \varepsilon_m(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned} d(\mathcal{T}_m \xi(x), \mathcal{T}_m \mu(x)) &= d\left(\frac{1}{a}\left[\xi((a+bm)x) - b\xi(mx)\right], \frac{1}{a}\left[\mu((a+bm)x) - b\mu(mx)\right]\right) \\ &= d(\xi((a+bm)x) - b\xi(mx), \mu((a+bm)x) - b\mu(mx)) \\ &\leq d(\xi((a+bm)x) - b\xi(mx), -b\xi(mx) + \mu((a+bm)x)) \\ &\quad + d(-b\xi(mx) + \mu((a+bm)x), \mu((a+bm)x) - b\mu(mx) \\ &= d(\xi((a+bm)x), \mu((a+bm)x) + d(b\xi(mx), b\mu(mx)) \\ &= d(\xi((a+bm)x), \mu((a+bm)x) + d(\xi(mx), \mu(mx)) \end{aligned} \tag{3.3.7}$$

for all $m \in \mathbb{N}, \xi, \mu \in \mathbb{R}^{\mathbb{R}}$, $x \in \mathbb{R}$. Consequently, for each $m \in \mathbb{N}$, (1.1.17) is valid with $X := \mathbb{R}$, $Y := \mathbb{R}$ and $\mathcal{T} := \mathcal{T}_m$.

Next, we show that

$$\Lambda_m^n \varepsilon_m(x) \leq (a+bs(m))h(x)(s(m)+s(a+bm))^n \tag{3.3.8}$$

for $n \in \mathbb{N}_0$ (nonnegative integers), $m \in M_0$ and $x \in \mathbb{R}$. Now, we have

$$\begin{aligned} \Lambda_m \varepsilon_m(x) &= \varepsilon_m((a+bm)x) + \varepsilon_m(mx) \\ &= [a+bs(m)]h((a+bm)x) + [a+bs(m)]h(mx) \\ &\leq [a+bs(m)][s(a+bm)+s(m)]h(x). \end{aligned} \tag{3.3.9}$$

From above relation, we obtain that

$$\begin{aligned}
\Lambda_m^2\varepsilon_m(x) &= \Lambda_m(\Lambda_m\varepsilon_m(x)) \\
&= \Lambda_m\varepsilon_m((a+bm)x)+\Lambda_m\varepsilon_m(mx) \\
&\leq [a+bs(m)][s(a+bm)+s(m)]h[(a+bm)x] \\
&\quad +[a+bs(m)][s(a+bm)+s(m)]h(mx) \\
&\leq [a+bs(m)][s(a+bm)+s(m)]s(a+bm)h(x) \\
&\quad +[a+bs(m)][s(a+bm)+s(m)]s(m)h(x) \\
&= [a+bs(m)]h(x)[s(a+bm)+s(m)]^2.
\end{aligned}$$

In the same way, we get

$$\Lambda_m^n\varepsilon_m(x) \leq [a+bs(m)]h(x)[s(m)+s(a+bm)]^n, \tag{3.3.10}$$

for $n \in \mathbb{N}_0$, $m \in M_0$ and $x \in \mathbb{R}$.

For $m \in M_0$ and $x \in \mathbb{R}$, we get

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x) \\
&\leq [a+bs(m)]h(x)\sum_{n=0}^{\infty}[s(m)+s(a+bm)]^n \\
&= \frac{[a+bs(m)]h(x)}{1-s(m)-s(a+bm)}.
\end{aligned}$$

Now, we can use Theorem 1.1.9 with $X := \mathbb{R}$, $Y := \mathbb{R}$ and $\varphi := f$. According to it, the limit

$$T_m(x) := \lim_{n\to\infty}(\mathcal{T}_m^n f)(x)$$

exists for each $m \in M_0$ and $x \in \mathbb{R}$, and

$$d(f(x),T_m(x)) \leq \frac{[a+bs(m)]h(x)}{1-s(m)-s(a+bm)}, \tag{3.3.11}$$

for all $m \in M_0$ and $x \in \mathbb{R}$.

Next, we show that

$$d(\mathcal{T}_m^n f(ax+by), a\mathcal{T}_m^n f(x)+b\mathcal{T}_m^n f(y))$$

$$\leq [s(m)+s(a+bm)]^n(ah(x)+bh(y)), \tag{3.3.12}$$

for all $x,y \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $m \in M_0$. Since the case $n=0$ is just (3.3.2), take $k \in \mathbb{N}_0$ and assume that (3.3.12) holds for $n=k$ and every $x,y \in \mathbb{R}, m \in M_0$. Then

$$
\begin{aligned}
& d\left(\mathcal{T}_m^{k+1} f(ax+by), a\mathcal{T}_m^{k+1} f(x) + b\mathcal{T}_m^{k+1} f(y)\right) \\
= \;& d\Big(\frac{1}{a}\Big[\mathcal{T}_m^k f((a+bm)(ax+by)) - b\mathcal{T}_m^k f(m(ax+by))\Big], \\
& \frac{1}{a}\Big[a\mathcal{T}_m^k f((a+bm)x) - ab\mathcal{T}_m^k f(mx) + b\mathcal{T}_m^k f((a+bm)y) - b^2\mathcal{T}_m^k f(my)\Big]\Big) \\
= \;& d\Big(\mathcal{T}_m^k f((a+bm)(ax+by)) - b\mathcal{T}_m^k f(m(ax+by)), \\
& a\mathcal{T}_m^k f((a+bm)x) - ab\mathcal{T}_m^k f(mx) + b\mathcal{T}_m^k f((a+bm)y) - b^2\mathcal{T}_m^k f(my)\Big) \\
\leq \;& d\Big(\mathcal{T}_m^k f((a+bm)(ax+by)) - b\mathcal{T}_m^k f(m(ax+by)), \\
& -b\mathcal{T}_m^k f(m(ax+by)) + a\mathcal{T}_m^k f((a+bm)x) + b\mathcal{T}_m^k f((a+bm)y)\Big) \\
& +d\Big(-b\mathcal{T}_m^k f(m(ax+by)) + a\mathcal{T}_m^k f((a+bm)x) + b\mathcal{T}_m^k f((a+bm)y), \\
& a\mathcal{T}_m^k f((a+bm)x) - ab\mathcal{T}_m^k f(mx) + b\mathcal{T}_m^k f((a+bm)y) - b^2\mathcal{T}_m^k f(my)\Big) \\
= \;& d\Big(\mathcal{T}_m^k f((a+bm)(ax+by)), a\mathcal{T}_m^k f((a+bm)x) + b\mathcal{T}_m^k f((a+bm)y)\Big) \\
& +d\Big(b\mathcal{T}_m^k f(m(ax+by)), ab\mathcal{T}_m^k f(mx) + b^2\mathcal{T}_m^k f(my)\Big) \\
= \;& d\Big(\mathcal{T}_m^k f((a+bm)(ax) + (a+bm)(by)), a\mathcal{T}_m^k f((a+bm)x) + b\mathcal{T}_m^k f((a+bm)y)\Big) \\
& +d\Big(b\mathcal{T}_m^k f(amx+bmy), ab\mathcal{T}_m^k f(mx) + b^2\mathcal{T}_m^k f(my)\Big) \\
= \;& d\Big(\mathcal{T}_m^k f((a+bm)(ax) + (a+bm)(by)), a\mathcal{T}_m^k f((a+bm)x) + b\mathcal{T}_m^k f((a+bm)y)\Big) \\
& +d\Big(\mathcal{T}_m^k f(amx+bmy), a\mathcal{T}_m^k f(mx) + b\mathcal{T}_m^k f(my)\Big) \\
\leq \;& [s(m)+s(a+bm)]^k[ah((a+bm)x) + bh((a+bm)y)] \\
& +[s(m)+s(a+bm)]^k[ah(mx)+bh(my)] \\
\leq \;& [s(m)+s(a+bm)]^k[as(a+bm)h(x) + bs(a+bm)h(y)] \\
& +[s(m)+s(a+bm)]^k[as(m)h(x)+bs(m)h(y)] \\
= \;& [s(m)+s(a+bm)]^k[s(m)+s(a+bm)](ah(x)+bh(y)) \\
= \;& [s(m)+s(a+bm)]^{k+1}[ah(x)+bh(y)].
\end{aligned}
$$

By induction, we have shown that (3.3.12) holds for every $x,y \in \mathbb{R}$, $n \in \mathbb{N}_0$ and

$m \in M_0$. Letting $n \to \infty$ in (3.3.12), we obtain the equality

$$T_m(ax+by) = aT_m(x) + bT_m(y), \tag{3.3.13}$$

for all $x, y \in \mathbb{R}$, $m \in M_0$ and the function $T_m : \mathbb{R} \to \mathbb{R}$, defined in this way, is a solution of the equation

$$aT(x) = T((a+bm)x) - bT(mx). \tag{3.3.14}$$

Next, we prove that each generalized additive Cauchy function $T : \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$d(f(x), T(x)) \leq Lh(x), \quad x \in \mathbb{R} \tag{3.3.15}$$

with some $L > 0$, is equal to $T_m$ for each $m \in M_0$. To this end, fix $m_0 \in M_0$ and $T : \mathbb{R} \to \mathbb{R}$ satisfying (3.3.15). Then observe that

$$\begin{aligned} d(T(x), T_{m_0}(x)) &\leq d(T(x), f(x)) + d(f(x), T_{m_0}(x)) \\ &\leq Lh(x) + (a+bs(m_0))h(x) \sum_{n=0}^{\infty} (s(m_0) + s(a+bm_0))^n \\ &= h(x)[L + (a+bs(m_0)) \sum_{n=0}^{\infty} (s(m_0) + s(a+bm_0))^n] \\ &= h(x)L_0 \sum_{n=0}^{\infty} (s(m_0) + s(a+bm_0))^n \end{aligned} \tag{3.3.16}$$

where $L_0 = L(1 - s(m_0) - s(a+bm_0)) + (a+bs(m_0))$ (the case $h(x) \equiv 0$ is trivial, so we exclude it here). Observe yet that $T$ and $T_{m_0}$ are solutions to equation (3.3.14) for all $m \in M_0$. Here we will show that for each $j \in \mathbb{N}_0$, we have

$$d(T(x), T_{m_0}(x)) \leq h(x)L_0 \sum_{n=j}^{\infty} (bs(m_0) + s(a+bm_0))^n, \quad x \in \mathbb{R}. \tag{3.3.17}$$

The case $j = 0$ is exactly (3.3.16). So, fix $l \in \mathbb{N}_0$ and assume that (3.3.17) hold for

$j=l$. For $x \in \mathbb{R}$, we get

$$\begin{aligned}
d(T(x),T_{m_0}(x)) &= d(aT(x),aT_{m_0}(x)) \\
&= d(T((a+bm_0)x)-bT(m_0x),T_{m_0}((a+bm_0)x)-bT_{m_0}(m_0x)) \\
&\le d(T((a+bm_0)x)-bT(m_0x),-bT(m_0x)+T_{m_0}((a+bm_0)x)) \\
&\quad +d(-bT(m_0x)+T_{m_0}((a+bm_0)x),bT_{m_0}(m_0x)-bT_{m_0}(m_0x)) \\
&= d(T((a+bm_0)x),T_{m_0}((a+bm_0)x))+d(bT(m_0x),bT_{m_0}(m_0x)) \\
&= d(T((a+bm_0)x),T_{m_0}((a+bm_0)x))+d(T(m_0x),T_{m_0}(m_0x)) \\
&\le h((a+bm_0)x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n \\
&\quad +h(m_0x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n \\
&\le s((a+bm_0))h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n \\
&\quad +s(m_0)h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n \\
&= [s(a+bm_0)+s(m_0)]h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n \\
&< h(x)L_0\sum_{n=l}^{\infty}[s(m_0)+s(a+bm_0)]^n \\
&< h(x)L_0\sum_{n=l+1}^{\infty}[bs(m_0)+s(a+bm_0)]^n.
\end{aligned}$$

Thus we have shown (3.3.17). Now, letting $j\to\infty$ in (3.3.17), we get

$$T=T_{m_0}. \tag{3.3.18}$$

Hence, we have also proved that $T_m=T_{m_0}$ for each $m\in M_0$, which (in view of (3.3.11)) yields

$$d(f(x),T_{m_0}(x)) \le \frac{(a+bs(m))h(x)}{(1-s(m)-s(a+bm))}, \quad x\in\mathbb{R}, m\in M_0. \tag{3.3.19}$$

This implies (3.3.3) with $T:=T_{m_0}$; clearly, equality (3.3.18) means the uniqueness of $T$ as well. □

**Corollary 3.3.2.** *Let $d$ be as in Theorem 3.3.1 and $h:\mathbb{R}\to(0,\infty)$ be such that*

$$\liminf_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{h(nx)+h((a+bn)x)}{h(x)}=0. \tag{3.3.20}$$

*Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies (3.3.2). Then there exists a unique generalized additive Cauchy function $T:\mathbb{R}\to\mathbb{R}$ such that*

$$d(f(x),T(x)) \le ah(x), \quad x\in\mathbb{R}. \tag{3.3.21}$$

*Proof.* From (3.3.20), there is a subsequence $\{n_k\}$ of positive integer such that

$$\lim_{k\to\infty}\sup_{x\in\mathbb{R}}\frac{h(n_kx)+h((a+bn_k)x)}{h(x)}=0. \tag{3.3.22}$$

By the property of infimum for $s(n_k)$ and $s(a+bn_k)$, we get

$$s(n_k)=\sup_{x\in\mathbb{R}}\frac{h(n_kx)}{h(x)}\le\sup_{x\in\mathbb{R}}\frac{h(n_kx)+h((a+bn_k)x)}{h(x)} \tag{3.3.23}$$

and

$$s(a+bn_k)=\sup_{x\in\mathbb{R}}\frac{h((a+bn_k)x)}{h(x)}\le\sup_{x\in\mathbb{R}}\frac{h(n_kx)+h((a+bn_k)x)}{h(x)}. \tag{3.3.24}$$

Taking limit as $k\to\infty$ in (3.3.23) and (3.3.24), we obtain that

$$\lim_{k\to\infty}s(n_k)=0 \tag{3.3.25}$$

and

$$\lim_{k\to\infty}s(a+bn_k)=0. \tag{3.3.26}$$

This implies that

$$\lim_{k\to\infty}\frac{a+bs(n_k)}{1-s(n_k)-s(a+bn_k)}=a.$$

It follows that $s_0=a$. □

It is easy to see that Theorem 3.3.1 and Corollary 3.3.2 yield the following results concerning the stability of additive Cauchy functional equation.

**Corollary 3.3.3.** *Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant and $h:\mathbb{R}\to\mathbb{R}_+$ be a function such that*

$$M_0:=\{n\in\mathbb{N}: s(n)+s(1+n)<1\}\ne\emptyset,$$

*where $s(n):=\inf\{t\in\mathbb{R}_+: h(nx)\le th(x) \text{ for all } x\in\mathbb{R}\}$ for $n\in\mathbb{N}$. Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies the following inequality*

$$d(f(x+y),f(x)+f(y))\le h(x)+h(y) \tag{3.3.27}$$

*for all $x, y \in \mathbb{R}$. Then there exists a unique additive Cauchy function $T : \mathbb{R} \to \mathbb{R}$ such that*

$$d(f(x), T(x)) \leq s_0 h(x), \quad x \in \mathbb{R}, \tag{3.3.28}$$

*with* $s_0 := \inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)} : n \in M_0\right\}$.

**Corollary 3.3.4.** *Let $d$ be as in Corollary 3.3.3 and $h : \mathbb{R} \to (0, \infty)$ be such that*

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{h(nx) + h((1+n)x)}{h(x)} = 0. \tag{3.3.29}$$

*Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies (3.3.27). Then there exists a unique additive Cauchy function $T : \mathbb{R} \to \mathbb{R}$ such that*

$$d(f(x), T(x)) \leq h(x), \quad x \in \mathbb{R}. \tag{3.3.30}$$

### 3.3.2 Stability of generalized additive Cauchy functional equation by Brzdęk's technique on arbitrary group

In this subsection, we present a new type of stability results for generalized Cauchy functional equation of the form

$$f(ax * by) = af(x) \diamond bf(y),$$

where $a, b \in \mathbb{N}$ and $f$ is a mapping from a commutative semigroup $(G_1, *)$ to a commutative group $(G_2, \diamond)$.

**Theorem 3.3.5.** *Let $(G_1, *)$ be a commutative semigroup, $(G_2, \diamond)$ be a commutative group, $(G_2, d)$ be a complete metric space such that $d$ is invariant, that is,*

$$d(x \diamond z, y \diamond z) = d(x, y)$$

*for all $x, y, z \in G_2$, and let $a, b$ be two fixed natural numbers and $h : G_1 \to \mathbb{R}_+$ be a function such that*

$$M_0 := \{n \in \mathbb{N} : s(n) + s(a + bn) < 1\} \neq \emptyset,$$

*where* $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x)$ *for all* $x \in G_1\}$ *for* $n \in \mathbb{N}$*. Suppose that*

$$d(x, ay) = d(x, y) \tag{3.3.31}$$

*and*

$$d(kx, ky) = d(x, y) \tag{3.3.32}$$

*for all* $x, y \in G_2$ *and for all* $k \in \{a, b\}$*. If* $f : G_1 \to G_2$ *satisfies the following inequality*

$$d(f(ax * by), af(x) \diamond bf(y)) \le ah(x) + bh(y) \tag{3.3.33}$$

*for all* $x, y \in G_1$*, then there exists a unique generalized Cauchy function* $T : G_1 \to G_2$ *such that*

$$d(f(x), T(x)) \le s_0 h(x) \tag{3.3.34}$$

*for all* $x \in G_1$*, where* $s_0 := \inf\left\{\dfrac{a + bs(n)}{1 - s(n) - s(a + bn)} : n \in M_0\right\}$.

*Proof.* For each $m \in \mathbb{N}$ and $x \in G_1$ note that (3.3.33) with $y = mx$ gives

$$d(f(a * bm)x), af(x) \diamond bf(mx)) \le (a + bs(m))h(x). \tag{3.3.35}$$

Define operators $\mathcal{T}_m : G_2^{G_1} \to G_2^{G_1}$ and $\Lambda_m : \mathbb{R}_+^{G_1} \to \mathbb{R}_+^{G_1}$ by

$$\begin{aligned} \mathcal{T}_m\xi(x) &:= \xi((a + bm)x) \diamond [b\xi(mx)]^{-1}, \quad m \in \mathbb{N}, x \in G_1, \xi \in G_2^{G_1}, \\ \Lambda_m\delta(x) &:= \delta((a + bm)x) + \delta(mx), \quad m \in \mathbb{N}, x \in G_1, \delta \in \mathbb{R}_+^{G_1}. \end{aligned} \tag{3.3.36}$$

Then it is easily seen that, for each $m \in \mathbb{N}$, $\Lambda := \Lambda_m$ has the form described in (1.1.18) with $f_1(x) = (a + bm)x$ and $f_2(x) = mx$. Since $d$ is invariant and $d$ satisfies condition (3.3.31), inequality (3.3.35) follows that

$$d(\mathcal{T}_m f(x), f(x)) = d(\mathcal{T}_m f(x), af(x)) \le (a + bs(m))h(x) =: \varepsilon_m(x) \tag{3.3.37}$$

for all $x \in G_1$, and

$$\begin{aligned} d(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x)) &= d(\xi((a + bm)x) \diamond [b\xi(mx)]^{-1}, \mu((a + bm)x) \diamond [b\mu(mx)]^{-1}) \\ &\le d(\xi((a + bm)x) \diamond [b\xi(mx)]^{-1}, [b\xi(mx)]^{-1} \diamond \mu((a + bm)x)) \\ &\quad + d([b\xi(mx)]^{-}1 \diamond \mu((a + bm)x), \mu((a + bm)x) \diamond [b\mu(mx)]^{-1}) \\ &= d(\xi((a + bm)x), \mu((a + bm)x) + d(b\xi(mx), b\mu(mx)) \\ &= d(\xi((a + bm)x), \mu((a + bm)x) + d(\xi(mx), \mu(mx)) \end{aligned} \tag{3.3.38}$$

for all $m \in \mathbb{N}, \xi, \mu \in G_2^{G_1}$, $x \in G_1$. Consequently, for each $m \in \mathbb{N}$, condition (1.1.17) is valid with $X := G_1$, $Y := G_2$ and $\mathcal{T} := \mathcal{T}_m$.

Next, we will show that

$$\Lambda_m^n \varepsilon_m(x) \leq (a+bs(m))h(x)(s(m)+s(a+bm))^n \tag{3.3.39}$$

for all $n \in \mathbb{N}_0$, $m \in M_0$ and $x \in G_1$. It is easy to se that condition (3.3.39) holds for $n = 0$. From (3.3.36), we have

$$\begin{aligned}
\Lambda_m \varepsilon_m(x) &:= \varepsilon_m((a+bm)x) + \varepsilon_m(mx) \\
&= [a+bs(m)]h((a+bm)x) + [a+bs(m)]h(mx) \\
&\leq [a+bs(m)][s(a+bm)+s(m)]h(x) \qquad (3.3.40)
\end{aligned}$$

for all $m \in M_0$ and $x \in G_1$. From the above relation, we obtain that

$$\begin{aligned}
\Lambda_m^2 \varepsilon_m(x) &:= \Lambda_m(\Lambda_m \varepsilon_m(x)) \\
&= \Lambda_m \varepsilon_m((a+bm)x) + \Lambda_m \varepsilon_m(mx) \\
&\leq [a+bs(m)][s(a+bm)+s(m)]h[(a+bm)x] \\
&\quad +[a+bs(m)][s(a+bm)+s(m)]h(mx) \\
&\leq [a+bs(m)][s(a+bm)+s(m)]s(a+bm)h(x) \\
&\quad +[a+bs(m)][s(a+bm)+s(m)]s(m)h(x) \\
&= [a+bs(m)]h(x)[s(a+bm)+s(m)]^2
\end{aligned}$$

for all $m \in M_0$ and $x \in G_1$. In the same way, we get

$$\Lambda_m^n \varepsilon_m(x) \leq [a+bs(m)]h(x)[s(m)+s(a+bm)]^n \tag{3.3.41}$$

for all $n \in \mathbb{N}_0$, $m \in M_0$ and $x \in G_1$.

For $m \in M_0$ and $x \in G_1$, we get

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&\leq [a+bs(m)]h(x) \sum_{n=0}^{\infty} [s(m)+s(a+bm)]^n \\
&= \frac{(a+bs(m))h(x)}{1-s(m)-s(a+bm)}.
\end{aligned}$$

Now, we can use Theorem 1.1.9 with $X = G_1$, $Y = G_2$ and $\varphi = f$. According to it, the limit

$$T_m(x) := \lim_{n\to\infty} (\mathcal{T}_m^n f)(x)$$

exists for each $m \in M_0$ and $x \in G_1$, and

$$d(f(x), T_m(x)) \leq \frac{[a+bs(m)]h(x)}{1-s(m)-s(a+bm)} \tag{3.3.42}$$

for all $m \in M_0$ and $x \in G_1$.

Next, we show that

$$d(\mathcal{T}_m^n f(ax*by), a\mathcal{T}_m^n f(x) \diamond b\mathcal{T}_m^n f(y)) \leq [s(m)+s(a+bm)]^n (ah(x)+bh(y)) \tag{3.3.43}$$

for all $x, y \in G_1$, $n \in \mathbb{N}_0$ and $m \in M_0$. Since the case $n = 0$ is just (3.3.33), take $k \in \mathbb{N}_0$ and assume that (3.3.43) holds for $n = k$ and every $x, y \in X, m \in M_0$. Then

$$
\begin{aligned}
&d\Big(\mathcal{T}_m^{k+1} f(ax*by), a\mathcal{T}_m^{k+1} f(x) \diamond b\mathcal{T}_m^{k+1} f(y)\Big) \\
&= d\Big(\mathcal{T}_m^k f((a+bm)(ax*by)) \diamond [b\mathcal{T}_m^k f(m(ax*by))]^{-1}, \\
&\quad a\mathcal{T}_m^k f((a+bm)x) \diamond [ab\mathcal{T}_m^k f(mx)]^{-1} \diamond b\mathcal{T}_m^k f((a+bm)y) \diamond [b^2\mathcal{T}_m^k f(my)]^{-1}\Big) \\
&\le d\Big(\mathcal{T}_m^k f((a+bm)(ax*by)) \diamond [b\mathcal{T}_m^k f(m(ax*by))]^{-1}, \\
&\quad [b\mathcal{T}_m^k f(m(ax*by))]^{-1} \diamond a\mathcal{T}_m^k f((a+bm)x) \diamond b\mathcal{T}_m^k f((a+bm)y)\Big) \\
&\quad + d\Big([b\mathcal{T}_m^k f(m(ax*by))]^{-1} \diamond a\mathcal{T}_m^k f((a+bm)x) \diamond b\mathcal{T}_m^k f((a+bm)y), \\
&\quad a\mathcal{T}_m^k f((a+bm)x) \diamond [ab\mathcal{T}_m^k f(mx)]^{-1} \diamond b\mathcal{T}_m^k f((a+bm)y) \diamond [b^2\mathcal{T}_m^k f(my)]^{-1}\Big) \\
&= d\Big(\mathcal{T}_m^k f((a+bm)(ax*by)), a\mathcal{T}_m^k f((a+bm)x) \diamond b\mathcal{T}_m^k f((a+bm)y)\Big) \\
&\quad + d\Big(b\mathcal{T}_m^k f(m(ax*by)), ab\mathcal{T}_m^k f(mx) \diamond b^2\mathcal{T}_m^k f(my)\Big) \\
&= d\Big(\mathcal{T}_m^k f((a+bm)(ax)*(a+bm)(by)), a\mathcal{T}_m^k f((a+bm)x) \diamond b\mathcal{T}_m^k f((a+bm)y)\Big) \\
&\quad + d\Big(\mathcal{T}_m^k f(amx*bmy), a\mathcal{T}_m^k f(mx) \diamond b\mathcal{T}_m^k f(my)\Big) \\
&\le [s(m)+s(a+bm)]^k [ah((a+bm)x)+bh((a+bm)y)] \\
&\quad + [s(m)+s(a+bm)]^k [ah(mx)+bh(my)] \\
&\le [s(m)+s(a+bm)]^k [as(a+bm)h(x)+bs(a+bm)h(y)] \\
&\quad + [s(m)+s(a+bm)]^k [as(m)h(x)+bs(m)h(y)] \\
&= [s(m)+s(a+bm)]^k [s(m)+s(a+bm)](ah(x)+bh(y)) \\
&= [s(m)+s(a+bm)]^{k+1} [ah(x)+bh(y)].
\end{aligned}
$$

By introduction, we have shown that (3.3.43) holds for every $x, y \in G_1$, $n \in \mathbb{N}_0$ and $m \in M_0$. Letting $n \to \infty$ in (3.3.43), we obtain the equality

$$T_m(ax*by) = aT_m(x) \diamond bT_m(y) \tag{3.3.44}$$

for all $x, y \in G_1$, $m \in M_0$ and the function $T_m : G_1 \to G_2$, defined in this way, is a solution of the equation

$$aT(x) = T((a+bm)x) \diamond [bT(mx)]^{-1}. \tag{3.3.45}$$

Next, we prove that each generalized Cauchy function $T : G_1 \to G_2$ satisfying the inequality

$$d(f(x),T(x)) \leq Lh(x), \quad x \in G_1 \tag{3.3.46}$$

with some $L > 0$, is equal to $T_m$ for each $m \in M_0$. To this end, fix $m_0 \in M_0$ and $T : G_1 \to G_2$ satisfying (3.3.46).

Then we observe that

$$\begin{aligned} d(T(x),T_{m_0}(x)) &\leq d(T(x),f(x)) + d(f(x),T_{m_0}(x)) \\ &\leq Lh(x) + (a+bs(m_0))h(x)\sum_{n=0}^{\infty}(s(m_0)+s(a+bm_0))^n \\ &= h(x)[L+(a+bs(m_0))\sum_{n=0}^{\infty}(s(m_0)+s(a+bm_0))^n] \\ &= h(x)L_0\sum_{n=0}^{\infty}(s(m_0)+s(a+bm_0))^n, \end{aligned} \tag{3.3.47}$$

where $L_0 = L(1-s(m_0)-s(a+bm_0))+(a+bs(m_0))$ (the case $h(x) \equiv 0$ is trivial, so we exclude it here). Observe yet that $T$ and $T_{m_0}$ are solutions to equation (3.3.45) for all $m \in M_0$. We will show that for each $j \in \mathbb{N}_0$,

$$d(T(x),T_{m_0}(x)) \leq h(x)L_0\sum_{n=j}^{\infty}(bs(m_0)+s(a+bm_0))^n, \quad x \in G_1. \tag{3.3.48}$$

The case $j = 0$ is exactly (3.3.47). So, fix $l \in \mathbb{N}_0$ and assume that (3.3.48) hold for

$j = l$. Then, in view of (3.3.47), for each $\quad a,b \in \mathbb{R}, \quad m,m_0 \in M_0$

$$
\begin{aligned}
d(T(x),T_{m_0}(x)) &= d(aT(x),aT_{m_0}(x)) \\
&= d(T((a+bm_0)x) \diamond [bT(m_0x)]^{-1}, T_{m_0}((a+bm_0)x) \diamond [bT_{m_0}(m_0x)]^{-1}) \\
&\leq d(T((a+bm_0)x) \diamond [bT(m_0x)]^{-1}, [bT(m_0x)]^{-1} \diamond T_{m_0}((a+bm_0)x)) \\
&\quad + d([bT(m_0x)]^{-1} \diamond T_{m_0}((a+bm_0)x), T_{m_0}((a+bm_0)x) \diamond [bT_{m_0}(m_0x)]^{-1}) \\
&= d(T((a+bm_0)x), T_{m_0}((a+bm_0)x)) + d(bT(m_0x), bT_{m_0}(m_0x)) \\
&= d(T((a+bm_0)x), T_{m_0}((a+bm_0)x)) + d(T(m_0x), T_{m_0}(m_0x)) \\
&\leq h((a+bm_0)x)L_0 \sum_{n=l}^{\infty} [s(m_0)+s(a+bm_0)]^n \\
&\quad + h(m_0x)L_0 \sum_{n=l}^{\infty} [s(m_0)+s(a+bm_0)]^n \\
&\leq s((a+bm_0))h(x)L_0 \sum_{n=l}^{\infty} [s(m_0)+s(a+bm_0)]^n \\
&\quad + s(m_0)h(x)L_0 \sum_{n=l}^{\infty} [s(m_0)+s(a+bm_0)]^n \\
&= [s(a+bm_0)+s(m_0)]h(x)L_0 \sum_{n=l}^{\infty} [s(m_0)+s(a+bm_0)]^n \\
&< h(x)L_0 \sum_{n=l}^{\infty} [s(m_0)+s(a+bm_0)]^n \\
&< h(x)L_0 \sum_{n=l+1}^{\infty} [s(m_0)+s(a+bm_0)]^n.
\end{aligned}
$$

Thus we have shown (3.3.48). Now, letting $j \to \infty$ in (3.3.48), we get

$$T = T_{m_0}. \tag{3.3.49}$$

Thus we have also proved that $T_m = T_{m_0}$ for each $m \in M_0$, which (in view of (3.3.42)) yields

$$d(f(x),T_{m_0}(x)) \leq \frac{(a+bs(m))h(x)}{(1-s(m)-s(a+bm))}, \quad x \in G_1, m \in M_0. \tag{3.3.50}$$

This implies (3.3.34) with $T := T_{m_0}$; clearly, equality (3.3.49) means the uniqueness of $T$ as well. This completes the proof. $\square$

**Remark 3.3.6.** A bit more involved example of an invariant metric $d$ on group $G_2$ satisfying conditions (3.3.31) and (3.3.32) we obtain taking $n \in \mathbb{N}$ and $a = b :=$

$n+1$, group $(G_2,\diamond) := (\mathbb{Z}_n,+)$, where $\mathbb{Z}_n$ is the set of integers mod $n$, and $d$ is a discrete metric on $G_2$.

**Corollary 3.3.7.** *Let $d$ be as in Theorem 3.3.5 and $h : G_1 \to (0,\infty)$ be such that*

$$\liminf_{n\to\infty} \sup_{x\in G_1} \frac{h(nx)+h((a+bn)x)}{h(x)} = 0. \tag{3.3.51}$$

*Assume that $f : G_1 \to G_2$ satisfies (3.3.34). Then there exists a unique generalized Cauchy function $T : G_1 \to G_2$ such that*

$$d(f(x),T(x)) \le ah(x) \tag{3.3.52}$$

*for all $x \in G_1$.*

*Proof.* From (3.3.51), there is a subsequence $\{n_k\}$ of positive integer such that

$$\lim_{k\to\infty} \sup_{x\in G_1} \frac{h(n_k x)+h((a+bn_k)x)}{h(x)} = 0. \tag{3.3.53}$$

By the property of infimum for $s(n_k)$ and $s(a+bn_k)$, we have

$$s(n_k) = \sup_{x\in G_1} \frac{h(n_k x)}{h(x)} \le \sup_{x\in G_1} \frac{h(n_k x)+h((a+bn_k)x)}{h(x)} \tag{3.3.54}$$

and

$$s(a+bn_k) = \sup_{x\in G_1} \frac{h((a+bn_k)x)}{h(x)} \le \sup_{x\in G_1} \frac{h(n_k x)+h((a+bn_k)x)}{h(x)}. \tag{3.3.55}$$

Taking limit as $k\to\infty$ in (3.3.54) and (3.3.55), we get

$$\lim_{k\to\infty} s(n_k) = 0$$

and

$$\lim_{k\to\infty} s(a+bn_k) = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{a+bs(n_k)}{1-s(n_k)-s(a+bn_k)} = a.$$

It follows that $s_0 = a$. This completes the proof. □

It is easy to see that Theorem 3.3.5 yields the recent result in [4].

**Corollary 3.3.8** ([4]). *Let $(G_1,*)$ be a commutative semigroup, $(G_2,\diamond)$ be a commutative group, $(G_2,d)$ be a complete metric space such that $d$ is invariant and $h : G_1 \to \mathbb{R}_+$ be a function such that*

$$M_0 := \{n \in \mathbb{N} : s(n)+s(1+n) < 1\} \neq \emptyset,$$

*where $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in G_1\}$ for $n \in \mathbb{N}$. Assume that $f : G_1 \to G_2$ satisfies the following inequality*

$$d(f(x*y), f(x) \diamond f(y)) \leq h(x)+h(y) \tag{3.3.56}$$

*for all $x,y \in G_1$. Then there exists a unique generalized Cauchy function $T : G_1 \to G_2$ such that*

$$d(f(x),T(x)) \leq s_0 h(x) \tag{3.3.57}$$

*for all $x \in G_1$, where $s_0 := \inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)} : n \in M_0\right\}$.*

**Corollary 3.3.9** ([4]). *Let $d$ be as in Corollary 3.3.8 and $h : G_1 \to (0,\infty)$ be such that*

$$\liminf_{n\to\infty} \sup_{x\in G_1} \frac{h(nx)+h((1+n)x)}{h(x)} = 0. \tag{3.3.58}$$

*Assume that $f : G_1 \to G_2$ satisfies (3.3.56). Then there exists a unique generalized Cauchy function $T : G_1 \to G_2$ such that*

$$d(f(x),T(x)) \leq h(x) \tag{3.3.59}$$

*for all $x \in G_1$.*

It has been pointed out in some studies that several results of the stability of many kinds Cauchy functional equations can be concluded from our main results in previous section related with commutative semigroup $(G_1,*)$ and commutative group $(G_2,\diamond)$. By using Corollary 3.3.8, we get the following result.

**Corollary 3.3.10.** *Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant and $h : \mathbb{R} \to \mathbb{R}_+$ be a function such that*

$$M_0 := \{n \in \mathbb{N} : s(n)+s(1+n) < 1\} \neq \emptyset,$$

*where* $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x) \text{ for all } x \in \mathbb{R}\}$ *for* $n \in \mathbb{N}$. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the following inequality*

$$d(f(x+y), f(x)+f(y)) \le h(x)+h(y) \tag{3.3.60}$$

*for all* $x, y \in \mathbb{R}$. *Then there exists a unique additive Cauchy function* $T : \mathbb{R} \to \mathbb{R}$ *such that*

$$d(f(x), T(x)) \le s_0 h(x) \tag{3.3.61}$$

*for all* $x \in \mathbb{R}$, *where* $s_0 := \inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)} : n \in M_0\right\}$.

*Proof.* This result can be obtained from Corollary 3.3.8 by take commutative semigroup $(G_1, *) = (\mathbb{R}, +)$ and commutative group $(G_2, \diamond) = (\mathbb{R}, +)$. $\square$

**Corollary 3.3.11.** *Let* $d$ *be as in Corollary 3.3.10 and* $h : \mathbb{R} \to (0, \infty)$ *be such that*

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}} \frac{h(nx)+h((1+n)x)}{h(x)} = 0. \tag{3.3.62}$$

*Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies (3.3.60). Then there exists a unique additive Cauchy function* $T : \mathbb{R} \to \mathbb{R}$ *such that*

$$d(f(x), T(x)) \le h(x) \tag{3.3.63}$$

*for all* $x \in \mathbb{R}$.

**Corollary 3.3.12.** *Let* $(\mathbb{R}\backslash\{0\}, d)$ *be a complete metric space such that* $d$ *is invariant and* $h : \mathbb{R} \to \mathbb{R}_+$ *be a function such that*

$$M_0 := \{n \in \mathbb{N} : s(n)+s(1+n) < 1\} \ne \emptyset,$$

*where* $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x) \text{ for all } x \in \mathbb{R}\}$ *for* $n \in \mathbb{N}$. *Assume that* $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ *satisfies the following inequality*

$$d(f(x+y), f(x)f(y)) \le h(x)+h(y) \tag{3.3.64}$$

*for all* $x, y \in \mathbb{R}$. *Then there exists a unique exponential function* $T : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ *such that*

$$d(f(x), T(x)) \le s_0 h(x) \tag{3.3.65}$$

*for all* $x \in \mathbb{R}$, *where* $s_0 := \inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)} : n \in M_0\right\}$.

*Proof.* This result can be obtained from Corollary 3.3.8 by take commutative semigroup $(G_1,*) = (\mathbb{R},+)$ and commutative group $(G_2,\diamond) = (\mathbb{R}\backslash\{0\},\cdot)$. $\square$

**Corollary 3.3.13.** *Let $d$ be as in Corollary 3.3.12 and $h:\mathbb{R}\to(0,\infty)$ be such that*

$$\liminf_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{h(nx)+h((1+n)x)}{h(x)}=0. \tag{3.3.66}$$

*Assume that $f:\mathbb{R}\to\mathbb{R}\backslash\{0\}$ satisfies (3.3.64). Then there exists a unique exponential function $T:\mathbb{R}\to\mathbb{R}\backslash\{0\}$ such that*

$$d(f(x),T(x))\le h(x) \tag{3.3.67}$$

*for all $x\in\mathbb{R}$.*

**Corollary 3.3.14.** *Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant and $h:\mathbb{R}\backslash\{0\}\to\mathbb{R}_+$ be a function such that*

$$M_0 := \{n\in\mathbb{N} : s(n)+s(1+n)<1\}\neq\emptyset,$$

*where $s(n):=\inf\{t\in\mathbb{R}_+ : h(nx)\le th(x) \text{ for all } x\in\mathbb{R}\}$ for $n\in\mathbb{N}$. Assume that $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ satisfies the following inequality*

$$d(f(xy),f(x)+f(y))\le h(x)+h(y) \tag{3.3.68}$$

*for all $x,y\in\mathbb{R}\backslash\{0\}$. Then there exists a unique logarithmic function $T:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ such that*

$$d(f(x),T(x))\le s_0h(x) \tag{3.3.69}$$

*for all $x\in\mathbb{R}\backslash\{0\}$, where* $s_0:=\inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)} : n\in M_0\right\}$.

*Proof.* This result can be obtained from Corollary 3.3.8 by take commutative semigroup $(G_1,*) = (\mathbb{R}\backslash\{0\},\cdot)$ and commutative group $(G_2,\diamond) = (\mathbb{R},+)$. $\square$

**Corollary 3.3.15.** *Let $d$ be as in Corollary 3.3.14 and $h:\mathbb{R}\backslash\{0\}\to(0,\infty)$ be such that*

$$\liminf_{n\to\infty}\sup_{x\in\mathbb{R}\backslash\{0\}}\frac{h(nx)+h((1+n)x)}{h(x)}=0. \tag{3.3.70}$$

*Assume that* $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *satisfies (3.3.68). Then there exists a unique logarithmic function* $T:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *such that*

$$d(f(x),T(x))\le h(x) \tag{3.3.71}$$

*for all* $x\in\mathbb{R}\backslash\{0\}$.

**Corollary 3.3.16.** *Let* $(\mathbb{R}\backslash\{0\},d)$ *be a complete metric space such that* $d$ *is invariant and* $h:\mathbb{R}\to\mathbb{R}_+$ *be a function such that*

$$M_0:=\{n\in\mathbb{N}:s(n)+s(1+n)<1\}\neq\emptyset,$$

*where* $s(n):=\inf\{t\in\mathbb{R}_+ : h(nx)\le th(x)$ *for all* $x\in\mathbb{R}\}$ *for* $n\in\mathbb{N}$. *Assume that* $f:\mathbb{R}\to\mathbb{R}\backslash\{0\}$ *satisfies the following inequality*

$$d(f(xy),f(x)f(y))\le h(x)+h(y) \tag{3.3.72}$$

*for all* $x,y\in\mathbb{R}$. *Then there exists a unique multiplicative function* $T:\mathbb{R}\to\mathbb{R}\backslash\{0\}$ *such that*

$$d(f(x),T(x))\le s_0h(x) \tag{3.3.73}$$

*for all* $x\in\mathbb{R}$, *where* $s_0:=\inf\left\{\dfrac{1+s(n)}{1-s(n)-s(1+n)}:n\in M_0\right\}$.

*Proof.* This result can be obtained from Corollary 3.3.8 by take commutative semigroup $(G_1,*)=(\mathbb{R},\cdot)$ and commutative group $(G_2,\diamond)=(\mathbb{R}\backslash\{0\},\cdot)$. $\square$

**Corollary 3.3.17.** *Let* $d$ *be as in Corollary 3.3.16 and* $h:\mathbb{R}\to(0,\infty)$ *be such that*

$$\liminf_{n\to\infty}\sup_{x\in\mathbb{R}}\frac{h(nx)+h((1+n)x)}{h(x)}=0. \tag{3.3.74}$$

*Assume that* $f:\mathbb{R}\to\mathbb{R}\backslash\{0\}$ *satisfies (3.3.72). Then there exists a unique multiplicative function* $T:\mathbb{R}\to\mathbb{R}\backslash\{0\}$ *such that*

$$d(f(x),T(x))\le h(x) \tag{3.3.75}$$

*for all* $x\in\mathbb{R}$.

# CHAPTER 4

# HYPERSTABILITY OF FUNCTIONAL EQUATIONS

The purpose of this chapter is to prove new hyperstability of general linear and Drygas functional equations via fixed point result due to Brzdęk (Theorem 1.1.10).

## 4.1 Hyperstability of general linear functional equation

### 4.1.1 Hyperstability results

Let $\mathbb{F},\mathbb{K}$ be two fields of real or complex numbers and $X,Y$ be two normed spaces over $\mathbb{F},\mathbb{K}$, respectively. In this subsection, we use a modification of Brzdęk's method in [4] to obtain two generalized hyperstability results for general linear equation of the form

$$g(ax+by) = Ag(x)+Bg(y), \tag{4.1.1}$$

where $g : X \to Y$ is a mapping and $a,b$Ą $\in \mathbb{F}\setminus\{0\}$, $A,B \in \mathbb{K}$. Our results are improvement and generalization of main results of Piszczek [10].

**Theorem 4.1.1.** *Let $X$ be a normed space over a field $\mathbb{F}$, $Y$ be a Banach space over a field $\mathbb{K}$, $a,b \in \mathbb{F}\setminus\{0\}$, $A,B \in \mathbb{K}$ and $h : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : |A|s\left(\frac{1}{a}(n+1)\right) + |B|s\left(-\frac{1}{b}n\right) < 1 \right\} \text{ is an infinite set,}$$

*where $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x) \text{ for all } x \in X\}$ for $n \in \mathbb{F}\setminus\{0\}$ such that*

$$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0. \tag{4.1.2}$$

*Suppose that $g : X \to Y$ satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \le h(x)+h(y), \quad x,y \in X\setminus\{0\}. \tag{4.1.3}$$

*Then $g$ satisfies the equation*

$$g(ax+by) = Ag(x)+Bg(y), \quad x,y \in X\setminus\{0\}. \tag{4.1.4}$$

*Proof.* Replacing $x$ by $\frac{1}{a}(m+1)x$ and $y$ by $-\frac{1}{b}mx$ for $m \in \mathbb{N}$ in (4.1.3), we get

$$\begin{aligned}\left\|g(x) - Ag\left(\tfrac{1}{a}(m+1)x\right) - Bg\left(-\tfrac{1}{b}mx\right)\right\| \\ \leq\ & h\left(\frac{1}{a}(m+1)x\right) + h\left(-\frac{1}{b}mx\right) \qquad (4.1.5)\end{aligned}$$

for all $x, y \in X \backslash \{0\}$. For each $m \in M_0$, we will define operator $\mathcal{T}_m : Y^{X\backslash\{0\}} \to Y^{X\backslash\{0\}}$ by

$$\mathcal{T}_m \xi(x) := A\xi\left(\frac{1}{a}(m+1)x\right) + B\xi\left(-\frac{1}{b}mx\right), \quad x \in X\backslash\{0\}, \xi \in Y^{X\backslash\{0\}}. \qquad (4.1.6)$$

Further put

$$\begin{aligned}\varepsilon_m(x) &:= h\left(\frac{1}{a}(m+1)x\right) + h\left(-\frac{1}{b}mx\right) \\ &\leq \left[s\left(\frac{1}{a}(m+1)\right) + s\left(-\frac{1}{b}m\right)\right] h(x), \quad x \in X\backslash\{0\}. \qquad (4.1.7)\end{aligned}$$

Then the inequality (4.1.5) takes the form

$$\|\mathcal{T}_m g(x) - g(x)\| \leq \varepsilon_m(x), \quad x \in X\backslash\{0\}.$$

For each $m \in M_0$, the operator $\Lambda_m : \mathbb{R}_+^{X\backslash\{0\}} \to \mathbb{R}_+^{X\backslash\{0\}}$ which is defined by

$$\Lambda_m \eta(x) := |A|\eta\left(\frac{1}{a}(m+1)x)\right) + |B|\eta\left(-\frac{1}{b}mx\right), \quad \eta \in \mathbb{R}_+^{X\backslash\{0\}}, x \in X\backslash\{0\}$$

has the form (1.1.20) with $k = 2$ and $f_1(x) = \frac{1}{a}(m+1)x$, $f_2(x) = -\frac{1}{b}mx$, $L_1(x) = |A|$, $L_2(x) = |B|$ for $x \in X$. Moreover, for every $\xi, \mu \in Y^{X\backslash\{0\}}$, $x \in X\backslash\{0\}$, we have $\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\|$

$$\begin{aligned}&= \left\|A\xi\left(\frac{1}{a}(m+1)x\right) + B\xi\left(-\frac{1}{b}mx\right) - A\mu\left(\frac{1}{a}(m+1)x\right) - B\mu\left(-\frac{1}{b}mx\right)\right\| \\ &\leq |A|\left\|\xi\left(\frac{1}{a}(m+1)x\right) - \mu\left(\frac{1}{a}(m+1)x\right)\right\| + |B|\left\|\xi\left(-\frac{1}{b}mx\right) - \mu\left(-\frac{1}{b}mx\right)\right\| \\ &= \sum_{i=1}^{2} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|.\end{aligned}$$

By using mathematical induction, we will show that for each $x \in X\backslash\{0\}$ we have

$$\Lambda_m^n \varepsilon_m(x) \leq \left[s\left(\frac{1}{a}(m+1)\right) + s\left(-\frac{1}{b}m\right)\right]\left[|A|s\left(\frac{1}{a}(m+1)\right) + |B|s\left(-\frac{1}{b}m\right)\right]^n h(x) \qquad (4.1.8)$$

for all $n\in\mathbb{N}_0$. From (4.1.7), we obtain that the inequality (4.1.8) holds for $n=0$. Next, we will assume that (4.1.8) holds for $n=k$, where $k\in\mathbb{N}_0$. Then we have

$$\begin{aligned}
\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m(\Lambda_m^k\varepsilon_m(x))\\
&= |A|\Lambda_m^k\varepsilon_m\left(\frac{1}{a}(m+1)x\right)+|B|\Lambda_m^k\varepsilon_m\left(-\frac{1}{b}mx\right)\\
&\le |A|\left[s\left(\frac{1}{a}(m+1)\right)+s\left(-\frac{1}{b}m\right)\right]\\
&\quad \left[|A|s\left(\frac{1}{a}(m+1)\right)+|B|s\left(-\frac{1}{b}m\right)\right]^k h\left(\frac{1}{a}(m+1)x\right)\\
&\quad +|B|\left[s\left(\frac{1}{a}(m+1)\right)+s\left(-\frac{1}{b}m\right)\right]\\
&\quad \left[|A|s\left(\frac{1}{a}(m+1)\right)+|B|s\left(-\frac{1}{b}m\right)\right]^k h\left(-\frac{1}{b}mx\right)\\
&\le \left[s\left(\frac{1}{a}(m+1)\right)+s\left(-\frac{1}{b}m\right)\right]\left[|A|s\left(\frac{1}{a}(m+1)\right)+|B|s\left(-\frac{1}{b}m\right)\right]^{k+1} h(x).
\end{aligned}$$

This shows that (4.1.8) holds for $n=k+1$. Now we can conclude that the inequality (4.1.8) holds for all $n\in\mathbb{N}_0$. From (4.1.8), we get

$$\begin{aligned}
\varepsilon^*(x) &= \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x)\\
&\le \sum_{n=0}^{\infty}\left[s\left(\frac{1}{a}(m+1)\right)+s\left(-\frac{1}{b}m\right)\right]\left[|A|s\left(\frac{1}{a}(m+1)\right)+|B|s\left(-\frac{1}{b}m\right)\right]^n h(x)\\
&= \frac{\left[s\left(\frac{1}{a}(m+1)\right)+s\left(-\frac{1}{b}m\right)\right]h(x)}{1-|A|s\left(\frac{1}{a}(m+1)\right)-|B|s\left(-\frac{1}{b}m\right)}
\end{aligned}$$

for all $x\in X\backslash\{0\}$ and $m\in M_0$. Thus, according to Theorem 1.1.10, for each $m\in M_0$ there exists a unique solution $G_m: X\backslash\{0\}\to Y$ of the equation

$$G_m(x)=AG_m\left(\frac{1}{a}(m+1)x)\right)+BG_m\left(-\frac{1}{b}mx\right)$$

such that

$$\|g(x)-G_m(x)\|\le\frac{\left[s\left(\frac{1}{a}(m+1)\right)+s\left(-\frac{1}{b}m\right)\right]h(x)}{1-|A|s\left(\frac{1}{a}(m+1)\right)-|B|s\left(-\frac{1}{b}m\right)},\quad x\in X\backslash\{0\}.$$

We now show that

$$\begin{aligned}
&\|\mathcal{T}_m^n g(ax+by)-A\mathcal{T}_m^n g(x)-B\mathcal{T}_m^n g(y)\|\\
&\qquad\le\left[|A|s\left(\frac{1}{a}(m+1)\right)+|B|s\left(-\frac{1}{b}m\right)\right]^n(h(x)+h(y)) \qquad (4.1.9)
\end{aligned}$$

for every $x, y \in X \setminus \{0\}$, $n \in \mathbb{N}_0$. If $n = 0$, then (4.1.9) is simply (4.1.3). So take $r \in \mathbb{N}_0$ and suppose that (4.1.9) holds for $n = r$ and $x, y \in X \setminus \{0\}$. Then we have

$$\begin{aligned}
&\|\mathcal{T}_m^{r+1} g(ax+by) - A\mathcal{T}_m^{r+1} g(x) - B\mathcal{T}_m^{r+1} g(y)\| \\
= \ & \bigg\| A\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)(ax+by)\right) + B\mathcal{T}_m^r g\left(-\frac{1}{b}m(ax+by)\right) \\
& -A^2\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)x\right) - AB\mathcal{T}_m^r g\left(-\frac{1}{b}mx\right) \\
& -BA\mathcal{T}_m^r g(\left(\frac{1}{a}(m+1)y\right) - B^2\mathcal{T}_m^r g\,(y) \bigg\| \\
\leq \ & |A| \left\| \mathcal{T}_m^r g\left(\frac{1}{a}(m+1)(ax+by)\right) - A\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)x\right) - B\mathcal{T}_m^r g(\left(\frac{1}{a}(m+1)y\right) \right\| \\
& +|B| \left\| \mathcal{T}_m^r g\left(-\frac{1}{b}m(ax+by)\right) - A\mathcal{T}_m^r g\left(-\frac{1}{b}mx\right) - B\mathcal{T}_m^r g\left(-\frac{1}{b}my\right) \right\| \\
\leq \ & |A| \left[|A| s\left(\frac{1}{a}(m+1)\right) + |B| s\left(-\frac{1}{b}m\right)\right]^n \left(\left(h(\frac{1}{a}(m+1)x\right) + h\left(\frac{1}{a}(m+1)y\right)\right) \\
& +|B| \left[|A| s\left(\frac{1}{a}(m+1)\right) + |B| s\left(-\frac{1}{b}m\right)\right]^n \left(h\left(-\frac{1}{b}mx\right) + h\left(-\frac{1}{b}my\right)\right) \\
\leq \ & \left[|A| s\left(\frac{1}{a}(m+1)\right) + |B| s\left(-\frac{1}{b}m\right)\right]^{n+1} (h(x) + h(y)).
\end{aligned}$$

Letting $n \to \infty$ in (4.1.9), we obtain that

$$G_m(ax+by) = AG_m(x) + BG_m(y), \quad x, y \in X \setminus \{0\}.$$

So, we have a sequence $\{G_m\}_{m \in M_0}$ of functions satisfying equation (4.1.4) such that

$$\|g(x) - G_m(x)\| \leq \frac{\left[s\left(\frac{1}{a}(m+1)\right) + s\left(-\frac{1}{b}m\right)\right] h(x)}{1 - |A| s\left(\frac{1}{a}(m+1)\right) - |B| s\left(-\frac{1}{b}m\right)}, \quad x \in X \setminus \{0\}.$$

It follows, with $m \to \infty$, that $g$ also satisfies (4.1.4) for $x, y \in X \setminus \{0\}$. $\square$

In 2015, Brzdęk [5] proved the following result.

**Lemma 4.1.2** ([5]). *Assume that $X$ is a linear space over a field $\mathbb{F}$, $Y$ is a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, and $g : X \to Y$ satisfies*

$$g(ax+by) = Ag(x) + Bg(y), \quad x, y \in X \setminus \{0\}.$$

*Then $g$ satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x, y \in X.$$

By using Theorem 4.1.1 and Lemma 4.1.2, we get the following result.

**Theorem 4.1.3.** *Let $X$ be a normed space over a field $\mathbb{F}$, $Y$ be a Banach space over a field $\mathbb{K}$, $a,b \in \mathbb{F}\backslash\{0\}$, $A,B \in \mathbb{K}$ and $h : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{n \in \mathbb{N} : |A|s\left(\frac{1}{a}(n+1)\right) + |B|s\left(-\frac{1}{b}n\right) < 1\right\} \text{ is an infinite set,}$$

*where $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in X\}$ for $n \in \mathbb{F}\backslash\{0\}$ such that*

$$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0. \tag{4.1.10}$$

*Suppose that $g : X \to Y$ satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leq h(x) + h(y), \quad x,y \in X\backslash\{0\}. \tag{4.1.11}$$

*Then $g$ satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x,y \in X. \tag{4.1.12}$$

**Remark 4.1.4.** If $g$ satisfies (4.1.11) with $A = B = 0$, then, by Theorem 4.1.3,

$$g(ax+by) = 0$$

for all $x,y \in X$, whence, it follows that

$$g(x) = 0$$

for $x \in X$.

By using Theorem 4.1.1, Lemma 4.1.2 and the same technique in the proof of Corollary 4.8 of Brzdęk [5], we get Corollary 4.1.5. Then, in order to avoid repetition, the details are omitted.

**Corollary 4.1.5.** *Let $X$ be a normed space over a field $\mathbb{F}$, $Y$ be a Banach space over a field $\mathbb{K}$, $a,b \in \mathbb{F}\backslash\{0\}$, $A,B \in \mathbb{K}$, $C : X \times X \to Y$ be a given mapping and $h : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{n \in \mathbb{N} : |A|s\left(\frac{1}{a}(n+1)\right) + |B|s\left(-\frac{1}{b}n\right) < 1\right\} \text{ is an infinite set,}$$

*where* $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x)$ *for all* $x \in X\}$ *for* $n \in \mathbb{F}\backslash\{0\}$ *with*

$$\lim_{n\to\infty} s(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} s(-n) = 0. \tag{4.1.13}$$

*Suppose that* $g : X \to Y$ *satisfies the following condition*

$$\|g(ax+by) - Ag(x) - Bg(y) - C(x,y)\| \le h(x) + h(y), \quad x,y \in X\backslash\{0\}$$

*and the functional equation*

$$f(ax+by) = Af(x) + Bf(y) + C(x,y), \quad x,y \in X. \tag{4.1.14}$$

*has a solution* $f_0 : X \to Y$*. Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y) + C(x,y), \quad x,y \in X.$$

Under the assumption of Corollary 4.1.5 it is easily see that in the case $A+B \ne 1$ and $C$ is a constant function $C(x,y) := C$, we get the function $f_0 : X \to Y$, which is defined by

$$f_0(x) = \frac{C}{1-A-B}, \quad x \in X,$$

satisfies the functional equation (4.1.14). Then we get the following result.

**Corollary 4.1.6.** *Let* $X$ *be a normed space over a field* $\mathbb{F}$*,* $Y$ *be a Banach space over a field* $\mathbb{K}$*,* $a,b \in \mathbb{F}\backslash\{0\}$*,* $A,B \in \mathbb{K}$ *with* $A+B \ne 1$*,* $C \in Y$ *and* $h : X \to \mathbb{R}_+$ *be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : |A| s\left(\frac{1}{a}(n+1)\right) + |B| s\left(-\frac{1}{b}n\right) < 1 \right\} \textit{ is an infinite set,}$$

*where* $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x)$ *for all* $x \in X\}$ *for* $n \in \mathbb{F}\backslash\{0\}$ *with*

$$\lim_{n\to\infty} s(n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} s(-n) = 0. \tag{4.1.15}$$

*Suppose that* $g : X \to Y$ *satisfies the following condition*

$$\|g(ax+by) - Ag(x) - Bg(y) - C\| \le h(x) + h(y), \quad x,y \in X\backslash\{0\}.$$

*Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y) + C, \quad x,y \in X.$$

Next, we prove new generalized hyperstability results for general linear equation along with the modified Brzdęk's technique in [4] Also, we show that the hyperstability results of Piszczek [11] can be derived from the following result.

**Theorem 4.1.7.** *Let $X$ and $Y$ be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a,b \in \mathbb{F}$, $A,B \in \mathbb{K}\backslash\{0\}$ and $u,v : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\} \text{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

*for $n \in \mathbb{F}\backslash\{0\}$ such that $s_1, s_2$ satisfies the following the conditions:*

*($W_1$)* $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

*($W_2$)* $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*Suppose that $g : X \to Y$ satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \le u(x)v(y), \quad x,y \in X\backslash\{0\}. \tag{4.1.16}$$

*Then $g$ satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x,y \in X\backslash\{0\}. \tag{4.1.17}$$

*Proof.* First we notice that without loss of generality we may assume that $Y$ is a Banach space, because otherwise we can replace it by its completion.

From the condition ($W_2$), we may assume that $\lim_{n\to\infty} s_2(n) = 0$. Replacing $y$ by $mx$ for $m \in \mathbb{N}$ in (4.1.16) we get

$$\|g(ax+bmx) - Ag(x) - Bg(mx)\| \le u(x)v(mx), \quad x \in X\backslash\{0\} \tag{4.1.18}$$

and

$$\left\|\frac{1}{A}g(ax+bmx) - g(x) - \frac{B}{A}g(mx)\right\| \le \left|\frac{1}{A}\right| u(x)v(mx), \quad x \in X\backslash\{0\} \tag{4.1.19}$$

We now define operator $\mathcal{T}_m : Y^{X\backslash\{0\}} \to Y^{X\backslash\{0\}}$ by

$$\mathcal{T}_m\xi(x) := \frac{1}{A}\xi((a+bm)x) - \frac{B}{A}\xi(mx), \quad x \in X\backslash\{0\}, \xi \in Y^{X\backslash\{0\}} \tag{4.1.20}$$

and put

$$\begin{aligned} \varepsilon_m(x) &:= \left|\frac{1}{A}\right| u(x)v(mx) \\ &\leq \left|\frac{1}{A}\right| s_2(m)u(x)v(x), \quad x \in X\backslash\{0\}. \end{aligned} \tag{4.1.21}$$

Then the inequality (4.1.19) takes the form

$$\|\mathcal{T}_m g(x) - g(x)\| \leq \varepsilon_m(x), \quad x \in X\backslash\{0\}.$$

For fixed natural number $m \in M_0$, the operator $\Lambda : \mathbb{R}_+^{X\backslash\{0\}} \to \mathbb{R}_+^{X\backslash\{0\}}$ which is defined by

$$\Lambda_m\eta(x) := \left|\frac{1}{A}\right|\eta((a+bm)x) + \left|\frac{B}{A}\right|\eta(mx), \quad \eta \in \mathbb{R}_+^{X\backslash\{0\}}, x \in X\backslash\{0\}$$

has the form described in (1.1.20) with $k = 2$ and $f_1(x) = (a+bm)x$, $f_2(x) = mx$, $L_1(x) = \left|\frac{1}{A}\right|$, $L_2(x) = \left|\frac{B}{A}\right|$ for all $x \in X$. Furthermore, for each $\xi, \mu \in Y^{X\backslash\{0\}}$ and $x \in X\backslash\{0\}$, we obtain that

$$\begin{aligned} \|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| &= \left\|\frac{1}{A}\xi((a+bm)x) - \frac{B}{A}\xi(mx) - \frac{1}{A}\mu((a+bm)x) + \frac{B}{A}\mu(mx)\right\| \\ &\leq \left|\frac{1}{A}\right| \|\xi((a+bm)x) - \mu((a+bm)x)\| + \left|\frac{B}{A}\right| \|\xi(mx) - \mu(mx)\| \\ &= \sum_{i=1}^{2} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\| \\ &= \sum_{i=1}^{2} L_i(x)\|(\xi - \mu)(f_i(x))\|. \end{aligned}$$

Next, we will show that for each $n \in \mathbb{N}_0$, we get

$$\Lambda_m^n\varepsilon_m(x) \leq \left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^n s_2(m)u(x)v(x), \tag{4.1.22}$$

for all $x \in X\backslash\{0\}$. It is easy to see that the inequality (4.1.22) holds for $n = 0$. By the definition of $\Lambda_m$ and $\varepsilon_m$, we can see that

$$\begin{aligned}
\Lambda_m\varepsilon_m(x) &= \left|\frac{1}{A}\right|\varepsilon_m((a+bm)x)+\left|\frac{B}{A}\right|\varepsilon_m(mx) \\
&\le \left|\frac{1}{A}\right|\cdot\left|\frac{1}{A}\right|s_2(m)u((a+bm)x)v((a+bm)x)+\left|\frac{B}{A}\right|\cdot\left|\frac{1}{A}\right|s_2(m)u(mx)v(mx) \\
&\le \left|\frac{1}{A}\right|\cdot\left|\frac{1}{A}\right|s_2(m)s_1(a+bm)s_2(a+bm)u(x)v(x) \\
&\quad +\left|\frac{B}{A}\right|\cdot\left|\frac{1}{A}\right|s_2(m)s_1(m)s_2(m)u(x)v(x) \\
&= \left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]s_2(m)u(x)v(x).
\end{aligned}$$

From the above relation, we get

$$\begin{aligned}
\Lambda_m^2\varepsilon_m(x) &= \Lambda_m(\Lambda_m\varepsilon_m(x)) \\
&= \left|\frac{1}{A}\right|\Lambda_m\varepsilon_m((a+bm)x)+\left|\frac{B}{A}\right|\Lambda_m\varepsilon_m(mx) \\
&\le \left|\frac{1}{A}\right|\left\{\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]\right. \\
&\quad \left. s_2(m)u((a+bm)x)v((a+bm)x)\right\} \\
&\quad +\left|\frac{B}{A}\right|\left\{\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)\right.\right. \\
&\quad \left.\left.+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]s_2(m)u(mx)v(mx)\right\} \\
&\le \left|\frac{1}{A}\right|\left\{\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]\right. \\
&\quad \left. s_1(m)s_2(a+bm)u(x)v(x)\right\} \\
&\quad +\left|\frac{B}{A}\right|\left\{\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\right.\right. \\
&\quad \left.\left.\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]s_1(m)s_2(m)u(x)v(x)\right\} \\
&= \left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^2 s_2(m)u(x)v(x).
\end{aligned}$$

In the same way, we have

$$\Lambda_m^n\varepsilon_m(x)=\left|\frac{1}{A}\right|\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^n s_2(m)u(x)v(x),\quad x\in X\backslash\{0\}$$

for all $n \in \mathbb{N}_0$ This yields that

$$\begin{aligned}
\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= \sum_{n=0}^{\infty} \left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^n s_2(m)u(x)v(x) \\
&= \left|\frac{1}{A}\right| \left[\frac{s_2(m)u(x)v(x)}{1-\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) - \left|\frac{B}{A}\right| s_1(m)s_2(m)}\right]
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and $m \in M_0$. By using Theorem 1.1.10, for each $m \in M_0$ there is a unique solution $G_m : X\backslash\{0\} \to Y$ of the equation

$$G_m(x) := \frac{1}{A} G_m((a+bm)x) - \frac{B}{A} G_m(mx)$$

such that

$$\|g(x) - G_m(x)\| \le \left|\frac{1}{A}\right| \left[\frac{s_2(m)u(x)v(x)}{1-\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) - \left|\frac{B}{A}\right| s_1(m)s_2(m)}\right], \quad x \in X\backslash\{0\}.$$

Here we will show that for each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\|\mathcal{T}_m^n g(ax+by) - A\mathcal{T}_m^n g(x) - B\mathcal{T}_m^n g(y)\| \\
&\qquad \le \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^n u(x)v(y) \qquad (4.1.23)
\end{aligned}$$

for every $x, y \in X\backslash\{0\}$. If $n = 0$, then (4.1.23) is simply (4.1.16). So take $r \in \mathbb{N}_0$ and suppose that (4.1.23) holds for $n = r$ and $x, y \in X\backslash\{0\}$. Then

$$
\begin{aligned}
&\|\mathcal{T}_m^{r+1}g(ax+by)-A\mathcal{T}_m^{r+1}g(x)-B\mathcal{T}_m^{r+1}g(y)\|\\
&\quad= \left\|\frac{1}{A}\mathcal{T}_m^r g\left((a+bm)(ax+by)\right)-\frac{B}{A}\mathcal{T}_m^r g\left(m(ax+by)\right)\right.\\
&\qquad -A\left(\frac{1}{A}\mathcal{T}_m^r g\left((a+bm)x\right)+\frac{B}{A}\mathcal{T}_m^r g\left(mx\right)\right)\\
&\qquad \left. -B\left(\frac{1}{A}\mathcal{T}_m^r g\left((a+bm)y\right)+\frac{B}{A}\mathcal{T}_m^r g\left(my\right)\right)\right\|\\
&\quad\le \left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^r\left|\frac{1}{A}\right|u((a+bm)x)v((a+bm)y)\\
&\qquad +\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^r\left|\frac{B}{A}\right|u(mx)v(my)\\
&\quad\le \left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^r\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)u(x)v(y)\\
&\qquad +\left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^r\left|\frac{B}{A}\right|s_1(mx)s_2(mx)u(x)v(y)\\
&\quad= \left[\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)+\left|\frac{B}{A}\right|s_1(m)s_2(m)\right]^{r+1}u(x)v(y).
\end{aligned}
$$

This shows that the inequality (4.1.23) holds for all $n\in\mathbb{N}_0$. Letting $n\to\infty$ in (4.1.23), we obtain that

$$G_m(ax+by)=AG_m(x)+BG_m(y),\quad x,y\in X\backslash\{0\}.$$

Then $\{G_m\}_{m\in M_0}$ is a sequence of functions satisfying equation

$$g(ax+by)=Ag(x)+Bg(y),\quad x,y\in X\backslash\{0\}$$

such that

$$\|g(x)-G_m(x)\|\le\left|\frac{1}{A}\right|\left[\frac{s_2(m)u(x)v(x)}{1-\left|\frac{1}{A}\right|s_1(a+bm)s_2(a+bm)-\left|\frac{B}{A}\right|s_1(m)s_2(m)}\right],\quad x\in X\backslash\{0\}.$$

It follows, with $m\to\infty$, that $g$ is a general linear equation.

On the other hand, we will assume that $\lim\limits_{n\to\infty}s_1(n)=0$. Replacing $x$ by $my$ for $m\in\mathbb{N}$ in (4.1.16) and using similar method in the first case, it follows that the same result. This completes the proof. □

Recently, Brzdęk [5] proved that if $g:X\to Y$ satisfies the general linear equation on $X\backslash\{0\}$, then $g$ satisfies the general linear equation on $X$. By using this observation and Theorem 4.1.7, we get the following result.

**Theorem 4.1.8.** *Let $X$ and $Y$ be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a,b \in \mathbb{F}$, $A,B \in \mathbb{K}\backslash\{0\}$ and $u,v : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\} \text{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

*for $n \in \mathbb{F}\backslash\{0\}$ such that $s_1, s_2$ satisfies the following the conditions:*

*($W_1$)* $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

*($W_2$)* $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*Suppose that $g : X \to Y$ satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \le u(x)v(y), \quad x,y \in X\backslash\{0\}.$$

*Then $g$ satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x,y \in X.$$

**Remark 4.1.9.** From Theorem 4.1.8, if $A = B = 0$ and $g$ satisfies (4.1.16), then

$$g(ax+by) = 0$$

for all $x,y \in X$. This implies that $g(x) = 0$ for all $x \in X$.

According to Theorem 4.1.7 and the same technique in the proof of Corollary 4.8 of Brzdęk [5], we get the following hyperstability results of inhomogeneous functional equations.

**Corollary 4.1.10.** *Let $X$ and $Y$ be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a,b \in \mathbb{F}$, $A,B \in \mathbb{K}\backslash\{0\}$, $C : X \times X \to Y$ be a given mapping and $u,v : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\} \text{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

*for* $n \in \mathbb{F}\backslash\{0\}$ *such that* $s_1, s_2$ *satisfies the following the conditions:*

($W_1$) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

($W_2$) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*Suppose that* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y) - C(x,y)\| \le u(x)v(y), \quad x,y \in X\backslash\{0\}$$

*and the functional equation*

$$f(ax+by) = Af(x) + Bf(y) + C(x,y), \quad x,y \in X. \tag{4.1.24}$$

*has a solution* $f_0 : X \to Y$. *Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y) + C(x,y), \quad x,y \in X.$$

**Corollary 4.1.11.** *Let* $X$ *and* $Y$ *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}$, $A, B \in \mathbb{K}\backslash\{0\}$, $C : X \times X \to Y$ *be a given mapping and* $u, v : X \to \mathbb{R}_+$ *be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\} \text{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

*for* $n \in \mathbb{F}\backslash\{0\}$ *such that* $s_1, s_2$ *satisfies the following the conditions:*

($W_1$) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

*($W_2$)* $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*Suppose that $g : X \to Y$ satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y) - C\| \le u(x)v(y), \quad x,y \in X\backslash\{0\}.$$

*Then $g$ satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y) + C, \quad x,y \in X.$$

*Proof.* Note that the function $f_0 : X \to Y$, which is defined by

$$f_0(x) = \frac{C}{1-A-B}, \quad x \in X,$$

satisfies the functional equation (4.1.24). By using Corollary 4.1.10, we get this result. $\square$

To the end of this subsection we give another simple application of Theorem 4.1.8.

**Corollary 4.1.12.** *Let $X$ and $Y$ be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a,b \in \mathbb{F}\backslash\{0\}$, $A,B \in \mathbb{K}\backslash\{0\}$ and $u,v : X \to \mathbb{R}_+$ be a function such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\} \text{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

*for $n \in \mathbb{F}\backslash\{0\}$ such that $s_1, s_2$ satisfies the following the conditions:*

*($W_1$)* $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

*($W_2$)* $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*Suppose that* $H : X \times X \to Y$ *is a mapping with* $H(w,z) \neq 0$ *for some* $w, z \in X$ *and it satisfies the following inequality*

$$\|H(x,y)\| \leq u(x)v(y), \quad x, y \in X \backslash \{0\}.$$

*Then the functional equation*

$$h(ax+by) = Ah(x) + Bh(y) + H(x,y), \quad x, y \in X \tag{4.1.25}$$

*has no solutions in the class of functions* $h : X \to Y$.

*Proof.* Suppose that $h : X \to Y$ is a solution to (4.1.25). Then (4.1.16) holds, and consequently, according to Theorem 4.1.8, $h$ is general linear. This implies that $H(w,z) = 0$ for all $w, z \in X$, which is a contradiction. This completes the proof. □

### 4.1.2 Some particular cases

According to Theorem 4.1.3, Corollary 4.1.5 and Corollary 4.1.6 with $h(x) := c\|x\|^p$ for all $x \in X$, where $c \geq 0$ and $p < 0$, we get the improvement of the main result of Piszczek [10] as follows.

**Corollary 4.1.13.** *Let* $X$ *be a normed space over a field* $\mathbb{F}$*,* $Y$ *be a Banach space over a field* $\mathbb{K}$*,* $a, b \in \mathbb{F}\backslash\{0\}$*,* $A, B \in \mathbb{K}$*,* $c \geq 0$*,* $p < 0$ *and* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X \backslash \{0\}. \tag{4.1.26}$$

*Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x, y \in X.$$

**Corollary 4.1.14.** *Let* $X$ *be a normed space over a field* $\mathbb{F}$*,* $Y$ *be a Banach space over a field* $\mathbb{K}$*,* $a, b \in \mathbb{F}\backslash\{0\}$*,* $A, B \in \mathbb{K}$*,* $c \geq 0$*,* $p < 0$ *and* $C : X \times X \to Y$ *be a given mapping. Suppose that* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y) - C(x,y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X \backslash \{0\}$$

*and the functional equation*

$$f(ax+by) = Af(x)+Bf(y)+C(x,y), \quad x,y \in X. \tag{4.1.27}$$

*has a solution* $f_0 : X \to Y$. *Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x)+Bg(y)+C(x,y), \quad x,y \in X.$$

**Corollary 4.1.15.** *Let* $X$ *be a normed space over a field* $\mathbb{F}$, $Y$ *be a Banach space over a field* $\mathbb{K}$, $a,b \in \mathbb{F}\backslash\{0\}$, $A,B \in \mathbb{K}$ *with* $A+B \neq 1$, $c \geq 0$, $p < 0$ *and* $C \in Y$. *Suppose that* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by)-Ag(x)-Bg(y)-C\| \leq c(\|x\|^p+\|y\|^p), \quad x,y \in X\backslash\{0\}.$$

*Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x)+Bg(y)+C, \quad x,y \in X.$$

Next, we show that hyperstability result of Piszczek (Theorem 2.1 in [11]) and hyperstability result of inhomogeneous general linear equations can be derived from our main results.

**Corollary 4.1.16** ([11])**.** *Let* $X$ *and* $Y$ *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a,b \in \mathbb{F}$, $A,B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p,q \in \mathbb{R}$ *with* $p+q < 0$ *and* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by)-Ag(x)-Bg(y)\| \leq c(\|x\|^p\|y\|^q), \quad x,y \in X\backslash\{0\}. \tag{4.1.28}$$

*Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x)+Bg(y), \quad x,y \in X.$$

*Proof.* Let $u,v : X \to \mathbb{R}_+$ be defined by

$$u(x) := s\|x\|^p \quad \text{and} \quad v(x) := r\|x\|^q,$$

where $s,t \in \mathbb{R}_+$ with $sr = c$. Now we have

$$\begin{aligned} s_1(n) &= \inf\{t \in \mathbb{R}_+ | u(nx) \leq tu(x) \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | s\|nx\|^p \leq ts\|x\|^p \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | |n|^p\|x\|^p \leq t\|x\|^p \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | |n|^p \leq t\} \\ &= |n|^p \end{aligned}$$

and

$$\begin{aligned} s_2(n) &= \inf\{t \in \mathbb{R}_+ | v(nx) \leq tv(x) \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | r\|nx\|^q \leq tr\|x\|^q \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | |n|^q\|x\|^q \leq t\|x\|^q \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | |n|^q \leq t\} \\ &= |n|^q. \end{aligned}$$

Then we obtain that

$$\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = \lim_{n\to\infty} |n|^{p+q} = 0.$$

Next, we will claim that $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0$. Since $p,q \in \mathbb{R}$ with $p+q < 0$, we get $p < 0$ or $q < 0$. If $p < 0$, we get

$$\lim_{n\to\infty} s_1(n) = \lim_{n\to\infty} |n|^p = 0.$$

On the other hand, if $q < 0$, then

$$\lim_{n\to\infty} s_2(n) = \lim_{n\to\infty} |n|^q = 0.$$

It is easy to see that $M_0$ is an infinite set. All conditions in Theorem 4.1.8 now hold. Therefore, we obtain this result. This completes the proof. $\square$

**Corollary 4.1.17.** *Let $X$ and $Y$ be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a,b \in \mathbb{F}$, $A,B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p,q \in \mathbb{R}$ with $p+q < 0$ and $C : X \times X \to Y$ be a given mapping. Suppose that $g : X \to Y$ satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y) - C(x,y)\| \leq c\|x\|^p\|y\|^q, \quad x,y \in X\backslash\{0\}.$$

*and the functional equation*

$$f(ax+by) = Af(x) + Bf(y) + C(x,y), \quad x,y \in X.$$

*has a solution* $f_0 : X \to Y$*. Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y) + C(x,y), \quad x,y \in X.$$

**Corollary 4.1.18.** *Let* $X$ *and* $Y$ *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$*, respectively,* $a,b \in \mathbb{F}$*,* $A,B \in \mathbb{K}\backslash\{0\}$*,* $c \geq 0$*,* $p,q \in \mathbb{R}$ *with* $p+q < 0$ *and* $C \in Y$*. Suppose that* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y) - C\| \leq c\|x\|^p\|y\|^q, \quad x,y \in X\backslash\{0\}.$$

*Then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y) + C, \quad x,y \in X.$$

### 4.1.3 Open problems

The following hyperstability also have been studied by Piszczek in [11].

**Theorem 4.1.19** ([11])**.** *Let* $X$ *and* $Y$ *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$*, respectively,* $a,b \in \mathbb{F}\backslash\{0\}$*,* $A,B \in \mathbb{K}\backslash\{0\}$*,* $c \geq 0$*,* $p,q \in \mathbb{R}$ *with* $p+q > 0$ *and* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p\|y\|^q), \quad x,y \in X\backslash\{0\}. \tag{4.1.29}$$

*If (*$q > 0$ *and* $|a|^{p+q} \neq |A|$*) or (*$p > 0$ *and* $|b|^{p+q} \neq |B|$*), then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x,y \in X.$$

**Theorem 4.1.20** ([11])**.** *Let* $X$ *and* $Y$ *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$*, respectively,* $a,b \in \mathbb{F}\backslash\{0\}$*,* $A,B \in \mathbb{K}\backslash\{0\}$*,* $c \geq 0$*,* $p,q > 0$ *and* $g : X \to Y$ *satisfies the following inequality*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p\|y\|^q), \quad x,y \in X. \tag{4.1.30}$$

*If* $|a|^{p+q} \neq |A|$ *or* $|b|^{p+q} \neq |B|$*, then* $g$ *satisfies the equation*

$$g(ax+by) = Ag(x) + Bg(y), \quad x,y \in X.$$

The study of the improvement of Theorems 4.1.19 and 4.1.20 along with the similar technique in this paper still open for interested mathematicians.

## 4.2 Hyperstability of Drygas functional equation

Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ is symmetric with respect to 0 (i.e., $x \in X$ implies that $-x \in X$ ) and $Y$ be a Banach space. The purpose of this work is to study two new generalized hyperstability results of the Drygas functional equation of the form

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y),$$

where $f$ maps from $X$ into $Y$ and $x,y \in X$ with $x+y, x-y \in X$. Our first main result in this section is an improvement of main results of Piszczek and Szczawińska [12] Moreover, the corresponding hyperstability results of inhomogeneous of Drygas functional equation can be derived from our main results.

### 4.2.1 Hyperstability results

In this subsection, we give two generalized hyperstability results of Drygas functional equation under the appropriate conditions of domain and codomain of unknown function. We now give the first main result in this work.

**Theorem 4.2.1.** *Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ is symmetric with respect to 0 and $Y$ be a Banach space. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and a function $h : X \to \mathbb{R}_+$ satisfying*

$$M_0 := \{n \in \mathbb{N}_{n_0} : 2s(n+1)+s(n)+s(-n)+s(2n+1) < 1\} \text{ is an infinite set},$$

*where*

$$s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in X\}$$

*and it satisfies the following condition for $n \in \mathbb{N}$:*

$$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0. \tag{4.2.1}$$

*If $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\| \leq h(x)+h(y) \tag{4.2.2}$$

*for all $x,y \in X$ with $x+y, x-y \in X$, then $f$ satisfies the equation*

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y) \tag{4.2.3}$$

*for all $x,y \in X$.*

*Proof.* Replacing $x$ by $(m+1)x$ and $y$ by $mx$ for $m \in M_0$ in (4.2.2), we get

$$\|2f((m+1)x)+f(mx)+f(-mx)-f((2m+1)x)-f(x)\| \leq h((m+1)x)+h(mx) \tag{4.2.4}$$

for all $x,y \in X$. For each $m \in M_0$, we will define operator $\mathcal{T}_m : Y^X \to Y^X$ by

$$\mathcal{T}_m\xi(x) := 2\xi((m+1)x)+\xi(mx)+\xi(-mx)-\xi((2m+1)x), \quad x \in X, \xi \in Y^X. \tag{4.2.5}$$

Further put

$$\begin{aligned} \varepsilon_m(x) &:= h((m+1)x)+h(mx) \\ &\leq [s(m+1)+s(m)]h(x), \quad x \in X. \end{aligned} \tag{4.2.6}$$

Then the inequality (4.2.4) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X.$$

For each $m \in M_0$, the operator $\Lambda_m : \mathbb{R}_+^X \to \mathbb{R}_+^X$ which is defined by

$$\Lambda_m\eta(x) := 2\eta((m+1)x)+\eta(mx)+\eta(-mx)+\eta((2m+1)x), \quad \eta \in \mathbb{R}_+^X, x \in X$$

has the form (1.1.20) with $k=4$ and $f_1(x) = (m+1)x$, $f_2(x) = mx$, $f_3(x) = -mx$, $f_4(x) = (2m+1)x$, $L_1(x) = 2$ and $L_2(x) = L_3(x) = L_4(x) = 1$ for $x \in X$. For each $\xi, \mu \in Y^X$, $x \in X$, we have

$$\begin{aligned} \|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| &= \|2\xi((m+1)x)+\xi(mx)+\xi(-mx)-\xi((2m+1)x) \\ &\quad -2\mu((m+1)x)-\mu(mx)-\mu(-mx)+\mu((2m+1)x)\| \\ &\leq 2\|(\xi-\mu)((m+1)x)\| + \|(\xi-\mu)(mx)\| \\ &\quad +\|(\xi-\mu)(-mx)\| + \|(\xi-\mu)((2m+1)x)\| \\ &= \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|. \end{aligned}$$

By using mathematical induction, we will show that for each $x \in X$ we have

$$\Lambda_m^n \varepsilon_m(x) \leq [s(m+1)+s(m)][2s(m+1)+s(m)+s(-m)+s(2m+1)]^n h(x) \quad (4.2.7)$$

for all $n \in \mathbb{N}_0$. From (4.2.6), we obtain that the inequality (4.2.7) holds for $n=0$. Next, we will assume that (4.2.7) holds for $n=k$, where $k \in \mathbb{N}_0$. Then we have

$$\begin{aligned}
\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m(\Lambda_m^k \varepsilon_m(x)) \\
&= 2\Lambda_m^k \varepsilon_m((m+1)x) + \Lambda_m^k \varepsilon_m(mx) + \Lambda_m^k \varepsilon_m(-mx) + \Lambda_m^k \varepsilon_m((2m+1)x) \\
&\leq [s(m+1)+s(m)][2s(m+1)+s(m)+s(-m)+s(2m+1)]^k \\
&\quad\ [2h((m+1)x)+h(mx)+h(-mx)+h((2m+1)x)] \\
&\leq [s(m+1)+s(m)][2s(m+1)+s(m)+s(-m)+s(2m+1)]^{k+1} h(x).
\end{aligned}$$

This shows that (4.2.7) holds for $n=k+1$. Now we can conclude that the inequality (4.2.7) holds for all $n \in \mathbb{N}_0$. From (4.2.7), we get

$$\begin{aligned}
\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&\leq \sum_{n=0}^{\infty} [s(m+1)+s(m)][2s(m+1)+s(m)+s(-m)+s(2m+1)]^n h(x) \\
&= \frac{[s(m+1)+s(m)]h(x)}{1-2s(m+1)-s(m)-s(-m)-s(2m+1)}
\end{aligned}$$

for all $x \in X$ and $m \in M_0$. Thus, according to Theorem 1.1.10, for each $m \in M_0$ there exists a unique solution $F_m : X \to Y$ of the equation

$$F_m(x) = 2F_m((m+1)x) + F_m(mx) + F_m(-mx) + F_m((2m+1)x)$$

such that

$$\|f(x) - F_m(x)\| \leq \frac{[s(m+1)+s(m)]h(x)}{1-2s(m+1)-s(m)-s(-m)-s(2m+1)}, \quad x \in X.$$

We now show that

$$\begin{aligned}
&\|\mathcal{T}_m^n f(x+y) + \mathcal{T}_m^n f(x-y) - 2\mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y) - \mathcal{T}_m^n f(-y)\| \\
&\quad \leq [2s(m+1)+s(m)+s(-m)+s(2m+1)]^n (h(x)+h(y)) \qquad (4.2.8)
\end{aligned}$$

for every $x,y \in X$ with $x+y, x-y \in X$ and $n \in \mathbb{N}_0$. If $n=0$, then (4.2.8) is simply (4.2.2). So take $r \in \mathbb{N}_0$ and suppose that (4.2.8) holds for $n=r$ and $x,y \in X$ such that $x+y, x-y \in X$. Then we have

$$\begin{aligned}
&\|\mathcal{T}_m^{r+1} f(x+y) + \mathcal{T}_m^{r+1} f(x-y) - 2\mathcal{T}_m^{r+1} f(x) - \mathcal{T}_m^{r+1} f(y) - \mathcal{T}_m^{r+1} f(-y)\| \\
= \;& \|2\mathcal{T}_m^r f((m+1)(x+y)) + \mathcal{T}_m^r f(m(x+y)) + \mathcal{T}_m^r f(-m(x+y)) - \mathcal{T}_m^r f((2m+1)(x+y)) \\
& +2\mathcal{T}_m^r f((m+1)(x-y)) + \mathcal{T}_m^r f(m(x-y)) + \mathcal{T}_m^r f(-m(x-y)) - \mathcal{T}_m^r f((2m+1)(x-y)) \\
& -2(2\mathcal{T}_m^r f((m+1)x) + \mathcal{T}_m^r f(mx) + \mathcal{T}_m^r f(-mx) - \mathcal{T}_m^r f((2m+1)x)) \\
& -2\mathcal{T}_m^r f((m+1)y) - \mathcal{T}_m^r f(my) - \mathcal{T}_m^r f(-my) + \mathcal{T}_m^r f((2m+1)y) \\
& -2\mathcal{T}_m^r f((m+1)(-y)) - \mathcal{T}_m^r f(m(-y)) - \mathcal{T}_m^r f(-m(-y)) + \mathcal{T}_m^r f((2m+1)(-y))\| \\
\leq \;& [2s(m+1) + s(m) + s(-m) + s(2m+1)]^r \\
& [2h((m+1)x) + 2h((m+1)y) + h(mx) + h(my) \\
& +h(-mx) + h(-my) + h((2m+1)x) + h((2m+1)y)] \\
= \;& [2s(m+1) + s(m) + s(-m) + s(2m+1)]^{r+1}(h(x) + h(y)).
\end{aligned}$$

Letting $n \to \infty$ in (4.2.8), we obtain that

$$F_m(x+y) + F_m(x-y) = 2F_m(x) + F_m(y) + F_m(-y)$$

for all $x,y \in X$ with $x+y, x-y \in X$. So, we have a sequence $\{F_m\}_{m \in M_0}$ of functions satisfying equation (4.2.3) such that

$$\|f(x) - F_m(x)\| \leq \frac{[s(m+1)+s(m)]h(x)}{1-2s(m+1)-s(m)-s(-m)-s(2m+1)}, \quad x \in X.$$

It follows, with $m \to \infty$, that $f$ also satisfies (4.2.3) for $x,y \in X$. □

Next, we give the second main result. The idea of the next theorem derived from [11] which Piszczek have studied hyperstability of the general linear functional equation.

**Theorem 4.2.2.** *Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ be symmetric with respect to $0$ and $Y$ be a Banach space. Assume that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and functions $u, v : X \to \mathbb{R}_+$ satisfying*

$$M_0 := \{n \in \mathbb{N}_{n_0} : 2s_1(n+1)s_2(n+1) + s_1(n)s_2(n) + s_1(-n)s_2(-n)$$

$$+s_1(2n+1)s_2(2n+1) < 1\} \textit{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \textit{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \textit{ for all } x \in X\}$$

*such that $s_1, s_2$ satisfy the following the conditions for all $n \in \mathbb{N}$:*

*($W_1$)* $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

*($W_2$)* $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*If $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\| \le u(x)v(y) \tag{4.2.9}$$

*for all $x,y \in X$ with $x+y, x-y \in X$, then $f$ satisfies the equation*

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y) \tag{4.2.10}$$

*for all $x,y \in X$.*

*Proof.* Replacing $x$ by $(m+1)x$ and $y$ by $mx$ for $m \in M_0$ in (4.2.9), we get

$$\|2f((m+1)x)+f(mx)+f(-mx)-f((2m+1)x)-f(x)\|$$

$$\le u((m+1)x)v(mx) \tag{4.2.11}$$

for all $x,y \in X$. For each $m \in M_0$, we will define operator $\mathcal{T}_m : Y^X \to Y^X$ by

$$\mathcal{T}_m\xi(x) := 2\xi((m+1)x)+\xi(mx)+\xi(-mx)-\xi((2m+1)x), \quad x \in X, \xi \in Y^X. \tag{4.2.12}$$

Further put

$$\begin{aligned} \varepsilon_m(x) &:= u((m+1)x)v(mx) \\ &\le [s_1(m+1)s_2(m)]u(x)v(x), \quad x \in X. \end{aligned} \tag{4.2.13}$$

Then the inequality (4.2.11) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X.$$

For each $m \in M_0$, the operator $\Lambda_m : \mathbb{R}_+^X \to \mathbb{R}_+^X$ which is defined by

$$\Lambda_m \eta(x) := 2\eta((m+1)x) + \eta(mx) + \eta(-mx) + \eta((2m+1)x), \quad \eta \in \mathbb{R}_+^X, x \in X$$

has the form (1.1.20) with $k = 4$ and $f_1(x) = (m+1)x$, $f_2(x) = mx$, $f_3(x) = -mx$, $f_4(x) = (2m+1)x$, $L_1(x) = 2$ and $L_2(x) = L_3(x) = L_4(x) = 1$ for $x \in X$. For each $\xi, \mu \in Y^X$, $x \in X$, we have

$$\begin{aligned} \|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x)\| &= \|2\xi((m+1)x) + \xi(mx) + \xi(-mx) - \xi((2m+1)x) \\ &\quad -2\mu((m+1)x) - \mu(mx) - \mu(-mx) + \mu((2m+1)x)\| \\ &\leq 2\|(\xi - \mu)((m+1)x)\| + \|(\xi - \mu)(mx)\| \\ &\quad +\|(\xi - \mu)(-mx)\| + \|(\xi - \mu)((2m+1)x)\| \\ &= \sum_{i=1}^{4} L_i(x)\|(\xi - \mu)(f_i(x))\|. \end{aligned}$$

By using mathematical induction, we will show that for each $x \in X$ we get

$$\begin{aligned} \Lambda_m^n \varepsilon_m(x) &\leq [2s_1(m+1)s_2(m+1) + s_1(m)s_2(m) + s_1(-m)s_2(-m) \\ &\quad +s_1(2m+1)s_2(2m+1)]^n [s_1(m+1)s_2(m)]u(x)v(x) \end{aligned} \tag{4.2.14}$$

for all $n \in \mathbb{N}_0$. From (4.2.13), we see that the inequality (4.2.14) holds for $n = 0$. Next, we will suppose that (4.2.14) holds for $n = k$, where $k \in \mathbb{N}_0$. Then we have

$$\begin{aligned} \Lambda_m^{k+1} \varepsilon_m(x) &= \Lambda_m(\Lambda_m^k \varepsilon_m(x)) \\ &= 2\Lambda_m^k \varepsilon_m((m+1)x) + \Lambda_m^k \varepsilon_m(mx) + \Lambda_m^k \varepsilon_m(-mx) + \Lambda_m^k \varepsilon_m((2m+1)x) \\ &\leq [s_1(m+1)s_2(m)][2u((m+1)x)v((m+1)x) \\ &\quad +u(mx)v(mx) + u(-mx)v(-mx) + u((2m+1)x)v((2m+1)x)] \\ &\quad [2s_1(m+1)s_2(m+1) + s_1(m)s_2(m) + s_1(-m)s_2(-m) \\ &\quad +s_1(2m+1)s_2(2m+1)]^k \\ &\leq [s_1(m+1)s_2(m)]u(x)v(x)[2s_1(m+1)s_2(m+1) + s_1(m)s_2(m) \\ &\quad +s_1(-m)s_2(-m) + s_1(2m+1)s_2(2m+1)]^{k+1}. \end{aligned}$$

This yields that (4.2.14) holds for $n = k+1$. Then we can summarize that the inequality (4.2.14) holds for all $n \in \mathbb{N}_0$. From (4.2.14), we get

$$\begin{aligned}
\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&\le [s_1(m+1)s_2(m)]u(x)v(x) \\
&\quad \sum_{n=0}^{\infty} [2s_1(m+1)s_2(m+1)+s_1(m)s_2(m)+s_1(-m)s_2(-m)+s_1(2m+1)s_2(2m+1)]^n \\
&= \frac{[s_1(m+1)s_2(m)]u(x)v(x)}{1-2s_1(m+1)s_2(m+1)-s_1(m)s_2(m)-s_1(-m)s_2(-m)-s_1(2m+1)s_2(2m+1)}
\end{aligned}$$

for all $x \in X$ and $m \in M_0$. Thus, according to Theorem 1.1.10, for each $m \in M_0$ there exists a unique solution $F_m : X \to Y$ of the equation

$$F_m(x) = 2F_m((m+1)x) + F_m(mx) + F_m(-mx) + F_m((2m+1)x)$$

such that

$$\|f(x) - F_m(x)\| \le \frac{[s(m+1)+s(m)]h(x)}{1-2s(m+1)-s(m)-s(-m)-s(2m+1)}, \quad x \in X.$$

Next, we will show that

$$\begin{aligned}
&\|\mathcal{T}_m^n f(x+y) + \mathcal{T}_m^n f(x-y) - 2\mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y) - \mathcal{T}_m^n f(-y)\| \\
&\quad \le [2s_1(m+1)s_2(m+1)+s_1(m)s_2(m)+s_1(-m)s_2(-m)+s_1(2m+1)s_2(2m+1)]^n \\
&\qquad u(x)v(y) \qquad (4.2.15)
\end{aligned}$$

for every $x, y \in X$ with $x+y, x-y \in X$ and $n \in \mathbb{N}_0$. If $n = 0$, then (4.2.15) is simply (4.2.9). Then take $r \in \mathbb{N}_0$ and assume that (4.2.15) holds for $n = r$ and $x, y \in X$ with $x+y, x-y \in X$. Then we have

$$
\begin{aligned}
&\|\mathcal{T}_m^{r+1}f(x+y)+\mathcal{T}_m^{r+1}f(x-y)-2\mathcal{T}_m^{r+1}f(x)-\mathcal{T}_m^{r+1}f(y)-\mathcal{T}_m^{r+1}f(-y)\| \\
&\quad = \|2\mathcal{T}_m^{r}f((m+1)(x+y))+\mathcal{T}_m^{r}f(m(x+y))+\mathcal{T}_m^{r}f(-m(x+y))-\mathcal{T}_m^{r}f((2m+1)(x+y)) \\
&\qquad +2\mathcal{T}_m^{r}f((m+1)(x-y))+\mathcal{T}_m^{r}f(m(x-y))+\mathcal{T}_m^{r}f(-m(x-y))-\mathcal{T}_m^{r}f((2m+1)(x-y)) \\
&\qquad -2(2\mathcal{T}_m^{r}f((m+1)x)+\mathcal{T}_m^{r}f(mx)+\mathcal{T}_m^{r}f(-mx)-\mathcal{T}_m^{r}f((2m+1)x)) \\
&\qquad -2\mathcal{T}_m^{r}f((m+1)y)-\mathcal{T}_m^{r}f(my)-\mathcal{T}_m^{r}f(-my)+\mathcal{T}_m^{r}f((2m+1)y) \\
&\qquad -2\mathcal{T}_m^{r}f((m+1)(-y))-\mathcal{T}_m^{r}f(m(-y))-\mathcal{T}_m^{r}f(-m(-y))+\mathcal{T}_m^{r}f((2m+1)(-y))\| \\
&\quad \le [2s_1(m+1)s_2(m+1)+s_1(m)s_2(m)+s_1(-m)s_2(-m)+s_1(2m+1)s_2(2m+1)]^r \\
&\qquad [2u((m+1)x)v((m+1)y)+u(mx)v(my) \\
&\qquad +u(-mx)v(-my)+u((2m+1)x)v((2m+1)y)] \\
&\quad = [2s_1(m+1)s_2(m+1)+s_1(m)s_2(m) \\
&\qquad +s_1(-m)s_2(-m)+s_1(2m+1)s_2(2m+1)]^{r+1}u(x)v(y).
\end{aligned}
$$

Letting $n\to\infty$ in (4.2.15), we obtain that

$$F_m(x+y)+F_m(x-y)=2F_m(x)+F_m(y)+F_m(-y)$$

for all $x,y\in X$ with $x+y, x-y\in X$. Then, we have a sequence $\{F_m\}_{m\in M_0}$ of functions satisfying equation (4.2.10) which

$$
\begin{aligned}
&\|f(x)-F_m(x)\| \\
&\le \frac{[s_1(m+1)s_2(m)]u(x)v(x)}{1-2s_1(m+1)s_2(m+1)-s_1(m)s_2(m)-s_1(-m)s_2(-m)-s_1(2m+1)s_2(2m+1)}
\end{aligned}
$$

for all $x\in X$. It follows, with $m\to\infty$, that $f$ also satisfies (4.2.10) for $x,y\in X$. □

By using Theorems 4.2.1, 4.2.2 and the same technique in the proof of Corollary 4.8 of Brzdęk [5], we get Corollaries 4.2.3 and 4.2.4, that is, the hyperstability results of inhomogeneous of Drygas functional equation. Then, in order to avoid repetition, the details are omitted.

**Corollary 4.2.3.** *Let $X$ be a nonempty subset of a normed space such that $0\notin X$ and $X$ be symmetric with respect to $0$, $Y$ be a Banach space and $C: X\times X\to Y$ be a given mapping. Suppose that there exist $n_0\in\mathbb{N}$ with $nx\in X$ for all $x\in X$,*

*$n \in \mathbb{N}_{n_0}$, and a function $h : X \to \mathbb{R}_+$ satisfying*

$$M_0 := \{n \in \mathbb{N}_{n_0} : 2s(n+1) + s(n) + s(-n) + s(2n+1) < 1\} \text{ is an infinite set,}$$

*where*

$$s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in X\}$$

*and it satisfies the following condition for all $n \in \mathbb{N}$:*

$$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0.$$

*If $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) - C(x,y)\| \leq h(x) + h(y)$$

*for all $x, y \in X$ with $x+y, x-y \in X$ and the functional equation*

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) + C(x,y) \tag{4.2.16}$$

*has a solution $g_0 : X \to Y$, then $f$ satisfies the equation*

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) + C(x,y)$$

*for all $x, y \in X$.*

**Corollary 4.2.4.** *Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ be symmetric with respect to 0 and $Y$ be a Banach space and $C : X \times X \to Y$ be a given mapping. Assume that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and functions $u, v : X \to \mathbb{R}_+$ satisfying*

$$M_0 := \{n \in \mathbb{N}_{n_0} : 2s_1(n+1)s_2(n+1) + s_1(n)s_2(n) + s_1(-n)s_2(-n)$$

$$+ s_1(2n+1)s_2(2n+1) < 1\} \text{ is an infinite set,}$$

*where*

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \leq tu(x) \text{ for all } x \in X\}$$

*and*

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \leq tv(x) \text{ for all } x \in X\}$$

*such that $s_1, s_2$ satisfy the following the conditions for all $n \in \mathbb{N}$:*

*($W_1$)* $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

*($W_2$)* $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

*If* $f : X \to Y$ *satisfies the following inequality*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)-C(x,y)\| \leq u(x)v(y) \tag{4.2.17}$$

*for all* $x,y \in X$ *with* $x+y, x-y \in X$ *and the functional equation*

$$g(x+y)+g(x-y) = 2g(x)+g(y)+g(-y)+C(x,y) \tag{4.2.18}$$

*has a solution* $g_0 : X \to Y$. *then* $f$ *satisfies the equation*

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y)+C(x,y) \tag{4.2.19}$$

*for all* $x,y \in X$.

### 4.2.2 Some particular cases

In this section, we show that some hyperstability results of Drygas functional equation and some hyperstability of inhomogeneous Drygas functional equation can be derived from our main results.

**Corollary 4.2.5** ([12]). *Let* $X$ *be a nonempty subset of a normed space such that* $0 \notin X$ *and* $X$ *be symmetric with respect to* $0$, $Y$ *be a Banach space,* $c \geq 0$, $p < 0$. *Suppose that there exist* $n_0 \in \mathbb{N}$ *with* $nx \in X$ *for all* $x \in X$, $n \in \mathbb{N}_{n_0}$ *and* $f : X \to Y$ *satisfies the following inequality*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\| \leq c(\|x\|^p+\|y\|^p) \tag{4.2.20}$$

*for all* $x,y \in X$ *with* $x+y, x-y \in X$. *Then* $f$ *satisfies the equation*

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y)$$

*for all* $x,y \in X$.

*Proof.* Let $h : X \to \mathbb{R}_+$ be defined by

$$h(x) := c\|x\|^p, \quad x \in X.$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned} s(n) &= \inf\{t \in \mathbb{R}_+ | h(nx) \leq th(x) \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ | c\|nx\|^p \leq tc\|x\|^p \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ || n|^p \|x\|^p \leq t\|x\|^p \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ || n|^p \leq t\} \\ &= |n|^p. \end{aligned}$$

In the same way, $s(-n) = |n|^p$ for all $n \in \mathbb{N}$. So, we have

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} |n|^p = 0 \text{ and } \lim_{n\to\infty} s(-n) = \lim_{n\to\infty} |n|^p = 0$$

for all $n \in \mathbb{N}$. Moreover, we can see that $M_0$ is an infinite set. Then all conditions in Theorem 4.2.1 hold. Therefore, we get this result. $\square$

According to Corollary 4.2.3 with Corollary 4.2.5, we get the next result.

**Corollary 4.2.6.** *Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ be symmetric with respect to $0$, $Y$ be a Banach space, $c \geq 0$, $p < 0$ and $C : X \times X \to Y$ be a given mapping. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) - C(x,y)\| \leq c(\|x\|^p + \|y\|^p)$$

*for all $x, y \in X$ with $x+y, x-y \in X$ and the functional equation*

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) + C(x,y)$$

*has a solution $g_0 : X \to Y$. Then $f$ satisfies the equation*

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) + C(x,y)$$

*for all $x, y \in X$.*

Next two corollaries can be derived from Theorem 4.2.2 and Corollary 4.2.4.

**Corollary 4.2.7.** *Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ is symmetric with respect to $0$ and $Y$ be a Banach space, $c \geq 0$, $p,q \in \mathbb{R}$ with $p+q<0$. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\| \leq c(\|x\|^p\|y\|^q) \tag{4.2.21}$$

*for all $x,y \in X$ with $x+y, x-y \in X$. Then $f$ satisfies the equation*

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y)$$

*for all $x,y \in X$.*

*Proof.* Let $u,v : X \to \mathbb{R}_+$ be defined by

$$u(x) := s\|x\|^p \quad \text{and} \quad v(x) := r\|x\|^q,$$

where $s,r \in \mathbb{R}_+$ with $sr = c$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
s_1(n) &= \inf\{t \in \mathbb{R}_+ | v(nx) \leq tv(x) \text{ for all } x \in X\} \\
&= \inf\{t \in \mathbb{R}_+ | s\|nx\|^p \leq ts\|x\|^p \text{ for all } x \in X\} \\
&= \inf\{t \in \mathbb{R}_+ | |n|^p\|x\|^p \leq t\|x\|^p \text{ for all } x \in X\} \\
&= \inf\{t \in \mathbb{R}_+ | |n|^p \leq t\} \\
&= |n|^p
\end{aligned}$$

and

$$\begin{aligned}
s_2(n) &= \inf\{t \in \mathbb{R}_+ | u(nx) \leq tu(x) \text{ for all } x \in X\} \\
&= \inf\{t \in \mathbb{R}_+ | r\|nx\|^q \leq tr\|x\|^q \text{ for all } x \in X\} \\
&= \inf\{t \in \mathbb{R}_+ | |n|^q\|x\|^q \leq t\|x\|^q \text{ for all } x \in X\} \\
&= \inf\{t \in \mathbb{R}_+ | |n|^q \leq t\} \\
&= |n|^q.
\end{aligned}$$

So, we have

$$\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = \lim_{n\to\infty} |n|^{p+q} = 0$$

for all $n \in \mathbb{N}$. Next, we will claim that $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0$ for each $n \in \mathbb{N}$. Since $p,q \in \mathbb{R}$ with $p+q < 0$, we get $p < 0$ or $q < 0$. If $p < 0$, we get

$$\lim_{n\to\infty} s_1(n) = \lim_{n\to\infty} |n|^p = 0.$$

On the other hand, if $q < 0$, then

$$\lim_{n\to\infty} s_2(n) = \lim_{n\to\infty} |n|^q = 0.$$

It is easy to see that $M_0$ is an infinite set. Then all conditions in Theorem 4.2.2 now hold. Therefore, we obtain this result. This completes the proof. □

**Corollary 4.2.8.** *Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ be symmetric with respect to 0 and $Y$ be a Banach space, $c \geq 0$, $p,q \in \mathbb{R}$ with $p+q < 0$ and $C : X \times X \to Y$ be a given mapping. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)-C(x,y)\| \leq c(\|x\|^p\|y\|^q) \quad (4.2.22)$$

*for all $x,y \in X$ with $x+y, x-y \in X$ and the functional equation*

$$g(x+y)+g(x-y) = 2g(x)+g(y)+g(-y)+C(x,y)$$

*has a solution $g_0 : X \to Y$.Then $f$ satisfies the equation*

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y)+C(x,y)$$

*for all $x,y \in X$.*

# CHAPTER 5

# CONCLUSION

The aim of this Chapter is to show all stability and hyperstability results in this thesis.

In Chapter 3, we obtain the following stability results which are generalization of several well-known stability results in the literature.

1. Let $d$ be a complete metric in $\mathbb{R}$ which is invariant (i.e., $d(x+z,y+z) = d(x,y)$ for $x,y,z \in \mathbb{R}$), and $h:\mathbb{R}\to\mathbb{R}_+$ be a function such that

$$M_0 := \{n\in\mathbb{N} : s(n^2)+s(n^2+1)<1\} \neq \emptyset,$$

where $s(n) := \inf\{t\in\mathbb{R}_+ : h(nx^2)\leq th(x^2) \text{ for all } x\in\mathbb{R}\}$ for $n\in\mathbb{N}$. Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies the following inequality

$$d\left(f\left(\sqrt{x^2+y^2}\right), f(x)+f(y)\right) \leq h(x^2)+h(y^2),$$

for all $x,y\in\mathbb{R}$. Then there exists a unique radical quadratic function $T:\mathbb{R}\to\mathbb{R}$ such that

$$d(f(x),T(x)) \leq s_0 h(x^2), \quad x\in\mathbb{R},$$

with $s_0 := \inf\left\{\dfrac{1+s(n^2)}{1-s(n^2)-s(n^2+1)} : n\in M_0\right\}$.

2. Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant (i.e., $d(x+z,y+z) = d(x,y)$ for $x,y,z\in\mathbb{R}$) and $a,b$ be two given positive integer numbers and $h:\mathbb{R}\to\mathbb{R}_+$ be a function such that

$$M_0 := \{n\in\mathbb{N} : s(n)+s(a+bn)<1\} \neq \emptyset,$$

where $s(n) := \inf\{t\in\mathbb{R}_+ : h(x^n)\leq th(x) \text{ for all } x\in\mathbb{R}\}$ for $n\in\mathbb{N}$. Suppose that

$$d(kx,ky) = d(x,y)$$

for all $x,y \in \mathbb{R}$ and for all $k \in \{a,b\}$. If $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality

$$d(f(x^a \cdot y^b), af(x)+bf(y)) \leq ah(x)+bh(y),$$

for all $x,y \in \mathbb{R}\backslash\{0\}$, then there exists a unique function $T : \mathbb{R} \to \mathbb{R}$ such that it satisfies the generalized logarithmic Cauchy functional equation (3.2.1) with respect to $a$ and $b$ for all $x,y \in \mathbb{R}\backslash\{0\}$ and

$$d(f(x),T(x)) \leq s_0 h(x), \quad x \in \mathbb{R}\backslash\{0\},$$

with $s_0 := \inf\left\{\dfrac{a+bs(n)}{1-s(n)-s(a+bn)} : n \in M_0\right\}$.

3. Let $(\mathbb{R},d)$ be a complete metric space such that $d$ is invariant (i.e., $d(x+z,y+z) = d(x,y)$ for $x,y,z \in \mathbb{R}$) and $a,b$ be two given positive integer numbers and $h : \mathbb{R} \to \mathbb{R}_+$ be a function such that

$$M_0 := \{n \in \mathbb{N} : s(n)+s(a+bn) < 1\} \neq \emptyset,$$

where $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in \mathbb{R}\}$ for $n \in \mathbb{N}$. Suppose that

$$d(kx,ky) = d(x,y)$$

for all $x,y \in \mathbb{R}$ and for all $k \in \{a,b\}$. If $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality

$$d(f(ax+by), af(x)+bf(y)) \leq ah(x)+bh(y)$$

for all $x,y \in \mathbb{R}$, then there exists a unique generalized additive Cauchy function $T : \mathbb{R} \to \mathbb{R}$ such that

$$d(f(x),T(x)) \leq s_0 h(x), \quad x \in \mathbb{R},$$

with $s_0 := \inf\left\{\dfrac{a+bs(n)}{1-s(n)-s(a+bn)} : n \in M_0\right\}$.

4. Let $(G_1,*)$ be a commutative semigroup, $(G_2,\diamond)$ be a commutative group, $(G_2,d)$ be a complete metric space such that $d$ is invariant, that is,

$$d(x\diamond z, y\diamond z) = d(x,y)$$

for all $x,y,z\in G_2$, and let $a,b$ be two fixed natural numbers and $h: G_1\to\mathbb{R}_+$ be a function such that

$$M_0 := \{n\in\mathbb{N} : s(n)+s(a+bn) < 1\} \neq \emptyset,$$

where $s(n) := \inf\{t\in\mathbb{R}_+ : h(nx)\le th(x) \text{ for all } x\in G_1\}$ for $n\in\mathbb{N}$. Suppose that

$$d(x,ay) = d(x,y)$$

and

$$d(kx,ky) = d(x,y)$$

for all $x,y\in G_2$ and for all $k\in\{a,b\}$. If $f: G_1\to G_2$ satisfies the following inequality

$$d(f(ax*by), af(x)\diamond bf(y)) \le ah(x)+bh(y)$$

for all $x,y\in G_1$, then there exists a unique generalized Cauchy function $T: G_1\to G_2$ such that

$$d(f(x),T(x)) \le s_0 h(x)$$

for all $x\in G_1$, where $s_0 := \inf\left\{\dfrac{a+bs(n)}{1-s(n)-s(a+bn)} : n\in M_0\right\}$.

In Chapter 4, we obtain the following hyperstability results which are generalization of several hyperstability results in the literature.

5. Let $X$ be a normed space over a field $\mathbb{F}$, $Y$ be a Banach space over a field $\mathbb{K}$, $a,b\in\mathbb{F}\backslash\{0\}$, $A,B\in\mathbb{K}$ and $h: X\to\mathbb{R}_+$ be a function such that

$$M_0 := \left\{n\in\mathbb{N} : |A|s\left(\frac{1}{a}(n+1)\right)+|B|s\left(-\frac{1}{b}n\right) < 1\right\} \text{ is an infinite set,}$$

where $s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x) \text{ for all } x \in X\}$ for $n \in \mathbb{F}\backslash\{0\}$ such that

$$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0.$$

Suppose that $g : X \to Y$ satisfies the following inequality

$$\|g(ax+by) - Ag(x) - Bg(y)\| \le h(x) + h(y), \quad x, y \in X\backslash\{0\}.$$

Then $g$ satisfies the equation

$$g(ax+by) = Ag(x) + Bg(y)$$

for all $x, y \in X\backslash\{0\}$.

6. Let $X$ and $Y$ be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$ and $u, v : X \to \mathbb{R}_+$ be a function such that $M_0 := \left\{n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1\right\}$ is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

for $n \in \mathbb{F}\backslash\{0\}$ such that $s_1, s_2$ satisfies the following the conditions:

$(W_1)$ $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0;$

$(W_2)$ $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0.$

Suppose that $g : X \to Y$ satisfies the following inequality

$$\|g(ax+by) - Ag(x) - Bg(y)\| \le u(x)v(y), \quad x, y \in X\backslash\{0\}.$$

Then $g$ satisfies the equation

$$g(ax+by) = Ag(x) + Bg(y)$$

for all $x, y \in X\backslash\{0\}$.

7. Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ is symmetric with respect to 0 and $Y$ be a Banach space. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and a function $h : X \to \mathbb{R}_+$ satisfying

   $M_0 := \{n \in \mathbb{N}_{n_0} : 2s(n+1) + s(n) + s(-n) + s(2n+1) < 1\}$ is an infinite set,

   where
   $$s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \leq th(x) \text{ for all } x \in X\}$$
   and it satisfies the following condition for $n \in \mathbb{N}$:
   $$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0.$$
   If $f : X \to Y$ satisfies the following inequality
   $$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq h(x) + h(y)$$
   for all $x, y \in X$ with $x+y, x-y \in X$, then $f$ satisfies the equation
   $$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$
   for all $x, y \in X$.

8. Let $X$ be a nonempty subset of a normed space such that $0 \notin X$ and $X$ be symmetric with respect to 0 and $Y$ be a Banach space. Assume that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$ and functions $u, v : X \to \mathbb{R}_+$ satisfying
   $$M_0 := \{n \in \mathbb{N}_{n_0} : 2s_1(n+1)s_2(n+1) + s_1(n)s_2(n) + s_1(-n)s_2(-n)$$
   $$+s_1(2n+1)s_2(2n+1) < 1\} \text{ is an infinite set,}$$
   where
   $$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \leq tu(x) \text{ for all } x \in X\}$$
   and
   $$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \leq tv(x) \text{ for all } x \in X\}$$
   such that $s_1, s_2$ satisfy the following the conditions for all $n \in \mathbb{N}$:

$(W_1)$ $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

$(W_2)$ $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0$.

If $f : X \to Y$ satisfies the following inequality

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\| \leq u(x)v(y)$$

for all $x, y \in X$ with $x+y, x-y \in X$, then $f$ satisfies the equation

$$f(x+y)+f(x-y) = 2f(x)+f(y)+f(-y)$$

for all $x, y \in X$.

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## Publications

1. Aiemsomboon, L. & Sintunavarat, W. (2015). TheBrzdęk's Fixed Point Method Approach to Stability of Generalized Additive Cauchy Functional Equations. In Silpakorn University, The $20^{th}$ Annual Meeting in Mathematics (AMM2015), (pp. 137–146). Nakhon Pathom.

2. Aiemsomboon, L. & Sintunavarat, W. (2015). On new stability results for generalized Cauchy functional equations on groups by using Brzdęk's fixed point theorem, *J. Fixed Point Theory Appl.*, *18*, 44–59.

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1. Aiemsomboon, L. & Sintunavarat, W. (2015). Stability of radical quadratic functional equations via fixed point methods, The ninth international conference on Nonlinear Analysis and Convex Analysis (NACA2015), 21-25 January 2015, Rimkok Resort Hotel, Chiang Rai, Thailand.

2. Aiemsomboon, L. & Sintunavarat, W. (2015). The Brzdek's fixed point method approach to stability of generalized Jensen functional equation, The

$7^{th}$ Science Confrerence, 30-31 March 2015, the faculty of science, Naresuan University, Thailand.

3. Aiemsomboon, L. & Sintunavarat, W. (2015). The Brzdęk's fixed point method approach to stability of generalized additive Cauchy functional equations, The $20^{th}$ Annual Meeting in Mathematics 2015, 27-29 May 2015, the faculty of science, Silpakorn University, Thailand.

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6. Aiemsomboon, L. & Sintunavarat, W. (2015). On the general solution and stability of the generalized logarithmic functional equations arising from fixed point theory, The $11^{th}$ IMT-GT International Conference on Mathematics, Statistics and Its Applications 2015 (ICMSA2015), 23-25 November 2015, Ambassador City Jomtien Hotel, Pattaya, Thailand.

7. Aiemsomboon, L. & Sintunavarat, W. (2016). Brzdęk's fixed point theorem approach to generalized hyperstability of the general linear equation, The $9^{th}$ Asian Conference on Fixed Point Theory and Optimization 2016 (ACFPTO2016), 18-20 May 2016, the faculty of science, King Mongkut's University of Technology.

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**Poster Presentations**

1. Aiemsomboon, L. & Sintunavarat, W. (2014). Stability of generalized additive Cauchy functional equations by using fixed point methods, The $40^{th}$ Congress on Science and Technology of Thailand (STT40), 2-4 December 2014, the faculty of science, Khon Kaen University, Thailand.