ON SEMIPOLYGROUPS
AND INVERSE SEMIPOLYGROUPS

BY

MISS SUTASINEE WANNUSIT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
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ABSTRACT

The purpose of this thesis is to examine the notions of semipolygroups and investigate their basic properties. Furthermore, we introduce the morphism on a semipolygroup and the (strongly) regular equivalence relation on a semipolygroup. Some results on equivalence relations are presented. Using the notions of semipolygroups and polygroups, we define the inverse semipolygroup. Also, we describe the properties of inverse semipolygroups and some results generalized from the inverse semigroup. We review the concepts of polygroups and some their properties are also described.

Keywords: Semipolygroups, (Strongly) Regular equivalence relations, Inverse semipolygroups, Polygroups

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Miss Sutasinee Wannusit

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CHAPTER 1

INTRODUCTION

In 1934, Marty [9] first presented the concept of the algebraic hypergroup. The class of hypergroups is generalized from group theory. In a group, the combination between elements is an element but in a hypergroup, the combination between elements is a set. The semihypergroups are an associative property of hypergroups which are based on the concept of hyperoperation.

In 1993, Corsini [2] introduced the fundamental of theory of hyperstructures, and many notion of hyperstructure can be found in his work. Other reseachers extended his idea to develop the concept of hyperstructure, for example, in 2000, Davvaz [5] proved new identities of strong regularity and fuzzy strong regularity on semihypergroups, and presented results on congruences on semihypergroups [6].

In 2012, Jafarabadi, Sarmin and Moleri introduced new kinds of hyperstructure called simple and completely simple semihypergroups, presented methods for constructing these new classes of hyperstructure and considered the regularity of semihypergroups [8].

The special subclasses of hypergroup called polygroups were studied by Comer [1]. He studied polygroups and applied hyperstructures with algebras and color schemes.

Davvaz considered the normal subpolygroups and homomorphisms between polygroups and identified the isomorphism theorems of polygroups [3]. He later discussed polygroup theory and related systems [4].

The outline of this thesis is as follows:

Chapter 2 gives a brief overview of some basic notions and results on polygroups theory related to this reseach. Some results on semipolygroups and inverse semipolygroups are presented in Chapter 3.

CHAPTER 2

BASIC NOTIONS AND PRELIMINARIES

2.1 Hyperstructures

In this chapter, we introduce some basic notions and results concerning polygroups.

Let H be a non-empty set and let $P^*(H)$ be the set of all non-empty subsets of H. A **hyperoperation** on H is a map $\circ : H \times H \to P^*(H)$ and the couple $(H, \circ)$ is called a **hypergroupoid**. If A and B are non-empty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,$$

$$x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

For all a, b, c in H, if $a \circ b$ equals the singleton set, for example, $a \circ b = \{c\}$, then we write the singleton set of c as c instead of $\{c\}$.

Definition 2.1.1. [4] A hypergroupoid $(H, \circ)$ is called a **semihypergroup** if, for all x, y, z of H, we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A hypergroupoid $(H, \circ)$ is called a **quasihypergroup** if, for all x of H, we have

$$x \circ H = H \circ x = H.$$

Definition 2.1.2. [4] A hypergroupoid $(H, \circ)$ that is both a semihypergroup and a quasihypergroup is called a **hypergroup**.

Example 2.1.3. [4] Let $H = \{a, b, c, d\}$. Let the hyperoperation $\circ$ on H be given by the following table:

$\circ$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
c	$\{c,d\}$	$\{c,d\}$	a	b
d	$\{c,d\}$	$\{c,d\}$	b	a

Then $(H,\circ)$ is a hypergroup.

A polygroup is a special case of a hypergroup.

Definition 2.1.4. [**1**] Let P be a non-empty set.
A **polygroup** is a system $\mathcal{P} = < P, \cdot, e, ^{-1} >$, where $e \in P$, $\cdot$ maps $P \times P$ into the non-empty subsets of P, i.e., $\emptyset \neq x \cdot y = \cdot(x,y) \subseteq P$ for all $x,y \in P$, and the following axioms hold for all x,y,z in P :

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(ii) $e \cdot x = x \cdot e = \{x\}$;

(iii) for each x there exists a unique $x^{-1} \in P$ such that

$$e \in x \cdot x^{-1} \text{ and } e \in x^{-1} \cdot x;$$

(iv) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

The following elementary facts about polygroups follow easily from the axioms: For all $x,y \in P$, $e \in x \cdot x^{-1} \cap x^{-1} \cdot x, e^{-1} = e, (x^{-1})^{-1} = x, (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, and $x \in y \cdot z$ implies $x^{-1} \in z^{-1} \cdot y^{-1}$.

Example 2.1.5. [**4**] Let $P = \{a,b,c,d\}$ with the following table:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	c	$\{a,b,d\}$	$\{c,d\}$
d	d	d	$\{c,d\}$	$\{a,b,c\}$

Then P is a polygroup.

Example 2.1.6. [**4**] Let $P = \{e, a, b, c, d\}$ with the following table:

$\cdot$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	b	c	d
b	b	b	$\{e,a\}$	d	c
c	c	c	d	$\{e,a\}$	b
d	d	d	c	b	$\{e,a\}$

Then P is a polygroup.

Example 2.1.7. Let G be a group. Define a system $< G, *, e, ^{-1} >$, where e is the identity of G and for all $g_1, g_2 \in G$

$$g_1 * g_2 = \{g_1 \cdot g_2\},$$

where $\cdot$ is a binary operation of G. For all $g_1, g_2, g_3 \in G$.

(i) $(g_1 * g_2) * g_3 = \{g_1 \cdot g_2\} * g_3 = \{(g_1 \cdot g_2) \cdot g_3\} = \{g_1 \cdot (g_2 \cdot g_3)\} = g_1 * \{g_2 \cdot g_3\} = g_1 * (g_2 * g_3)$.

(ii) $e * g_1 = \{e \cdot g_1\} = \{g_1\} = \{g_1 \cdot e\} = g_1 * e$.

(iii) For each $g \in G$ there exists a unique $g^{-1} \in G$ such that $e \in g * g^{-1} = \{g \cdot g^{-1}\} = \{e\}$ and $e \in g^{-1} * g = \{g^{-1} \cdot g\} = \{e\}$.

(iv) If $g_1 \in g_2 * g_3$, then $g_1 \in \{g_2 \cdot g_3\}$ and so $g_1 = g_2 g_3$.

It follows that $g_2 = g_1 g_3^{-1} \in \{g_1 \cdot g_3^{-1}\} = g_1 * g_3^{-1}$ and $g_3 = g_2^{-1} g_1 \in \{g_2^{-1} \cdot g_1\} = g_2^{-1} * g_1$.

Hence $< G, *, e, ^{-1} >$ is a polygroup.

Example 2.1.8. [**3**] *Conjugacy class polygroups.*

In dealing with a symmetry group, two symmetric operations belong to the same class if they present the same map with respect to (possibly) different coordinate systems, where one coordinate system is converted into the other by a member of the group. In the group theory, this means that the elements a, b in a symmetric group G belong to the same class if there exists a $g \in G$ such that $a = gbg^{-1}$, i.e., a and b are conjugate. The collecion of all conjugacy classes of a group G in denoted by $\overline{G}$ and the system $< \overline{G}, *, \{e\}, ^{-1} >$ is a polygroup where e is the identity of G and the product $A * B$ of conjugacy classes A and B consists of all conjugacy classes contained in the elementwise product AB.

Now, we illustrate constructions using the dihedral group D_4,

$$D_4 =< r, h | h^2 = 1 = r^4 \text{ and } rh = hr^{-1} >.$$

This group is generated by a counter-clockwise rotation r of $90°$ and a horizontal reflection h. The group consists of the following eight symmetries:

$$\{1 = r^0, r, r^2 = s, r^3 = t, h, hr = d, hr^2 = v, hr^3 = f\}.$$

The dihedral groups occur frequently in art and nature. Many of the decorative designs used on floor coverings, pottery, and buildings have one of the dihedral groups as a group of symmetry. In the case of D_4 there are five conjugacy classes: $\{1\}, \{s\}, \{r, t\}, \{d, f\}$ and $\{h, v\}$. Let us denote these classes by $C_1, \ldots, C_5$ respectively. Then, the polygroup $\overline{D_4}$ is

$*$	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2	C_2	C_1	C_3	C_4	C_5
C_3	C_3	C_3	$C_1 \cup C_2$	C_5	C_4
C_4	C_4	C_4	C_5	$C_1 \cup C_2$	C_3
C_5	C_5	C_5	C_4	C_3	$C_1 \cup C_2$

As a sample of how the table entries are calculated, consider $C_3 * C_3$. To determine this product, compute the elementwise product of the conjugacy classes

$$\{r, t\}\{r, t\} = \{r^2, rt, tr, t^2\} = \{s, rr^3 = r^4 = 1, r^3 r = r^4 = 1, r^6 = s\} = \{s, 1\} = C_1 \cup C_2.$$

Thus, $C_3 * C_3$ consists of the two conjugacy classes C_1, C_2. Furthermore, for example,

$C_2 * C_2 = \{s\}\{s\} = \{s^2\} = \{r^4\} = \{1\} = C_1,$
$C_5 * C_5 = \{h,v\}\{h,v\} = \{1, h(hr^2), (hr)^2h, (hr^2)^2\} = \{1,s\} = C_1 \cup C_2.$

Example 2.1.9. [4] *Extensions of polygroups by polygroups.*
Suppose that $\mathbb{A} =< A, \cdot, e, ^{-1} >$ and $\mathbb{B} =< B, \cdot, e, ^{-1} >$ are two polygroups with $A \cap B = \{e\}$. A new system $\mathbb{A}[\mathbb{B}] =< M, *, e, ^I >$ called the *extension of* $\mathbb{A}$ *by* $\mathbb{B}$ is formed in the following way: Set $M = A \cup B$ and let $e^I = e, x^I = x^{-1}, e * x = x * e = x$ for all $x \in M$, and for all $x, y \in M - \{e\}$, then

$$x * y = \begin{cases} x \cdot y & \text{if } x, y \in A \\ x & \text{if } x \in B, y \in A \\ y & \text{if } x \in A, y \in B \\ x \cdot y & \text{if } x, y \in B, y \neq x^{-1} \\ x \cdot y \cup A & \text{if } x, y \in B, y = x^{-1}. \end{cases}$$

In this case, $\mathbb{A}[\mathbb{B}]$ is a polygroup which is called the extension of $\mathbb{A}$ by $\mathbb{B}$.

In the last case, e occurs in both $x \cdot y$ and A. If $A = \{e, a_1, a_2, \ldots\}$ and $B = \{e, b_1, b_2, \ldots\}$, the table for $*$ in $\mathbb{A}[\mathbb{B}]$ has the form

	e	a_1	a_2	$\cdots$	b_1	b_2	$\cdots$
e	e	a_1	a_2	$\cdots$	b_1	b_2	$\cdots$
a_1	a_1	a_1a_1	a_1a_2	$\cdots$	b_1	b_2	$\cdots$
a_2	a_2	a_2a_1	a_2a_2	$\cdots$	b_1	b_2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_1	b_1	b_1	b_1	$\cdots$	$b_1 * b_1$	$b_1 * b_2$	$\cdots$
b_2	b_2	b_2	b_2	$\cdots$	$b_2 * b_1$	$b_2 * b_2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Definition 2.1.10. A polygroup P is called **commutative** if $a \cdot b = b \cdot a$ for all a, b in P.

Definition 2.1.11. [3] A non-empty subset K of a polygroup P is said to be a **subpolygroup** of P if, under the hyperoperation in P, K itself forms a polygroup.

Lemma 2.1.12. [3] *A non-empty subset K of a polygroup P is a* ***subpolygroup*** *of P if and only if*

(i) $a, b \in K$ implies $a \cdot b \subseteq K$;

(ii) $a \in K$ implies $a^{-1} \in K$.

Example 2.1.13. Let $A = \{e, a, b\}$ with the following table:

$\cdot$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a\}$

From Example 2.1.6, it is clear that A is a subpolygroup of P where $a^{-1} = a$ and $b^{-1} = b$.

Definition 2.1.14. [4] Let $< P_1, \cdot, e_1, ^{-1} >$ and $< P_2, *, e_2, ^{-I} >$ be polygroups. Let φ be a mapping from P_1 into P_2 such that $\varphi(e_1) = e_2$. Then, φ is called
(i) an **inclusion homomorphism** if

$$\varphi(a \cdot b) \subseteq \varphi(a) * \varphi(b), \quad \text{for all } a, b \in P_1;$$

(ii) a **strong homomorphism** or a **good homomorphism** if

$$\varphi(a \cdot b) = \varphi(a) * \varphi(b), \quad \text{for all } a, b \in P_1.$$

Definition 2.1.15. [4] A strong homomorphism φ is an **isomorphism** if φ is one to one correspondence. We write $P_1 \cong P_2$ if P_1 is isomorphic to P_2.

Example 2.1.16. Let $< P, \cdot, e, ^{-1} >$ and $< P, *, e, ^{-1} >$ be polygroups with the following tables:

$\cdot$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, b\}$

$*$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{e, a\}$

Define $\varphi : P \to P$ by $\varphi(x) = x * x$ for all $x \in P$. As $\varphi(e) = e * e = e, \varphi$ is an inclusion homomorphism.

Example 2.1.17. From Example 2.1.16, if we define $\varphi : P \to P$ by $\varphi(x) = x^{-1}$ for all $x \in P$, such that $\varphi(e) = e^{-1} = e$, then we also have that φ is an inclusion homomorphism.

Example 2.1.18. Let P be a commutative polygroup. Define $\varphi : P \to P$ by $\varphi(x) = x \cdot x$ for all $x \in P$. Then φ is a strong homomorphism.

2.2 Binary Relations; Equivalences

A **(binary) relation** on a set X, by which we simply mean a subset ρ of the Cartesian product $X \times X$. At this stage, it is convenient to develop the theory of relations in a some what more general and abstract way. Intuitively, we think of elements x and y for which $(x,y) \in \rho$ as being **related**, and we frequently prefer writing $x\rho y$ instead of $(x,y) \in \rho$. The empty set ϕ of $X \times X$ is included among the binary relations on X; other special relations worthy of mention are the **universal** relation $X \times X$, in which everything is related to everything else, and the **equality** relation

$$1_X = \{(x,x) : x \in X\}, \tag{2.2.1}$$

also known as the **diagonal relation** on X, in which two elements are related if and only if they are equal.

Let us denote the set of all binary relations on X by $\mathcal{B}_X$. A binary operation $\circ$ is defined on $\mathcal{B}_X$ by the rule that, for all ρ, σ in $\mathcal{B}_X$,

$$\rho \circ \sigma = \{(x,y) \in X \times X : (\exists z \in X)\ (x,z) \in \rho \text{ and } (z,y) \in \sigma\}. \tag{2.2.2}$$

It is easy to see that, for all ρ, σ, τ in $\mathcal{B}_X$,
if we suppose that $\rho \subseteq \sigma$, for

$$\begin{aligned} (x,y) \in \rho \circ \tau \quad &\Rightarrow \quad (\exists z \in X)\ (x,z) \in \rho \text{ and } (z,y) \in \tau, \\ &\Rightarrow \quad (\exists z \in X)\ (x,z) \in \sigma \text{ and } (z,y) \in \tau, \\ &\Rightarrow \quad (x,y) \in \sigma \circ \tau, \end{aligned}$$

and for

$$\begin{aligned}(x,y)\in\tau\circ\rho &\Rightarrow (\exists z\in X)\ \ (x,z)\in\tau \text{ and } (z,y)\in\rho,\\ &\Rightarrow (\exists z\in X)\ \ (x,z)\in\tau \text{ and } (z,y)\in\sigma,\\ &\Rightarrow (x,y)\in\tau\circ\sigma,\end{aligned}$$

$\rho\circ\tau\subseteq\sigma\circ\tau$ and $\tau\circ\rho\subseteq\tau\circ\sigma$.

The proof that the operation is associative as follows: for all $\rho,\sigma,\tau\in\mathcal{B}_X$,

$$(\rho\circ\sigma)\circ\tau=\rho\circ(\sigma\circ\tau),$$

for $$\begin{aligned}(x,y)\in(\rho\circ\sigma)\circ\tau &\Leftrightarrow (\exists z\in X)\ \ (x,z)\in\rho\circ\sigma \text{ and } (z,y)\in\tau,\\ &\Leftrightarrow (\exists z\in X)(\exists u\in X)\ \ (x,u)\in\rho,(u,z)\in\sigma \text{ and } (z,y)\in\tau,\\ &\Leftrightarrow (\exists u\in X)\ \ (x,u)\in\rho \text{ and } (u,y)\in\sigma\circ\tau,\\ &\Leftrightarrow (x,y)\in\rho\circ(\sigma\circ\tau).\end{aligned}$$

Whilst we shall not normally revert to simple multiplicative notation when discussing the semipolygroup $(\mathcal{B}_X,\circ)$, we shall allow ourselves to write ρ^2,ρ^3, etc., instead of $\rho\circ\rho,\rho\circ\rho\circ\rho$, etc.

For each $\rho\in\mathcal{B}_X$, we define the **domain** dom ρ by

$$\text{dom }\rho=\{x\in X:(\exists y\in X)\ \ (x,y)\in\rho\},\tag{2.2.3}$$

and the **image** im ρ by

$$\text{im }\rho=\{y\in X:(\exists x\in X)\ \ (x,y)\in\rho\}.\tag{2.2.4}$$

For each x in X and ρ in $\mathcal{B}_X$ we define a subset $\rho(x)$ of X by

$$\rho(x)=\{y\in X:(x,y)\in\rho\};\tag{2.2.5}$$

thus $\rho(x)\neq\emptyset$ if and only if $x\in$ dom ρ. If A is a subset of X, we define

$$\rho(A)=\bigcup_{a\in A}\{\rho(a)\}.\tag{2.2.6}$$

For each ρ in $\mathcal{B}_X$, we define ρ^{-1}, the **converse** of ρ, by

$$\rho^{-1}=\{(x,y)\in X\times X:(y,x)\in\rho\}.\tag{2.2.7}$$

Certainly, $\rho^{-1}\in\mathcal{B}_X$.

Proposition 2.2.1. [7] *Let $\mathcal{B}_X$ be the set of all binary relations on a set X. Then, for all ρ, σ in $\mathcal{B}_X$,*

(i) $\rho \subseteq \sigma \Rightarrow$ *dom* $\rho \subseteq$ *dom* σ *and im* $\rho \subseteq$ *im* σ;

(ii) $(\rho^{-1})^{-1} = \rho$;

(iii) $(\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1}$;

(iv) if $\rho \subseteq \sigma$, *then* $\rho^{-1} \subseteq \sigma^{-1}$;

(v) dom $(\rho^{-1}) =$ *im* ρ *and im* $(\rho^{-1}) =$ *dom* ρ;

(vi) $\rho^{-1}(x) \neq \emptyset$ *if and only if* $x \in$ *im* ρ.

Proof. Let $\rho, \sigma \in \mathcal{B}_X$.

(i) Suppose that $\rho \subseteq \sigma$ and let $x \in \text{dom } \rho$,

$$\begin{aligned} x \in \text{ dom } \rho &\Rightarrow (\exists y \in X)\ (x,y) \in \rho, \\ &\Rightarrow (\exists y \in X)\ (x,y) \in \sigma, \\ &\Rightarrow x \in \text{ dom } \sigma. \end{aligned}$$

That is, $\text{dom } \rho \subseteq \text{ dom } \sigma$.

Let $y \in \text{im } \rho$,
$$\begin{aligned} y \in \text{ im } \rho &\Rightarrow (\exists x \in X)\ (x,y) \in \rho, \\ &\Rightarrow (\exists x \in X)\ (x,y) \in \sigma, \\ &\Rightarrow y \in \text{ im } \sigma. \end{aligned}$$

So $\text{im } \rho \subseteq \text{im } \sigma$.

(ii) Since $(x,y) \in \rho \Leftrightarrow (y,x) \in \rho^{-1} \Leftrightarrow (x,y) \in (\rho^{-1})^{-1}$, we have $(\rho^{-1})^{-1} = \rho$.

(iii)
$$\begin{aligned} (x,y) \in (\rho \circ \sigma)^{-1} &\Leftrightarrow (y,x) \in \rho \circ \sigma, \\ &\Leftrightarrow (\exists z \in X)\ (y,z) \in \rho \text{ and } (z,x) \in \sigma, \\ &\Leftrightarrow (\exists z \in X)\ (z,y) \in \rho^{-1} \text{ and } (x,z) \in \sigma^{-1}, \\ &\Leftrightarrow (x,y) \in \sigma^{-1} \circ \rho^{-1}. \end{aligned}$$

(iv) Assume that $\rho \subseteq \sigma$. Then

$$\begin{aligned}(x,y) \in \rho^{-1} &\Leftrightarrow (y,x) \in \rho,\\ &\Rightarrow (y,x) \in \sigma,\\ &\Leftrightarrow (x,y) \in \sigma^{-1}.\end{aligned}$$

(v) We see that

$$\begin{aligned}x \in \text{ dom } (\rho^{-1}) &\Leftrightarrow (\exists y \in X)\ (x,y) \in \rho^{-1},\\ &\Leftrightarrow (\exists y \in X)\ (y,x) \in \rho,\\ &\Leftrightarrow x \in \text{ im } \rho.\\ \text{and } y \in \text{ im } (\rho^{-1}) &\Leftrightarrow (\exists x \in X)\ (x,y) \in \rho^{-1},\\ &\Leftrightarrow (\exists x \in X)\ (y,x) \in \rho,\\ &\Leftrightarrow y \in \text{ dom } \rho.\end{aligned}$$

(vi) Suppose that $\rho^{-1}(x) \neq \emptyset$.

Then there exists $y \in \rho^{-1}(x)$ if and only if $y \in X$ such that $(x,y) \in \rho^{-1}$.

It follows that $(y,x) \in \rho$. Therefore, $x \in \text{im } \rho$.

Conversely, if $x \in \text{im } \rho$ then $(y,x) \in \rho$ for some $y \in X$.

This implies that $(x,y) \in \rho^{-1}$ and $y \in \rho^{-1}(x)$. Thus, $\rho^{-1}(x) \neq \emptyset$. $\square$

A relation ϕ is called a **map**, or a **function**, if dom $\phi = X$. Then, a relation ϕ on X is a map if and only if $|\phi(x)| = 1$ for every x in X.
A relation ρ on a set X is

reflexive if and only if $1_X \subseteq \rho$,

symmetric if and only if $(\forall x, y \in X)\ (x,y) \in \rho \Rightarrow (y,x) \in \rho$,

anti-symmetric if and only if $\rho \cap \rho^{-1} = 1_X$, and

transitive if and only if $\rho \circ \rho \subseteq \rho$.

An **equivalence** ρ on a set X to be a relation that is reflexive, symmetric, and transitive.

By symmetric property, this property is expressed as $\rho \subseteq \rho^{-1}$. Notice that, by (iv) and (ii) it follows that $\rho^{-1} \subseteq (\rho^{-1})^{-1} = \rho$; thus the symmetry condition can equally well be expressed as $\rho^{-1} = \rho$. On the same theme, if ρ is an equivalence, then we can deduce that

$$\rho = 1_X \circ \rho \subseteq \rho \circ \rho.$$

Thus the transitivity condition can be replaced by $\rho \circ \rho = \rho$.
If ρ is an equivalence on X then, by (i) of Proposition (2.2.1),

$$\text{dom } \rho \supseteq \text{dom } 1_X = X, \text{ im } \rho \supseteq \text{im } 1_X = X,$$

hence dom $\rho =$ im $\rho = X$.
A family $\pi = \{A_i : i \in I\}$ of the subsets of a set X is said to form a **partition** of X if

(P1) each A_i is non-empty;

(P2) for all i, j in I, either $A_i = A_j$ or $A_i \cap A_j = \emptyset$;

(P3) $\bigcup\{A_i : i \in I\} = X$.

On the face of it, the notions of partition and equivalence are quite different, but in fact they are closely related.

If ρ is an equivalence on X, we shall sometimes write $x\rho y$ or $x \equiv y \pmod{\rho}$ as alternatives to $(x,y) \in \rho$. The sets $\rho(x)$ that form the partition associated with the equivalence are called **ρ-classes**, or **equivalence classes**. The set of ρ-classes, whose elements are the subsets $\rho(x)$, is called the **quotient set of X by ρ**, and is denoted by X/ρ. In the next section we shall have occasion to examine the natural map $\rho^\natural$ (read ρ natural) from X onto X/ρ by

$$\rho^\natural(x) = \rho(x) \qquad \text{for all } x \in X. \tag{2.2.8}$$

Remark *Let ρ be an equivalence on X and let $x, y \in X$. $\rho(x) = \rho(y)$ if and only if $x\rho y$.*

Proposition 2.2.2. [7] *If $\phi : X \to Y$ is a map, then $\phi \circ \phi^{-1}$ is an equivalence.*

Proof. The easiest way to see this is to note that

$$\begin{aligned}
\phi\circ\phi^{-1} &= \{(x,y)\in X\times X : (\exists z\in X)\quad (x,z)\in\phi \text{ and } (z,y)\in\phi^{-1}\}\\
&= \{(x,y)\in X\times X : (\exists z\in X)\quad (x,z)\in\phi \text{ and } (y,z)\in\phi\}\\
&= \{(x,y)\in X\times X : \phi(x)=\phi(y)\}.
\end{aligned}$$

Let $x,y\in X$.

Then, for all $(x,x)\in 1_X, (x,x)\in\phi\circ\phi^{-1}$, so it is reflextive.

Suppose that $(x,y)\in\phi\circ\phi^{-1}$.

There exists $z\in X$ such that $(x,z)\in\phi$ and $(z,y)\in\phi^{-1}$.

This implies that $(z,x)\in\phi^{-1}$ and $(y,z)\in\phi$.

Hence $(y,x)\in\phi\circ\phi^{-1}$, and thus it is symmetric.

Finally, we assume that $(x,y)\in(\phi\circ\phi^{-1})\circ(\phi\circ\phi^{-1})$.

This means that there is $z\in X$ such that $(x,z)\in\phi\circ\phi^{-1}$ and $(z,y)\in\phi\circ\phi^{-1}$.

It follows that $\phi(x)=\phi(z)$ and $\phi(z)=\phi(y)$.

Therefore, $\phi(x)=\phi(y)$ and $(x,y)\in\phi\circ\phi^{-1}$. So $\phi\circ\phi^{-1}$ is transitive. $\square$

We call the equivalence $\phi\circ\phi^{-1}$ the **kernel** of ϕ, and write $\phi\circ\phi^{-1}=\ker\phi$. Notice that $\ker\rho^{\natural}=\rho$.

In the next chapter, we will introduce the notions of semipolygroups and inverse semipolygroups that main results of this reseach.

CHAPTER 3

SEMIPOLYGROUPS AND INVERSE SEMIPOLYGROUPS

In this chapter, some properties of semipolygroups and inverse semipolygroups are described.

3.1 Semipolygroups

First, we present some basic notions and results concerning semipolygroups.

Definition 3.1.1. A **polygroupoid** is a system $(S,\circ)$ where $\circ$ is a hyperoperation, i.e., $\emptyset \neq x\circ y = \circ(x,y) \subseteq S$ for all x, y of S.

A polygroupoid $(S,\circ)$ is called a **semipolygroup** if, for all x, y, z of S, we have $(x\circ y)\circ z = x\circ(y\circ z)$, which means that

$$\bigcup_{u\in x\circ y} u\circ z = \bigcup_{v\in y\circ z} x\circ v.$$

Here, we behold that semipolygroup is a semihypergroup.
We desire to present the concernment between this property and definition of polygroup, so we called this property as semipolygroup instead of semihypergroup.

Example 3.1.2. Let $(S,\cdot)$ be a semigroup. Define system $(S,*)$ by

$$s_1 * s_2 = \{s_1\cdot s_2\} \qquad \text{for all } s_1, s_2 \in S.$$

For any $s_1, s_2, s_3 \in S, s_1 * s_2 = \{s_1\cdot s_2\} \subseteq S$ and

$$\begin{aligned}(s_1 * s_2) * s_3 &= \{s_1\cdot s_2\} * s_3 = \{(s_1\cdot s_2)\cdot s_3\} = \{s_1\cdot(s_2\cdot s_3)\} = s_1 * \{s_2\cdot s_3\} \\ &= s_1 * (s_2 * s_3).\end{aligned}$$

Therefore, $(S,*)$ is a semipolygroup.

Definition 3.1.3. If a semipolygroup $(S,\circ)$ has the property that,
for all x,y in S,

$$x\circ y=y\circ x,$$

we shall say that S is a **commutative semipolygroup**.
If a semipolygroup $(S,\circ)$ contains an element e with the property that, for all x in S,

$$e\circ x=x\circ e=\{x\},$$

we say that e is an **identity element** of S, and that S is a **semipolygroup with identity**.

A semipolygroup $(S,\circ)$ has at most one identity element, since if e' also has the property that $e'\circ x=x\circ e'=\{x\}$ for all x in S, then

$$\begin{aligned}\{e'\} &= e\circ e' && \text{(since } e \text{ is an identity)}\\ &= \{e\} && \text{(since } e' \text{ is an identity)}\end{aligned}$$

and so $e'=e$.
If a semipolygroup $(S,\circ)$ has no identity element, then it is very easy to adjoin an extra element e to S to form a semipolygroup with identity. We define a hyperoperation on S which has an adjoined element e as

$$e\circ x=x\circ e=\{x\} \text{ for all } x \text{ in } S, \text{ and } e\circ e=\{e\},$$

and it is a routine matter to check that $S\cup\{e\}$ becomes a semipolygroup with identity.
We now define

$$S^1=\begin{cases} S & \text{if } S \text{ has an identity element;}\\ S\cup\{e\} & \text{otherwise.}\end{cases}$$

We refer to S^1 as the **semipolygroup with identity obtained from S by adjoining an identity if necessary**.

Definition 3.1.4. If a semipolygroup $(S, \circ)$ with at least two elements contains an element 0 such that, for all x in S,

$$0 \circ x = \{0\},$$

we say that 0 is a **left zero element** of S, and S is called a **semipolygroup with left zero**. Similarly, 0 is called a **right zero element** of S, and S is a **semipolygroup with right zero**, if for all x in S,

$$x \circ 0 = \{0\}.$$

We say that 0 is a **zero element** (or just a **zero**) of S if it is both a left and a right zero element of S, and that S is a **semipolygroup with zero** if it is both a semipolygroup with left zero and right zero.

Again, if a semipolygroup $(S, \circ)$ has no zero, then it is easy to adjoin an extra element 0 and we define

$$0 \circ x = x \circ 0 = 0 \circ 0 = \{0\} \text{ for all } x \text{ in } S.$$

It is then a routine matter to check that associativity survives in the extended set $S \cup \{0\}$.
By analogy with the case of S^1, we define

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element;} \\ S \cup \{0\} & \text{otherwise,} \end{cases}$$

and refer to S^0 as the **semipolygroup obtained from S by adjoining a zero if necessary**.

Example 3.1.5. Let S be a non-empty set with at least two elements and let $a \in S$. If we define a hyperoperation on S by

$$x \circ y = y \circ x = \{a\} \text{ for all } x, y \text{ in } S,$$

we have $(x \circ y) \circ z = \{a\} \circ z = \{a\} = x \circ \{a\} = x \circ (y \circ z)$ for all $x, y, z \in S$. Then, S is a semipolygroup. Since $a \circ x = x \circ a = \{a\}$ for all $x \in S$, so a is a zero element of S, and hence S is a semipolygroup with zero.

Example 3.1.6. Let S be a non-empty set with at least two elements. If we define a hyperoperation on S by

$$x \circ y = \{x\} \text{ for all } x, y \text{ in } S,$$

then $(x \circ y) \circ z = \{x\} \circ z = \{x\} = x \circ \{y\} = x \circ (y \circ z)$ for all $x, y, z \in S$. Thus, S is a semipolygroup. For any $a \in S, a \circ x = \{a\}$ for all $x \in S$, so a is a left zero element of semipolygroup S and S is called a **left zero semipolygroup**.

Example 3.1.7. Let S be a non-empty set with at least two elements. If we define a hyperoperation on S by

$$x \circ y = \{y\} \text{ for all } x, y \text{ in } S,$$

we have $(x \circ y) \circ z = \{y\} \circ z = \{z\} = x \circ \{z\} = x \circ (y \circ z)$ for all $x, y, z \in S$. Therefore, S is a semipolygroup. For any $b \in S, x \circ b = \{b\}$ for all $x \in S$, so b is a right zero element of semipolygroup S and S is called a **right zero semipolygroup**.

Example 3.1.8. Define a hyperoperation on the closed interval $I = [0, 1]$ by

$$x \circ y = \{\min\{x, y\}\} \text{ for all } x, y \in I.$$

Let $x, y, z \in I$.
If $x = y = z$, then $x \circ (y \circ z) = x \circ (x \circ x) = x \circ \{x\} = \{x\} = \{x\} \circ x = (x \circ x) \circ x = (x \circ y) \circ z$.
If $x = y$ and $y \neq z$, then

$$x \circ (y \circ z) = \begin{cases} x \circ \{y\} = \{y\} = \{y\} \circ z & \text{if } y < z \\ x \circ \{z\} = \{z\} = \{y\} \circ z & \text{if } y > z \end{cases} = (x \circ y) \circ z.$$

If $x \neq y$ and $y = z$, then

$$x \circ (y \circ z) = x \circ \{y\} = \begin{cases} \{x\} = \{x\} \circ z & \text{if } x < y \\ \{y\} = \{z\} = \{x\} \circ z & \text{if } x > y \end{cases} = (x \circ y) \circ z.$$

We now assume that $x \neq y \neq z$.
If $x < y < z$, then $x \circ (y \circ z) = x \circ \{y\} = \{x\} = \{x\} \circ z = (x \circ y) \circ z$.
If $x < z < y$, then $x \circ (y \circ z) = x \circ \{z\} = \{x\} = \{x\} \circ z = (x \circ y) \circ z$.
If $y < x < z$, then $x \circ (y \circ z) = x \circ \{y\} = \{y\} = \{y\} \circ z = (x \circ y) \circ z$.
If $y < z < x$, then $x \circ (y \circ z) = x \circ \{y\} = \{y\} = \{y\} \circ z = (x \circ y) \circ z$.
If $z < x < y$, then $x \circ (y \circ z) = x \circ \{z\} = \{z\} = \{x\} \circ z = (x \circ y) \circ z$.
If $z < y < x$, then $x \circ (y \circ z) = x \circ \{z\} = \{z\} = \{x\} \circ z = (x \circ y) \circ z$.
Then I is a semipolygroup.
Since $0 \circ x = \{\min\{0, x\}\} = \{0\} = \{\min\{x, 0\}\} = x \circ 0$, so that 0 is a zero element.
We have $x \circ 1 = \{\min\{x, 1\}\} = \{x\} = \{\min\{1, x\}\} = 1 \circ x$, so that 1 is an identity element.

Example 3.1.9. Define a multiplication on the closed interval $I = [0, 1]$ by

$$x \circ y = \{\max\{x, y\}\} \text{ for all } x, y \in I.$$

It is easy to see that I is a semipolygroup.
Since $0 \circ x = \{\max\{0, x\}\} = \{x\} = \{\max\{x, 0\}\} = x \circ 0$, 0 is an identity element, and since $x \circ 1 = \{\max\{x, 1\}\} = \{1\} = \{\max\{1, x\}\} = 1 \circ x$, 1 is a zero element.

Example 3.1.10. Define a multiplication on the closed interval $I = [a, b]$, where a, b are real numbers such that $a < b$:

$$x \circ y = \{\min\{x, y\}\} \text{ for all } x, y \in I.$$

Then I is a semipolygroup with a as a zero element and b as an identity element.

Similarly, if we define $x \circ y = \{\max\{x, y\}\}$ for all $x, y \in I$, then b is a zero element and a is an identity element of semipolygroup I.

Example 3.1.11. Let $S = \{a, b, c\}$ with the following table:

$\cdot$	a	b	c
a	a	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	b	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	c

Then S is a semipolygroup.

Example 3.1.12. Let $S = \{a,b,c\}$ with the following table:

$\cdot$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	c
b	$\{a,b\}$	a	c
c	c	c	$\{a,b\}$

Then S is a semipolygroup.

Example 3.1.13. Let $(\mathbb{Z},+)$ be a group. Define a hyperoperation $\circ$ on $\mathbb{Z}$ by

$$x \circ y = \{x+y, x-y\} \qquad \text{for all } x,y \in \mathbb{Z}.$$

Let $x,y,z \in \mathbb{Z}$. We see that

$$\begin{aligned}
(x \circ y) \circ z &= \{x+y, x-y\} \circ z = \bigcup_{u \in \{x+y,x-y\}} u \circ z \\
&= \{(x+y)+z, (x+y)-z, (x-y)+z, (x-y)-z\} \\
&= \{x+(y+z), x-(y+z), x+(y-z), x-(y-z)\} \\
&= \bigcup_{v \in \{y+z,y-z\}} x \circ v = x \circ \{y+z, y-z\} = x \circ (y \circ z).
\end{aligned}$$

Therefore, $(\mathbb{Z},\circ)$ is a semipolygroup.

Example 3.1.14. Let S be a non-empty set. Define a hyperoperation $\circ$ on S by

$$x \circ y = \{x,y\} \text{ for all } x,y \text{ in } S.$$

Let $x,y,z \in S$. We obtain

$$\begin{aligned}
(x \circ y) \circ z &= \{x,y\} \circ z = \bigcup_{u \in \{x,y\}} u \circ z = (x \circ z) \cup (y \circ z) = \{x,y\} \cup \{y,z\} = \{x,y,z\} \\
&= \{x,y\} \cup \{x,z\} = (x \circ y) \cup (x \circ z) = \bigcup_{v \in \{y,z\}} x \circ v = x \circ \{y,z\} = x \circ (y \circ z).
\end{aligned}$$

So, $(S,\circ)$ is a semipolygroup.

Example 3.1.15. Define a hyperoperation $\circ$ on $\mathbb{Z}$ by

$$x \circ y = \{xy, -xy\} \text{ for all } x, y, z \in \mathbb{Z}.$$

Let $x, y, z \in \mathbb{Z}$. Then

$$\begin{aligned} (x \circ y) \circ z &= \{xy, -xy\} \circ z = \{(xy)z, -((xy)z), (-xy)z, -((-xy)z)\} = \{xyz, -xyz\} \\ &= \{x(yz), -(x(yz)), x(-yz) - (x(-yz))\} = x \circ \{yz, -yz\} = x \circ (y \circ z). \end{aligned}$$

Thus, $(\mathbb{Z}, \circ)$ is a semipolygroup.

Example 3.1.16. Define a hyperoperation $\circ$ on $\mathbb{R}$ by

$$x \circ y = \{x, -x, y, -y\} \text{ for all } x, y, z \in \mathbb{R}.$$

Let $x, y, z \in \mathbb{R}$. We have

$$\begin{aligned} (x \circ y) \circ z &= \{x, -x, y, -y\} \circ z \\ &= \{x, -x, z, -z, -x, -(-x), z, -z, y, -y, z, -z, -y, -(-y), z, -z\} \\ &= \{x, -x, y, -y, z, -z\} \\ &= \{x, -x, y, -y, x, -x, -y, -(-y), x, -x, z, -z, x, -x, -z, -(-z)\} \\ &= x \circ \{y, -y, z, -z\} = x \circ (y \circ z), \end{aligned}$$

and $(\mathbb{R}, \circ)$ is a semipolygroup.

Example 3.1.17. Let $(\mathbb{Z}_n, \cdot)$ be a group. Define a hyperoperation $\circ$ on $\mathbb{Z}_n$ by

$$[a] \circ [b] = \{[a], [b], [a] \cdot_n [b]\} \text{ for all } [a], [b] \in \mathbb{Z}_n.$$

Let $[a], [b], [c] \in \mathbb{Z}_n$. Then

$$\begin{aligned} ([a] \circ [b]) \circ [c] &= \{[a], [b], [a] \cdot_n [b]\} \circ [c] \\ &= \{[a], [b], [c], [a] \cdot_n [c], [b] \cdot_n [c], [a] \cdot_n [b], ([a] \cdot_n [b]) \cdot_n [c]\} \\ &= \{[a], [b], [c], [a] \cdot_n [b], [a] \cdot_n [c], [b] \cdot_n [c], [a] \cdot_n ([b] \cdot_n [c])\} \\ &= [a] \circ \{[b], [c], [b] \cdot_n [c]\} \\ &= [a] \circ ([b] \circ [c]), \end{aligned}$$

and thus $(\mathbb{Z}_n, \circ)$ is a semipolygroup.

If A and B are non-empty subsets of a semipolygroup S, then we write AB to mean $\bigcup\limits_{a\in A, b\in B} a\circ b$. For all non-empty subsets A, B, and C of S,

$$\begin{aligned}
(AB)C &= \bigcup_{x\in AB, c\in C} x\circ c \\
&= \bigcup_{x\in\bigcup_{a\in A,b\in B} a\circ b, c\in C} x\circ c \\
&= \bigcup_{c\in C}\left(\bigcup_{a\in A,b\in B} a\circ b\right)\circ c \\
&= \bigcup_{a\in A,b\in B,c\in C} (a\circ b)\circ c \\
&= \bigcup_{a\in A,b\in B,c\in C} a\circ (b\circ c) \\
&= \bigcup_{a\in A} a\circ\left(\bigcup_{b\in B,c\in C} b\circ c\right) \\
&= \bigcup_{a\in A, y\in\bigcup_{b\in B,c\in C} b\circ c} a\circ y \\
&= \bigcup_{a\in A, y\in BC} a\circ y \\
&= A(BC).
\end{aligned}$$

Hence, once again, notations such as ABC and $A_1A_2\cdots A_n$ are meaningful. When dealing with singleton sets we shall use the notational simplifications that are customary in algebra, writing Ab and bA rather than $A\{b\}$ and $\{b\}A$, respectively.

If a is an element of a semipolygroup S without identity, then Sa need not contain a. The following notations will be standard:

$$S^1a = Sa\cup\{a\},$$
$$aS^1 = aS\cup\{a\}.$$

For $x,y\in S$, we write the product of x,y as xy instead of $x\circ y$.

Definition 3.1.18. Let S be a semipolygroup. For $a\in S$ and $k\in\mathbb{N}$,

$$a^1 = a,$$
$$a^{k+1} = a^k a.$$

Lemma 3.1.19. *Let S be a semipolygroup. For $a \in S$ and $m, n \in \mathbb{N}$,*

(i) $a^m a^n = a^{m+n}$*;*

(ii) $(a^m)^n = a^{mn}$.

Proof. The proof is by induction on n.

(i) Let $P(n)$ be a statement $a^m a^n = a^{m+n}$ for all $m \in \mathbb{N}$.

If $n = 1$, then $a^m a^1 = a^{m+1}$ for all $m \in \mathbb{N}$. Thus, $P(1)$ is true.

Suppose that $P(k)$ is true where $k \in \mathbb{N}$. Then $a^m a^k = a^{m+k}$ for all $m \in \mathbb{N}$.

Therefore, $a^m a^{k+1} = a^m(a^k a) = (a^m a^k)a = a^{m+k}a = a^{(m+k)+1} = a^{m+(k+1)}$.

Hence, $P(k+1)$ is true and it holds that $a^m a^n = a^{m+n}$ for all $m, n \in \mathbb{N}$.

(ii) Let $P(n)$ be a statement $(a^m)^n = a^{mn}$ for all $m \in \mathbb{N}$.

If $n = 1$, then $(a^m)^1 = a^m = a^{m1}$ for all $m \in \mathbb{N}$. So $P(1)$ is true.

Suppose that $P(k)$ is true where $k \in \mathbb{N}$. Then $(a^m)^k = a^{mk}$ for all $m \in \mathbb{N}$.

Then $(a^m)^{k+1} = (a^m)^k a^m = a^{mk} a^m = a^{mk+m} = a^{m(k+1)}$.

Therefore, $P(k+1)$ is true and $(a^m)^n = a^{mn}$ for all $m, n \in \mathbb{N}$. □

Definition 3.1.20. A non-empty subset T of a semipolygroup S is called a **subsemipolygroup** if

$$xy \subseteq T \qquad \text{for all } x, y \in T.$$

Example 3.1.21. Let $T = \{a, b\}$ with the following table:

$\cdot$	a	b
a	a	$\{a, b\}$
b	$\{a, b\}$	b

From Example 3.1.11, we see that T is a subsemipolygroup of a semipolygroup S.

Definition 3.1.22. [8] An element e of a semipolygroup S is called an **idempotent** if $e \in e^2$.

Example 3.1.23. Let $S = \{a,b,c\}$ and $\circ$ be a hyperoperation on S defined by the following table:

$\circ$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	S
b	$\{a,b\}$	a	S
c	S	S	c

We see that $(S,\circ)$ is a semipolygroup and a,c are idempotents of S.
Even though a,c are idempotents of S, every element in ac need not be an idempotent element. Here, b is not an idempotent.

Definition 3.1.24. Let S be a non-empty set. The number of elements in S is denoted by $|S|$.

Definition 3.1.25. [8] The element a of a semipolygroup S is called a **scalar** if

$$|ax| = |xa| = 1 \qquad \text{for all } x \in S.$$

Example 3.1.26. Let S be a semipolygroup and $e \in S$ be a scalar idempotent, for which $e^2 = ee = \{e\}$. Then $\{e\}$ is a subsemipolygroup of S.

Definition 3.1.27. A semipolygroup S is a **rectangular band** if $aba = \{a\}$ for all a,b in S.

From Examples 3.1.6 and 3.1.7, we can see that the left zero semipolygroup and the right zero semipolygroup are rectangular bands.

Definition 3.1.28. Let I be a non-empty subset of a semipolygroup S. We say that I is a **right ideal** of S if $xs \subseteq I$ for all $s \in S, x \in I$. A **left ideal** is defined analogously. We call I is a **(two-sided) ideal** if it is both a left and a right ideal of S.

Lemma 3.1.29. *Let a,b be an elements of a semipolygroup S. Then $aS = S$ and $Sa = S$ if and only if there exists x,y in S such that $ax \supseteq \{b\}$ and $ya \supseteq \{b\}$.*

Proof. Let a, b be an elements of a semipolygroup S.

Suppose first that $aS = S$ and $Sa = S$ for all $a \in S$.

Then, $aS = S, Sa = S$, and $bS = S, Sb = S$.

So, $b \in S = aS$ implies $b \in ax$ or $\{b\} \subseteq ax$ for some $x \in S$.

We also have that $b \in S = Sa$, and so $b \in ya$ or $\{b\} \subseteq ya$ for some $y \in S$.

Conversely, assume that $\{b\} \subseteq ax$ and $\{b\} \subseteq ya$ for some $x, y \in S$.

Let $x \in aS$, where $a \in S$. Then $x \in as$ for some $s \in S$.

Since $a \in S, s \in S$, $as \subseteq S$ and thus $x \in S$. Hence $aS \subseteq S$.

Let $p \in S$. By hypothesis, $\{p\} \subseteq an$ for some $n \in S$.

Since $n \in S$, $an \subseteq aS$. This implies that $\{p\} \subseteq aS$, that is $p \in aS$.

It follows that $S \subseteq aS$, and hence $aS = S$.

Next, we will show that $Sa = S$. Let $x \in Sa$. Then $x \in sa$ for some $s \in S$.

Since $a \in S, s \in S, sa \subseteq S$ and so $x \in S$. Therefore, $Sa \subseteq S$.

Let $p \in S$. By hypothesis again, $\{p\} \subseteq ma$ for some $m \in S$.

Because $m \in S$, $ma \subseteq Sa$. This means that $\{p\} \subseteq Sa$, and $p \in Sa$.

It follows that $S \subseteq Sa$. Then $Sa = S$. $\square$

Definition 3.1.30. A map $\phi : S \to T$, where $(S, \circ)$ and $(T, *)$ are semipolygroups, is called a **morphism** (or **homomorphism**) if, for all x, y in S

$$\phi(x \circ y) = \phi(x) * \phi(y).$$

If S and T are semipolygroups with identity elements e_S and e_T, respectively, then ϕ will be called a morphism only if we have the additional property

$$\phi(e_S) = e_T.$$

We refer to S as the **domain** of ϕ and T as the **codomain**.
The **image** (or **range**) of ϕ is defined as $\{\phi(s) : s \in S\}$. If ϕ is one to one we shall call it a **monomorphism**.

Example 3.1.31. Let $\phi : S \to T$ be a map, where $(S,\circ)$ and $(T,*)$ are left zero semipolygroups. It is easy to see that, for all x, y in S, $\phi(x \circ y) = \phi(\{x\}) = \{\phi(x)\} = \phi(x) * \phi(y)$.Therefore, ϕ is a morphism.
Similarly, if $(S,\circ)$ and $(T,*)$ are right zero semipolygroups, then ϕ is a morphism.

Proposition 3.1.32. *Let $\phi : S \to T$ be a morphism, where S is a left (right) semipolygroup and T is a semipolygroup. Suppose that, for all semipolygroup U and for all morphisms $\alpha, \beta : U \to S$,*

$$\phi \circ \alpha = \phi \circ \beta \text{ implies } \alpha = \beta.$$

Then ϕ is a monomorphism.

Proof. Let $\phi : S \to T$ be a morphism, where S is a left (right) zero semipolygroup and T is a semipolygroup.

Suppose that for all semipolygroup U and for all morphisms $\alpha, \beta : U \to S$, $\phi \circ \alpha = \phi \circ \beta$ implies $\alpha = \beta$.

Assume that $\phi(x_1) = \phi(x_2)$, where $x_1, x_2 \in S$.

Let $U = \{p\}$ be a singleton set and $pp = \{p\}$. Then U is a semipolygroup.

Let $\alpha, \beta : U \to S$ be defined by $\alpha(p) = x_1$ and $\beta(p) = x_2$.

That is, $\alpha(pp) = \alpha(\{p\}) = \{\alpha(p)\} = \{x_1\} = x_1 \circ x_1 = \alpha(p) \circ \alpha(p)$ and $\beta(pp) = \beta(\{p\}) = \{\beta(p)\} = \{x_2\} = x_2 \circ x_2 = \beta(p) \circ \beta(p)$.

Thus, α, β are morphisms.

Then, $(\phi \circ \alpha)(p) = \phi(\alpha(p)) = \phi(x_1) = \phi(x_2) = \phi(\beta(p)) = (\phi \circ \beta)(p)$.

Therefore, $\phi \circ \alpha = \phi \circ \beta$ implies that $\alpha = \beta$.

Thus $x_1 = \alpha(p) = \beta(p) = x_2$, and so ϕ is a monomorphism. $\square$

Lemma 3.1.33. *Let $\phi : S \to T$ be a morphism from a semipolygroup S into a semipolygroup T. Then the following:*

(i) If e is an idempotent in S, then $\phi(e)$ is an idempotent in T;

(ii) If a is a scalar idempotent in S, then $\phi(a)$ is a scalar idempotent in T.

Proof. (i) Suppose that e is an idempotent in S. Then $e \in e^2$.

Since $e \in S$, $\phi(e) \in \phi(S) \subseteq T$.

So, we have $\phi(e) \in \phi(e^2) = \phi(ee) = \phi(e)\phi(e) = \phi(e)^2$.

Hence, $\phi(e)$ is an idempotent in T.

(ii) Suppose that a is a scalar idempotent in S. Then $a^2 = aa = \{a\}$.

Because ϕ is a map from S into T, $\phi(a) \in \phi(S) \subseteq T$.

Hence, $\phi(a)\phi(a) = \phi(aa) = \phi(\{a\}) = \{\phi(a)\}$, and so $\phi(a)$ is a scalar idempotent in T. $\square$

Lemma 3.1.34. *Let $\phi : S \to T$ be a morphism from a semipolygroup S into a semipolygroup T. Then $\phi(S)$ is a semipolygroup.*

Proof. Let $\phi(x_1), \phi(x_2), \phi(x_3) \in \phi(S)$. Then $x_1, x_2, x_3 \in S$.

First of all, $x_1x_2 \subseteq S$ because S is a semipolygroup.

Let $z \in \phi(x_1)\phi(x_2) = \phi(x_1x_2)$. Then $z = \phi(x)$ for some $x \in x_1x_2 \subseteq S$.

It holds that $z = \phi(x) \in \phi(x_1x_2) \subseteq \phi(S)$. Hence, $\phi(x_1)\phi(x_2) \subseteq \phi(S)$.

Let $y \in [\phi(x_1)\phi(x_2)]\phi(x_3) = [\phi(x_1x_2)]\phi(x_3)$.

Hence, $y \in \phi(z)\phi(x_3) = \phi(zx_3)$ for some $z \in x_1x_2 \subseteq S$.

That is, $y = \phi(m)$ for some $m \in zx_3 \subseteq (x_1x_2)x_3 = x_1(x_2x_3)$.

Thus, there exists $n \in x_2x_3$ such that $m \in x_1n$. This implies that

$$y = \phi(m) \in \phi(x_1n) = \phi(x_1)\phi(n) \subseteq \phi(x_1)\phi(x_2x_3) = \phi(x_1)[\phi(x_2)\phi(x_3)].$$

Therefore, $[\phi(x_1)\phi(x_2)]\phi(x_3) \subseteq \phi(x_1)[\phi(x_2)\phi(x_3)]$.

Similarly, let $u \in \phi(x_1)[\phi(x_2)\phi(x_3)] = \phi(x_1)[\phi(x_2x_3)]$.

It follows that $u \in \phi(x_1)\phi(k) = \phi(x_1k)$ for some $k \in x_2x_3 \subseteq S$.

It also have that $u = \phi(p)$ for some $p \in x_1k \subseteq x_1(x_2x_3) = (x_1x_2)x_3$.

Then, there exists $q \in x_1x_2$ such that $p \in qx_3$.

Consequently,

$$u = \phi(p) \in \phi(qx_3) = \phi(q)\phi(x_3) \subseteq \phi(x_1x_2)\phi(x_3) = [\phi(x_1)\phi(x_2)]\phi(x_3).$$

It concludes that $\phi(S)$ is a semipolygroup. □

Theorem 3.1.35. *Let $\phi : S \to T$ be a morphism from a rectangular band S into a semipolygroup T. Then $\phi(S)$ is a rectangular band.*

Proof. Let $\phi(s_1), \phi(s_2) \in \phi(S)$. Then $s_1, s_2 \in S$.

By rectangular band property, $s_1s_2s_1 = \{s_1\}$.

By Lemma 3.1.34, $\phi(S)$ is a semipolygroup.

Now, we have $\phi(s_1)\phi(s_2)\phi(s_1) = \phi(s_1s_2s_1) = \phi(\{s_1\}) = \{\phi(s_1)\}$, as required. □

Lemma 3.1.36. *Let $\phi : S \to T$ be a morphism from a semipolygroup with identity S into a semipolygroup T. If e is an identity element of S, then $\phi(e)$ is an identity element of $\phi(S)$.*

Proof. Suppose that e is an identity element in S.

Then, for all $x \in S, ex = xe = \{x\}$.

Let $\phi(x) \in \phi(S)$. Then $x \in S$. By Lemma 3.1.34, $\phi(S)$ is a semipolygroup.

By the morphism property, it is obvious to verify that

$$\phi(e)\phi(x) = \phi(ex) = \phi(\{x\}) = \{\phi(x)\} = \phi(\{x\}) = \phi(xe) = \phi(x)\phi(e).$$

Consequently, $\phi(e)$ is an identity element of $\phi(S)$. □

Lemma 3.1.37. *Let $\phi : S \to T$ be a morphism from a semipolygroup with left zero S into a semipolygroup T. If 0 is a left zero element of S, then $\phi(0)$ is a left zero element of $\phi(S)$.*

Proof. Assume that 0 is a left zero element S. Then, for all $x \in S, 0x = \{0\}$.

By Lemma 3.1.34, $\phi(S)$ is a semipolygroup.

Let $\phi(x) \in \phi(S)$. Then $x \in S$ and $\phi(0)\phi(x) = \phi(0x) = \phi(\{0\}) = \{\phi(0)\}$.

Hence, $\phi(0)$ is a left zero element of $\phi(S)$. □

Similarly, we now obtain the next Lemma:

Lemma 3.1.38. *Let $\phi : S \to T$ be a morphism from a semipolygroup with right zero S into a semipolygroup T. If 0 is a right zero element of S, then $\phi(0)$ is a right zero element of $\phi(S)$.*

By Lemma 3.1.37 and Lemma 3.1.38, we have proved the following Lemma:

Lemma 3.1.39. *Let $\phi : S \to T$ be a morphism from a semipolygroup with zero S into a semipolygroup T. If 0 is a zero of S, then $\phi(0)$ is a zero of $\phi(S)$.*

Theorem 3.1.40. *Let $\phi : S \to T$ be a morphism from a commutative semipolygroup S into a semipolygroup T. Then $\phi(S)$ is a commutative semipolygroup.*

Proof. Let $\phi(s_1), \phi(s_2) \in \phi(S)$. Then $s_1, s_2 \in S$.

By Lemma 3.1.34, $\phi(S)$ is a semipolygroup.

It obtains $\phi(s_1)\phi(s_2) = \phi(s_1 s_2) = \phi(s_2 s_1) = \phi(s_2)\phi(s_1)$, and thus

$\phi(S)$ is a commutative semipolygroup. □

Definition 3.1.41. A morphism $\phi : S \to T$ is called an **isomorphism** if it is invertible, that is to say, if there exists a morphism $\phi^{-1} : T \to S$ such that $\phi^{-1} \circ \phi$ is the identity map of S and $\phi \circ \phi^{-1}$ is the identity map of T.
If there exists an isomorphism $\phi : S \to T$ we say that S and T are **isomorphic**, and write $S \simeq T$.
A morphism ϕ from S into S is called an **endomorphism** of S, and if it is one to one correspondence it is called an **automorphism**.

Proposition 3.1.42. [8] *If $(S, \cdot)$ and $(T, \circ)$ are semipolygroups, then the cartesian product $S \times T$ becomes a semipolygroup if we define*

$$(s,t) \diamond (s',t') = (s \cdot s') \times (t \circ t') = \bigcup_{x \in s \cdot s', y \in t \circ t'} \{(x,y)\}.$$

We refer to this semipolygroup as the ***direct product*** *of* S *and* T.

Proof. Define $\diamond : (S \times T) \times (S \times T) \to S \times T$ as follows:

$$(s,t) \diamond (s',t') = \bigcup_{x \in s \cdot s', y \in t \circ t'} \{(x,y)\} \text{ for all } (s,t),(s',t') \in S \times T.$$

Since $s, s' \in S$ and $t, t' \in T$, $s \cdot s' \subseteq S$ and $t \circ t' \subseteq T$.

Hence, $(s \cdot s') \times (t \circ t') = \bigcup_{x \in s \cdot s', y \in t \circ t'} \{(x,y)\} = (s,t) \diamond (s',t') \subseteq S \times T$.

Suppose that $(s,t) = (a,b)$ and $(s',t') = (a',b')$.

$$\begin{aligned} (s,t) \diamond (s',t') &= \bigcup_{x \in s \cdot s', y \in t \circ t'} \{(x,y)\} \\ &= \bigcup_{x \in a \cdot a', y \in b \circ b'} \{(x,y)\} \\ &= (a,b) \diamond (a',b'). \end{aligned}$$

This implies that $\diamond$ is well-defined.

Let $(s_1,t_1),(s_2,t_2),(s_3,t_3) \in S \times T$.

$$\begin{aligned} \Big[(s_1,t_1) \diamond (s_2,t_2)\Big] \diamond (s_3,t_3) &= \left[\bigcup_{x \in s_1 \cdot s_2, y \in t_1 \circ t_2} \{(x,y)\}\right] \diamond (s_3,t_3) \\ &= \bigcup_{x \in s_1 \cdot s_2, y \in t_1 \circ t_2} (x,y) \diamond (s_3,t_3) \\ &= \bigcup_{x \in s_1 \cdot s_2, y \in t_1 \circ t_2} \left[\bigcup_{u \in x \cdot s_3, v \in y \circ t_3} \{(u,v)\}\right] \\ &= \bigcup_{u \in (s_1 \cdot s_2) \cdot s_3, v \in (t_1 \circ t_2) \circ t_3} \{(u,v)\} \\ &= \bigcup_{u \in s_1 \cdot (s_2 \cdot s_3), v \in t_1 \circ (t_2 \circ t_3)} \{(u,v)\} \\ &= (s_1,t_1) \diamond \bigcup_{p \in s_2 \cdot s_3, q \in t_2 \circ t_3} \{(p,q)\} \\ &= (s_1,t_1) \diamond \Big[(s_2,t_2) \diamond (s_3,t_3)\Big]. \end{aligned}$$

Hence, $S \times T$ is a semipolygroup. $\square$

Example 3.1.43. Let S be a semipolygroup with at least two elements. If A is a left zero semipolygroup of S and B is a right zero semipolygroup of S, then $A\times B$ is a semipolygroup whose hyperoperation is given by $(a_1,b_1)(a_2,b_2) = \bigcup_{x\in a_1a_2, y\in b_1b_2} \{(x,y)\} = \{(a_1,b_2)\}$.

Theorem 3.1.44. *Let S be a semipolygroup. Then the following conditions are equivalent:*

(i) S is a rectangular band;

(ii) every element of S is a scalar idempotent, and $abc = ac$ for all a,b,c in S;

(iii) there exists a left zero semipolygroup L and a right zero semipolygroup R such that $S \simeq L\times R$;

(iv) S is isomorphic to a semipolygroup of the form $A\times B$, where A and B are non-empty sets, and where hyperoperation is given by $(a_1,b_1)(a_2,b_2) = \{(a_1,b_2)\}$.

Proof. (i)⇒(ii). Let $a\in S$.

Then, by (i), we get $aaa = \{a\}$, and so $a^4 = a^3a = \{a\}a = a^2$.

Again, by (i), we have $\{a\} = aa^2a = a^4$. Hence, $a^2 = \{a\}$ as required.

Now, let $a,b,c\in S$. From (i), we have $\{a\} = aba, \{c\} = cbc$, and $\{b\} = b(ac)b$.

Hence, $ac = (aba)(cbc) = a(bacb)c = a\{b\}c = abc$, as required.

(ii)⇒ (iii). Choose and fix an element c of S. Let $L = Sc$ and $R = cS$.

Then, using (ii), we see that, for all $\{x\} = zc$ and $\{y\} = tc$ in L,

$$xy = (zc)(tc) = z(ctc) = zcc = z\{c\} = zc = \{x\},$$

and so L is a left zero semipolygroup.

Similarly, R is a right zero semipolygroup.

Define $\phi : S\to L\times R$ by $\phi(x) = \bigcup_{a\in xc, b\in cx} \{(a,b)\}$ for all $x\in S$.

Suppose that $\bigcup_{a\in xc, b\in cx} \{(a,b)\} = \bigcup_{m\in yc, n\in cy} \{(m,n)\}$, then

$$\{x\} = x^2 = xcx \text{ by (ii) } = ycx = ycy = yy = y^2 = \{y\},$$

that is $x = y$, and so ϕ is one-to-one.

Also, ϕ is onto, since for all $\bigcup\limits_{a\in xc, b\in cy} \{(a,b)\}$ in $L \times R$, we see that

$$\bigcup_{a\in xc, b\in cy} \{(a,b)\} = \bigcup_{a\in xyc, b\in cxy} \{(a,b)\} = \phi(xy).$$

Finally, ϕ is a morphism, since , for all $x, y \in S$,

$$\begin{aligned}
\phi(x)\phi(y) &= \bigcup_{a\in xc, b\in cx} \{(a,b)\} \bigcup_{m\in yc, n\in cy} \{(m,n)\} \\
&= \bigcup_{p\in (xc)(yc), q\in (cx)(cy)} \{(p,q)\} = \bigcup_{p\in xcc, q\in ccy} \{(p,q)\} \\
&= \bigcup_{p\in xc, q\in cy} \{(p,q)\} = \bigcup_{p\in xyc, q\in cxy} \{(p,q)\} \\
&= \phi(xy).
\end{aligned}$$

(iii)$\Rightarrow$ (iv). Suppose that $S \simeq L \times R$, where L is a left zero semipolygroup and R is a right zero semipolygroup.

Then, the product of two elements (a,b) and (c,d) in S is given by

$$(a,b)(c,d) = \bigcup_{x\in ac, y\in bd} \{(x,y)\} = \bigcup_{x\in\{a\}, y\in\{d\}} \{(x,y)\} = \{(a,d)\}.$$

Thus, we take $A = L$ and $B = R$ as required.

(iv)$\Rightarrow$ (i). Let $A \times B \simeq S$, with the given hyperoperation.

Let $a, b, c \in S$. Then, we have $a = \phi(x,y), b = \phi(p,q)$, and $c = \phi(m,n)$ for some $(a,b), (p,q), (m,n) \in A \times B$,

$a^2 = \phi((x,y))\phi((x,y)) = \phi((x,y)(x,y)) = \phi(\{(x,y)\}) = \{\phi((x,y))\} = \{a\}$.

Therefore, every element of S is a scalar idempotent, and we also have that

$$\begin{aligned}
aba &= \phi((x,y))\phi((p,q))\phi((x,y)) = \phi((x,y)(p,q))\phi((x,y)) \\
&= \phi(\{(x,q)\})\phi((x,y)) = \phi(\{(x,y)\}) = \{\phi((x,y))\} = \{a\}.
\end{aligned}$$

$\square$

3.2 Regular and Strongly Regular Relations

Let $(S,\circ)$ be a semipolygroup and $\rho \subseteq S\times S$ be an equivalence relation, we set

$$\begin{aligned} A\bar{\rho}B &\Leftrightarrow \forall a\in A\ \exists b\in B, a\rho b \text{ and } \forall b'\in B\ \exists a'\in A, a'\rho b',\\ A\bar{\bar{\rho}}B &\Leftrightarrow a\rho b,\ \forall a\in A\ \ \forall b\in B, \end{aligned}$$

where A and B are non-empty subsets of S. We see that if $A\bar{\bar{\rho}}B$, then $A\bar{\rho}B$.

Definition 3.2.1. [2] The equivalence relation ρ on S is called
(i) **regular on the left (on the right)** if, for all x, y, a of S,

$$x\rho y \Rightarrow (ax)\bar{\rho}(ay)\ \ ((xa)\bar{\rho}(ya), \text{ respectively});$$

(ii) **strongly regular on the left (on the right)** if, for all x, y, a of S,

$$x\rho y \Rightarrow (ax)\bar{\bar{\rho}}(ay)\ \ ((xa)\bar{\bar{\rho}}(ya), \text{ respectively});$$

(iii) ρ is called **regular (strongly regular)** if it is regular (strongly regular) on the left and on the right.

Remark *If ρ is a strongly regular relation, then ρ is a regular relation.*

Theorem 3.2.2. [4] *If S is a semipolygroup and ρ is a regular relation on S, then the quotient set S/ρ is a semipolygroup with respect to the following hyperoperation:*

$$\rho(x)\otimes\rho(y) = \{\rho(z) : z\in xy\}.$$

Proof. First, we will show that the hyperoperation $\otimes$ is well-defined on S/ρ.

Let $x, y, x', y' \in S$ be such that $\rho(x) = \rho(x')$ and $\rho(y) = \rho(y')$.

This implies that $x\rho x'$ and $y\rho y'$.

It follows that $(xy)\bar{\rho}(x'y)$ and $(x'y)\bar{\rho}(x'y')$, because ρ is regular.

By transitivity, $(xy)\bar{\rho}(x'y')$.

Then, for all $z \in xy$ there exists $z' \in x'y'$ such that $z\rho z'$.

This means that $\rho(z) = \rho(z')$. It follows that $\rho(x) \otimes \rho(y) \subseteq \rho(x') \otimes \rho(y')$.

For all $w' \in x'y'$ there exists $w \in xy, w\rho w'$ and so $\rho(w') = \rho(w)$.

Then, $\rho(x') \otimes \rho(y') \subseteq \rho(x) \otimes \rho(y)$. Therefore, $\rho(x) \otimes \rho(y) = \rho(x') \otimes \rho(y')$.

Now, we check the associativity of $\otimes$.

Let $\rho(x), \rho(y), \rho(z)$ be arbitrary elements in S/ρ.

Assume that

$\rho(u) \in (\rho(x) \otimes \rho(y)) \otimes \rho(z) = \{\rho(a) : a \in xy\} \otimes \rho(z) = \bigcup_{a \in xy} \{\rho(a) \otimes \rho(z)\}$.

This means that there exists $\rho(a) \in \rho(x) \otimes \rho(y)$ such that

$$\rho(u) \in \rho(a) \otimes \rho(z) = \{\rho(b) : b \in az\}.$$

In other words, there exist $a_1 \in xy$ and $u_1 \in az$ such that $\rho(a) = \rho(a_1)$ and $\rho(u) = \rho(u_1)$, that is, $a\rho a_1$ and $u\rho u_1$.

It follows that $(az)\bar{\rho}(a_1 z)$ and so there exists $u_2 \in a_1 z \subseteq (xy)z = x(yz)$ such that $u_1 \rho u_2$.

From here, we hold that there exists $u_3 \in yz$ such that $u_2 \in xu_3$.

We have $\rho(u) = \rho(u_1) = \rho(u_2) \in \rho(x) \otimes \rho(u_3) \subseteq \rho(x) \otimes (\rho(y) \otimes \rho(z))$.

It follows that $(\rho(x) \otimes \rho(y)) \otimes \rho(z) \subseteq \rho(x) \otimes (\rho(y) \otimes \rho(z))$.

In a similar way, we can obtain the converse inclusion. □

Theorem 3.2.3. [4] *Let S be a semipolygroup and ρ be an equivalence relation on S. If the hyperoperation defined by Theorem 3.2.2 is well-defined on S/ρ, then ρ is regular.*

Proof. Let $x\rho y$ and a be an arbitrary element of S. Then $\rho(x) = \rho(y)$.

If $u \in xa$, then $\rho(u) \in \rho(x) \otimes \rho(a) = \rho(y) \otimes \rho(a) = \{\rho(b) : b \in ya\}$.

Hence, there exists $v \in ya$ such that $\rho(u) = \rho(v)$ or $u\rho v$.

If $v' \in ya$, then $\rho(v') \in \rho(y) \otimes \rho(a) = \rho(x) \otimes \rho(a) = \{\rho(c) : c \in xa\}$.

Thus, there exists $u' \in xa$ such that $\rho(v') = \rho(u')$, i.e., $u'\rho v'$. Hence $(xa)\bar{\rho}(ya)$.

Similarly, we obtain that ρ is regular on the left. $\square$

Theorem 3.2.4. **[6]** *Let S be a semipolygroup, and let ρ be a regular on S. Then S/ρ is a semipolygroup with respect to the hyperoperation defined by Theorem 3.2.2 and the map $\rho^\natural$ from S onto S/ρ given by (2.2.8) is a morphism.*
Now, let T be a semipolygroup and let $\phi : S \to T$ be a morphism. Then the relation

$$\ker\phi = \phi \circ \phi^{-1} = \{(a,b) \in S \times S : \phi(a) = \phi(b)\}$$

is regular on S, and there is a monomorphism $\alpha : S/\ker\phi \to T$ such that $\mathrm{im}\ \alpha =$ $\mathrm{im}\ \phi$ *and the diagram*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "(\ker\phi)^\natural"'] & T \\ S/\ker\phi \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

Proof. Let S be a semipolygroup, and let ρ be regular on S.

The natural map $\rho^\natural$ from S onto S/ρ is defined by

$$\rho^\natural(x) = \rho(x) \qquad \text{for all } x \in S.$$

Let $x, y \in S$. We have

$\rho^\natural(x) \otimes \rho^\natural(y) = \rho(x) \otimes \rho(y) = \{\rho(z) : z \in xy\} = \bigcup\limits_{z \in xy} \{\rho(z)\} = \rho(xy) = \rho^\natural(xy)$,

so $\rho^\natural$ is a morphism.

For the second part, suppose that T is a semipolygroup and let $\phi : S \to T$ be a morphism.

Thus, that $\ker\phi$ is an equivalence follows from Proposition 2.2.2.

Let $\ker\phi(x) = \ker\phi(x_1)$ and $\ker\phi(y) = \ker\phi(y_1)$.

We check that $\ker\phi(x) \otimes \ker\phi(y) = \ker\phi(x_1) \otimes \ker\phi(y_1)$.

We have $x \ker\phi\, x_1$ and $y \ker\phi\, y_1$.

Then, $(x, x_1), (y, y_1) \in \ker\phi$ and $\phi(x) = \phi(x_1), \phi(y) = \phi(y_1)$.

For all $\ker\phi(z) \in \ker\phi(x) \otimes \ker\phi(y)$, we have $z \in xy$.

Thus, $\phi(z) \in \phi(xy) = \phi(x)\phi(y) = \phi(x_1)\phi(y_1) = \phi(x_1 y_1)$.

Then, there exists $z_1 \in x_1 y_1$ such that $\phi(z) = \phi(z_1)$.

This gives $(z, z_1) \in \ker\phi$ or $z \ker\phi\, z_1$.

That is, $\ker\phi(z) = \ker\phi(z_1) \in \ker\phi(x_1) \otimes \ker\phi(y_1)$.

It follows that $\ker\phi(x) \otimes \ker\phi(y) \subseteq \ker\phi(x_1) \otimes \ker\phi(y_1)$ and, in the similar way we obtain the converse inclusion.

Hence, the hyperoperation $\otimes$ is well-defined on $S/\ker\phi$.

By Theorem 3.2.3, $\ker\phi$ is regular.

For brevity, we denote $\ker\phi$ by κ, and define $\alpha : S/\kappa \to T$ by

$$\alpha(\kappa(a)) = \phi(a) \text{ for all } a \in S.$$

Then α is both well-defined and one to one, since

$$\kappa(a) = \kappa(b) \Leftrightarrow (a, b) \in \kappa \Leftrightarrow \phi(a) = \phi(b).$$

It is also a morphism, since, for all a, b in S,

$$\alpha[\kappa(a) \otimes \kappa(b)] = \alpha[\kappa(ab)] = \phi(ab) = \phi(a)\phi(b) = \alpha(\kappa(a))\alpha(\kappa(b)).$$

Clearly, im $\alpha =$ im ϕ, and from the definition of α it is clear that, for all a in S,

$$\alpha(\kappa^{\natural}(a)) = \alpha(\kappa(a)) = \phi(a).$$ □

Theorem 3.2.5. **[6]** *Let ρ be a regular on a semipolygroup S, and let $\phi : S \to T$ be a morphism such that $\rho \subseteq \ker\phi$. Then there is a unique morphism $\beta : S/\rho \to T$*

such that im $\beta =$ im ϕ and such that the diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^\natural"'] & T \\ S/\rho \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes.

Proof. We define $\beta : S/\rho \to T$ by

$$\beta(\rho(a)) = \phi(a) \quad \text{for all } a \in S. \tag{3.2.1}$$

Then β is well-defined, since, for all a,b in S,

$$\rho(a) = \rho(b) \Rightarrow (a,b) \in \rho \Rightarrow (a,b) \in \ker\phi \Rightarrow \phi(a) = \phi(b).$$

It is now a routine matter to show that β is a morphism, that im $\beta =$ im ϕ, and that $\beta \circ \rho^\natural = \phi$.

The uniqueness of β is also clear, since any morphism satisfying $\beta \circ \rho^\natural = \phi$ must be defined by the rule (3.2.1). $\square$

Theorem 3.2.6. *Let ρ, σ be regular relations on a semipolygroup S such that $\rho \subseteq \sigma$. Then*

$$\sigma/\rho = \{(\rho(x), \rho(y)) \in (S/\rho) \times (S/\rho) : (x,y) \in \sigma\}$$

is a regular on S/ρ and $(S/\rho)/(\sigma/\rho) \simeq S/\sigma$.

Proof. The above theorem implies that there is a morphism β from S/ρ onto S/σ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\phi^\natural"] \arrow[d, "\rho^\natural"'] & S/\sigma \\ S/\rho \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes. The morphism β is given by $\beta(\rho(a)) = \sigma(a)$ for all $a \in S$, and the regular $\ker\beta$ on S/ρ is given by

$$\begin{aligned} \ker\beta &= \{(\rho(a),\rho(b)) \in S/\rho \times S/\rho : \beta(\rho(a)) = \beta(\rho(b))\} \\ &= \{(\rho(a),\rho(b)) \in S/\rho \times S/\rho : \sigma(a) = \sigma(b)\} \\ &= \{(\rho(a),\rho(b)) \in S/\rho \times S/\rho : (a,b) \in \sigma\}. \end{aligned}$$

It is usual to write $\ker\beta$ as σ/ρ.

From Theorem 3.2.4, it now follows that there is a monomorphism

$\alpha : (S/\rho)/(\sigma/\rho) \to S/\sigma$ defined by

$$\alpha(\sigma/\rho(\rho(a))) = \sigma(a) \quad \text{for all } a \in S. \tag{3.2.2}$$

For all, $\sigma(a) \in S/\sigma$,

since $a \in S$, we have $\rho(a) \in S/\rho$, so there is $\sigma/\rho(\rho(a)) \in (S/\rho)/(\sigma/\rho)$ such

that $\alpha(\sigma/\rho(\rho(a))) = \sigma(a)$ and so α is surjective.

Hence, α is an isomorphism such that the diagram

$$\begin{tikzcd} S \arrow[r, "\phi^\natural"] \arrow[d, "\rho^\natural"'] & S/\sigma \\ S/\rho \arrow[ur, "\beta"] \arrow[r, "(\sigma/\rho)^\natural"'] & (S/\rho)/(\sigma/\rho) \arrow[u, "\alpha"'] \end{tikzcd}$$

commutes. $\square$

Proposition 3.2.7. *For an arbitrary equivalence E on a semipolygroup S, we define*

$$E^\flat = \{(a,b) \in S \times S : (xay)\bar{\bar{E}}(xby) \ \forall x,y \in S^1\}.$$

Then $E^\flat$ is a equivalence on S.

Proof. Let $x,y \in S^1$ and $a,b,s \in S$.

For all $u \in xay$ we have uEu, i.e., $(xay)\bar{\bar{E}}(xay)$. It follows that $(a,a) \in E^\flat$.

Suppose that $(a,b) \in E^\flat$. That is, for all $x,y \in S^1$, we have $(xay)\bar{\bar{E}}(xby)$.

Then, for all $u \in xay$, for all $v \in xby, uEv$ implies vEu by symmetry.

Hence, $(xby)\bar{\bar{\bar{E}}}(xay)$, and so $E^\flat$ is symmetric.

Next, we assume that $(a,c) \in E^\flat \circ E^\flat$.

Thus, for some $b \in S, (a,b) \in E^\flat$ and $(b,c) \in E^\flat$.

That is, $(xay)\bar{\bar{\bar{E}}}(xby)$ and $(xby)\bar{\bar{\bar{E}}}(xcy)$ for all $x,y \in S^1$.

Therefore, for all $u \in xay$, $v \in xby$, $w \in xcy$, we have uEv and vEw.

By transitivity, uEw and $(xay)\bar{\bar{\bar{E}}}(xcy)$. We conclude that $(a,c) \in E^\flat$. □

If a is an element of semipolygroup S, the smallest left ideal of S containing a is $Sa \cup \{a\}$, which, as noted in Section 3.1, it is convenient to denote by S^1a. We shall call it the **principal left ideal generated by** a. An equivalence $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if and only if a and b generate the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly, we define equivalence $\mathcal{R}$ by the rule that $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.

Proposition 3.2.8. *Let a,b be elements of a semipolygroup S. Then $a\mathcal{L}b$ if and only if there exist x,y in S^1 such that $xa \supseteq \{b\}, yb \supseteq \{a\}$ and $a\mathcal{R}b$ if and only if there exist u,v in S^1 such that $au \supseteq \{b\}, bv \supseteq \{a\}$.*

Proof. Suppose that $a\mathcal{L}b$. Then $S^1a = S^1b$, i.e., $Sa \cup \{a\} = Sb \cup \{b\}$.

If $a = b$, then there exist $x = e, y = e \in S^1$ such that $\{b\} = eb = xa$ and $\{a\} = ea = yb$.

If $a \neq b$,then $b \in Sa$ and $a \in Sb$.

That is, $\{b\} \subseteq xa$ for some $x \in S \subseteq S^1$ and $\{a\} \subseteq yb$ for some $y \in S \subseteq S^1$.

Conversely, we assume that $\{b\} \subseteq xa$ and $\{a\} \subseteq yb$ for some $x,y \in S^1$.

If $x = e$, then $\{b\} \subseteq ea = \{a\}$ implies $b = a$.

Similarly, if $y = e$ we get $a = b$.

Therefore, for $x = e$ or $y = e$, we have $Sa = Sb$ and also

$$S^1a = Sa \cup \{a\} = Sb \cup \{b\} = S^1b.$$

Next, we assume that $x \neq e$ and $y \neq e$.

If S has no identity element, then $x, y \in S$, and so $\{b\} \subseteq Sa$ and $\{a\} \subseteq Sb$.

Let $p \in Sa \cup \{a\}$.

If $p = a$ we get $p \in Sb \subseteq Sb \cup \{b\}$, because $\{a\} \subseteq Sb$.

If $p \neq a$, then $p \in Sa$ and for some

$$s \in S, p \in sa \subseteq s(Sb) = (sS)b \subseteq Sb \subseteq Sb \cup \{b\}.$$

Therefore, $Sa \cup \{a\} \subseteq Sb \cup \{b\}$.

Let $q \in Sb \cup \{b\}$.

If $q = b$ we get $q \in Sa \subseteq Sa \cup \{a\}$, because $\{b\} \subseteq Sa$.

If $q \in Sb$ then, for some $s \in S, q \in sb \subseteq s(Sa) = (sS)a \subseteq Sa \subseteq Sa \cup \{a\}$.

Therefore, $Sb \cup \{b\} \subseteq Sa \cup \{a\}$, and so $Sa \cup \{a\} = Sb \cup \{b\}$.

Similarly, if S has an identity element, then $x, y \in S - \{e\}$ and $S^1 = S$.

If $p' \in S^1 a = Sa$, then there is $s_1 \in S$ such that

$$p' \in s_1 a \subseteq s_1(yb) = (s_1 y)b \subseteq Sb = S^1 b.$$

If $q' \in S^1 b = Sb$, then there is $s_2 \in S$ such that

$$q' \in s_2 b \subseteq s_2(xa) = (s_2 x)a \subseteq Sa = S^1 a.$$

Hence, $S^1 a = S^1 b$.

In the same way, we can prove that $a\mathcal{R}b$ if and only if there exist $u, v \in S^1$ such that $\{b\} \subseteq au$ and $\{a\} \subseteq bv$. $\square$

Our final equivalence is the two-sided analogue of $\mathcal{L}$ and $\mathcal{R}$. The principal two-sided ideal of S generated by a is $S^1 a S^1$, and we define the equivalence $\mathcal{J}$ by the rule that $a\mathcal{J}b$ if and only if $S^1 a S^1 = S^1 b S^1$, that is to say, if and only if there exist x, y, u, v in S^1 such that

$$\{b\} \subseteq xay, \{a\} \subseteq ubv.$$

If $a\mathcal{L}b$, then there exist $x,y \in S^1$ such that $\{b\} \subseteq xa$ and $\{a\} \subseteq yb$. We can choose $k = e \in S^1$ such that $\{b\} \subseteq xa = (xa)e = xak$ and $\{a\} \subseteq yb = (yb)e = ybk$. Similarly, if $a\mathcal{R}b$, then there exist $u,v \in S^1$ such that $\{b\} \subseteq au$ and $\{a\} \subseteq bv$. Hence, $\{b\} \subseteq au = (au)e = auk$ and $\{a\} \subseteq bv = (bv)e = bvk$. So $a\mathcal{J}b$. Hence, $\mathcal{L} \subseteq \mathcal{J}$. Similarly, $\mathcal{R} \subseteq \mathcal{L}$.

Definition 3.2.9. A semipolygroup S is called **equidivisible** if, for all s,t,u,v in S, $st = uv$ implies that either

(i) there exists $x \in S^1$ such that $s \in ux$ and $v \in xt$; or

(ii) there exists $y \in S^1$ such that $u \in sy$ and $t \in yv$.

Notice that every polygroup is equidivisible simply by defining $x \in u^{-1}s$ in (i), or $y \in s^{-1}u$ in (ii).

Definition 3.2.10. [8] An element a of semipolygroup S is called **regular** if there exists x in S such that $axa \ni a$.

Definition 3.2.11. [8] The semipolygroup S is called **regular** if all its elements are regular.

Polygroups are of course regular semipolygroups, but the class of regular semipolygroups is vastly more extensive than the class of polygroups. For example, every rectangular band B is regular, since $\{a\} = aba$ for all a,b in B. We also have that the left (right) zero semipolygroup S is regular because $xyx = \{x\}$ for all $x,y \in S$.

Lemma 3.2.12. *Let $\phi : S \to T$ be a monomorphism from a regular semipolygroup S into a semipolygroup T. Then $im\ \phi$ is regular. If f is an idempotent in $im\ \phi$ then there exists an idempotent e in S such that $\phi(e) = f$.*

Proof. Let $\phi(s) \in im\ \phi = \{\phi(s) : s \in S\}$.

There exists x in S such that $s \in sxs$, because S is a regular.

Thus, $\phi(s) \in \phi(sxs) = \phi(s)\phi(x)\phi(s)$, and so $im\ \phi$ is regular.

Suppose that f is an idempotent in $im\ \phi$.

Since $f \in im\ \phi$, there exists $e \in S$ such that $\phi(e) = f$.

We have $\phi(e) = f \in ff = \phi(e)\phi(e) = \phi(ee)$, that is, $\phi(e) \in \phi(ee) = \bigcup_{a \in ee} \{\phi(a)\}$.

Then, there exists $a \in ee$ such that $\phi(a) = \phi(e)$.

It follows that $a = e$, because ϕ is a monomorphism. Hence, $e \in ee$.

Therefore, e is an idempotent, as proved. □

3.3 Inverse Semipolygroups

Definition 3.3.1. If a is an element of a semipolygroup S, we say that a' is an **inverse** of a if

$$a \in aa'a \text{ and } a' \in a'aa'.$$

Notice that an element a may well have more than one inverse. Indeed, in a rectangular band, every element is an inverse of every other element and every inverse semigroup has an inverse. We also have that every element of a left (right) zero semipolygroup has an inverse.

Definition 3.3.2. A semipolygroup S will be called an **inverse semipolygroup** if there exists a unique unary operation $a \mapsto a^{-1}$ on S with the properties

$$(a^{-1})^{-1} = a, \quad aa^{-1}a \ni a,$$
$$a \in bc \text{ implies } b \in ac^{-1} \text{ and } c \in b^{-1}a \qquad \text{for all } a, b, c \text{ in } S.$$

Notice that, since these equations are to hold for every element of S it follows that $a^{-1}aa^{-1} = a^{-1}(a^{-1})^{-1}a^{-1} \ni a^{-1}$, and so a^{-1} is the inverse of a.

Remark Every inverse semipolygroup is regular.

Example 3.3.3. Let $S = \{a, b, c\}$ with the following table:

$\cdot$	a	b	c
a	$\{a,b,c\}$	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	$\{a,b,c\}$	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$

Then $(S,\cdot)$ is an inverse semipolygroup.

Example 3.3.4. Let $(G,\cdot)$ be a group. Define a semipolygroup $(G,\circ)$ by

$$g_1 \circ g_2 = \{g_1 \cdot g_2\} \text{ for all } g_1, g_2 \in G,$$

Because G is a group, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = e = a^{-1}a$ for all $a \in G$, where e is the identity element of G. It follows that $(a^{-1})^{-1} = a$.
Moreover, $a \circ a^{-1} \circ a = \{a \cdot a^{-1}\} \circ a = \{e\} \circ a = \{e \cdot a\} = \{a\}$.
Next, we suppose that $a \in b \circ c$. That is, $a \in \{b \cdot c\}$.
It follows that $a \circ c^{-1} \subseteq \{b \cdot c\} \circ c^{-1} = \{(b \cdot c) \cdot c^{-1}\} = \{b \cdot (c \cdot c^{-1})\} = \{b \cdot e\} = \{b\}$, so we get $a \circ c^{-1} = \{b\}$. Similarly, $b^{-1} \circ a \subseteq b^{-1} \circ \{b \cdot c\} = \{b^{-1} \cdot b \cdot c\} = \{e \cdot c\} = \{c\}$, so $b^{-1} \circ a = \{c\}$. Hence, $(G,\circ)$ is an inverse semipolygroup.

Example 3.3.5. Let P be a polygroup.
Let $a, b, c \in P$. We have $(a^{-1})^{-1} = a$ and $a \in bc$ implies $b \in ac^{-1}, c \in b^{-1}a$.
Since $e \in aa^{-1}$, it follows that $\{a\} = ea \subseteq aa^{-1}a$, i.e., $a \in aa^{-1}a$.
Therefore, P is an inverse semipolygroup.

Example 3.3.6. Define a hyperoperation $\circ$ on $\mathbb{R}^+$ by

$$x \circ y = \{xy\} \text{ for all } x, y \in \mathbb{R}^+.$$

Let $x, y, z \in \mathbb{R}^+$. First,

$$(x \circ y) \circ z \;=\; \{xy\} \circ z = \{(xy)z\} = \{x(yz)\} = x \circ \{yz\} = x \circ (y \circ z).$$

Thus, $(\mathbb{R}^+,\circ)$ is a semipolygroup.
Since $x \in \mathbb{R}^+$, there exists $x^{-1} = \dfrac{1}{x} \in \mathbb{R}^+$ such that

$$x \circ x^{-1} \circ x \;=\; x \circ \frac{1}{x} \circ x = \left\{x\left(\frac{1}{x}\right)\right\} \circ x = \{1\} \circ x = \{x\} \ni x,$$

and $(x^{-1})^{-1} = \left(\dfrac{1}{x}\right)^{-1} = x$. Suppose that w is an inverse of x.
Then $x \in x \circ w \circ x = \{xwx\}$.
It follows that $x = xwx$, and hence $w = \dfrac{1}{x}$. That is, $x^{-1} = \dfrac{1}{x}$ is unique.

Suppose that $x \in y \circ z = \{yz\}$ for all $x, y, z \in \mathbb{R}^+$.
It holds that $x = yz$, so that $y = \dfrac{x}{z}$ and $z = \dfrac{x}{y}$. It follows that

$$\begin{aligned} y &\in \{y\} = \left\{\frac{x}{z}\right\} = \left\{x\left(\frac{1}{z}\right)\right\} = x \circ \frac{1}{z} = x \circ z^{-1} \text{ and} \\ z &\in \{z\} = \left\{\frac{x}{y}\right\} = \left\{\left(\frac{1}{y}\right)x\right\} = \frac{1}{y} \circ x = y^{-1} \circ x. \end{aligned}$$

Therefore, $(\mathbb{R}^+, \circ)$ is an inverse semipolygroup.

Example 3.3.7. Define a hyperoperation $\circ$ on $\mathbb{R}^-$ by

$$x \circ y = \{-xy\} \text{ for all } x, y \in \mathbb{R}^-.$$

Let $x, y, z \in \mathbb{R}^-$.
Then, there exists $r_1, r_2, r_3 \in \mathbb{R}^+$ such that $x = -r_1, y = -r_2$ and $z = -r_3$.

$$\begin{aligned} (x \circ y) \circ z &= ((-r_1) \circ (-r_2)) \circ (-r_3) = \{-(-r_1)(-r_2)\} \circ (-r_3) = \{-r_1r_2\} \circ (-r_3) \\ &= \{-(-r_1r_2)(-r_3)\} = \{-r_1r_2r_3\} = \{-(-r_1)(-r_2r_3)\} = (-r_1) \circ \{-r_2r_3\} \\ &= (-r_1) \circ \{-(-r_2)(-r_3)\} = (-r_1) \circ ((-r_2) \circ (-r_3)) = x \circ (y \circ z). \end{aligned}$$

Thus, $(\mathbb{R}^-, \circ)$ is a semipolygroup.
Since $x \in \mathbb{R}^-$, there exists $x^{-1} = (-r_1)^{-1} = \dfrac{1}{-r_1} = \dfrac{1}{x} \in \mathbb{R}^-$ such that

$$\begin{aligned} x \circ x^{-1} \circ x &= (-r_1) \circ \frac{1}{-r_1} \circ (-r_1) = \left\{-(-r_1)\left(\frac{1}{-r_1}\right)\right\} \circ (-r_1) = \{-1\} \circ (-r_1) \\ &= \{-(-1)(-r_1)\} = \{-r_1\} = \{x\} \ni x, \end{aligned}$$

and $(x^{-1})^{-1} = \left(\dfrac{1}{x}\right)^{-1} = x$. Suppose that p is an inverse of x.
Then, there exists $a \in \mathbb{R}^+$ such that $p = -a$.

$$\begin{aligned} x = -r_1 &\in x \circ p \circ x = (-r_1) \circ (-a) \circ (-r_1) = \{-(-r_1)(-a)\} \circ (-r_1) = \{-r_1a\} \circ (-r_1) \\ &= \{-(-r_1a)(-r_1)\} = \{-r_1ar_1\}. \end{aligned}$$

This implies that $-r_1 = -r_1ar_1$, and hence $\dfrac{1}{-r_1} = -a = p$. Thus, $x^{-1} = \dfrac{1}{-r_1}$ is unique.
Suppose that $x \in y \circ z = \{-yz\}$ for all $x, y, z \in \mathbb{R}^-$.

Then, $x = -yz$ and $-r_1 = -(-r_2)(-r_3)$.
It follows that $-r_2 = \dfrac{r_1}{-r_3}$ and $-r_3 = \dfrac{r_1}{-r_2}$. Thus,

$$\begin{aligned} y &\in \{y\} = \{-r_2\} = \left\{\frac{r_1}{-r_3}\right\} = \left\{\frac{-(-r_1)}{-r_3}\right\} = \left\{-(-r_1)\left(\frac{1}{-r_3}\right)\right\} = (-r_1)\circ\left(\frac{1}{-r_3}\right) \\ &= x\circ\frac{1}{z} = x\circ z^{-1}. \text{ and} \\ z &\in \{z\} = \{-r_3\} = \left\{\frac{r_1}{-r_2}\right\} = \left\{\frac{-(-r_1)}{-r_2}\right\} = \left\{-\left(\frac{1}{-r_2}\right)(-r_1)\right\} = \left(\frac{1}{-r_2}\right)\circ(-r_1) \\ &= \frac{1}{y}\circ x = y^{-1}\circ x. \end{aligned}$$

Therefore, $(\mathbb{R}^-,\circ)$ is an inverse semipolygroup.

Example 3.3.8. Let $S = \{a,b\}$ with the following table:

$\cdot$	a	b
a	a	$\{a,b\}$
b	$\{a,b\}$	b

Then $(S,\cdot)$ is an inverse semipolygroup where $a^{-1} = b$ and $b^{-1} = a$.

Consider the system $< S,\cdot,e,^{-1} >$, we have $aa^{-1}\cap a^{-1}a = ab\cap ba = S$ and $bb^{-1}\cap b^{-1}b = ba\cap ab = S$.

If $e = a$, then $e = a \in aa^{-1}\cap a^{-1}a$. In addition, $ea = aa = \{a\} = aa = ae$, but $eb = ab = S \neq \{b\}$ and $be = ba = S \neq \{b\}$. Similarly, if $e = b$, then $ea = ba = S \neq \{a\}$ and $ae = ab = S \neq \{b\}$. Consequently, $(S,\circ)$ is an inverse semipolygroup, but it isn't a polygroup.

Lemma 3.3.9. *Let S be an inverse semipolygroup and $x \in S$. Every element in xx^{-1} and $x^{-1}x$ is regular.*

Proof. Let S be an inverse semipolygroup and $x \in S$.

Then, there exists a unique $x^{-1} \in S$ such that $xx^{-1} \subseteq S$.

Let $a \in xx^{-1}$. This implies that $x^{-1} \in x^{-1}a$ and $x \in a(x^{-1})^{-1} = ax$.

Thus, we have $a \in xx^{-1} \subseteq (ax)(x^{-1}a) = a(xx^{-1})a$.

Hence, there exists $b \in xx^{-1}$ such that $a \in aba$, and hence every element in

xx^{-1} is regular. Similarly, every element in $x^{-1}x$ is regular. $\square$

Lemma 3.3.10. *Let S be an inverse semipolygroup. Then*
(i) if e is an idempotent of S, then $e^{-1} = e$;
(ii) if e is a scalar idempotent of S, then $ee^{-1} = \{e\}$.

Proof. (i) Let e be an idempotent of S.

Because S is an inverse semipolygroup, there is a unique $e^{-1} \in S$ such that $e \in ee^{-1}e$. Since $e \in e^2 = ee \subseteq eee$, e is an inverse of e.

That is, $e^{-1} = e$.

(ii) Let e be a scalar idempotent of S. By (i), $ee^{-1} = ee = \{e\}$. $\square$

Definition 3.3.11. A non-empty subset K of an inverse semipolygroup S is said to be an **inverse subsemipolygroup** of S if, under the hyperoperation in S, K itself forms an inverse semipolygroup.

Lemma 3.3.12. *A non-empty subset K of an inverse semipolygroup S is an inverse subsemipolygroup of S if and only if*

(i) $a, b \in K$ implies $ab \subseteq K$;

(ii) $a \in K$ implies $a^{-1} \in K$.

Proof. First, suppose that K is an inverse subsemipolygroup and let $a, b \in K$.

Then K is an inverse semipolygroup.

Thus, there exists a unique $a^{-1} \in K$ and we have $ab \subseteq K$.

Conversely, we assume that (i), (ii) are true and let $a, b, c \in K$.

Thus, we have $ab \subseteq K$ and $a^{-1}, b^{-1} \in K$.

Since $K \subseteq S$, $(ab)c = a(bc)$, that is, K is a semipolygroup.

Because S is an inverse semipolygroup, there exists a unique $a^{-1}, b^{-1} \in S$ such that $(a^{-1})^{-1} = a, a \in aa^{-1}a$.

Because $K \subseteq S$, the inverse is unique and thus $a^{-1}, b^{-1} \in K$.

Suppose that $a \in bc$. Since $K \subseteq S$, this implies that $b \in ac^{-1}$ and $c \in b^{-1}a$.

It follows that K is an inverse subsemipolygroup. $\square$

Example 3.3.13. Let S be an inverse semipolygroup and let a be a scalar idempotent of S. Then we have $\{a\} = a^2$ and $a^{-1} \in S$. Let $x, y \in aa^{-1}$.
Now, consider $xy \subseteq (aa^{-1})(aa^{-1}) = aaaa = a^2a^2 = aa = aa^{-1}$ by Lemma 3.3.10(i).
Since $x \in aa^{-1}, x^{-1} \in (aa^{-1})^{-1} = aa^{-1}$.
Hence, aa^{-1} is an inverse subsemipolygroup of inverse semipolygroup S.

Proposition 3.3.14. *Let a, b be an elements of an inverse semipolygroup S. Then*

(i) $(ab)^{-1} = b^{-1}a^{-1}$;

(ii) if $a^{-1}a = b^{-1}b$ then $a\mathcal{L}b$;

(iii) if $aa^{-1} = bb^{-1}$ then $a\mathcal{R}b$.

Proof. Let $a, b \in S$.

Then, there exist $a^{-1}, b^{-1} \in S$ such that $a \in aa^{-1}a$ and $b \in bb^{-1}b$.

(i) We have $ab \subseteq S$. Let $x^{-1} \in (ab)^{-1} = \{x^{-1} : x \in ab\}$.

Then, $x \in ab$ implies $x^{-1} \in b^{-1}a^{-1}$. Hence, $(ab)^{-1} \subseteq b^{-1}a^{-1}$

Since $a^{-1}, b^{-1} \in S$, $b^{-1}a^{-1} \subseteq S$. Then, let $y \in b^{-1}a^{-1}$.

This implies that $a^{-1} \in (b^{-1})^{-1}y = by \Rightarrow b \in a^{-1}y^{-1} \Rightarrow y^{-1} \in (a^{-1})^{-1}b = ab$.

Thus, $y^{-1} \in ab$. It follows that $(y^{-1})^{-1} \in (ab)^{-1}$, i.e., $y \in (ab)^{-1}$.

Therefore, we get $b^{-1}a^{-1} \subseteq (ab)^{-1}$. Consequently, $(ab)^{-1} = b^{-1}a^{-1}$.

(ii) Suppose that $a^{-1}a = b^{-1}b$.

We have $a \in aa^{-1}a = ab^{-1}b = (ab^{-1})b$ and $b \in bb^{-1}b = ba^{-1}a = (ba^{-1})a$.

Then, there exists $x \in ab^{-1} \subseteq S \subseteq S^1, y \in ba^{-1} \subseteq S \subseteq S^1$ such that $a \in xb$ and $b \in ya$. From Proposition 3.2.8, it follows that $a\mathcal{L}b$.

(iii) Similarly, $aa^{-1} = bb^{-1}$ implies that $a\mathcal{R}b$. $\square$

Lemma 3.3.15. *Let A, B be inverse subsemipolygroups of an inverse semipolygroup S. Then, for all $x, y, a, b \in S$,* *(i)* $x \in Ab$ *implies* $b \in A^{-1}x$*;*

(ii) $y \in aB$ *implies* $a \in yB^{-1}$*.*

Proof. Suppose that $x \in Ab$. Then $x \in a_1 b$ for some $a_1 \in A$.

It holds that $b \in a_1^{-1}x \subseteq A^{-1}x$.

In the same way, we can prove that $y \in aB$ implise $a \in yB^{-1}$. □

Lemma 3.3.16. *Let* A, B *be inverse subsemipolygroups of an inverse semipolygroup* S*. Then*

$$(AB)^{-1} = B^{-1}A^{-1}, \text{ where } A^{-1} = \{a^{-1} : a \in A\}.$$

Proof. Consider $(AB)^{-1} = \{y^{-1} : y \in AB\}$. Let $y^{-1} \in (AB)^{-1}$. Then $y \in AB$.

Then, $y \in ab$ for some $a \in A$ and $b \in B$ which implies that

$$y^{-1} \in b^{-1}a^{-1} \subseteq B^{-1}A^{-1}.$$

Let $x^{-1} \in B^{-1}A^{-1}$.

Thus, there are $b^{-1} \in B^{-1}$ and $a^{-1} \in A^{-1}$ such that $x^{-1} \in b^{-1}a^{-1}$.

This implies that $x \in ab$, and we get $x^{-1} \in (AB)^{-1}$.

Hence, $(AB)^{-1} = B^{-1}A^{-1}$. □

Corollary 3.3.17. *Let* $a_1, a_2, \ldots, a_n$ *be elements of an inverse semipolygroups. Then*

$$(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1}a_{n-1}^{-1} \cdots a_1^{-1} \text{ for every positive integer } n.$$

Proof. The proof is by induction on $n \in \mathbb{N}$.

Let $P(n)$ be the statement $(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1}a_{n-1}^{-1} \cdots a_1^{-1}$.

If $n = 1$, then $P(1)$ is true.

Suppose that $P(k)$ is true where $k \in \mathbb{N}$. Then $(a_1 a_2 \cdots a_k)^{-1} = a_k^{-1}a_{k-1}^{-1} \cdots a_1^{-1}$.

Thus,

$$
\begin{aligned}
a_{k+1}^{-1}a_k^{-1}a_{k-1}^{-1}\cdots a_1^{-1} &= a_{k+1}^{-1}(a_k^{-1}a_{k-1}^{-1}\cdots a_1^{-1}) \\
&= a_{k+1}^{-1}(a_1a_2\cdots a_k)^{-1} \\
&= ((a_1a_2\cdots a_k)a_{k+1})^{-1} \qquad ; \text{by Lemma 3.3.16} \\
&= (a_1a_2\cdots a_ka_{k+1})^{-1}
\end{aligned}
$$

Hence, $P(k+1)$ is true.

By Mathematical Induction, we get

$$(a_1a_2\cdots a_n)^{-1} = a_n^{-1}a_{n-1}^{-1}\cdots a_1^{-1} \text{ for all } n \in \mathbb{N}.$$

□

Definition 3.3.18. Let S be an inverse semipolygroup and $a \in S$. For all $k \in \mathbb{N}$,

$$a^{-k} = (a^{-1})^k.$$

Lemma 3.3.19. *Let S be an inverse semipolygroup and $a \in S$. Then $a^m a^n = a^{m+n}$ for all $m,n \in \mathbb{Z}^-$.*

Proof. For $m,n \in \mathbb{Z}^-$, we have $n = -p$ and $m = -q$ for some $p,q \in \mathbb{Z}^+$. Then

$$a^m a^n = a^{-q}a^{-p} = (a^{-1})^q(a^{-1})^p = (a^{-1})^{q+p} = a^{-(q+p)} = a^{(-q)+(-p)} = a^{m+n}.$$

□

For all $m \in \mathbb{Z}^+, n \in \mathbb{Z}^-$ or $m \in \mathbb{Z}^-, n \in \mathbb{Z}^+$, Lemma 3.3.19 isn't true, because if we suppose $P = \{e,a,b,c\}$, and consider the commutative polygroup $< P, \cdot, e, ^{-1} >$ ([4]), where $\cdot$ is defined on P as follows:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	P	$\{a,b,c\}$	$\{a,b,c\}$
b	b	$\{a,b,c\}$	P	$\{a,b,c\}$
c	c	$\{a,b,c\}$	$\{a,b,c\}$	P

then P is an inverse semipolygroup where $a^{-1} = a, b^{-1} = b$, and $c^{-1} = c$, because P is a polygroup. Consider $a^{-3}a^2 = (a^{-1})^3a^2 = a^5 = P$, but $a^{(-3)+2} = a^{-1} = a$. Thus, $a^{-3}a^2 \neq a^{(-3)+2}$.

Lemma 3.3.20. *Let S be an inverse semipolygroup and $a \in S$. Then, for all $m, n \in \mathbb{Z} - \{0\}$,*

(i) $a^{-n} = (a^{-1})^n = (a^n)^{-1}$;

(ii) $(a^m)^n = a^{mn}$.

Proof. (i) For $n \in \mathbb{Z}^+$, we have $a^{-n} = (a^{-1})^n$ by Definition 3.3.18.

Let $P(n)$ be the statement $a^{-n} = (a^n)^{-1}$ for all $n \in \mathbb{Z}^+$.

If $n = 1$, then $a^{-1} = (a^1)^{-1}$. So $P(1)$ is true.

Suppose that $P(k)$ is true, where $k \in \mathbb{Z}^+$. That is, $a^{-k} = (a^k)^{-1}$.

Since $k \in \mathbb{Z}^+$, $-k \in \mathbb{Z}^-$.

By Lemma 3.3.19 and Proposition 3.3.14 (i), we have

$$a^{-(k+1)} = a^{-k}a^{-1} = (a^k)^{-1}a^{-1} = (aa^k)^{-1} = (a^{k+1})^{-1}.$$

Hence, $P(k+1)$ is true.

By Mathematical Induction, we get $a^{-n} = (a^n)^{-1}$ for all $n \in \mathbb{Z}^+$.

For $n \in \mathbb{Z}^-$, we have $n = -p$ for some $p \in \mathbb{Z}^+$.

Thus, $a^{-n} = a^{-(-p)} = a^p = ((a^p)^{-1})^{-1} = (a^{-p})^{-1} = (a^n)^{-1}$ and

$a^{-n} = a^{-(-p)} = a^p = ((a^{-1})^{-1})^p = (a^{-1})^{-p} = (a^{-1})^n$.

Thus, $a^{-n} = (a^{-1})^n = (a^n)^{-1}$ as required.

(ii) For $n \in \mathbb{Z}^+$, let $P(n)$ be a statement $(a^m)^n = a^{mn}$ for all $m \in \mathbb{Z} - \{0\}$.

If $n = 1$, then $P(1)$ is $(a^m)^1 = a^m = a^{m1} = a^{mn}$ for all $m \in \mathbb{Z} - \{0\}$.

Suppose that $P(k)$ is true, where $k \in \mathbb{Z}^+$.

That is, $(a^m)^k = a^{mk}$ for all $m \in \mathbb{Z} - \{0\}$.

If $m \in \mathbb{Z}^+$, then $mk \in \mathbb{Z}^+$, and so by Lemma 3.1.19 (i), (ii) we get

$a^{m(k+1)} = a^{mk+m} = a^{mk}a^m = (a^m)^k a^m = (a^m)^{k+1}$.

If $m \in \mathbb{Z}^-$, then $mk \in \mathbb{Z}^-$.

Hence, $a^{m(k+1)} = a^{mk+m} = a^{mk}a^m = (a^m)^k(a^m) = (a^m)^{k+1}$ by Lemma 3.3.19.

Therefore, we get $(a^m)^n = a^{mn}$ for all $m \in \mathbb{Z} - \{0\}$ for all $n \in \mathbb{Z}^+$.

For $n \in \mathbb{Z}^-$, we have $n = -p$ for some $p \in \mathbb{Z}^+$.

If $m \in \mathbb{Z}^+$, then

$$a^{mn} = a^{m(-p)} = (a^{-(mp)}) = (a^{mp})^{-1} = ((a^m)^p)^{-1} = (a^m)^{-p} = (a^m)^n.$$

If $m \in \mathbb{Z}^-$, then we have $m = -q$ for some $q \in \mathbb{Z}^+$.

Since $(a^m)^n = a^{mn}$ for all $m \in \mathbb{Z} - \{0\}$ for all $n \in \mathbb{Z}^+$,

$(a^m)^n = (a^{-q})^{-p} = ((a^q)^{-1})^{-p} = (a^q)^{-(-p)} = (a^q)^p = a^{qp} = a^{(-q)(-p)} = a^{mn}$.

So, our claim holds. □

Proposition 3.3.21. *Let S be an inverse semipolygroup, and let $a, b \in S$. The following statements are (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (v), (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (vi) and (v) $\Leftrightarrow$ (vi):*

(i) $aa^{-1} = ba^{-1}$; *(ii)* $aa^{-1} = ab^{-1}$;

(iii) $a^{-1}a = b^{-1}a$; *(iv)* $a^{-1}a = a^{-1}b$;

(v) $a \in ab^{-1}a$; *(vi)* $a \in aa^{-1}b$.

Proof. (i)$\Leftrightarrow$(ii). $aa^{-1} = ba^{-1}$ iff $(aa^{-1})^{-1} = (ba^{-1})^{-1}$ iff $aa^{-1} = ab^{-1}$.

(i)$\Rightarrow$(v). Suppose that $aa^{-1} = ba^{-1}$.

Since $a \in S, a^{-1} \in S$ and $a^{-1} \in a^{-1}aa^{-1} = a^{-1}ba^{-1}$. It obtains $a \in ab^{-1}a$.

(iv)$\Leftrightarrow$(iii). $a^{-1}a = a^{-1}b$ iff $(a^{-1}a)^{-1} = (a^{-1}b)^{-1}$ iff $a^{-1}a = b^{-1}a$.

(iii)$\Rightarrow$(vi). Suppose that $a^{-1}a = b^{-1}a$.

Since $a \in S, a^{-1} \in a^{-1}aa^{-1} = b^{-1}aa^{-1}$. This implies that $a \in aa^{-1}b$.

(vi)$\Rightarrow$(v). Suppose that $a \in aa^{-1}b$. It follows that $a^{-1} \in b^{-1}aa^{-1}$.

By Lemma 3.3.15 (i), we obtain $a^{-1} \in (b^{-1}a)^{-1}a^{-1} = a^{-1}ba^{-1}$.

Thus, $a \in ab^{-1}a$.

(v)$\Rightarrow$(vi). Suppose that $a \in ab^{-1}a$. This implies that $a^{-1} \in a^{-1}ba^{-1}$.

By Lemma 3.3.15 (i), we obtain $a^{-1} \in (a^{-1}b)^{-1}a^{-1} = b^{-1}aa^{-1}$.

Then, $a \in aa^{-1}b$. □

Theorem 3.3.22. *Let $\phi : S \to T$ be a morphism from an inverse semipolygroup S into semipolygroup T. Then $\phi(S)$ is an inverse semipolygroup.*

Proof. Assume that t is an element of $\phi(S)$.

Then, there exists an element s in S such that $\phi(s) = t$.

Because S is an inverse semipolygroup, there exists a unique $s^{-1} \in S$ such that $s \in ss^{-1}s$, and so $t = \phi(s) \in \phi(ss^{-1}s) = \phi(s)\phi(s^{-1})\phi(s)$.

We also have that $s^{-1} \in s^{-1}ss^{-1}, \phi(s^{-1}) \in \phi(s^{-1})\phi(s)\phi(s^{-1})$.

So, we obtains $\phi(s^{-1})$ is an inverse of $\phi(s)$ and $\phi(s)$ is an inverse of $\phi(s^{-1})$.

Then $\phi(s^{-1}) = (\phi(s))^{-1} = t^{-1}$ and $t = \phi(s) = (\phi(s^{-1}))^{-1}$.

It follows that $(t^{-1})^{-1} = ((\phi(s))^{-1})^{-1} = (\phi(s^{-1}))^{-1} = \phi(s) = t$.

Assume that m is an inverse of t.

Then $m = \phi(a)$ for some $a \in S$ and $t \in tmt$.

Therefore, $\phi(s) \in \phi(s)\phi(a)\phi(s) = \phi(sas)$.

So $s \in sas$, it follows that $a = s^{-1}$ because s^{-1} is unique.

Thus, $m = \phi(a) = \phi(s^{-1}) = (\phi(s))^{-1} = t^{-1}$.

We now conclude that t^{-1} is unique.

Let $t_1, t_2, t_3 \in \phi(S)$. Suppose now that $t_1 \in t_2t_3$.

Then, we have $t_1 = \phi(s_2), t_2 = \phi(s_2)$ and $t_3 = \phi(s_3)$ for some $s_1, s_2, s_3 \in S$.

Thus, $\phi(s_1) \in \phi(s_2)\phi(s_3) = \phi(s_2s_3)$, this means that $s_1 \in s_2s_3$.

It implies that $s_2 \in s_1s_3^{-1}$ and $s_3 \in s_2^{-1}s_1$, so

$$\begin{aligned} t_2 &= \phi(s_2) \in \phi(s_1s_3^{-1}) = \phi(s_1)\phi(s_3^{-1}) = \phi(s_1)(\phi(s_3))^{-1} = t_1t_3^{-1} \text{ and} \\ t_3 &= \phi(s_3) \in \phi(s_2^{-1}s_1) = \phi(s_2^{-1})\phi(s_1) = (\phi(s_2))^{-1}\phi(s_1) = t_2^{-1}t_1, \end{aligned}$$

and so $\phi(S)$ is an inverse semipolygroup, as required. □

By Theorem 3.3.22, we have the additional property that $\phi(s^{-1}) = (\phi(s))^{-1}$ for all s in S.

Theorem 3.3.23. *Let $\phi : S \to T$ be a morphism from a commutative inverse semipolygroup S into semipolygroup T. Then $\phi(S)$ is a commutative inverse semipolygroup.*

Proof. As a Theorem 3.1.40, $\phi(S)$ is a commutative semipolygroup.

Again, by Theorem 3.3.22, it holds that $\phi(S)$ is a commutative inverse semipolygroup, as required. □

Theorem 3.3.24. *Let $(S,\cdot)$ and $(T,\circ)$ be two inverse semipolygroups. Then the product $S\times T$ with respect to hyperoperation defined by Proposition 3.1.42 is an inverse semipolygroup, where $(s,t)^{-1} = (s^{-1},t^{-1})$ for all $(s,t) \in S\times T$.*

Proof. By Proposition 3.1.42, it obtains that $S\times T$ is a semipolygroup.

Let $(s,t),(s_1,t_1),(s_2,t_2),(s_3,t_3) \in S\times T$.

It obtains $((s,t)^{-1})^{-1} = (s^{-1},t^{-1})^{-1} = ((s^{-1})^{-1},(t^{-1})^{-1}) = (s,t)$.

Now, $(s,t)\diamond(s,t)^{-1}\diamond(s,t) = \bigcup_{x\in s\cdot s^{-1}\cdot s,\, y\in t\circ t^{-1}\circ t} \{(x,y)\}$.

Since $s \in s\cdot s^{-1}\cdot s$ and $t \in t\circ t^{-1}\circ t$,

$$(s,t) \in \bigcup_{x\in s\cdot s^{-1}\cdot s,\, y\in t\circ t^{-1}\circ t} \{(x,y)\} = (s,t)\diamond(s,t)^{-1}\diamond(s,t).$$

Suppose now that $(s_1,t_1) \in (s_2,t_2)\diamond(s_3,t_3) = \bigcup_{x\in s_2\cdot s_3,\, y\in t_2\circ t_3} \{(x,y)\}$.

Thus, $s_1 \in s_2\cdot s_3$ and $t_1 \in t_2\circ t_3$.

This implies that $s_2 \in s_1\cdot s_3^{-1}, s_3 \in s_2^{-1}\cdot s_1, t_2 \in t_1\circ t_3^{-1}$ and $t_3 \in t_2^{-1}\circ t_1$.

Hence,

$$\begin{aligned}
(s_1,t_1)\diamond(s_3,t_3)^{-1} &= (s_1,t_1)\diamond(s_3^{-1},t_3^{-1}) = \bigcup_{p\in s_1\cdot s_3^{-1},\, q\in t_1\circ t_3^{-1}} \{(p,q)\} \ni (s_2,t_2) \text{ and} \\
(s_2,t_2)^{-1}\diamond(s_1,t_1) &= (s_2^{-1},t_2^{-1})\diamond(s_1,t_1) = \bigcup_{u\in s_2^{-1}\cdot s_1,\, v\in t_2^{-1}\circ t_1} \{(u,v)\} \ni (s_3,t_3).
\end{aligned}$$

So, our claim holds. □

Let S be an inverse semipolygroup and let K be an inverse subsemipolygroup of S. If $s \in S$ then the subset Ks is not necessary contain s, but it definitely contains s if $ss^{-1} \subseteq K$ bacause $s \in ss^{-1}s \subseteq Ks$. We define a **right coset** of K to be a set Ks $(s \in S)$ for which $ss^{-1} \subseteq K$. If $s^{-1}s \subseteq K$ then $s \in ss^{-1}s \subseteq sK$. We define a **left coset** of K to be a set sK $(s \in S)$ for which $s^{-1}s \subseteq K$.

3.4 Normal Subsemipolygroups

Definition 3.4.1. A non-empty subset N of an inverse semipolygroup S is a **normal subsemipolygroup** in S if

(i) N is an inverse subsemipolygroup.

(ii) If e is idempotent of S then $e \in N$.

(iii) If $a \in N$ then $x^{-1}ax \subseteq N$ for all $x \in S$.

Example 3.4.2. Define a hyperoperation $\circ$ on $\mathbb{Q}^+$ by

$$x \circ y = \{xy\} \text{ for all } x, y \in \mathbb{Q}^+.$$

First, we see that $\mathbb{Q}^+ \subseteq \mathbb{R}^+$. Let $x, y, z \in \mathbb{Q}^+$.
Then, $x = \dfrac{m}{n}, y = \dfrac{p}{q}$ and $z = \dfrac{u}{v}$ where $m, n, p, q, u, v \in \mathbb{Z}^+$.
Thus, we have $x \circ y = \dfrac{m}{n} \circ \dfrac{p}{q} = \left\{\dfrac{mp}{nq}\right\}$.
Since $m, n, p, q \in \mathbb{Z}^+$, $mp, nq \in \mathbb{Z}^+$. Thus, $x \circ y \subseteq \mathbb{Q}^+$.
Since $x = \dfrac{m}{n} \in \mathbb{Q}^+$, there exists $x^{-1} = \dfrac{n}{m} \in \mathbb{Q}^+$ such that

$$x \circ x^{-1} \circ x \;=\; \frac{m}{n} \circ \frac{n}{m} \circ \frac{m}{n} = \left\{\left(\frac{m}{n}\right)\left(\frac{n}{m}\right)\left(\frac{m}{n}\right)\right\} = \left\{\frac{m}{n}\right\} = \{x\} \ni x,$$

and $(x^{-1})^{-1} = \left(\dfrac{n}{m}\right)^{-1} = \dfrac{m}{n} = x$. Suppose that w is an inverse of x.
Then, $x = \dfrac{m}{n} \in x \circ w \circ x = \dfrac{m}{n} \circ w \circ \dfrac{m}{n} = \left\{\left(\dfrac{m}{n}\right) w \left(\dfrac{m}{n}\right)\right\}$. Hence, $\dfrac{m}{n} = \left(\dfrac{m}{n}\right) w \left(\dfrac{m}{n}\right)$, and so $w = \dfrac{n}{m}$. Therefore, $x^{-1} = \dfrac{m}{n}$ is unique.
Suppose that $x \in y \circ z = \dfrac{p}{q} \circ \dfrac{u}{v} = \left\{\dfrac{pu}{qv}\right\}$ for all $x, y, z \in \mathbb{Q}^+$.

Then, $x=\dfrac{pu}{qv}$ implies that $y=\dfrac{mv}{nu}$ and $z=\dfrac{qm}{pn}$.
Thus, $y\in\left\{\dfrac{mv}{nu}\right\}=\dfrac{m}{n}\circ\dfrac{v}{u}=x\circ z^{-1}$ and $z\in\left\{\dfrac{qm}{pn}\right\}=\dfrac{q}{p}\circ\dfrac{m}{n}=y^{-1}\circ x$.
Therefore, $(\mathbb{Q}^+,\circ)$ is an inverse semipolygroup.

From Example 3.3.6, we have $1\in 1\circ 1=\{1\}$.

Suppose that a is an idempotent of $\mathbb{R}^+$. This means that $a\in a\circ a=\{a^2\}$.

Hence, $a=a^2$, and thus $a=1$.

Consequently, there is only 1 is an idempotents in $\mathbb{R}^+$ and it is obvious to see that 1 is an idempotents in $\mathbb{Q}^+$.

Let $r\in\mathbb{R}^+$. Then there exists a unique $r^{-1}=\dfrac{1}{r}\in\mathbb{R}^+$ such that

$$r^{-1}\circ x\circ r \;=\; \frac{1}{r}\circ\frac{m}{n}\circ r=\left\{\left(\frac{1}{r}\right)\left(\frac{m}{n}\right)r\right\}=\left\{\frac{m}{n}\right\}=\{x\}\subseteq\mathbb{Q}^+.$$

From Example 3.3.6, we have $\mathbb{Q}^+$ is a normal subsemipolygroup of an inverse semipolygroup $\mathbb{R}^+$.

Example 3.4.3. Define a hyperoperation $\circ$ on $\mathbb{Q}^-$ by

$$x\circ y=\{-xy\}\text{ for all }x,y\in\mathbb{Q}^-.$$

First, we have $\mathbb{Q}^-\subseteq\mathbb{R}^-$. Let $x,y,z\in\mathbb{Q}^-$.
Then, $x=-\dfrac{m}{n}, y=-\dfrac{p}{q}$ and $z=-t\dfrac{u}{v}$ where $m,n,p,q,u,v\in\mathbb{Z}^+$.
We get that $x\circ y=\left(-\dfrac{m}{n}\right)\circ\left(-\dfrac{p}{q}\right)=\left\{-\left(-\dfrac{m}{n}\right)\left(-\dfrac{p}{q}\right)\right\}=\left\{-\dfrac{mp}{nq}\right\}$.
Since $m,n,p,q\in\mathbb{Z}^+$, $mp,nq\in\mathbb{Z}^+$. Thus, $x\circ y\subseteq\mathbb{Q}^-$.
Since $x=-\dfrac{m}{n}\in\mathbb{Q}^-$, there exists $x^{-1}=-\dfrac{n}{m}\in\mathbb{Q}^-$ such that

$$\begin{aligned} x\circ x^{-1}\circ x &= \left(-\frac{m}{n}\right)\circ\left(-\frac{n}{m}\right)\circ\left(-\frac{m}{n}\right)=\left\{-\left(-\frac{m}{n}\right)\left(-\frac{n}{m}\right)\right\}\circ\left(\frac{m}{n}\right)\\ &= \{-1\}\circ\left(-\frac{m}{n}\right)=\left\{-(-1)\left(-\frac{m}{n}\right)\right\}=\left\{-\frac{m}{n}\right\}=\{x\}\ni x, \end{aligned}$$

and $(x^{-1})^{-1}=\left(-\dfrac{n}{m}\right)^{-1}=-\dfrac{m}{n}=x$. Suppose that w is an inverse of x.
Thus, $w\in\mathbb{Q}^-$, and there exists $a,b\in\mathbb{Z}^+$ such that $w=-\dfrac{a}{b}$. Then

$$\begin{aligned} x &= -\frac{m}{n} \in x \circ w \circ x = \left(-\frac{m}{n}\right) \circ \left(-\frac{a}{b}\right) \circ \left(-\frac{m}{n}\right) = \left\{-\left(-\frac{m}{n}\right)\left(-\frac{a}{b}\right)\right\} \circ \left(-\frac{m}{n}\right) \\ &= \left\{-\frac{ma}{nb}\right\} \circ \left(-\frac{m}{n}\right) = \left\{-\left(-\frac{ma}{nb}\right)\left(-\frac{m}{n}\right)\right\} = \left\{-\frac{m^2a}{n^2b}\right\}. \end{aligned}$$

Hence, $-\dfrac{m}{n} = -\dfrac{m^2a}{n^2b}$, and it implies that $-\dfrac{a}{b} = -\dfrac{n}{m}$. Then, $w = -\dfrac{n}{m}$.
Therefore, $x^{-1} = -\dfrac{n}{m}$ is unique.

Next, we suppose that $x \in y \circ z = \left(-\dfrac{p}{q}\right) \circ \left(-\dfrac{u}{v}\right) = \left\{-\dfrac{pu}{qv}\right\}$ for all $x, y, z \in \mathbb{Q}^-$.
Then, $x = -\dfrac{pu}{qv}$ implies that $y = -\dfrac{mv}{nu}$ and $z = -\dfrac{qm}{pn}$. Thus,

$$\begin{aligned} y &\in \{y\} = \left\{-\frac{p}{q}\right\} = \left\{-\frac{mv}{nu}\right\} = \left\{-\left(-\frac{m}{n}\right)\left(-\frac{v}{u}\right)\right\} = \left(-\frac{m}{n}\right) \circ \left(-\frac{v}{u}\right) = x \circ z^{-1} \text{ and} \\ z &\in \{z\} = \left\{-\frac{u}{v}\right\} = \left\{-\frac{mq}{np}\right\} = \left\{-\left(-\frac{q}{p}\right)\left(-\frac{m}{n}\right)\right\} = \left(-\frac{q}{p}\right) \circ \left(-\frac{m}{n}\right) = y^{-1} \circ x. \end{aligned}$$

Therefore, $(\mathbb{Q}^-, \circ)$ is an inverse semipolygroup.

From Example 3.3.7, we have $-1 \in (-1) \circ (-1) = \{-(-1)(-1)\} = \{-1\}$.

Suppose that a is an idempotent of $\mathbb{R}^-$. This means that $a \in a \circ a = \{-a^2\}$.

Hence, $a = -a^2$, and thus $a = -1$.

Consequently, there is only -1 is an idempotents in $\mathbb{R}^-$ and it is clearly that -1 is an idempotents in $\mathbb{Q}^-$.

Let $r \in \mathbb{R}^- - \{0\}$. Then there exists $b \in \mathbb{R}^+$ such that $r = -b$.
Again, there exists a unique $r^{-1} = (-b)^{-1} = \dfrac{1}{-b} = \dfrac{1}{r} \in \mathbb{R}^-$ such that

$$\begin{aligned} r^{-1} \circ x \circ r &= \left(\frac{1}{-b}\right) \circ \left(-\frac{m}{n}\right) \circ (-b) = \left\{-\left(\frac{1}{-b}\right)\left(-\frac{m}{n}\right)\right\} \circ (-b) \\ &= \left\{-\frac{m}{bn}\right\} \circ (-b) = \left\{-\left(-\frac{m}{bn}\right)(-b)\right\} = \left\{-\frac{m}{n}\right\} = \{x\} \subseteq \mathbb{Q}^-. \end{aligned}$$

From Example 3.3.7, we have $\mathbb{Q}^-$ is a normal subsemipolygroup of an inverse semipolygroup $\mathbb{R}^-$.

Corollary 3.4.4. *Let N be a normal subsemipolygroup of an inverse semipolygroup S and let $a \in S$. Then $Na = Nb$ for all $b \in Na$.*

Proof. Let $a \in S$ and let $x \in Na$. Then $x \in n_1 a$ for some $n_1 \in N$.

Since $b \in Na$, so $b \in n_2 a$ for some $n_2 \in N$.

It now implies that $a \in n_2^{-1} b$, and so $x \in n_1 a \subseteq n_1(n_2^{-1} b) = (n_1 n_2^{-1}) b \subseteq Nb$.

Similarly, we let $y \in Nb$. Thus, $y \in n_3 b$ for some $n_3 \in N$.

It is immediate that $y \in n_3 b \subseteq n_3(n_2 a) = (n_3 n_2) a \subseteq Na$, and so $Na = Nb$.

□

Corollary 3.4.5. *Let K and N be inverse subsemipolygroups of an inverse semipolygroup S with N normal in S. Then $N \cap K$ is a normal subsemipolygroup of K if $N \cap K$ is a non-empty set.*

Proof. Suppose that $N \cap K$ is a non-empty set. Let $a, b \in N \cap K$.

Then $a, b \in N$ and $a, b \in K$.

It follows easyily that $ab \subseteq N$ and $ab \subseteq K$, that is, $ab \subseteq N \cap K$.

Now, we get that $N \cap K$ is a subsemipolygroup of K.

Since $a \in N$ and $a \in K$, we also have that $a^{-1} \in N, a^{-1} \in K$, so $a^{-1} \in N \cap K$.

By using Lemma 3.3.12, $N \cap K$ is an inverse subsemipolygroup.

Suppose now that e is an idempotent of K.

Because N is normal in S and $e \in K \subseteq S$, it implies that $e \in N$.

Then $e \in N \cap K$.

Suppose that $a \in N \cap K$ and $k \in K$. Thus, we have that $a \in N$ and $a \in K$.

Because N is normal, this implies that $k^{-1} a k \subseteq N$.

Since $a, k, k^{-1} \in K$, we also implies that $k^{-1} a k \subseteq K$.

That is, $k^{-1} a k \subseteq N \cap K$.

Hence, $N \cap K$ is a normal subsemipolygroup of K as reqiured. □

Definition 3.4.6. Let N be a normal subsemipolygroup of an inverse semipolygroup S. Define the relation τ on S by

$$(x, y) \in \tau \text{ (or } x \,\tau\, y) \text{ if and only if } xy^{-1} \cap N \neq \emptyset.$$

Lemma 3.4.7. *The relation τ is an equivalence relation on an inverse semipolygroup S.*

Proof. Suppose that $x \in S$. Let $a \in S$ and let $n \in N$. Then $x^{-1} \in S$ and $n^{-1} \in N$.

We have $xn \subseteq S$.

Then, there exists $a \in xn$ such that $x \in an^{-1}$ and $x^{-1} \in na^{-1}$.

$xx^{-1} \subseteq (an^{-1})(na^{-1}) = a(n^{-1}n)a^{-1} \subseteq aNa^{-1} \subseteq N$ because N is normal.

Hence, $xx^{-1} \cap N \neq \emptyset$. Therefore, $x \,\tau\, x$, and so τ is reflexive.

Suppose that $x \,\tau\, y$, where $x, y \in S$. Then $xy^{-1} \cap N \neq \emptyset$.

That is, there exists $a \in xy^{-1} \cap N$,i.e., $a \in xy^{-1}$ and $a \in N$.

Since $a \in xy^{-1}$, this implies that $a^{-1} \in yx^{-1}$.

Since $a \in N$, $a^{-1} \in N$.

Hence $a^{-1} \in yx^{-1} \cap N$, that is, $yx^{-1} \cap N \neq \emptyset$ or $y \,\tau\, x$, and so τ is symmetric.

Assume that $x \,\tau\, y$ and $y \,\tau\, z$, where $x, y, z \in S$.

Then there are $a \in xy^{-1} \cap N$ and $b \in yz^{-1} \cap N$, that is, $a \in xy^{-1}, a \in N$ and $b \in yz^{-1}, b \in N$. So we get $x \in ay$ and $z^{-1} \in y^{-1}b$.

It follows that $z^{-1}x \subseteq (y^{-1}b)(ay) = y^{-1}(ba)y \subseteq N$ because $ba \subseteq N$

We see that $x^{-1}z = x^{-1}(z^{-1})^{-1} = (z^{-1}x)^{-1} = \{u^{-1} : u \in z^{-1}x\} \subseteq N$ because for all $u \in z^{-1}x \subseteq N$, so $u^{-1} \in N$.

Let $v \in x^{-1}z$. This implies that $z \in (x^{-1})^{-1}v = xv \Rightarrow x \in zv^{-1}$.

Then we have show that $xz^{-1} \subseteq zv^{-1}z^{-1} \subseteq N$, we deduce that $xz^{-1} \cap N \neq \emptyset$ or $x \,\tau\, z$. Hence, τ is transitive, as required. □

Lemma 3.4.8. *The equivalence relation τ on an inverse semipolygroup S is a strongly regular.*

Proof. Let $x, y, a \in S$. Suppose that $x\tau y$. So we have $xy^{-1} \cap N \neq \emptyset$.

Let $u \in xa$ and let $v \in ya$. It follows that $v^{-1} \in a^{-1}y^{-1}$.

We obtains $v^{-1}u \subseteq (a^{-1}y^{-1})(xa) = a^{-1}(y^{-1}x)a$.

Since $xy^{-1} \cap N \neq \emptyset$, there exists $k \in xy^{-1}$ and $k \in N$ such that $x \in ky$.

Now, we get that $y^{-1}x \subseteq y^{-1}ky \subseteq N$, because N is normal.

It is obvious to see that $v^{-1}u \subseteq N$.

For any $r \in v^{-1}u \subseteq N$, we have $r^{-1} \in N$.

This implies that $uv^{-1} = (u^{-1})^{-1}v^{-1} = (v^{-1}u)^{-1} = \{r^{-1} : r \in v^{-1}u\} \subseteq N$,

that is, $uv^{-1} \cap N \neq \emptyset$ and so $u\,\tau\,v$.

Therefore, τ is a strongly regular on the right.

Let $m \in ax$ and let $n \in ay$. Then $m^{-1} \in x^{-1}a^{-1}$ and $n^{-1} \in y^{-1}a^{-1}$.

It holds that $m^{-1}\,\tau\,n^{-1}$, because τ is a strongly regular on the right.

Thus, $m^{-1}n \cap N \neq \emptyset$ and there exists $p \in m^{-1}n \cap N$, i.e., $p \in m^{-1}n$ and $p \in N$. It follows that $p^{-1} \in n^{-1}m$ and $p^{-1} \in N$ and that $p^{-1} \in n^{-1}m \cap N$.

Hence, $n^{-1}m \cap N \neq \emptyset$ and so $m\,\tau\,n$. Thus, our claim holds. □

Let $\tau(x)$ be the equivalence class of the element x of an inverse semipolygroup S. Assume that the quotient set

$$S/\tau = \{\tau(x) : x \in S\}.$$

On S/τ we consider the hyperoperation $\otimes$ defined as follows:

$$\tau(x) \otimes \tau(y) = \{\tau(z) : z \in xy\}.$$

Lemma 3.4.9. *Let N be a normal subsemipolygroup of an inverse semipolygroup S and let τ be an equivalence relation on S. Then $\tau(x) = Nx$ for all $x \in S$.*

Proof. Suppose that $y \in Nx$.

Thus, there exists $n \in N$ such that $y \in nx$, which implies that $n \in yx^{-1}$.

Hence, $yx^{-1} \cap N \neq \emptyset$. That is, $y\tau x$ and so $y \in \tau(x)$.

Now, we have $Nx \subseteq \tau(x)$.

Next, we let $y \in \tau(x)$. Then $x\,\tau\,y$ or $xy^{-1} \cap N \neq \emptyset$. So, there exists $a \in xy^{-1}$ and $a \in N$ such that $y^{-1} \in x^{-1}a$ and $y \in a^{-1}x \subseteq Nx$.

Therefore, $\tau(x) \subseteq Nx$, and thus $\tau(x) = Nx$. $\square$

Lemma 3.4.10. *The quotient set S/τ is an inverse semipolygroup with respect to the hyperoperation $\otimes$.*

Proof. By Theorem 3.2.2, it concludes that S/τ is a semipolygroup.

First, we want to show that $((\tau(x))^{-1})^{-1} = \tau(x)$ for all $\tau(x) \in S/\tau$.

Since $(\tau(x))^{-1} = \{y^{-1} : y \in \tau(x)\}$, so

$$\begin{aligned} ((\tau(x))^{-1})^{-1} &= \{(y^{-1})^{-1} : y^{-1} \in (\tau(x))^{-1}\} \\ &= \{(y : y^{-1} \in (\tau(x))^{-1}\} \\ &= \{(y : y \in \tau(x)\} = \tau(x). \end{aligned}$$

Next, we will prove that $\tau(x) \in \tau(x) \otimes (\tau(x))^{-1} \otimes \tau(x)$.

We have

$$\begin{aligned} (\tau(x) \otimes (\tau(x))^{-1}) \otimes \tau(x) &= \{\tau(m) : m \in xx^{-1}\} \otimes \tau(x) \\ &= \bigcup_{\tau(m) \in \tau(x) \otimes (\tau(x))^{-1}} \tau(m) \otimes \tau(x) \\ &= \bigcup_{m \in xx^{-1}} \{\tau(n) : n \in mx\}. \end{aligned}$$

Since $x \in xx^{-1}x = (xx^{-1})x$, so there exists $n \in xx^{-1}$ such that $x \in nx$.

Hence, $\tau(x) \in \bigcup_{m \in xx^{-1}} \{\tau(n) : n \in mx\} = \tau(x) \otimes (\tau(x))^{-1} \otimes \tau(x)$.

Let $\tau(x), \tau(y), \tau(z) \in S/\tau$. Suppose that $\tau(x) \in \tau(y) \otimes \tau(z)$.

Then $x \in yz$ implies $y \in xz^{-1}$ and $z \in y^{-1}x$.

It holds that $\tau(y) \in \tau(x) \otimes (\tau(z))^{-1}$ and $\tau(z) \in (\tau(y))^{-1} \otimes \tau(x)$.

Therefore, S/τ is an inverse semipolygroup. $\square$

Lemma 3.4.11. *Let S be an inverse semipolygroup and N be a normal subsemipolygroup of S. Then*

$$\text{Ker } \tau = \bigcup_{e \in E} \tau(e)$$

is a normal subsemipolygroup of S*, where* $E = \{f \in S : f \text{ is an idempotent of } S\}$.

Proof. Let $x,y \in$ Ker τ. Then, there exist $e,f \in E$ such that $x \in \tau(e)$ and $y \in \tau(f)$.

This means that $ex^{-1} \cap N \neq \emptyset$ and $fy^{-1} \cap N \neq \emptyset$.

Thus, there exist $m \in ex^{-1} \cap N$ and $n \in fy^{-1} \cap N$, i.e., $m \in ex^{-1}, n \in fy^{-1}$ and $m,n \in N$. It implies that $x^{-1} \in e^{-1}m = em$ and $y^{-1} \in f^{-1}n = fn$.

Let $a \in xy$. Then $a^{-1} \in y^{-1}x^{-1} \subseteq (fn)(em) \subseteq N$.

Hence, $ea^{-1} \subseteq eN \subseteq N$ and thus $ea^{-1} \cap N \neq \emptyset$.

We obtains $a \in \tau(e)$, that is, $a \in$ Ker τ and we get $xy \subseteq$Ker τ.

Next, suppose that $k \in$ Ker τ. Then we have $ek^{-1} \cap N \neq \emptyset$ for some $e \in E$.

So, there exists $u \in ek^{-1} \cap N$ and that $u \in ek^{-1}$ and $u \in N$.

This also implies that $k^{-1} \in e^{-1}u = eu$ and then $k \in u^{-1}e^{-1} = u^{-1}e$.

It easily follows that $ek \subseteq eu^{-1}e \subseteq N$, so $ek \cap N = e(k^{-1})^{-1} \cap N \neq \emptyset$.

Certainly, $k^{-1} \in$ Ker τ, and thus Ker τ is an inverse subsemipolygroup.

Let $p \in S$ and let $q \in p^{-1}kp$. Then, we also have that $q^{-1} \in p^{-1}k^{-1}p$.

Since $u \in ek^{-1}, k^{-1} \in e^{-1}u = eu$.

We deduce that $q^{-1} \in p^{-1}k^{-1}p \subseteq p^{-1}(eu)p \subseteq p^{-1}Np \subseteq N$ and it also follows that $eq^{-1} \subseteq eN \subseteq N$, so $eq^{-1} \cap N \neq \emptyset$ and $q \in \tau(e)$.

Therefore, $q \in$ Ker τ and thus $p^{-1}kp \subseteq$ Ker τ for all $p \in S$.

Then we have established that Ker τ is a normal subsemipolygroup of S. $\square$

If A and B are inverse semipolygroups of an inverse semipolygroup S, $(A,B) \in \tau$ if and only if (a,b) for all $a \in A$ and $b \in B$. If $A = \{a\}$ then $(A,B) \in \tau$ if and only if (a,B).

For $a \in \tau(e)$, where $e \in E$ implies that $x^{-1}ax \subseteq N$ for all $x \in S$ because $ea^{-1} \cap N \neq \emptyset$, that is, there exists $m \in ea^{-1} \cap N$. It implies that $a^{-1} \in e^{-1}m$ and

$a \in m^{-1}e$. So, $x^{-1}ax \subseteq x^{-1}m^{-1}ex \subseteq N$.

3.5 Polygroups

In this section, some results on polygroups are presented.

Example 3.5.1. From Example 3.3.6, we obtain (i), (iv) of Definition 2.1.4.
Let $x \in \mathbb{R}^+$.
(ii) There exists $e = 1 \in \mathbb{R}^+$ such that

$$e \circ x = 1 \circ x = \{1x\} = \{x\} = \{x1\} = x \circ 1 = x \circ e.$$

Suppose that m is an identity of x. Then $m \circ x = \{mx\} = \{x\} = \{xm\} = x \circ m$.
This means that $x = mx$, that is, $m = 1$. Hence, $e = 1$ is unique.
(iii) By Example 3.3.6, there exists a unique $x^{-1} = \dfrac{1}{x} \in \mathbb{R}^+$ such that

$$\begin{aligned} x \circ x^{-1} &= x \circ \frac{1}{x} = \left\{x\left(\frac{1}{x}\right)\right\} = \{1\} \ni 1 = e \text{ and} \\ x^{-1} \circ x &= \frac{1}{x} \circ x = \left\{\left(\frac{1}{x}\right)x\right\} = \{1\} \ni 1 = e. \end{aligned}$$

Therefore, $< \mathbb{R}^+, \circ, e, ^{-1} >$ is a polygroup.

Example 3.5.2. From Example 3.3.7, we gain (i), (iv) of Definition 2.1.4.
Let $x \in \mathbb{R}^-$. Then, there exists $p \in \mathbb{R}^+$ such that $x = -p$.
(ii) There exists $e = -1 \in \mathbb{R}^-$ such that

$$\begin{aligned} e \circ x &= (-1) \circ (-p) = \{-(-1)(-p)\} = \{-p\} = \{x\} \\ &= \{-p\} = \{-(-p)(-1)\} = (-p) \circ (-1) = x \circ e. \end{aligned}$$

Suppose that m is an identity of x.
Thus, there exists $q \in \mathbb{R}^+$ such that $m = -q$ and $m \circ x = \{x\} = x \circ m$. We have

$$\begin{aligned} m \circ x &= (-q) \circ (-p) = \{-(-q)(-p)\} = \{-qp\}, \text{ and} \\ x \circ m &= (-p) \circ (-q) = \{-(-p)(-q)\} = \{-pq\}. \end{aligned}$$

It concludes that $-p=-pq$, and hence $m=-1$. It holds that $e=-1$ is unique.
(iii) By Example 3.3.7, there exists a unique $x^{-1}=\dfrac{1}{x}=-\dfrac{1}{p}\in\mathbb{R}^-$ such that

$$\begin{aligned} x\circ x^{-1} &= (-p)\circ\left(-\frac{1}{p}\right)=\left\{-(-p)\left(-\frac{1}{p}\right)\right\}=\{-1\}\ni -1=e \text{ and} \\ x^{-1}\circ x &= \left(-\frac{1}{p}\right)\circ(-p)=\left\{-\left(-\frac{1}{p}\right)(-p)\right\}=\{-1\}\ni -1=e. \end{aligned}$$

Consequently, $<\mathbb{R}^-,\circ,e,^{-1}>$ is a polygroup.

Definition 3.5.3. If P is a polygroup, then $P^0=P\cup\{0\}$ is a semipolygroup, where 0 is a zero. We shall call a semipolygroup formed in this way a **0-polygroup**, or **polygroup with zero**.

Lemma 3.5.4. *If S is a polygroup, then $aS=S$ and $Sa=S$ for all $a\in S$.*

Proof. Suppose that S be a polygroup and let $a\in S$.

Let $x\in aS$. Then $x\in as$ for some $s\in S$.

Since $a\in S, s\in S$, so $as\subseteq S$ and thus $x\in S$. Therefore, $aS\subseteq S$.

Let $y\in S$. Then, there exists a unique $y^{-1}\in S$ such that $e\in yy^{-1}$.

It follows that $a\in ea\subseteq(yy^{-1})a=y(y^{-1}a)$. Then $a\in yb$ for some $b\in y^{-1}a$.

This implies that $y\in ab^{-1}\subseteq a(a^{-1}y)$.

Since $a^{-1}\in S, y\in S$, $a^{-1}y\subseteq S$, and hence $y\in a(a^{-1}y)\subseteq aS$.

So, $S\subseteq aS$, and thus $aS=S$. Similarly, $Sa=S$ for all $a\in S$. □

Notice that the converse of Lemma 3.5.4 is not true, for example if $S=\{a,b\}$ with the following table:

$\cdot$	a	b
a	$\{a\}$	$\{a,b\}$
b	$\{b\}$	$\{a,b\}$

then $aS=S=Sa$ and $bS=S=Sb$, but S is not a polygroup.

Proposition 3.5.5. *If semipolygroup with zero S is a 0-polygroup, then for all $a\in S-\{0\}, aS=S$ and $Sa=S$.*

Proof. Suppose first that $S = P^0$, a 0-polygroup, and let $a \in P = S - \{0\}$.

Certainly $aP = Pa = P$.

Since $aS = aP \cup \{0\}$ and $Sa = Pa \cup \{0\}$, it follows that $aS = Sa = S$. □

Lemma 3.5.6. *Let S be a semipolygroup satisfies the following conditions:*

(i) there exists $e \in S$ such that $xe = \{x\}$ for all $x \in S$;

(ii) for each $x \in S$ there exists $x^{-1} \in S$ such that $e \in xx^{-1}$;

(iii) for all $x, y, z \in S$, $x \in yz$ implies $y \in xz^{-1}$ and $z \in y^{-1}x$.

Then S is a polygroup.

Proof. First, let $x, y \in S$. We will show that $e^{-1} = e$ and $x = (x^{-1})^{-1}$.

From (i), we have $e \in ee$. Then by (iii), $e \in e^{-1}e = \{e^{-1}\}$, so $e^{-1} = e$.

Since $x \in xe$, $e \in x^{-1}x$.

By (iii) $x \in (x^{-1})^{-1}e = \{(x^{-1})^{-1}\}$ and we then have $x = (x^{-1})^{-1}$.

Suppose thats $p \in S$ such that $e \in xp$ and $e \in px$.

This implies that $p \in x^{-1}e = \{x^{-1}\}$, that is $p = x^{-1}$. Hence x^{-1} is unique.

Finally, we prove that $ex = \{x\}$.

Since $e \in xx^{-1}$, so we also have that $x \in e(x^{-1})^{-1} = ex$. Hence, $\{x\} \subseteq ex$.

Let $a \in ex$. Then $e \in ax^{-1}$ and $x^{-1} \in a^{-1}e = \{a^{-1}\}$.

Certainly, we obtain that $x^{-1} = a^{-1}$.

It is immediately that $e \in xa^{-1}$ and $e \in a^{-1}x$.

Consider $e \in a^{-1}x$, we see that $x \in (a^{-1})^{-1}e = ae$, and finally we get

$a \in xe^{-1} = xe = \{x\}$.

Thus, we have established that $ex \subseteq \{x\}$, and so $ex = \{x\}$.

Therefore, S is a polygroup. □

Notice that if S is an inverse semipolygroup which satisfies the following conditions: (i) and (ii) then S is a polygroup.

Lemma 3.5.7. *A non-empty subset K of a polygroup P is a subpolygroup of P if and only if*

$$ab^{-1} \subseteq K \qquad \textit{for all } a, b \in K.$$

Proof. Suppose that K is a subpolygroup of polygroup P.

Let $a, b \in K$.

By Lemma 2.1.12, we get $b^{-1} \in K$, and thus $ab^{-1} \subseteq K$.

Assume that $ab^{-1} \subseteq K$ for all $a, b \in K$.

Let $k_1, k_2 \in K$. Then $k_1 k_1^{-1} \subseteq K$.

Since $k_1 \in K \subseteq P$, so $e \in k_1 k_1^{-1} \subseteq K$.

By hypothesis, it follows that $ek_1^{-1} \subseteq K$.

But $ek_1^{-1} = \{k_1^{-1}\}, \{k_1^{-1}\} \subseteq K$, and so $k_1^{-1} \in K$.

Hence $k_2^{-1} \in K$.

We obtain $k_1 k_2 = k_1 (k_2^{-1})^{-1} \subseteq K$.

Therfore, K is a subpolygroup of polygroup P. □

Corollary 3.5.8. *If $< S, \circ, e, ^{-1} >$ is a polygroup and ρ is an equivalence relation on S,then ρ is regular if and only if $< S/\rho, \otimes, \rho(e), ^{-I} >$ is a polygroup, where $\rho(a)^{-I} = \rho(a^{-1})$.*

Proof. Let S be a polygroup and ρ be an equivalence relation on S.

Suppose that ρ is regular. We have $(S/\rho, \otimes)$ is a semipolygroup.

Next, we show that $\rho(e)$ is an identity element of S/ρ. Let $x \in S$. Then

$$\begin{aligned} \rho(x) \otimes \rho(e) &= \{\rho(a) : a \in xe = \{x\}\} \\ &= \{\rho(a) : a \in \{x\}\} \\ &= \{\rho(x)\}, \end{aligned}$$

and

$$\begin{aligned}\rho(e)\otimes\rho(x) &= \{\rho(b) : b\in ex=\{x\}\}\\ &= \{\rho(b) : b\in\{x\}\}\\ &= \{\rho(x)\}.\end{aligned}$$

Hence, $\rho(e)$ is an identity element of S/ρ.

For each $\rho(x)\in S/\rho$, we have that $x\in S$.

Then, there exists a unique $x^{-1}\in S$ such that $e\in xx^{-1}$ and $e\in x^{-1}x$.

That is, there is $\rho(x)^{-I}=\rho(x^{-1})\in S/\rho$ such that

$$\begin{aligned}\rho(e) &\in \rho(x)\otimes\rho(x)^{-I}=\rho(x)\otimes\rho(x^{-1})=\{\rho(a) : a\in xx^{-1}\}\\ \text{and}\quad \rho(e) &\in \rho(x)^{-I}\otimes\rho(x)=\rho(x^{-1})\otimes\rho(x)=\{\rho(b) : b\in x^{-1}x\}.\end{aligned}$$

Next, let $\rho(y)\in S/\rho$ be such that $\rho(e)\in\rho(x)\otimes\rho(y)$ and $\rho(e)\in\rho(y)\otimes\rho(x)$.

Then $e\in xy$ and $e\in yx$.

It follows that $y=x^{-1}$ and so $\rho(x^{-1})=\rho(x)^{-I}$ is unique.

So, we get $\rho(x)^{-I}=\rho(x^{-1})$ is an inverse of $\rho(x)$ in S/ρ.

Now, we show that $\rho(x)\in\rho(y)\otimes\rho(z)$ implies $\rho(y)\in\rho(x)\otimes\rho(z)^{-I}$ and $\rho(z)\in\rho(y)^{-I}\otimes\rho(x)$.

If $\rho(x)\in\rho(y)\otimes\rho(z)$, then $x\in yz$.

This implies that $y\in xz^{-1}$ and $z\in y^{-1}x$.

It follows that
$$\begin{aligned}\rho(y) &\in \rho(x)\otimes\rho(z^{-1})=\rho(x)\otimes\rho(z)^{-I} \text{ and}\\ \rho(z) &\in \rho(y^{-1})\otimes\rho(x)=\rho(y)^{-I}\otimes\rho(x).\end{aligned}$$

Therefore, S/ρ is a polygroup.

Conversely, we let $a\rho b$ and x be an arbitrary element of S.

If $u\in ax$, then $\rho(u)\in\rho(a)\otimes\rho(x)=\rho(b)\otimes\rho(x)=\{\rho(v) : v\in bx\}$.

Therefore, there exists $v\in bx$ such that $\rho(u)=\rho(v)$, i.e., $u\,\rho\,v$.

If $v' \in bx$, then $\rho(v') \in \rho(b) \otimes \rho(x) = \rho(a) \otimes \rho(x)$.

There exists $u' \in ax$ such that $\rho(u') = \rho(v')$, i.e., $u' \, \rho \, v'$.

Hence, $(ax)\bar{\rho}(bx)$.

Similarly, we obtain that ρ is regular on the left.

Therefore, ρ is regular. $\square$

Theorem 3.5.9. *Let $\phi : P \to S$ be a morphism from a polygroup P into an inverse semipolygroup S. Then $\phi(P)$ is a polygroup.*

Proof. Suppose that s is an element in $\phi(P)$.

Then there exists $p \in P$ such that $\phi(p) = s$.

We have P is an inverse semipolygroup, because P is a polygroup.

By Theorem 3.3.22, $\phi(P)$ is an inverse semipolygroup.

We obtain $\{s\} = \{\phi(p)\} = \phi(\{p\}) = \phi(ep) = \phi(e)\phi(p) = e's$ and

$\{s\} = \{\phi(p)\} = \phi(\{p\}) = \phi(pe) = \phi(p)\phi(e) = se'$, where $e' = \phi(e) \in \phi(P)$.

Since $p \in P$, $ep = pe = \{p\}$ and there exists a unique $p^{-1} \in P$ such that $e \in pp^{-1}$ and $e \in p^{-1}p$.

Because P is an inverse semipolygroup,

$$p \in pp^{-1}p, \text{ and so } \phi(p) \in \phi(pp^{-1}p) = \phi(p)\phi(p^{-1})\phi(p).$$

Since $s \in \phi(P), s \in ss^{-1}s$ which is follows that $\phi(p) \in \phi(p)(\phi(p))^{-1}\phi(p)$.

It holds that $\phi(p^{-1}) = (\phi(p))^{-1} = s^{-1}$, because an inverse is unique.

Since $e \in pp^{-1}$ and $e \in p^{-1}p$, $e' = \phi(e) \in \phi(pp^{-1}) = \phi(p)\phi(p^{-1}) = ss^{-1}$ and $e' = \phi(e) \in \phi(p^{-1}p) = \phi(p^{-1})\phi(p) = s^{-1}s$.

Hence, $\phi(P)$ is a polygroup. $\square$

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