INVOLUTIVE WEAK GLOBULAR HIGHER CATEGORIES

BY

MR. PARATAT BEJRAKARBUM

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
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ENTITLED

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ABSTRACT

In this thesis, we investigate the notion of involutive weak globular ω-categories via the Penon's approach. In particular, we give the constructions of a free reflexive self-dual globular ω-magma and a free involutive globular ω-category over a modified ω-globular set. The monadic definition of involutive weak globular ω-categories is defined by the existence of a certain adjunction. Some examples of involutive weak globular ω-categories are also provided.

Keywords: Higher Category, Involutive Category, Monad

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CHAPTER 1

INTRODUCTION

1.1 Literature Review

Category theory has currently been an interesting subject in mathematics for several decades. It has various powerful connections not only with algebraic topology, but also with logic, computer science, foundation of mathematics, mathematical physics, and so on (see for instance [BS, L2, UFP] and also [B] for some theoretical applications to relational quantum theory). Category theory is a vigorous language or conceptual system letting us to consider what universal components a collection of certain structures has and how distinct structures are correlated.

In 1945, the first abstract definition of categories was introduced by S. Eilenberg - S. Mac Lane [EM] in the similar fashion of the group axioms to define the notions of functors between categories and natural transformations between functors. Later, the more practical definition of categories was used by A. Grothendieck in 1957 and P. J. Freyd in 1964 using the terminology in set theory. For further discussion on the history of category theory, the reader is required to see [SEP].

As a category (or 1-category) consists of objects and morphisms, the notion of a 2-category[1] generalizes this idea by adding 2-arrows between the 1-arrows. Continuing this process up to n-arrows between $(n-1)$-arrows yields an n-category. Strict n-categories were originally formalized by C. Ehresmann in both cubical forms [E1] in 1963 and globular forms [E2] in 1965. Furthermore, enriched categories of M. Kelly - S. Eilenburg [EK] permit an iterative construction of strict higher categories.

Sometimes the notion of strict higher categories is “too strict” for some

[1]Since natural transformations between functors are an example of globular 2-arrows in a strict 2-category, the notion of higher category theory implicitly existed in the same period as category theory.

structures; for example, in the category of topological spaces, composition of paths satisfies associativity and unitality only up to reparametrization. In 1963, weak categories, which are categories whose axioms of associativity and unitality are satisfied only up to higher-level isomorphisms, were first introduced in the definition of weak monoidal categories [Be1, M]. Later, in 1967, the concept of monoidal categories was generalized to bicategories by J. Bénabou [Be2].

In 1979, strict globular ω-categories were first introduced by J. Roberts, who connects category theory with the study of algebraic quantum field theory [R]. During 1977-1981, R. Brown - P. Higgins considered the notions of strict cubical ω-groupoids and categories in a series of works [BH]. In 1983, strict globular ω-categories were described and the usage of weak ω-groupoids was proposed as a way to capture the homotopy content of spaces in A. Grothendieck's well-known manuscript "Pursuing Stacks" [G].

In 1987, the first definition of weak ω-categories was introduced by R. Street [S] based on the algebra of "symplexes". Later, J. Baez - J. Dolan were motivated by this concept to obtain the "opetopic" approach to weak n-categories [BD]. In addition, T. Trimble considered a partially algebraic approach to weak n-categories via "enrichment" [Tr] in 1999. The definitions of weak n-categories and weak ω-categories have been a progressing process with various alternative definitions under discussion[2]. Recently, M. Batanin [Ba1, Ba2], J. Penon [P], and T. Leinster [L2] developed algebraic definitions of weak globular ω-categories as "algebras" for certain "monads".

In category theory, it is normal to supplement some additional structures to usual categories, such as involutions. However, the concepts of involutions in category theory have been used in different aspects. Strict involutions have appeared in several works; for example, M. Burgin [Bu] (1970), P. Ghez - R. Lima - J. Roberts [GLR] (1985), P. Freyd - A. Scedrov [FS] (1993), J. Lambek [La] (1999), S. Abramsky - B. Coecke [AC] (2004), P. Selinger [Se] (2005), and many more.

[2] For further discussion of comparison among several possible definitions, we refer to [CL, L2].

In 2014, P. Bertozzini et. al. described involutions for strict globular n-categories as covariant/contravariant endofuctors for compositions in [BCLS] and for the case of strict cubical 2-categories in [BCM]. Moreover, they also considered the notion of weak monoidal categories in [BCL3].

1.2 Overview

In order to obtain a possible treatment of weak higher C*-categories, our main objective is to propose a definition of weak involutive higher categories in the spirit of J. Penon's definition of weak globular ω-categories [P] and its variants [Ba2, ChM, L1, K]. Next, we are going to describe the content of the thesis.

In sections 2.1 and 2.2, we briefly review some basic notions of elementary algebraic and topological structures including groups, rings, vector spaces, modules, topological spaces, and their structure-preserving functions in order to get an idea of fundamental examples of categories which will be defined in the following section.

In section 2.3, some elementary concepts of category theory are provided; for example, some equivalent definitions of categories, functors between categories, free structures, natural transformations between functors, and adjunctions between functors. In addition, the notions of monads and algebras for monads are discussed in such a way that we can give a definition of weak involutive globular ω-categories as an algebra for an appropriate monad.

In section 2.4, we briefly recall the fundamental notions on strict higher categories which are necessary for our work. In order to contact with J.Penon's approach, we define strict higher categories via "higher quivers", whose definition is recalled in subsection 2.4.1. A previous work on higher categories [BCLS] used an algebraic definition of strict higher categories via "partial monoids on n-arrows"; a discussion of the categorical equivalence between the two descriptions is referred to [Pu, Proposition 2.4.3]. In our work, we restrict our attention to the case of globular higher quivers and globular higher categories based on them[3].

Since only strict n-categories are discussed in [BCLS], in this work we also provide the more general case of strict globular ω-categories. The definition of strict involutive n-category from [BCLS] is extended in the similar fashion to the case of strict involutive globular ω-categories. We remark that for strict (involutive) globular ω-categories it is also possible to replace the "usual exchange" axiom with the more relaxed "non-commutative exchange" property discussed in [BCLS].

In order to fix the notation and to make the thesis self-contained, the essential features of J.Penon's construction are recalled in subsection 2.4.3. In our case, ω-globular sets will not be required to be reflexive (thus avoiding the already known problems described in [ChM]).

The main subject of this work is described as follows. The existence of a free reflexive self-dual globular ω-magma and a free involutive Penon contraction over an ω-globular set is discussed in sections 3.1 and 3.2. The concept and some examples of involutive weak globular ω-categories are described in section 3.3. The notion and some properties of globular cones are described in section 4.1. An explicit definition and construction of free reflexive self-dual globular-cone ω-magmas is discussed in section 4.3 and free strict involutive globular-cone ω-categories over a globular cone is presented in detail in section 4.4. Moreover, a similar construction of the free involutive Penon cone-contraction over a globular cone is given. In section 4.5 we prove that the forgetful functor from the category $\hat{\mathscr{Q}}^*$ of involutive Penon cone-contractions to the category of globular cones admits a left-adjoint and then we give the monadic definition of involutive weak globular-cone ω-categories as an algebra for such monad.

[3]The treatment of cubical higher categories will be an objective of a further separate investigation.

CHAPTER 2

PRELIMINARIES

This chapter is devoted to recall some definitions, theorems, and examples of elementary algebraic and topological structures and basic category theory. Later in this chapter, we discuss the Penon's definition of weak ω-categories and the notion of strict involutive globular higher categories.

2.1 Elementary Algebraic Structures

In this section we recollect some basic definitions and examples of binary operations, groups, rings, vector spaces, and R-modules as some of these concepts will be important examples later on.

2.1.1 Binary Operations

Throughout this thesis we denote the sets of integers, positive integers, negative integers, natural numbers, natural numbers including zero, rational numbers, positive rational numbers, real numbers, positive real numbers, and complex numbers by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{Q}^+$,$\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$, respectively.

Definition 2.1.1.1. Let A be a nonempty set. We say that $*$ is a **binary operation** on A if $*$ is a function from $A \times A$ into A.

Remark 2.1.1.2. For convenience, we usually denote the image of $(x,y) \in A \times A$ under the binary operation $*$ by $*((x,y)) := x * y$.

Example 2.1.1.3. Let $n \in \mathbb{N}$ and $n\mathbb{Z} := \{\dots, -2n, -n, 0, n, 2n, \dots\}$. We see that

1. $+$, $-$, and $\cdot$ are binary operations on $\mathbb{Z}$, $n\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$,

2. $\div$ is a binary operation on $\mathbb{Q}\setminus\{0\}$, $\mathbb{R}\setminus\{0\}$, and $\mathbb{C}\setminus\{0\}$,

3. $-$ is not a binary operation on $\mathbb{Z}^+$, $\mathbb{Q}^+$, and $\mathbb{R}^+$,

4. $\div$ is not a binary operation on $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

Example 2.1.1.4. Let $n \in \mathbb{N}$ and consider the set $\mathbb{Z}_n := \{[k]_n \subseteq \mathbb{Z} \mid k = 0, 1, \ldots, n-1\}$, where $[k]_n := \{x \in \mathbb{Z} \mid x \equiv k(mod\ n)\}$. Define, for each $[a]_n, [b]_n \in \mathbb{Z}_n$,

- $[a]_n +_n [b]_n := [a+b]_n$,
- $[a]_n \cdot_n [b]_n := [a \cdot b]_n$.

Thus, $+_n$ and $\cdot_n$ are binary operations on $\mathbb{Z}_n$.

Example 2.1.1.5. Let $m, n \in \mathbb{N}$ and $\mathbb{M}_{m\times n}(\mathbb{R})$ be the set of $m \times n$ matrices with real entries. Define

1. $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} := [a_{ij} + b_{ij}]_{m\times n}$ for all $[a_{ij}]_{m\times n}, [b_{ij}]_{m\times n} \in \mathbb{M}_{m\times n}(\mathbb{R})$,

2. $[a_{ij}]_{n\times n} \cdot [b_{ij}]_{n\times n} := [c_{ij}]_{n\times n}$, where $c_{ij} = \sum\limits_{k=1}^{n} a_{ik}b_{kj}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, for all $[a_{ij}]_{n\times n}, [b_{ij}]_{n\times n} \in \mathbb{M}_{n\times n}(\mathbb{R})$.

Hence, $+$ and $\cdot$ are binary operations on $\mathbb{M}_{m\times n}(\mathbb{R})$ and $\mathbb{M}_{n\times n}(\mathbb{R})$, respectively.

Example 2.1.1.6. Let S be a set and $P(S)$ the power set of S. Define, for all $A, B \in P(S)$,

- $\cup((A,B)) := A \cup B$,
- $\cap((A,B)) := A \cap B$,
- $\setminus((A,B)) := A \setminus B$,
- $\triangle((A,B)) := A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Notice that $\cup$, $\cap$, $\setminus$, and $\triangle$ are binary operations on $P(S)$.

Example 2.1.1.7. Let A be a nonempty set. We see that $\circ$ (composition of functions) is a binary operation on $\mathrm{End}(A) := \{f \mid f : A \to A\}$ and $\mathrm{Aut}(A) := \{f \mid f : A \underset{onto}{\overset{1-1}{\to}} A\}$.

2.1.2 Groups and Group Homomorphisms

We begin this section by the notions of sets endowed with a binary operation. For more details, see Grillet P. A. [Gr]. For convenience, we let $m,n \in \mathbb{N}$ in this section.

Definition 2.1.2.1. A **magma** (or **groupoid**) $(A,*)$ is a nonempty set A equipped with a binary operation $*$, i.e. $* : A \times A \to A$ defined by $(x,y) \mapsto x*y$ for every $x,y \in A$.

Example 2.1.2.2. $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{Z},\cdot)$, $(\mathbb{Q},\cdot)$, $(\mathbb{R},\cdot)$, $(\mathbb{Z},-)$, $(\mathbb{Q}^+,\div)$, $(\mathbb{Z}_n,+_n)$, $(\mathbb{Z}_n,\cdot_n)$, $(n\mathbb{Z},+)$, $(\mathbb{M}_{m\times n}(\mathbb{R}),+)$, $(\mathbb{M}_{m\times n}(\mathbb{R}),-)$, $(\mathbb{M}_{n\times n}(\mathbb{R}),\cdot)$, $(P(S),\cup)$, $(P(S),\cap)$, $(P(S),\setminus)$, $(P(S),\triangle)$, $(\mathrm{End}(A),\circ)$, and $(\mathrm{Aut}(A),\circ)$ are magmas (or groupoids) while $(\mathbb{Z}^+,-)$ and $(\mathbb{Z}^+,\div)$ are not.

It is natural to ask when two magmas are related via a function which preserves their structure.

Definition 2.1.2.3. Let $(M,*)$ and $(N,\diamond)$ be magmas. A function $f : M \to N$ is called a **homomorphism of magmas** if $f(x*y) = f(x) \diamond f(y)$ for all $x,y \in M$.

Example 2.1.2.4. The following functions are homomorphisms of magmas:

1. $f : (\mathbb{Z},-) \to (\mathbb{Z},-)$ defined by $f(x) := -x$ for all $x \in \mathbb{Z}$,
2. $g : (\mathbb{Q}^+,\div) \to (\mathbb{Q}^+,\div)$ defined by $g(\frac{a}{b}) := \frac{b}{a}$ for each $\frac{a}{b} \in \mathbb{Q}^+$,
3. $h : (P(S),\cup) \to (P(P(S)),\cap)$ defined by $h(A) := A^c$ for every $A \in P(S)$.

Proof. Let $x,y \in \mathbb{Z}$, $\frac{a}{b},\frac{c}{d} \in \mathbb{Q}^+$ and $A,B \in P(S)$.

We have the following equalities:

1. $f(x-y) = -(x-y) = -x-(-y) = f(x)-f(y)$,
2. $g(\frac{a}{b} \div \frac{c}{d}) = g(\frac{ad}{bc}) = \frac{bc}{ad} = \frac{b}{a} \div \frac{d}{c} = g(\frac{a}{b}) \div g(\frac{c}{d})$,
3. $h(A \cup B) = (A \cup B)^c = A^c \cap B^c = h(A) \cap h(B)$.

Hence, f, g and h are homomorphisms of magmas. $\square$

Definition 2.1.2.5. Let $(S,*)$ be a magma. We say that $(S,*)$ is a **semigroup** if $x*(y*z) = (x*y)*z$ for all $x,y,z \in S$.

Example 2.1.2.6. $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{Z},\cdot)$, $(\mathbb{Q},\cdot)$, $(\mathbb{R},\cdot)$, $(\mathbb{Z}_n,+_n)$, $(\mathbb{Z}_n,\cdot_n)$, $(n\mathbb{Z},+)$, $(\mathbb{M}_{m\times n}(\mathbb{R}),+)$, $(\mathbb{M}_{n\times n}(\mathbb{R}),\cdot)$, $(P(S),\cup)$, $(P(S),\cap)$, $(P(S),\triangle)$, $(\mathrm{End}(A),\circ)$, and $(\mathrm{Aut}(A),\circ)$ are semigroups while $(\mathbb{Z},-)$, $(\mathbb{Q}^+,\div)$, $(\mathbb{M}_{m\times n}(\mathbb{R}),-)$, and $(P(S),\setminus)$ are not.

Definition 2.1.2.7. Let $(S,*)$ and $(T,\diamond)$ be semigroups. A function $f: S \to T$ is called a **homomorphism of semigroups** if $f(x*y) = f(x)\diamond f(y)$ for all $x,y \in S$.

Example 2.1.2.8. The following functions are homomorphisms of semigroups:

1. $h: (2\mathbb{Z},+) \to (3\mathbb{Z},+)$ defined by $h(2x) := 3(2x)$ for every $2x \in 2\mathbb{Z}$,
2. $k: (P(S),\cap) \to (P(P(S)),\cap)$ defined by $k(A) := P(A)$ for each $A \in P(S)$.

Proof. Let $2x, 2y \in 2\mathbb{Z}$ and $A, B \in P(S)$.

We get the following assertions:

1. $h(2x+2y) = 3(2x+2y) = 3(2x)+3(2y) = h(2x)+h(2y)$,
2. $k(A\cap B) = P(A\cap B) = P(A)\cap P(B) = k(A)\cap k(B)$.

Thus, h and k are homomorphisms of semigroups. $\square$

Definition 2.1.2.9. Let $(M,*)$ be a semigroup. We say that $(M,*)$ is a **monoid** if there exists $e \in M$ such that $x*e = x = e*x$ for all $x \in M$.

Example 2.1.2.10. $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{Z},\cdot)$, $(\mathbb{Q},\cdot)$, $(\mathbb{R},\cdot)$, $(\mathbb{Z}_n,+_n)$, $(\mathbb{Z}_n,\cdot_n)$, $(n\mathbb{Z},+)$, $(\mathbb{M}_{m\times n}(\mathbb{R}),+)$, $(\mathbb{M}_{n\times n}(\mathbb{R}),\cdot)$, $(P(S),\cup)$, $(P(S),\cap)$, $(P(S),\triangle)$, $(\mathrm{End}(A),\circ)$, and $(\mathrm{Aut}(A),\circ)$ are monoids while $(2\mathbb{Z},\cdot)$ is not.

Definition 2.1.2.11. Let $(M,*)$ and $(N,\diamond)$ be monoids. A function $f: M \to N$ is called a **homomorphism of monoids** if $f(x*y) = f(x)\diamond f(y)$ for all $x,y \in M$ and if also $f(1_M) = 1_N$, then f is **unital**.

Example 2.1.2.12. Let $f, g : (\mathbb{R}, \cdot) \to (\mathbb{M}_{2\times 2}(\mathbb{R}), \cdot)$ be defined by

$$f(a) := \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \text{ and } g(a) := \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that f is a unital homomorphism of monoids while g is a homomorphism of monoids but not unital.

Proof. Let $a, b \in \mathbb{R}$.

We obtain the following equalities:

1. $f(a \cdot b) = \begin{bmatrix} a \cdot b & 0 \\ 0 & a \cdot b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = f(a) \cdot f(b),$

2. $g(a \cdot b) = \begin{bmatrix} a \cdot b & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = g(a) \cdot g(b).$

This yields that f and g are homomorphisms of monoids.

Notice that 1 is the identity of $(\mathbb{R}, \cdot)$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity of $(\mathbb{M}_{2\times 2}(\mathbb{R}), \cdot)$.

Since $f(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but $g(1) \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, f is unital but g is not unital. $\square$

Definition 2.1.2.13. Let $(G, *)$ be a monoid. We say that $(G, *)$ is a **group** if for each $x \in G$ there exists $y \in G$ such that $x * y = e = y * x$.

Example 2.1.2.14. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Z}_n, +_n)$, $(\mathbb{Z}_n, \cdot_n)$, $(n\mathbb{Z}, +)$, $(\mathbb{M}_{m\times n}(\mathbb{R}), +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{R} \setminus \{0\}, \cdot)$, $(P(S), \triangle)$, and $(\mathrm{Aut}(A), \circ)$ are groups while $(\mathbb{Z}, \cdot)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{M}_{n\times n}(\mathbb{R}), \cdot)$, $(P(S), \cup)$, $(P(S), \cap)$, and $(\mathrm{End}(A), \circ)$ are not, where $S \neq \varnothing$.

Definition 2.1.2.15. Let $(G, *)$ and $(H, \diamond)$ be groups. A function $f : G \to H$ is called a **group homomorphism** if $f(x * y) = f(x) \diamond f(y)$ for all $x, y \in G$.

Remark 2.1.2.16. We do not require that group homomorphisms preserve identities and inverses as this is an immediate consequence of their definition.

Example 2.1.2.17. The following functions are group homomorphisms:

1. $f:(\mathbb{R},+)\to(\mathbb{R}^+,\cdot)$ defined by $f(x):=3^x$ for all $x\in\mathbb{R}$,
2. $g:(\mathbb{Z},+)\to(\mathbb{Q},+)$ defined by $g(x):=\frac{x}{y}$ for all $x\in\mathbb{Z}$, where $y\in\mathbb{R}\setminus\{0\}$,
3. $h:(\mathbb{R},+)\to(\mathbb{R},+)$ defined by $h(x):=-\pi x$ for all $x\in\mathbb{R}$.

Proof. Let $y\in\mathbb{R}\setminus\{0\}$, $a,b\in\mathbb{R}$ and $r,s\in\mathbb{Z}$.

We have the following equalities:

1. $f(a+b)=3^{a+b}=3^a\cdot 3^b=f(a)\cdot f(b)$,
2. $g(r+s)=\frac{r+s}{y}=\frac{r}{y}+\frac{s}{y}=g(r)+g(s)$,
3. $h(a+b)=-\pi(a+b)=-\pi(a)+(-\pi(b))=h(a)+h(b)$.

Thus, f,g and h are group homomorphisms. □

Example 2.1.2.18. The following functions are not group homomorphisms:

1. $f:(\mathbb{Z},+)\to(\mathbb{Z},+)$ defined by $f(x):=x+1$ for all $x\in\mathbb{Z}$,
2. $g:(\mathbb{Q}^+,\cdot)\to(\mathbb{Q}^+,\cdot)$ defined by $g(x):=\frac{1}{x+1}$ for each $x\in\mathbb{Q}^+$.

Proof. Consider the following arguments:

1. $f(0+1)=f(1)=2\neq 3=1+2=f(0)+f(1)$,
2. $g(2\cdot 1)=g(2)=\frac{1}{3}\neq\frac{1}{6}=\frac{1}{3}\cdot\frac{1}{2}=g(2)\cdot g(1)$.

Thus, f and g are not group homomorphisms. □

Definition 2.1.2.19. Let $(G,*)$ be a group. We say that $(G,*)$ is an **abelian group** if $x*y=y*x$ for all $x,y\in G$.

Example 2.1.2.20. $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{Z}_n,+_n)$, $(\mathbb{Z}_n,\cdot_n)$, $(n\mathbb{Z},+)$, $(\mathbb{M}_{m\times n}(\mathbb{R}),+)$, $(\mathbb{Q}\setminus\{0\},\cdot)$, $(\mathbb{R}\setminus\{0\},\cdot)$, and $(P(S),\triangle)$ are abelian groups while $(\mathrm{Aut}(A),\circ)$ is not.

2.1.3 Rings and Ring Homomorphisms

We now consider some algebraic structures equipped with two binary operations which are based on the notion of abelian groups. For further discussion of rings, the reader is required to see Grillet P. A. [Gr].

Definition 2.1.3.1. Let R be a nonempty set and $+$ and $\cdot$ be binary operations on R. We say that $(R,+,\cdot)$ is a **ring** if

1. $(R,+)$ is an abelian group,
2. $(R,\cdot)$ is a semigroup,
3. $z\cdot(x+y)=(z\cdot x)+(z\cdot y)$ and $(x+y)\cdot z=(x\cdot z)+(y\cdot z)$ for all $x,y,z\in R$.

Example 2.1.3.2. $(\mathbb{Z},+,\cdot),(\mathbb{Q},+,\cdot),(\mathbb{R},+,\cdot),(\mathbb{C},+,\cdot),(n\mathbb{Z},+,\cdot),(\mathbb{Z}_n,+_n,\cdot_n)$, and $(\mathbb{M}_{n\times n}(\mathbb{R}),+,\cdot)$ are rings.

Example 2.1.3.3. Let $X\neq\varnothing$ and $(R,+,\cdot)$ be a ring. Consider $R^X:=\{f\mid f:X\to R\}$. Define binary operations $\oplus$ and $\otimes$ on R^X as follows: for every $f,g\in R^X$ and $x\in X$,

1. $(f\oplus g)(x):=f(x)+g(x)$,
2. $(f\otimes g)(x):=f(x)\cdot g(x)$.

It is easy to see that $(R^X,\oplus,\otimes)$ is a ring.

Definition 2.1.3.4. A ring $(R,+,\cdot)$ is said to be **unital** if $(R,\cdot)$ is a monoid.

Example 2.1.3.5. $(\mathbb{Z},+,\cdot),(\mathbb{Q},+,\cdot),(\mathbb{R},+,\cdot),(\mathbb{C},+,\cdot),(\mathbb{Z}_n,+_n,\cdot_n)$, and $(\mathbb{M}_{n\times n}(\mathbb{R}),+,\cdot)$ are unital rings while $(2\mathbb{Z},+,\cdot)$ is not.

Definition 2.1.3.6. A ring $(R,+,\cdot)$ is **commutative** if $x\cdot y=y\cdot x$ for all $x,y\in R$.

Example 2.1.3.7. $(\mathbb{Z},+,\cdot),(\mathbb{Q},+,\cdot),(\mathbb{R},+,\cdot),(\mathbb{C},+,\cdot),(n\mathbb{Z},+,\cdot)$, and $(\mathbb{Z}_n,+_n,\cdot_n)$ are commutative rings while $(\mathbb{M}_{n\times n}(\mathbb{R}),+,\cdot)$ is not.

Similar to group homomorphisms, functions which preserves the structures of rings are called ring homomorphisms.

Definition 2.1.3.8. Let $(R,+,\cdot)$ and $(S,\oplus,\otimes)$ be two rings.
A function $f:(R,+,\cdot)\to(S,\oplus,\otimes)$ is called a **ring homomorphism** if

1. $f(x+y)=f(x)\oplus f(y)$,
2. $f(x\cdot y)=f(x)\otimes f(y)$

for all $x,y\in R$. And we call it a **unital ring homomorphism** if it is a ring homomorphism and $f(1_R)=1_S$.

Example 2.1.3.9. Let $\mathbb{Z}(i)=\{a+bi\mid a,b\in\mathbb{Z}\}$. We see that $(\mathbb{Z}(i),+,\cdot)$ is a ring.
A function $\theta:\mathbb{Z}(i)\to\mathbb{Z}(i)$ defined by $\theta(a+bi):=a-bi$, for all $a+bi\in\mathbb{Z}(i)$, is a unital ring homomorphism.

Definition 2.1.3.10. Let R be a ring and $\varnothing\neq I\subseteq R$. We say that I is an **ideal** of R if

1. $a-b\in I$ for all $a,b\in I$,
2. $ar\in I$ and $ra\in I$ for all $a\in I$ and $r\in R$.

Example 2.1.3.11. For each $a\in\mathbb{Z}$, we have $(a):=\{na\mid n\in\mathbb{Z}\}$ is an ideal of $(\mathbb{Z},+,\cdot)$.

Definition 2.1.3.12. Let I be an ideal of a ring R. We say that I is a **prime ideal** of R if, for each $a,b\in R$, $ab\in I$ implies $a\in I$ or $b\in I$.

Example 2.1.3.13. For each prime number p, (p) is a prime ideal of $(\mathbb{Z},+,\cdot)$.

Definition 2.1.3.14. A unital ring $(R,+,\cdot)$ is a **division ring** if $(R\setminus\{0_R\},\cdot)$ is a group.

Example 2.1.3.15. $(\mathbb{Q},+,\cdot),(\mathbb{R},+,\cdot),(\mathbb{C},+,\cdot)$, and $(\mathbb{Z}_p,+_p,\cdot_p)$, where p is a prime number, are division rings while $(\mathbb{Z},+,\cdot)$ is not.

Definition 2.1.3.16. A ring $(R,+,\cdot)$ is a **field** if $(R,+,\cdot)$ is a commutative division ring.

Example 2.1.3.17. $(\mathbb{Q},+,\cdot),(\mathbb{R},+,\cdot),(\mathbb{C},+,\cdot)$, and $(\mathbb{Z}_p,+_p,\cdot_p)$, where p is a prime number, are fields.

2.1.4 Vector Spaces and Linear Transformations

In this subsection we discuss another structure based on abelian groups. Unlike rings, *vector spaces* are sets equipped with a binary operation and a scalar multiplication satisfying some properties. For more discussion of vector spaces, see Sunder V. S. [Su].

Definition 2.1.4.1. Let $\mathbb{K}$ be a field and $(V,+)$ an abelian group. We say that $(V,+,\cdot)$ is a **vector space** over the field $\mathbb{K}$ if there exists a scalar multiplication $\cdot : \mathbb{K} \times V \to V$ defined by $(\alpha,x) \mapsto \alpha \cdot x$, for all $\alpha \in \mathbb{K}$ and $x \in V$, satisfying the following properties:

1. $\alpha \cdot (x+y) = (\alpha \cdot x) + (\alpha \cdot y)$,
2. $(\alpha+\beta) \cdot x = (\alpha \cdot x) + (\beta \cdot y)$,
3. $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$,
4. $1_{\mathbb{K}} \cdot x = x$,

for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in V$.

Some typical examples of vector spaces are given as follows.

Example 2.1.4.2. Let $\mathbb{R}^n := \{(x_1,x_2,\ldots,x_n) \mid x_1,x_2,\ldots,x_n \in \mathbb{R}\}$, where $n \in \mathbb{N}$. Define, for all $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}$,

1. $(x_1,x_2,\ldots,x_n)+(y_1,y_2,\ldots,y_n) := (x_1+y_1,x_2+y_2,\ldots,x_n+y_n)$,
2. $r \cdot (x_1,x_2,\ldots,x_n) := (r \cdot x_1, r \cdot x_2, \ldots, r \cdot x_n)$.

Then $(\mathbb{R}^n,+,\cdot)$ is a vector space over $\mathbb{R}$.

Example 2.1.4.3. Let $\mathbb{M}_{m\times n}(\mathbb{C})$ be the set of $m \times n$ matrices with complex entries. Define, for all $[a_{ij}]_{m\times n}, [b_{ij}]_{m\times n} \in \mathbb{M}_{m\times n}(\mathbb{C})$ and $k \in \mathbb{C}$,

1. $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} := [a_{ij}+b_{ij}]_{m\times n}$,

2. $k \cdot [a_{ij}]_{m\times n} := [k \cdot a_{ij}]_{m\times n}$.

Then $(\mathbb{M}_{m\times n}(\mathbb{C}), +, \cdot)$ is a vector space over $\mathbb{C}$.

Example 2.1.4.4. Let $C[a,b] := \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$.
Define, for each $f, g \in C[a,b]$ and $c \in \mathbb{R}$,

1. $(f+g)(x) := f(x) + g(x)$,

2. $(c \cdot f)(x) := cf(x)$.

Then $(C[a,b], +, \cdot)$ is a vector space over $\mathbb{R}$.

Example 2.1.4.5. Let $p \in \mathbb{N}$ and $l^p := \{\{x_n\} \subseteq \mathbb{C} \mid \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$.
Define, for each $x = \{x_1, x_2, \ldots\}, y = \{y_1, y_2, \ldots\} \in l^p$ and $c \in \mathbb{C}$,

1. $x + y := \{x_1 + y_1, x_2 + y_2, \ldots\}$,

2. $c \cdot x := \{cx_1, cx_2, \ldots\}$.

Hence, $(l^p, +, \cdot)$ is a vector space over $\mathbb{C}$.

Like group and ring homomorphisms, linear transformations play the role of property-preserving maps from a vector space to the other.

Definition 2.1.4.6. Let V and W be vector spaces over the same field $\mathbb{K}$. We call a function $T : V \to W$ a **linear transformation** from V to W if $T(x+y) = T(x) + T(y)$ and $T(cx) = cT(x)$ for all $x, y \in V$ and $c \in \mathbb{K}$.

Here we provide some examples of linear transformations.

Example 2.1.4.7. For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_\theta(x,y) := (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$$

for all $(x,y) \in \mathbb{R}^2$. Then T_θ is a linear transformation.

Example 2.1.4.8. A function $f : \mathbb{M}_{m\times n}(\mathbb{K}) \to \mathbb{M}_{n\times m}(\mathbb{K})$ defined by $f(A) := A^T$, where A^T is the *transpose* of A, for all $A \in \mathbb{M}_{m\times n}(\mathbb{K})$, is a linear transformation.

2.1.5 R-Modules and R-Linear Maps

This subsection discusses mainly about one of the possible generalizations of vector spaces, namely modules, in the sense that scalars in vector spaces are elements of the underlying fields but scalars in modules come from the underlying rings. For further information about modules, see Aluffi P. [A].

Definition 2.1.5.1. Let R be a ring and M an abelian group. We say that

- ${}_RM$ is a **left R-module** if there exists an external multiplication $\cdot : R \times M \to M$ defined by $(r,x) \mapsto r \cdot x$ satisfying the following properties:
  for all $x, y \in M$ and $r, s \in R$,

  1. $r \cdot (x+y) = (r \cdot x) + (r \cdot y)$,
  2. $(r+s) \cdot x = (r \cdot x) + (s \cdot x)$,
  3. $r \cdot (s \cdot x) = (rs) \cdot x$.

  If R is unital, then we also require $1_R \cdot x = x$.

- M_R is a **right R-module** if there exists an external multiplication $\cdot : M \times R \to M$ defined by $(x, r) \mapsto x \cdot r$ satisfying the following properties:
  for all $x, y \in M$ and $r, s \in R$,

  1. $(x+y) \cdot r = (x \cdot r) + (y \cdot r)$,
  2. $x \cdot (r+s) = (x \cdot r) + (x \cdot s)$,
  3. $(x \cdot r) \cdot s = x \cdot (rs)$.

  If R is unital, then we also require $x \cdot 1_R = x$.

Example 2.1.5.2. ${}_{\mathbb{Z}}\mathbb{Z}$ is a left $\mathbb{Z}$-module, ${}_{\mathbb{R}}\mathbb{R}^n$ is a left $\mathbb{R}$-module, and ${}_{\mathbb{C}}\mathbb{M}_{n\times n}(\mathbb{C})$ is a left $\mathbb{C}$-module.

Example 2.1.5.3. $\mathbb{Z}_{\mathbb{Z}}$ is a right $\mathbb{Z}$-module, $\mathbb{R}^n_{\mathbb{R}}$ is a right $\mathbb{R}$-module, and $\mathbb{M}_{n\times n}(\mathbb{C})_{\mathbb{C}}$ is a right $\mathbb{C}$-module.

Similar to linear transformations, R-linear maps are structure-preserving maps from a left/right R-module to the other.

Definition 2.1.5.4. Let R be a ring and ${}_RM$ and ${}_RN$ be left R-modules. A function $\phi :_R M \to_R N$ is said to be a **homomorphism of left R-modules** if $\phi(x+y) = \phi(x)+\phi(y)$ and $\phi(r\cdot x) = r\cdot\phi(x)$ for all $x, y \in_R M$ and $r \in R$.

Definition 2.1.5.5. Let R be a ring and M_R and N_R be right R-modules. A function $\phi : M_R \to N_R$ is said to be a **homomorphism of right R-modules** if $\phi(x+y) = \phi(x)+\phi(y)$ and $\phi(x\cdot r) = \phi(x)\cdot r$ for all $x, y \in M_R$ and $r \in R$.

Remark 2.1.5.6. We call a function ϕ in these definitions an **R-linear map**.

Example 2.1.5.7. An $\mathbb{R}$-linear map $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = mx$ with $m = f(1)$.

When both left and right module structures are present simultaneously, it is usual to further require their mutual compatibility as in the following definition.

Definition 2.1.5.8. Let R and S be rings and M an abelian group. We say that ${}_RM_S$ is a **left-R right-S bimodule** if ${}_RM$ is a left R-module and M_S is a right S-module such that $(r\cdot x)\cdot s = r\cdot(x\cdot s)$ for all $r \in R$, $x \in M$, and $s \in S$.

Example 2.1.5.9. ${}_{\mathbb{Z}}\mathbb{Z}_{\mathbb{Z}}$ is a left-$\mathbb{Z}$ right-$\mathbb{Z}$ bimodule, ${}_{\mathbb{R}}\mathbb{R}^n{}_{\mathbb{R}}$ is left-$\mathbb{R}$ right-$\mathbb{R}$ bimodule, and ${}_{\mathbb{C}}\mathbb{M}_{n\times n}(\mathbb{C})_{\mathbb{C}}$ is a left-$\mathbb{C}$ right-$\mathbb{C}$ bimodule.

Definition 2.1.5.10. Let R be a commutative ring and M_R, N_R, and P_R be R-modules. An **R-bilinear map** from $M\times N$ into P is a function $\phi : M\times N \to P$ such that

1. $\phi(r_1x_1 + r_2x_2, y) = r_1\phi(x_1, y) + r_2\phi(x_2, y)$,
2. $\phi(x, r_1y_1 + r_2y_2) = r_1\phi(x, y_1) + r_2\phi(x, y_2)$

for all $r_1, r_2 \in R, x, x_1, x_2 \in M_R$, and $y, y_1, y_2 \in N_R$.

Remark 2.1.5.11. We call a function ϕ in this definition an **R-bilinear map**.

Example 2.1.5.12. An $\mathbb{R}$-bilinear map $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is of the form $\phi(x, y) = axy$ with $a = \phi(1, 1)$.

2.2 Elementary Topological Structures

In this section, we turn our attention to the notion of sets together with subsets of their power sets satisfying some requirements called *topological spaces*. For more details, the reader is suggested to see Willard S. [Sp].

2.2.1 Topological Spaces and Continuous Maps

We begin this section by the most fundamental definition of this branch of study.

Definition 2.2.1.1. A **topology** on a set X is a collection τ of subsets of X, called the **open sets**, satisfying:

1. $\varnothing$ and X belong to τ,
2. any finite intersection of elements of τ belongs to τ,
3. any union of elements of τ belongs to τ.

We say (X,τ) is a **topological space**, sometimes abbreviated "X is a topological space" when no confusion can result about τ.

Some typical examples of topological spaces are given below.

Example 2.2.1.2. Let X be a set.

1. $P(X)$ is always a topology on X which is called the **discrete topology**.
2. $\{\varnothing, X\}$ is always a topology on X which is called the **indiscrete topology**.

Example 2.2.1.3. A collection $\tau := \{A \subseteq \mathbb{R} \mid \forall x_0 \in A\ \exists \varepsilon_{x_0} > 0 : (x_0 - \varepsilon_{x_0}, x_0 + \varepsilon_{x_0}) \subseteq A\}$ is a topology on $\mathbb{R}$ which is called the **standard topology on** $\mathbb{R}$.

Like homomorphisms of algebraic structures, *continuous maps* play the role of structure-preserving maps from a topological space to the other.

Definition 2.2.1.4. Let X and Y be topological spaces and let $f : X \to Y$. Then f is **continuous** on X if and only if for each open set B in Y, $f^{-1}(B)$ is an open set in X.

Example 2.2.1.5. Let $(\mathbb{R}, \tau)$ be the standard topological space. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := 4x - 1$ for all $x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

The following theorems are fundamentally significant; however, we do not prove in this situation.

Theorem 2.2.1.6. *[Sp, Theorem 7.3] Composition of continuous functions gives another continuous function.*

2.2.2 Homotopies

Once we study the relationship between two topological spaces via continuous functions, it is natural to ask what kind of relationships between two continuous functions this special function should be. One of the most intuitive notions of this is called a *homotopy* as defined in the following way.

Definition 2.2.2.1. Let X and Y be topological spaces and $f, g : X \to Y$ continuous functions. A **homotopy** between f and g is defined to be a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$. If such a homotopy exists, we say that f is **homotopic** to g, and denote this by $f \simeq g$.

Example 2.2.2.2. If $f, g : \mathbb{R} \to \mathbb{R}$ are any continuous functions, then $f \simeq g$.
Indeed, a homotopy between f and g is a function $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ defined by $H(x,t) := (1-t)f(x) + tg(x)$ for every $x \in \mathbb{R}$ and $t \in [0,1]$. It is easy to see that H is a continuous function because it is a composite of continuous functions.

Example 2.2.2.3. Given an annulus $A := \{(x,y) \mid 1 \leq x^2 + y^2 \leq 2\}$ and a circle $C := \{z \in \mathbb{C} \mid |z| = 1\}$, define functions $f, g : C \to A$ by $f(e^{i\theta}) := (2, \theta)$ and $g(e^{i\theta}) := (1, \theta)$ for every $e^{i\theta} \in C$ in the sense of polar coordinates (r, θ). A homotopy between f and g is a function $H : C \times [0,1] \to A$ defined by $H(z,t) := (z+1, t)$ for each $z \in C$ and $t \in [0,1]$.

The following theorem is a significant property of homotopies; however, we do not prove in details but give its sketch proof instead.

Theorem 2.2.2.4. *Given two topological spaces X and Y, homotopy is an equivalence relation on* $\mathrm{Hom}(X,Y) := \{f \mid f : X \to Y\}$.

Remark 2.2.2.5. In this remark, we give some ideas of how homotopy is reflexive, symmetric, and transitive. Firstly, for each $f \in \mathrm{Hom}(X,Y)$, the function $F : X \times [0,1] \to Y$ defined by $F(x,t) := f(x)$ for all $x \in X$ and $t \in [0,1]$ is a homotopy from f to f. Secondly, assume that $F : X \times [0,1] \to Y$ is a homotopy from f to g. It is easy to prove that $G : X \times [0,1] \to Y$ defined by $G(x,t) := F(x,1-t)$ for all $x \in X$ and $t \in [0,1]$ is a homotopy from g to f. Finally, suppose that $F : X \times [0,1] \to Y$ is a homotopy from f to g and $G : X \times [0,1] \to Y$ is a homotopy from g to h. We see that the map

$$H(x,t) := \begin{cases} F(x,2t) & ,0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & ,\frac{1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy from f to h.

2.2.3 Bundles

In the future discussion of categories and ω-categories over an ω-quiver, it is convenient to make use of the alternative language of bundles and fibers as defined by the following.

Definition 2.2.3.1. A **bundle** is a triple (E,π,B), where E and B are sets and $\pi : E \to B$ is a function. In this case, E is called the **total space**, B is called the **base space** of the bundle, and π is called the **projection**. Moreover, for each $b \in B$, $\pi^{-1}(b)$ is called the **fiber** of the bundle over b.

Remark 2.2.3.2. Since the definition of bundles is unrestrictive, we may further specify each component by adding some additional structures. For instance, we may assume that E and B are topological spaces and π is a continuous function with some appropriate properties (we will use the term *topological bundle* in this case). If E and B are categories and π is a suitable functor, we will use the term *categorical bundle*).

2.3 Basic Category Theory

In this section we recall some basic concepts in category theory; for example, functors, free structures, natural transformations, adjunctions, etc. For more details, see Aluffi P. [A], Borceux F. [Bo], and MacLane S. [ML].

2.3.1 Definitions and Examples of Categories

Definition 2.3.1.1. A **quiver** Q consists of a set Q^0 of **objects** of Q, a set Q^1 of **morphisms** of Q, and two functions $Q^0 \underset{t}{\overset{s}{\leftleftarrows}} Q^1$ giving the **source** and **target** of morphisms.

Definition 2.3.1.2. A **category** $\mathscr{C}$ is a quiver $\mathscr{C}^0 \underset{t}{\overset{s}{\leftleftarrows}} \mathscr{C}^1$ with an identity map $\mathscr{C}^0 \xrightarrow{\iota} \mathscr{C}^1$ and a partially defined composition $\circ : \mathscr{C}^1 \times_0 \mathscr{C}^1 \to \mathscr{C}^1$, where

$$\mathscr{C}^1 \times_0 \mathscr{C}^1 := \{(f,g) \in \mathscr{C}^1 \times \mathscr{C}^1 | s(g) = t(f)\},$$

defined by $(f,g) \mapsto g \circ f$, such that the following compatibilities hold:

1. (compatibility of source and target with composition)
   $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$,
2. (compatibility of source and target with identity)
   $s(\iota_A) = A = t(\iota_A)$ for every $A \in \mathscr{C}^0$

and such that the following algebraic axioms are satisfied:

1. (associativity) $h \circ (g \circ f) = (h \circ g) \circ f$,
2. (unitality) $f \circ \iota_A = f$ and $\iota_B \circ f = f$

whenever these compositions make sense.

Remark 2.3.1.3. For any category $\mathscr{C}$, we denote $\mathscr{C}^0$ and $\mathscr{C}^1$ to be the class of objects of $\mathscr{C}$ and the class of morphisms of $\mathscr{C}$, respectively.

Remark 2.3.1.4. This definition of categories is based on quivers. However, we have another equivalent definition of a category in terms of bundles as follows. A **category** $\mathscr{C}$ consists of:

- a class $\mathrm{Ob}_{\mathscr{C}}$ of **objects** of the category, and
- for each $A, B \in \mathrm{Ob}_{\mathscr{C}}$, a disjoint set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of **morphisms** or **arrows** from A to B with the following properties:
  - there exists one morphism ι_A for each $A \in \mathrm{Ob}_{\mathscr{C}}$ called an **identity map**,
  - for all $A, B, C \in \mathrm{Ob}_{\mathscr{C}}$, a function $\circ_{ABC} : \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$
    defined by $(f, g) \mapsto g \circ_{ABC} f$ or $g \circ f$ or simply gf called **composition**,
  - for each $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathscr{C}}(C, D)$,
    $h \circ (g \circ f) = (h \circ g) \circ f$ called **associativity**,
  - for all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $f \circ \iota_A = f$ and $\iota_B \circ f = f$ called **unitality**.

Remark 2.3.1.5. As bundles are defined in the subsection 2.2.3, for a category $\mathscr{C}$ the set $\mathscr{C}^0 \times \mathscr{C}^0$ can be seen to be the base space of the bundle and $\mathscr{C}^1$ is then the total space. Furthermore, the projection is a function $\pi : \mathscr{C}^1 \to \mathscr{C}^0 \times \mathscr{C}^0$ which is defined by $\pi : f \mapsto (s(f), t(f))$ for every $f \in \mathscr{C}^1$. Also, the fiber over $(A, B) \in \mathscr{C}^0 \times \mathscr{C}^0$ is $\mathscr{C}^1_{BA} := \{f \in \mathscr{C}^1 \mid s(f) = A, t(f) = B\} = \mathrm{Hom}_{\mathscr{C}}(A, B)$.

Some typical examples are given by the following.

Example 2.3.1.6. (Category of functions between sets: **Set**)

Let $\mathscr{C}^0 := \{S \mid S \text{ is a set}\} = \mathscr{U}$, the universal class of sets.

For all $A, B \in \mathscr{C}^0$, $\mathrm{Hom}_{\mathscr{C}}(A, B) := \{f \mid f \text{ is a function from } A \text{ to } B\} =: B^A$.

Thus, $\mathscr{C}^1 := \{f \mid f \text{ is a function between sets }\} = \bigcupdot\limits_{A, B \in \mathscr{C}^0} B^A$.

Define $\circ : B^A \times C^B \to C^A$ by $(f, g) \mapsto g \circ f$, composition of functions.

We see that composition of functions are always associative.

Moreover, $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for every $f \in B^A$.

Hence, this forms a category of functions between sets.

Example 2.3.1.7. (Monoid as a category)

Let $(X,*)$ be a monoid, i.e. $*: X \times X \to X$ is associative and there exists a unique element $\iota_X \in X$ such that $x * \iota_X = x = \iota_X * x$ for every $x \in X$.

Take $\mathscr{C}^0 := \{X\}$ and $\mathscr{C}^1 := X$.

Define[1] $y \circ x := x * y$ for all $x, y \in X$.

Composition is associative due to the associativity of $*$.

Unitality of composition is satisfied thanks to the existence of the identity of $(X,*)$.

Hence, a monoid is a category with one object.

Example 2.3.1.8. (Pre-ordered set as a category)

Let $(X, \preceq)$ be a pre-ordered set, i.e. $\preceq$ is a reflexive and transitive relation.

Take $\mathscr{C}^0 := X$ and $\mathscr{C}^1 := \preceq$.

Composition is defined by $(y,z) \circ (x,y) := (x,z)$ for all $(x,y),(y,z) \in \mathscr{C}^1$.

Associativity of composition follows from transitivity.

Unitality of composition is induced by reflexivity.

Hence, a pre-ordered set is a category with at most one morphism between two objects.

Example 2.3.1.9. (Matrices as a category)

Let $\mathscr{C}^0 := \mathbb{N}$ be the set of natural numbers.

For all $m, n \in \mathbb{N}$, $\mathrm{Hom}_{\mathscr{C}}(n,m) := \mathbb{M}_{m \times n}(\mathbb{R})$, the set of real $m \times n$ matrices.

[1] For the matter of notations, we have two different ways of writing notations of binary operations. For basic algebra we usually use the Polish notation; that is, the notation is defined forwardly. On the other hand, for category theory we define the notation reversedly.

Thus, $\mathscr{C}^1 := \bigcup\limits_{m,n\in\mathbb{N}} \mathbb{M}_{m\times n}(\mathbb{R}) = \mathbb{M}(\mathbb{R})$ is the set of matrices of real numbers.

For each $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n\times p}(\mathbb{R})$, define composition by $B \circ A := A \times B$, a line-by-column multiplication.

Moreover, for each $n \in \mathbb{N}$, there exists $I_n := [\delta_{ij}] \in \mathbb{M}_{n\times n}(\mathbb{R})$, the identity matrix of dimension n, where $\delta_{ij} = \begin{cases} 1 & ,i = j \\ 0 & ,i \neq j \end{cases}$.

It is easy to see that $\mathscr{C}$ is a category.

Example 2.3.1.10. (More examples of categories)

The following table lists some basic examples of categories.

Categories	**Objects**	**Morphisms**
Set	sets	functions
Mag	magmas	homomorphisms of magmas
Sem	semigroups	homomorphisms of semigroups
Mon	monoids	unital homomorphisms of monoids
Grp	groups	group homomorphisms
Rng	rings	ring homomorphisms
Ring	unital rings	unital ring homomorphisms
Vect	vector spaces	linear transformations
***R*-Mod**	*R*-modules	*R*-linear maps
Top	topological spaces	continuous maps

Example 2.3.1.11. Let $(\mathscr{C}, \circ)$ be a category. We can construct another category $(\mathscr{C}^{op}, \circ_{op})$ from the category $\mathscr{C}$ as follows:

1. $\mathrm{Ob}_{\mathscr{C}^{op}} := \mathrm{Ob}_{\mathscr{C}}$
2. for each $A, B \in \mathrm{Ob}_{\mathscr{C}^{op}}$, $\mathrm{Hom}_{\mathscr{C}^{op}}(A,B) := \mathrm{Hom}_{\mathscr{C}}(B,A)$
3. for each $A,B,C \in \mathrm{Ob}_{\mathscr{C}^{op}}$, composition $\circ_{op} : \mathrm{Hom}_{\mathscr{C}^{op}}(B,A) \times \mathrm{Hom}_{\mathscr{C}^{op}}(C,B) \to \mathrm{Hom}_{\mathscr{C}^{op}}(C,A)$ defined by $(f,g) \mapsto (g \circ f)^{op}$.

2.3.2 Initial and Final Objects

A category consists of a class of objects and a class of morphisms. In this subsection, we concentrate on some specific properties of objects, namely *initial* and *final* objects.

Definition 2.3.2.1. Let $\mathscr{C}$ be a category. We say that an object I of $\mathscr{C}$ is **initial** in $\mathscr{C}$ if for every object A of $\mathscr{C}$ there exists a unique morphism $I \to A$ in $\mathscr{C}$.

Proposition 2.3.2.2. *The initial object in the category* ***Set*** *is the empty set* $\varnothing$.

Proof. Let A be a set and f a function from $\varnothing$ into A.

This implies that $f \subseteq \varnothing \times A = \varnothing$ and so $f = \varnothing$.

Hence, $\varnothing$ is the initial object in **Set**. □

Proposition 2.3.2.3. *The initial objects in the category* ***Grp*** *are the trivial groups.*

Proof. Let $G := \{*\}$ and $\cdot : G \times G \to G$ be defined by $* \cdot * := *$.

Then $(G, \cdot)$ is a group with $1_G = *$ and $*^{-1} = *$.

Let H be a group and $f : G \to H$ be defined by $f(*) := 1_H$.

We see that this is the only group homomorphism we can define.

Thus, the trivial groups are initial in **Grp**. □

Proposition 2.3.2.4. *The initial object in the category* ***Ring*** *is isomorphic to the ring* $(\mathbb{Z}, +, \cdot)$.

Proof. Let $(R, +, \cdot)$ be any unital ring.

Define $f : (\mathbb{Z}, +, \cdot) \to (R, +, \cdot)$ by

$$f(n) := \begin{cases} \overbrace{1_R + \cdots + 1_R}^{n \text{ times}}, & n \in \mathbb{Z}^+; \\ 0_R, & n = 0; \\ \underbrace{(-1_R) + \cdots + (-1_R)}_{-n \text{ times}}, & n \in \mathbb{Z}^-. \end{cases}$$

Without loss of generality, assume that $m, n \in \mathbb{Z}^+$.

1. $f(m+n) = \overbrace{1_R + \cdots + 1_R}^{m+n \text{ times}} = (\overbrace{1_R + \cdots + 1_R}^{m \text{ times}}) + (\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}) = f(m) + f(n)$,

2. $f(m \cdot n) = \overbrace{1_R + \cdots + 1_R}^{m \cdot n \text{ times}} = (\overbrace{1_R + \cdots + 1_R}^{m \text{ times}}) \cdot (\overbrace{1_R + \cdots + 1_R}^{n \text{ times}}) = f(m) \cdot f(n)$,

3. $f(1_{\mathbb{Z}}) = 1_R$.

Hence, f is a unital ring homomorphism.

We see that this is the only unital ring homomorphism we can define.

Therefore, $(\mathbb{Z}, +, \cdot)$ is an initial object in **Ring**. $\square$

Definition 2.3.2.5. Let $\mathscr{C}$ be a category. We say that an object F of $\mathscr{C}$ is **final** in $\mathscr{C}$ if for every object A of $\mathscr{C}$ there exists a unique morphism $A \to F$ in $\mathscr{C}$.

Remark 2.3.2.6. The term **terminal objects** may be used to denote either final objects or both initial and final objects. Thus, in this proposal we will not use this terminology to make confusion.

Proposition 2.3.2.7. *Singletons play the role of final objects in the category **Set**.*

Proof. Let A be any set and $B := \{*\}$.

Define a function $f : A \to B$ by $f(a) := *$ for all $a \in A$.

We see that this is the only function we can define.

Thus, $B = \{*\}$ is a final object in **Set**. $\square$

Proposition 2.3.2.8. *The final objects in the category **Grp** are the trivial groups.*

Proof. Let $H := \{*\}$ be the trivial group and G any group.

Define $f : G \to H$ by $f(x) := *$ for each $x \in G$.

Since $f(x \cdot y) = * = * \cdot * = f(x) \cdot f(y)$ for $x, y \in G$, f is a group homomorphism.

We see that this is the only group homomorphism we can define.

Thus, the trivial groups are final in **Grp**. $\square$

Proposition 2.3.2.9. *The final objects in the category* ***Ring*** *are the trivial rings.*

Proof. Let $S := \{*\}$ and R be any unital ring.

Define $+ : S \times S \to S$ and $\cdot : S \times S \to S$ by $* + * := *$ and $* \cdot * := *$.

Then $(S, +, \cdot)$ is a unital ring with $0_S = * = 1_S$ and $-* = *$.

Define $f : R \to S$ by $f(x) := *$ for each $x \in R$.

We have the following equalities: for each $x, y \in R$

1. $f(x+y) = * = * + * = f(x) + f(y)$,
2. $f(x \cdot y) = * = * \cdot * = f(x) \cdot f(y)$,
3. $f(1_R) = * = 1_S$.

It follows that f is a unital ring homomorphism.

We see that this is the only unital ring homomorphism we can define.

Thus, the trivial rings are final in **Ring**. □

A category need not have initial or final objects as described in the following example.

Example 2.3.2.10. Consider the category obtained by endowing $\mathbb{Z}$ with the relation $\leq$.

1. $(\mathbb{Z}, \leq)$ has no initial objects. Indeed, an initial object in this category would be an integer z such that $z \leq x$ for all $x \in \mathbb{Z}$, but there is no such integer.
2. $(\mathbb{Z}, \leq)$ has no initial objects. A final object would be an integer y such that $x \leq y$ for all $x \in \mathbb{Z}$, but there is no such integer.

Remark 2.3.2.11. [A, Proposition 5.4] If initial or final objects exist, then they are unique up to isomorphism.

2.3.3 Functors

In a category, if one wants to study relationships between two objects, one needs to know about morphisms between them. But if we would like to study relationships between two categories, we need the notions of *covariant functors* and *contravariant functors*. Since a category is made up of the information of objects and morphisms, functors will have to send both objects and morphisms from one category to the other.

Definition 2.3.3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **covariant functor** $F : \mathscr{C} \to \mathscr{D}$ consists of

1. a function $\mathrm{Ob}_{\mathscr{C}} \to \mathrm{Ob}_{\mathscr{D}}$; the image of $A \in \mathrm{Ob}_{\mathscr{C}}$ is denoted $F(A)$ or simply FA,
2. for every pair $A, B \in \mathrm{Ob}_{\mathscr{C}}$, a function $\mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(F(A),F(B))$; the image of $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$ is denoted $F(f)$ or simply Ff

such that the following conditions are satisfied:

- $F(\iota_A) = \iota_{F(A)}$ for all $A \in \mathrm{Ob}_{\mathscr{C}}$,
- $F(g \circ_{\mathscr{C}} f) = F(g) \circ_{\mathscr{D}} F(f)$ for all $A, B, C \in \mathrm{Ob}_{\mathscr{C}}$, $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, and $g \in \mathrm{Hom}_{\mathscr{C}}(B,C)$.

Definition 2.3.3.2. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A **contravariant functor** $F : \mathscr{C} \to \mathscr{D}$ consists of

1. a function $\mathrm{Ob}_{\mathscr{C}} \to \mathrm{Ob}_{\mathscr{D}}$; the image of $A \in \mathrm{Ob}_{\mathscr{C}}$ is denoted $F(A)$ or simply FA,
2. for every pair $A, B \in \mathrm{Ob}_{\mathscr{C}}$, a function $\mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(F(B),F(A))$; the image of $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$ is denoted $F(f)$ or simply Ff

such that the following conditions are satisfied:

- $F(\iota_A) = \iota_{F(A)}$ for all $A \in \mathrm{Ob}_{\mathscr{C}}$,
- $F(g \circ_{\mathscr{C}} f) = F(f) \circ_{\mathscr{D}} F(g)$ for all $A, B, C \in \mathrm{Ob}_{\mathscr{C}}$, $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, and $g \in \mathrm{Hom}_{\mathscr{C}}(B,C)$.

The following example is an example of a covariant functor.

Example 2.3.3.3. If R is a unital ring, we denote by R^* the group of units (elements with multiplicative inverses) in R. Every unital ring homomorphism $\phi : R \to S$ induces a group homomorphism $\phi^* : R^* \to S^*$. We see that F : **Ring** $\to$ **Grp** which is defined by $F : R \mapsto R^*$ and $F : \phi \mapsto \phi^*$ is a covariant functor .

Proof. Let R be a unital ring and $R^* := \{x \in R \mid \exists y \in R, xy = 1_R = yx\}$.

Note that $x, y \in R^*$ imply $x \cdot y \in R^*$ since $(xy)(y^{-1}x^{-1}) = 1_R = (y^{-1}x^{-1})(xy)$.

So a binary operation $\cdot : R^* \times R^* \to R^*$ defined by $(x, y) \mapsto x \cdot y$, for all $x, y \in R^*$, is well-defined.

First, let $x, y, z \in R^*$.

This implies that there exist $x^{-1}, y^{-1} \in R$ such that $xx^{-1} = 1_R = x^{-1}x$ and $yy^{-1} = 1_R = y^{-1}y$.

We see that the following assertions hold:

1. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ because $(R, \cdot)$ is a semigroup,
2. that $1_R \in R^*$ follows from the fact that $1_R 1_R = 1_R$,
3. $x^{-1} \in R^*$ as $x^{-1}x = 1_R = xx^{-1}$.

Thus, $(R^*, \cdot)$ is a group.

Suppose now that $\phi : R \to S$ is a unital ring homomorphism.

If $x \in R^*$, then $\phi(x) \in S^*$ because $\phi(x)\phi(x') = 1_S = \phi(x')\phi(x)$ for some $x' \in R$.

This yields $\phi^* : R^* \to S^*$ defined by $x \mapsto \phi(x)$, for each $x \in R^*$, is well-defined.

Notice that $\phi^* = \phi\mid_{R^*}$.

It also follows that $\phi^*(x \cdot y) = \phi(x \cdot y) = \phi(x)\phi(y) = \phi^*(x)\phi^*(y)$ for all $x, y \in R^*$.

Hence, ϕ^* is a group homomorphism.

Next, we will show that F is a covariant functor.

Assume that $\phi : R \to S$ and $\psi : S \to T$ are two unital ring homomorphisms.

By definition of induced group homomorphisms, we obtain:

1. $F(\mathrm{Id}_R) = (\mathrm{Id}_R)^* = \mathrm{Id}_{R^*} = \mathrm{Id}_{F(R)}$,

2. $F(\psi \circ \phi) = (\psi \circ \phi)^* = \psi \circ \phi = \psi^* \circ \phi^* = F(\psi) \circ F(\phi)$.

Therefore, F is a covariant functor. □

The following example is an example of a contravariant functor.

Example 2.3.3.4. Let R be a commutative ring. Define the *spectrum* of R, $\mathrm{Sp}R$, as the set of prime ideals of R. Note that if $\phi : R \to S$ is a homomorphism of commutative rings, then $\phi^{-1}(P)$ is a prime ideal of R for each prime ideal P of S. So, ϕ induces a function $\phi^\circ : \mathrm{Sp}S \to \mathrm{Sp}R$. We see that $\mathrm{Sp} : \mathbf{Rng} \to \mathbf{Set}$ defined by $\mathrm{Sp} : R \mapsto \mathrm{Sp}R$ and $\mathrm{Sp} : \phi \mapsto \phi^\circ$ is a contravariant functor.

Proof. Let R be a commutative ring and $\mathrm{Sp}R := \{I \mid I \text{ is a prime ideal of } R\}$.

Firstly, suppose that $\phi : R \to S$ is a homomorphism of commutative rings.

Let P be any prime ideal of S and $ab \in \phi^{-1}(P)$.

Then there exists $c \in P$ such that $ab = \phi^{-1}(c)$, i.e. $\phi(a)\phi(b) = c$.

Since P is prime and $\phi(a)\phi(b) \in P$, $\phi(a) \in P$ or $\phi(b) \in P$.

That is, $a \in \phi^{-1}(P)$ or $b \in \phi^{-1}(P)$ and so $\phi^{-1}(P)$ is a prime ideal of R.

Thus, $\phi^\circ : \mathrm{Sp}S \to \mathrm{Sp}R$ defined by $P \mapsto \phi^{-1}(P)$ is a well-defined function.

Assume that $\psi : S \to T$ is another homomorphism of commutative rings.

We have

1. $\phi^\circ \circ \psi^\circ = \phi^{-1} \circ \psi^{-1} = (\psi \circ \phi)^{-1} = (\psi \circ \phi)^\circ$,

2. $(\mathrm{Id}_R)^\circ = (\mathrm{Id}_R)^{-1} = \mathrm{Id}_R = \mathrm{Id}_{\mathrm{Sp}R}$.

Therefore, Sp is a contravariant functor. □

Forgetful functors will play a prominent role in the definitions of free structures and adjunctions.

Example 2.3.3.5. (Forgetful functor: $\mathscr{U}$) Forgetful functors are obtained by forgetting part of the structure of a given object. For example, **Grp** $\to$ **Set** (forget the operation on groups), **Rng** $\to$ **Set** (forget the operations on rings), R-**Mod** $\to$ **Grp** (forget the scalar multiplication on R-modules), etc.

2.3.4 Free Structures

One may think that forgetful functors send objects with more structure to the underlying objects with less structure. Free functors will do the opposite. This means that free functors will send objects with less structure to free objects with more structure possessing no further constraints, the latter are called *free structures*. The formal definition of free structures is given by the following.

Definition 2.3.4.1. Let $\mathscr{C}$ and $\mathscr{S}$ be categories and $\mathscr{U} : \mathscr{C} \to \mathscr{S}$ a forgetful functor. A **free structure** in $\mathscr{C}$ over an object S in $\mathscr{S}$ is given by $S \xrightarrow{j} \mathscr{U}(A)$, where $A \in \mathrm{Ob}_{\mathscr{C}}$ and a morphism j from S to an object $\mathscr{U}(A)$ in $\mathscr{S}$ is *initial* for such morphisms, i.e. for all objects $\mathscr{U}(A')$ in $\mathscr{S}$, where $A' \in \mathrm{Ob}_{\mathscr{C}}$, and morphisms $S \xrightarrow{j'} \mathscr{U}(A')$, there exists a unique morphism of $\phi : A \to A'$ in $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{U}(A) \arrow[r, dotted, "\exists!\mathscr{U}(\phi)"] & \mathscr{U}(A') \\ S \arrow[u, "j"] \arrow[ur, "\forall j'"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & \end{tikzcd}$$

that is, $j' = \mathscr{U}(\phi) \circ j$.

Remark 2.3.4.2. The word "structure" in the above definition can be replaced by any structures; for instance, magma, semigroup, monoid, group, module, etc.

In the following five theorems we provide the results of the existence of some of the "free" algebraic structures over a given set. However, we do not spend time on proving them in details. But in order to give some ideas of their constructions, we decide to give here their sketch proofs in the remark after each of its theorem.

Theorem 2.3.4.3. *A free magma over a set S exists.*

Remark 2.3.4.4. Let $S := \{x_1, x_2, \dots, x_n\}$ for some $n \in \mathbb{N}$.
Define a binary operation $*$ recursively as follows: for each $i, j, k, l \in \{1, 2, \dots, n\}$

$$x_i * x_j := (x_i, x_j),$$

$$x_i * (x_j, x_k) := (x_i, (x_j, x_k)),$$

$$(x_i, x_j) * x_k := ((x_i, x_j), x_k),$$

$$(x_i, x_j) * (x_k, x_l) := \big((x_i, x_j), (x_k, x_l)\big),$$

$$\vdots$$

Now let M be the set of union of all such ordered pairs. It can be easily seen that $(M, *)$ gives a magma. Then set $j : S \to M$ to be a function defined by $j : x \mapsto (x)$ for every $x \in S$. This forms a free magma over the set S.

Remark 2.3.4.5. From this construction we see that $(M, *)$ does not satisfy any further properties; for example, associativity or unitality.

Theorem 2.3.4.6. *A free semigroup over a set S exists.*

Remark 2.3.4.7. Let $M := \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in S\}$ be the set of finite sequences of elements of S. Define, for all $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_m) \in M$,

$$(x_1, x_2, \dots, x_n) * (y_1, y_2, \dots, y_m) := (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m).$$

We see that $(M, *)$ is a semigroup. Now we define $j : S \to M$ by $j : x \mapsto (x)$.
This forms a free semigroup over the set S.

Theorem 2.3.4.8. *A free monoid over a set S exists.*

Remark 2.3.4.9. Let $M := \{(x_1,x_2,\ldots,x_n) \mid \ x_1,x_2,\ldots,x_n \in S\} \cup \{(\ )\}$ be the set of finite sequences of elements of S. Define, for all $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_m) \in M$,

$$(x_1,x_2,\ldots,x_n) * (y_1,y_2,\ldots,y_m) := (x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m).$$

It is easy to see that $(M,*)$ is a monoid. Now we define $j : S \to M$ by $j : x \mapsto (x)$. This forms a free monoid over the set S.

Theorem 2.3.4.10. *[A, Proposition 5.2] A free group over a set S exists.*

Remark 2.3.4.11. Consider the disjoint union $S \uplus S := (S \times \{0\}) \cup (S \times \{1\})$, where an element $x' \in S \times \{1\}$ is considered as the inverse of an element $x \in S \times \{0\}$. Next, we construct the free monoid $F(S \uplus S)$ over $S \uplus S$. Then we define an equivalence relation $\sim$ which cancels the terms $(\ldots,x,x',\ldots)$ and $(\ldots,x',x,\ldots)$ for every $x \in S \uplus S$. Now we let $G := F(S \uplus S)/\sim$ with a binary operation

$$[(x_1,x_2,\ldots,x_n)] * [(y_1,y_2,\ldots,y_m)] := [(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m)],$$

identity $[(\ )]$, and inverse $[(x_1,x_2,\ldots,x_n)]^{-1} = [(x_n',\ldots,x_2',x_1')]$. Furthermore, a function j is defined by $j : x \mapsto [(x)]$. This forms a free group over the set S.

Theorem 2.3.4.12. *[A, Section 6.3] Given a unital ring R, a free left R-module over a set S exists.*

Remark 2.3.4.13. Consider $R^{\bigoplus S} := \bigoplus\limits_{\gamma \in S} R := \{(r_\gamma) \mid r_\gamma \neq 0_R \text{ for a finite number of } \gamma \in S\}$ with a function $j : \gamma \to \delta_\gamma$, where $\delta_\gamma(\gamma_1) := \begin{cases} 1_R & ,\gamma = \gamma_1 \\ 0_R & ,\gamma \neq \gamma_1 \end{cases}$.
Define $(r_\gamma^1) + (r_\gamma^2) := (r_\gamma^1 + r_\gamma^2)$ and $r \cdot (r_\gamma^1) := (r \cdot r_\gamma^1)$ for all $(r_\gamma^1),(r_\gamma^2) \in R^{\bigoplus S}$ and $r \in R$. This forms a free left R-module over the set S.

Remark 2.3.4.14.

1. If we define $(r_\gamma^1) \cdot r := (r_\gamma^1 \cdot r)$ for all $(r_\gamma^1) \in R^{\bigoplus S}$ and $r \in R$ instead of $r \cdot (r_\gamma^1) := (r \cdot r_\gamma^1)$, we get a free right R-module.
2. If we define together $r \cdot (r_\gamma^1) := (r \cdot r_\gamma^1)$ with $(r_\gamma^1) \cdot r := (r_\gamma^1 \cdot r)$, we obtain a free left-$R$ right-R bimodule.

2.3.5 Limits and Colimits

Before ending this subsection, we would like to recall some of the most fundamental notions which will be necessary for our work later, namely *limit* and *colimit*. But before we head directly to their definition, we need some elementary concepts, called *product* and *coproduct*, which are their least complicated examples first.

Definition 2.3.5.1. Let $\mathscr{C}$ be a category and $A, B \in \mathrm{Ob}_{\mathscr{C}}$. A **product** of A and B is a diagram

$$\begin{tikzcd} & P \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

such that for every other diagram

$$\begin{tikzcd} & P' \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A & & B \end{tikzcd}$$

there exists a unique morphism $\gamma : P' \to P$ such that both diagrams commute:

$$\begin{tikzcd} & P' \arrow[ddl, bend right, "\alpha"'] \arrow[ddr, bend left, "\beta"] \arrow[d, dotted, "\exists!\gamma"] & \\ & P \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A & & B \end{tikzcd}$$

that is, $\alpha = \pi_A \circ \gamma$ and $\beta = \pi_B \circ \gamma$.

Example 2.3.5.2. In the category **Set**, a product of A and B is given by

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

where $A \times B := \{(a,b) \mid a \in A, b \in B\}$, $\pi_A : (a,b) \mapsto a$, and $\pi_B : (a,b) \mapsto b$.

Proof. Let a pair of functions be given

$$\begin{tikzcd} & P \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A & & B \end{tikzcd}$$

Define a function $\gamma : P \to A \times B$ by $\gamma(x) := (\alpha(x), \beta(x))$ for each $x \in P$.

It follows that, for every $x \in P$

1. $\pi_A \circ \gamma(x) = \pi_A(\gamma(x)) = \pi_A((\alpha(x), \beta(x))) = \alpha(x)$,

2. $\pi_B \circ \gamma(x) = \pi_B(\gamma(x)) = \pi_B((\alpha(x), \beta(x))) = \beta(x)$.

As a result, $\pi_A \circ \gamma = \alpha$ and $\pi_B \circ \gamma = \beta$.

We see that this is the only way we can define this function.

Therefore, a product of A and B is given by

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

□

Lemma 2.3.5.3. *Let $(M, +_M, \cdot_M)$ and $(N, +_N, \cdot_N)$ be modules over a ring R. Define binary operations on $M \bigoplus N := \{(m,n) \mid m \in M, n \in N\}$ as follows:*

1. $+ : M \bigoplus N \times M \bigoplus N \to M \bigoplus N$ by $(m_1, n_1) + (m_2, n_2) := (m_1 +_M m_2, n_1 +_N n_2)$,

2. $\cdot : R \times M \bigoplus N \to M \bigoplus N$ by $r \cdot (m,n) := (r \cdot_M m, r \cdot_N n)$.

Then $(M \bigoplus N, +, \cdot)$ becomes an R-module.

Example 2.3.5.4. In the category R-**Mod**, a product of $(M, +_M, \cdot_M)$ and $(N, +_N, \cdot_N)$ is

$$\begin{tikzcd} & M \bigoplus N \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M & & N \end{tikzcd}$$

where $M \bigoplus N := \{(m,n) \mid m \in M, n \in N\}$ equipped with binary operations $+$ and $\cdot$ defined above is an R-module, $\pi_M : (m,n) \mapsto m$, and $\pi_N : (m,n) \mapsto n$.

Proof. By Lemma 2.3.5.3, $(M \bigoplus N, +, \cdot)$ is an R-module.

Define functions $\pi_M : M \bigoplus N \to M$ and $\pi_N : M \bigoplus N \to N$ by $\pi_M((m,n)) := m$ and $\pi_N((m,n)) := n$ for each $(m,n) \in M \bigoplus N$.

We first show that both π_M and π_N are R-linear maps.

Let $(m,n), (m_1,n_1), (m_2,n_2) \in M \bigoplus N$ and $r \in R$.

1. $$\begin{aligned}\pi_M((m_1,n_1)+(m_2,n_2)) = \pi_M((m_1 +_M m_2, n_1 +_N n_2)) &= m_1 +_M m_2 \\ &= \pi_M((m_1,n_1)) + \pi_M((m_2,n_2)),\end{aligned}$$
2. $$\begin{aligned}\pi_N((m_1,n_1)+(m_2,n_2)) = \pi_N((m_1 +_M m_2, n_1 +_N n_2)) &= n_1 +_N n_2 \\ &= \pi_N((m_1,n_1)) + \pi_N((m_2,n_2)),\end{aligned}$$
3. $\pi_M(r \cdot (m,n)) = \pi_M((r \cdot_M m, r \cdot_N n)) = r \cdot_M m = r \cdot_M \pi_M((m,n))$,
4. $\pi_N(r \cdot (m,n)) = \pi_N((r \cdot_M m, r \cdot_N n)) = r \cdot_N n = r \cdot_M \pi_N((m,n))$.

This implies that π_M and π_N are R-linear maps.

Now let P be an R-module and $\mu : P \to M$ and $\nu : P \to N$ R-linear maps:

$$\begin{tikzcd} & P \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M & & N \end{tikzcd}$$

Define a function $\gamma : P \to M \bigoplus N$ by $\gamma(p) := (\mu(p), \nu(p))$ for each $p \in P$.

To check that γ is R-linear, let $p, q \in P$ and $r \in R$.

1. $$\begin{aligned}\gamma(p+q) &= (\mu(p+q), \nu(p+q)) = (\mu(p) +_M \mu(q), \nu(p) +_N \nu(q)) \\ &= (\mu(p), \nu(p)) + (\mu(q), \nu(q)) = \gamma(p) + \gamma(q),\end{aligned}$$
2. $\gamma(r \cdot p) = (\mu(r \cdot p), \nu(r \cdot p)) = (r \cdot_M \mu(p), r \cdot_N \nu(p)) = r \cdot (\mu(p), \mu(p)) = r \cdot \gamma(p)$.

This yields that γ is an R-linear map.

We have the following equalities: for every $p \in P$

1. $\pi_M \circ \gamma(p) = \pi_M(\gamma(p)) = \pi_M((\mu(p), \nu(p))) = \mu(p)$,

2. $\pi_N \circ \gamma(p) = \pi_N(\gamma(p)) = \pi_N((\mu(p), \nu(p))) = \nu(p)$,

It follows that γ satisfies $\pi_M \circ \gamma = \mu$ and $\pi_N \circ \gamma = \nu$.

We see that this is the only way to define such R-linear map.

Therefore, a product of M and N is given by

$$\begin{tikzcd} & M \bigoplus N \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M & & N \end{tikzcd}$$

$\square$

Definition 2.3.5.5. Let $\mathscr{C}$ be a category and $A, B \in \mathrm{Ob}_{\mathscr{C}}$. A **coproduct** of A and B is a diagram

$$\begin{tikzcd} & S & \\ A \arrow[ur, "i_A"] & & B \arrow[ul, "i_B"'] \end{tikzcd}$$

such that for every other diagram

$$\begin{tikzcd} & S' & \\ A \arrow[ur, "\alpha"] & & B \arrow[ul, "\beta"'] \end{tikzcd}$$

there exists a unique morphism $\gamma : S \to S'$ such that both diagrams commute:

$$\begin{tikzcd} & S' & \\ & S \arrow[u, dotted, "\exists!\gamma"] & \\ A \arrow[ur, "i_A"] \arrow[uur, bend left, "\alpha"] & & B \arrow[ul, "i_B"'] \arrow[uul, bend right, "\beta"'] \end{tikzcd}$$

that is, $\alpha = \gamma \circ i_A$ and $\beta = \gamma \circ i_B$.

Example 2.3.5.6. In the category **Set**, a coproduct of A and B is given by

$$\begin{tikzcd} & A \dot{\cup} B & \\ A \arrow[ur, "i_A"] & & B \arrow[ul, "i_B"'] \end{tikzcd}$$

where $A \dot{\cup} B := (A \times \{0\}) \cup (B \times \{1\})$, $i_A : a \mapsto (a, 0)$, and $i_B : b \mapsto (b, 1)$.

Proof. Let a pair of functions be given

$$\begin{tikzcd} & S & \\ A \arrow[ur, "\alpha"] & & B \arrow[ul, "\beta"'] \end{tikzcd}$$

Define $\gamma : A \dot{\cup} B \to S$ by $\gamma(x) := \begin{cases} \alpha(a), & x = (a, 0); \\ \beta(b), & x = (b, 1). \end{cases}$ for each $x \in A \dot{\cup} B$.

This yields that, for every $x \in A \dot{\cup} B$

1. $\gamma \circ i_A(x) = \gamma(i_A(x)) = \gamma((x, 0)) = \alpha(x)$,

2. $\gamma \circ i_B(x) = \gamma(i_B(x)) = \gamma((x, 1)) = \beta(x)$.

Thus, $\gamma \circ i_A = \alpha$ and $\gamma \circ i_B = \beta$.

We see that this is the only function satisfying such properties.

Hence, a coproduct of A and B is given by

$$\begin{tikzcd} & A \dot{\cup} B & \\ A \arrow[ur, "i_A"] & & B \arrow[ul, "i_B"'] \end{tikzcd}$$

$\square$

Example 2.3.5.7. In the category R-**Mod**, a product of $(M, +_M, \cdot_M)$ and $(N, +_N, \cdot_N)$ is

$$\begin{tikzcd} & M \bigoplus N & \\ M \arrow[ur, "i_M"] & & N \arrow[ul, "i_N"'] \end{tikzcd}$$

where $M \bigoplus N := \{(m, n) \mid m \in M, n \in N\}$ equipped with binary operations $+$ and $\cdot$ is an R-module, $i_M : m \mapsto (m, 0_N)$, and $i_N : n \mapsto (0_M, n)$.

Proof. Define functions $i_M : M \to M \bigoplus N$ and $i_N : N \to M \bigoplus N$ by $i_M(m) := (m, 0_N)$ and $i_N(n) := (0_M, n)$ for each $m \in M$ and $n \in N$.

We first verify that both i_M and i_N are R-linear maps.

Let $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

1. $i_M(m_1 +_M m_2) = (m_1 +_M m_2, 0_N) = (m_1, 0_N) + (m_2, 0_N) = i_M(m_1) + i_M(m_2)$,
2. $i_N(n_1 +_N n_2) = (0_M, n_1 +_N n_2) = (0_M, n_1) + (0_M, n_2) = i_N(n_1) + i_N(n_2)$,
3. $i_M(r \cdot_M m) = (r \cdot_M m, 0_N) = r \cdot (m, 0_N) = r \cdot i_M(m)$,
4. $i_N(r \cdot_N n) = (0_M, r \cdot_N n) = r \cdot (0_M, n) = r \cdot i_N(n)$.

Thus, i_M and i_N are R-linear maps.

Now let S be an R-module and $\mu : M \to P$ and $\nu : N \to P$ R-linear maps:

$$\begin{tikzcd} & S & \\ M \arrow[ur, "\mu"] & & N \arrow[ul, "\nu"'] \end{tikzcd}$$

Define $\gamma : M \bigoplus N \to S$ by $\gamma((m,n)) := \mu(m) + \nu(n)$ for all $(m,n) \in M \bigoplus N$.

To see that γ is R-linear, let $(m,n), (m_1,n_1), (m_2,n_2) \in M \bigoplus N$ and $r \in R$.

1. $$\begin{aligned} \gamma((m_1,n_1) + (m_2,n_2)) = \gamma((m_1 +_M m_2, n_1 +_N n_2)) &= \mu(m_1 +_M m_2) + \nu(n_1 +_N n_2) \\ &= (\mu(m_1) + \mu(m_2)) + (\nu(n_1) + \nu(n_2)) \\ &= (\mu(m_1) + \nu(n_1)) + (\mu(m_2) + \nu(n_2)) \\ &= \gamma((m_1,n_1)) + \gamma((m_2,n_2)), \end{aligned}$$
2. $$\begin{aligned} \gamma(r \cdot (m,n)) = \gamma((r \cdot_M m, r \cdot_N n)) &= \mu(r \cdot_M m) + \nu(r \cdot_N n) \\ &= r \cdot (\mu(m) + \nu(n)) = r \cdot \gamma((m,n)). \end{aligned}$$

Notice that μ and ν are R-linear imply $\mu(0_M) = 0_S = \nu(0_N)$.

Consider the following arguments:

1. $\gamma \circ i_M(m) = \gamma(i_M(m)) = \gamma((m, 0_N)) = \mu(m) + \nu(0_N) = \mu(m)$ for all $m \in M$,

2. $\gamma \circ i_N(n) = \gamma(i_N(n)) = \gamma((0_M, n)) = \mu(0_M) + \nu(n) = \nu(n)$ for all $n \in N$.

This means that $\gamma \circ i_M = \mu$ and $\gamma \circ i_N = \nu$.

We see that this is the only R-linear map satisfying such properties.

Therefore, a coproduct of M and N is given by

$$\begin{tikzcd} & M \bigoplus N & \\ M \arrow[ur, "i_M"] & & N \arrow[ul, "i_N"'] \end{tikzcd}$$

□

Definition 2.3.5.8. Let $\mathscr{I}$ and $\mathscr{C}$ be categories and $F : \mathscr{I} \to \mathscr{C}$ a covariant functor, where one thinks of $\mathscr{I}$ as of a category of indices. A **cone** on F is an object N of $\mathscr{C}$ endowed with morphisms $\psi_X : N \to F(X)$ for all $X \in \mathrm{Ob}_{\mathscr{I}}$ such that if $f : X \to Y$ is a morphism in $\mathscr{I}$ then the following diagram commutes:

$$\begin{tikzcd} & N \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(X) \arrow[rr, "F(f)"'] & {} & F(Y) \end{tikzcd}$$

that is, $\psi_Y = F(f) \circ \psi_X$.

Definition 2.3.5.9. Let $\mathscr{I}$ and $\mathscr{C}$ be categories and $F : \mathscr{I} \to \mathscr{C}$ a covariant functor, where one thinks of $\mathscr{I}$ as of a category of indices. A **co-cone** on F is an object M of $\mathscr{C}$ endowed with morphisms $\phi_X : F(X) \to M$ for all $X \in \mathrm{Ob}_{\mathscr{I}}$ such that if $f : X \to Y$ is a morphism in $\mathscr{I}$ then the following diagram commutes:

$$\begin{tikzcd} & M \arrow[d, phantom, "\circlearrowleft"] & \\ F(X) \arrow[ur, "\phi_X"] \arrow[rr, "F(f)"'] & {} & F(Y) \arrow[ul, "\phi_Y"'] \end{tikzcd}$$

that is, $\phi_X = \phi_Y \circ F(f)$.

Definition 2.3.5.10. Let $\mathscr{I}$ and $\mathscr{C}$ be categories and $F : \mathscr{I} \to \mathscr{C}$ a covariant functor, where one thinks of $\mathscr{I}$ as of a category of indices. The **limit** of F is (if it exists) an object L of $\mathscr{C}$, endowed with morphisms $\lambda_I : L \to F(I)$ for all $I \in \mathrm{Ob}_{\mathscr{I}}$, such that

1. if $\alpha : I \to J$ is a morphism in $\mathscr{I}$, then $\lambda_J = F(\alpha) \circ \lambda_I$:

$$\begin{tikzcd} & L \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(I) \arrow[rr, "F(\alpha)"'] & {} & F(J) \end{tikzcd}$$

2. L is *final* with respect to this property: that is, if M is another object, endowed with morphisms μ_I, also satisfying the previous requirement:

$$\begin{tikzcd} & M \arrow[dl, "\mu_I"'] \arrow[dr, "\mu_J"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(I) \arrow[rr, "F(\alpha)"'] & {} & F(J) \end{tikzcd}$$

   then there exists a unique morphism $\gamma : M \to L$ making all relevant diagrams commute:

$$\begin{tikzcd} & M \arrow[d, dotted, "\exists!\gamma"] \arrow[ddl, bend right, "\mu_I"'] \arrow[ddr, bend left, "\mu_J"] & \\ \circlearrowleft & L \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] \arrow[d, phantom, "\circlearrowleft"] & \circlearrowleft \\ F(I) \arrow[rr, "F(\alpha)"'] & {} & F(J) \end{tikzcd}$$

   that is, $\mu_I = \lambda_I \circ \gamma$ and $\mu_J = \lambda_J \circ \gamma$.

Remark 2.3.5.11. The limit is the final object in the category of cones.

Remark 2.3.5.12. If the limit exists, it is unique up to isomorphism.

Example 2.3.5.13. Let $\mathscr{I}$ be the discrete category, a category in which every morphism admits an inverse, consisting of two objects A and B with only identity morphisms and let F be a functor from $\mathscr{I}$ to any category $\mathscr{C}$. A limit of F is simply a product of $F(A)$ and $F(B)$. Notice that this limit exists if and only if a product of $F(A)$ and $F(B)$ exists.

Definition 2.3.5.14. Let $\mathscr{I}$ and $\mathscr{C}$ be categories and $F : \mathscr{I} \to \mathscr{C}$ a covariant functor, where one thinks of $\mathscr{I}$ as of a category of indices. The **colimit** of F is (if it exists) an object L of $\mathscr{C}$, endowed with morphisms $\lambda_I : F(I) \to L$ for all $I \in \mathrm{Ob}_{\mathscr{I}}$, such that

1. if $\alpha : I \to J$ is a morphism in $\mathscr{I}$, then $\lambda_I = \lambda_J \circ F(\alpha)$:

$$\begin{tikzcd}
 & L & \\
F(I) \arrow[ur, "\lambda_I"] \arrow[rr, "F(\alpha)"'] & \arrow[u, phantom, "\circlearrowleft"] & F(J) \arrow[ul, "\lambda_J"']
\end{tikzcd}$$

2. L is *initial* with respect to this property: that is, if M is another object, endowed with morphisms μ_I, also satisfying the previous requirement:

$$\begin{tikzcd}
 & M & \\
F(I) \arrow[ur, "\mu_I"] \arrow[rr, "F(\alpha)"'] & \arrow[u, phantom, "\circlearrowleft"] & F(J) \arrow[ul, "\mu_J"']
\end{tikzcd}$$

then there exists a unique morphism $\gamma : L \to M$ making all relevant diagrams commute:

$$\begin{tikzcd}
 & M & \\
 & L \arrow[u, dashed, "\exists!\gamma"] & \\
F(I) \arrow[ur, "\lambda_I"] \arrow[uur, bend left, "\mu_I"] & \arrow[u, phantom, "\circlearrowleft"] & F(J) \arrow[ul, "\lambda_J"'] \arrow[uul, bend right, "\mu_J"'] \arrow[ll, "F(\alpha)"]
\end{tikzcd}$$

that is, $\mu_I = \gamma \circ \lambda_I$ and $\mu_J = \gamma \circ \lambda_J$.

Remark 2.3.5.15. The colimit is the initial object in the category of co-cones.

Remark 2.3.5.16. If the colimit exists, it is unique up to isomorphism.

Example 2.3.5.17. Let $\mathscr{I}$ be the discrete category, a category in which every morphism admits an inverse, consisting of two objects A and B with only identity morphisms and let F be a functor from $\mathscr{I}$ to any category $\mathscr{C}$. A colimit of F is simply a coproduct of $F(A)$ and $F(B)$. Notice that this colimit exists if and only if a coproduct of $F(A)$ and $F(B)$ exists.

2.3.6 Natural Transformations and Adjunctions

As before, we can also study relationships between two functors. One of these is called a *natural transformation* and another is called an *adjunction*. First of all, the notion of *isomorphisms* is given below because it is preliminary for adjunctions.

Definition 2.3.6.1. Let $\mathscr{C}$ be a category and $f \in \mathscr{C}^1$. We say that f is an **isomorphism** if there exists $g \in \mathscr{C}^1$ such that $g \circ f = \iota_{s(f)}$ and $f \circ g = \iota_{t(f)}$.

Example 2.3.6.2. Some typical examples are as follows:

1. In the category **Set** of sets, isomorphisms are simply invertible functions.

2. In the category **Grp** of groups, isomorphisms are bijective homomorphisms.

Definition 2.3.6.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F, G : \mathscr{C} \to \mathscr{D}$ be covariant functors. A **natural transformation** between F and G is given by a map $\eta : \mathrm{Ob}_{\mathscr{C}} \to \mathrm{Hom}_{\mathscr{D}}$ such that, for each $A, B \in \mathrm{Ob}_{\mathscr{C}}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the following diagram is commutative:

$$\boxed{\begin{array}{c} A \\ {\scriptstyle f}\big\downarrow \\ B \end{array}} \quad \rightsquigarrow \quad \boxed{\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}}$$

$$\mathscr{C} \qquad\qquad\qquad \mathscr{D}$$

that is, $\eta_B \circ F(f) = G(f) \circ \eta_A$. A **natural isomorphism** is a natural transformation η such that η_X is an isomorphism for every X.

Example 2.3.6.4. Let R be a ring and let $\mathbb{M}_n(R)$ be the ring of $n \times n$ matrices over R. The inclusion map $\iota_R : R \to \mathbb{M}_n(R)$ sends an element $r \in R$ to the scalar matrix $\mathrm{diag}(r, \ldots, r)$. If $f : R \to S$ is a ring homomorphism, there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\iota_R"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{M}_n(R) \arrow[d, "\mathbb{M}_n(f)"] \\ S \arrow[r, "\iota_S"'] & \mathbb{M}_n(S) \end{tikzcd}$$

of ring homomorphisms. Thus, ι is a natural transformation from the identity functor on the the category **Rng** to the functor $\mathbb{M}_n(-)$ on **Rng**.

Proof. First of all, define $\mathrm{Id} : \mathbf{Rng} \to \mathbf{Rng}$ by

$$\begin{array}{ccc} R & & R \\ f\downarrow & \overset{\mathrm{Id}}{\mapsto} & \downarrow f \\ S & & S \end{array}$$

It is easy to see that Id is the identity functor.

Now we define $\mathbb{M}_n(-) : \mathbf{Rng} \to \mathbf{Rng}$ by

$$\begin{array}{ccc} R & & \mathbb{M}_n(R) \\ f\downarrow & \overset{\mathbb{M}_n(-)}{\mapsto} & \downarrow \mathbb{M}_n(f) \\ S & & \mathbb{M}_n(S) \end{array}$$

where $\mathbb{M}_n(f) : \mathbb{M}_n(R) \to \mathbb{M}_n(S)$ is defined by $[a_{ij}]_{n\times n} \mapsto [f(a_{ij})]_{n\times n}$.

Let $f : R \to S$ and $g : S \to T$ be ring homomorphisms and $[a_{ij}]_{n\times n} \in \mathbb{M}_n(R)$.

1. $\mathbb{M}_n(g\circ f)([a_{ij}]_{n\times n}) = [g\circ f(a_{ij})]_{n\times n} = [g(f(a_{ij}))]_{n\times n} = \mathbb{M}_n(g)([f(a_{ij})]_{n\times n})$

   $= \mathbb{M}_n(g)\circ \mathbb{M}_n(f)([a_{ij}]_{n\times n})$,

2. $\mathbb{M}_n(id_R)([a_{ij}]_{n\times n}) = [a_{ij}]_{n\times n} = id_{\mathbb{M}_n(R)}([a_{ij}]_{n\times n})$.

This yields that $\mathbb{M}_n(g\circ f) = \mathbb{M}_n(g)\circ \mathbb{M}_n(f)$ and $\mathbb{M}_n(id_R) = id_{\mathbb{M}_n(R)}$.

That is, $\mathbb{M}_n(-)$ becomes a functor.

Next, let $\iota : \mathrm{Ob}_{\mathbf{Rng}} \to \mathrm{Hom}_{\mathbf{Rng}}$ be defined by $R \mapsto \iota_R$ for each $R \in \mathrm{Ob}_{\mathbf{Rng}}$, where $\iota_R : R \to \mathbb{M}_n(R)$ is defined by $r \mapsto [r\delta_{ij}]_{n\times n}$.

Consider the following diagram

$$\begin{tikzcd} \mathrm{Id}(R) \arrow[r, "\iota_R"] \arrow[d, "\mathrm{Id}(f)"'] & \mathbb{M}_n(R) \arrow[d, "\mathbb{M}_n(f)"] \\ \mathrm{Id}(S) \arrow[r, "\iota_S"'] & \mathbb{M}_n(S) \end{tikzcd}$$

which is equivalent to the diagram

$$\begin{tikzcd} R \arrow[r, "\iota_R"] \arrow[d, "f"'] & \mathbb{M}_n(R) \arrow[d, "\mathbb{M}_n(f)"] \\ S \arrow[r, "\iota_S"'] & \mathbb{M}_n(S) \end{tikzcd}$$

To prove that this diagram is commutative, we let $r \in R$.

Consider $\mathbb{M}_n(f) \circ \iota_R(r) = \mathbb{M}_n(f)([r\delta_{ij}]_{n\times n}) = [f(r\delta_{ij})]_{n\times n}$

and $\iota_S \circ f(r) = \iota(f(r)) = [f(r)\delta_{ij}]_{n\times n} = [f(r\delta_{ij})]_{n\times n}$.

This means that $\mathbb{M}_n(f) \circ \iota_R = \iota_S \circ f$.

Therefore, ι is a natural transformation. $\square$

Example 2.3.6.5. Given a matrix $A = [a_{ij}]$ over a ring R, the *transpose* of A is defined to be the matrix $A^t = [a_{ji}]$, whose entries are in the opposite ring R°. We see that a map $(-)^t_R : \mathbb{M}_n(R^\circ) \to (\mathbb{M}_n(R))^\circ$ is a ring isomorphism. Given a ring homomorphism $f : R \to S$, there exists a commutative diagram

$$\begin{tikzcd} \mathbb{M}_n(R^\circ) \arrow[r, "(-)^t_R"] \arrow[d, "\mathbb{M}_n(f^\circ)"'] \arrow[dr, phantom, "\circlearrowleft"] & (\mathbb{M}_n(R))^\circ \arrow[d, "(\mathbb{M}_n(f))^\circ"] \\ \mathbb{M}_n(S^\circ) \arrow[r, "(-)^t_S"'] & (\mathbb{M}_n(S))^\circ \end{tikzcd}$$

so that $(-)^t$ is a natural isomorphism between the functors $\mathbb{M}_n((-)^\circ)$ and $(\mathbb{M}_n(-))^\circ$.

Proof. Recall that the *opposite* of a ring $(R, +, \cdot)$ is the ring $(R, +, \star)$ whose multiplication $\star$ is defined by $a \star b := b \cdot a$ for every $a, b \in R$.

Claim that a ring homomorphism $f : R \to S$ induces a ring homomorphism $f^\circ : R^\circ \to S^\circ$.

Since $R^\circ = R$ and $S^\circ = S$ as sets, we can define $f^\circ : r^\circ \mapsto f^\circ(r)$, where $r^\circ := r$ and $f^\circ(r) := f(r)$.

To verify that f° is a ring homomorphism, suppose that $a, b \in R^\circ$.

1. $f^\circ(a+b) = f(a+b) = f(a) + f(b) = f^\circ(a) + f^\circ(b)$,

2. $f^\circ(a \star b) = f(b \cdot a) = f(b) \cdot f(a) = f^\circ(a) \star f^\circ(b)$.

This means that f° is a ring homomorphism.

Now we define $\mathbb{M}_n(-) : \mathbf{Rng} \to \mathbf{Rng}$ by

$$\begin{tikzcd} R \arrow[d, "f"'] & \overset{\mathbb{M}_n(-)}{\mapsto} & \mathbb{M}_n(R) \arrow[d, "\mathbb{M}_n(f)"] \\ S & & \mathbb{M}_n(S) \end{tikzcd}$$

where $\mathbb{M}_n(f) : \mathbb{M}_n(R) \to \mathbb{M}_n(S)$ is defined by $[a_{ij}]_{n\times n} \mapsto [f(a_{ij})]_{n\times n}$.

We have already shown that $\mathbb{M}_n(-)$ is a functor in Example 2.3.6.4.

Now we define $(-)^\circ : \mathbf{Rng} \to \mathbf{Rng}$ by

$$\begin{tikzcd} R \arrow[d, "f"'] & \overset{(-)^\circ}{\mapsto} & R^\circ \arrow[d, "f^\circ"] \\ S & & S^\circ \end{tikzcd}$$

where $f^\circ : r^\circ \mapsto f(r)^\circ$ with $r^\circ := r$ and $f(r)^\circ := f(r)$.

The fact that $(-)^\circ$ is also a functor follows from the definition of f°.

Since composition of functors is another functor, both $\mathbb{M}_n((-)^\circ)$ and $(\mathbb{M}_n(-))^\circ$ are also functors.

Let $(-)^t : \mathrm{Ob}_{\mathbf{Rng}} \to \mathrm{Hom}_{\mathbf{Rng}}$ be defined by $R \mapsto (-)^t_R$ for every $R \in \mathrm{Ob}_{\mathbf{Rng}}$, where $(-)^t_R : \mathbb{M}_n(R^\circ) \to (\mathbb{M}_n(R))^\circ$ is defined by $[a^\circ_{ij}]_{n\times n} \mapsto [a_{ij}]^\circ_{n\times n}$.

Now consider the following diagram

$$\begin{tikzcd} \mathbb{M}_n(R^\circ) \arrow[r, "(-)^t_R"] \arrow[d, "\mathbb{M}_n(f^\circ)"'] & (\mathbb{M}_n(R))^\circ \arrow[d, "(\mathbb{M}_n(f))^\circ"] \\ \mathbb{M}_n(S^\circ) \arrow[r, "(-)^t_S"'] & (\mathbb{M}_n(S))^\circ \end{tikzcd}$$

To prove that $(-)^t$ is a natural transformation, let $[a^\circ_{ij}]_{n\times n} \in \mathbb{M}_n(R^\circ)$.

Consider $(\mathbb{M}_n(f))^\circ \circ (-)^t_R([a^\circ_{ij}]_{n\times n}) = (\mathbb{M}_n(f))^\circ([a_{ij}]^\circ_{n\times n}) = [f(a_{ij})]^\circ_{n\times n}$

and $(-)^t_S \circ \mathbb{M}_n(f^\circ)([a^\circ_{ij}]_{n\times n}) = (-)^t_S([f(a^\circ_{ij})]_{n\times n}) = [f(a_{ij})]^\circ_{n\times n}$.

This implies that $(\mathbb{M}_n(f))^\circ \circ (-)^t_R = (-)^t_S \circ \mathbb{M}_n(f^\circ)$.

Hence, $(-)^t$ is a natural transformation.

In addition, if we define $(-)^T_R : (\mathbb{M}_n(R))^\circ \to \mathbb{M}_n(R^\circ)$ by $[a_{ij}]^\circ_{n\times n} \mapsto [a^\circ_{ij}]_{n\times n}$, then $(-)^t_R \circ (-)^T_R = \mathrm{Id}_{(\mathbb{M}_n(R))^\circ}$ and $(-)^T_R \circ (-)^t_R = \mathrm{Id}_{\mathbb{M}_n(R^\circ)}$.

So $(-)^t_R$ is an isomorphism for every R.

Therefore, $(-)^t$ is a natural isomorphism. $\square$

As promised, we now consider the concept of adjunctions which will later relate our main results.

Definition 2.3.6.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. We say that the functor F is **left adjoint** to the functor G or the functor G is **right adjoint** to the functor F, denoted by $F \dashv G$ or $G \vdash F$, if there exist natural transformations $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$ making the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \arrow[dl, phantom, "\circlearrowleft"] \\ & G \end{tikzcd}$$

that is, $\varepsilon F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta G = 1_G$.

We also have an equivalent definition of adjunctions as stated in the following theorem.

Theorem 2.3.6.7. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. The functor F is left adjoint to G if and only if there exists a natural isomorphism* $\mathrm{Hom}_{\mathscr{C}}(A, G(B)) \simeq \mathrm{Hom}_{\mathscr{D}}(F(A), B)$ *for all $A \in \mathrm{Ob}_{\mathscr{C}}$ and $B \in \mathrm{Ob}_{\mathscr{D}}$.*

Example 2.3.6.8. Let **Set** and **Mon** be the categories of sets and monoids, respectively. Then the free monoid functor $\mathscr{F}$: **Set** $\to$ **Mon** is left adjoint to the forgetful functor $\mathscr{U}$: **Mon** $\to$ **Set**.

Proof. Remember that the forgetful functor $\mathscr{U}$ sends monoids to their underlying sets forgetting the binary operation and the identity.

On the other hand, the free monoid functor $\mathscr{F}$ maps sets to monoids whose binary operation is the concatenation of elements of those sets.

First of all, let $A \in \mathrm{Ob}_{\mathbf{Set}}$ and $M \in \mathrm{Ob}_{\mathbf{Mon}}$.

Consider $\mathscr{F}\mathscr{U}(M) = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathscr{U}(M)\} \cup \{()_M\}$.

Suppose that $f : M \to N$ is a homomorphism of monoids.

We get a function $\mathscr{F}\mathscr{U}(f) : \mathscr{F}\mathscr{U}(M) \to \mathscr{F}\mathscr{U}(N)$ defined by $(x_1, x_2, \ldots, x_n) \mapsto (f(x_1), f(x_2), \ldots, f(x_n))$ and $()_M \mapsto ()_N$.

This function is a homomorphism of monoids because

$$\begin{aligned}
& \mathscr{F}\mathscr{U}(f)\big((x_1, \ldots, x_n) *_{\mathscr{F}\mathscr{U}(M)} (y_1, \ldots, y_m)\big) \\
= \;& \mathscr{F}\mathscr{U}(f)\big((x_1, \ldots, x_n, y_1, \ldots, y_m)\big) \\
= \;& (f(x_1), \ldots, f(x_n), f(y_1), \ldots, f(y_m)) \\
= \;& (f(x_1), \ldots, f(x_n)) *_{\mathscr{F}\mathscr{U}(N)} (f(y_1), \ldots, f(y_m)) \\
= \;& \mathscr{F}\mathscr{U}(f)\big((x_1, \ldots, x_n)\big) *_{\mathscr{F}\mathscr{U}(N)} \mathscr{F}\mathscr{U}(f)\big((y_1, \ldots, y_m)\big)
\end{aligned}$$

for each $(x_1, \ldots, x_n), (y_1, \ldots, y_m) \in \mathscr{F}\mathscr{U}(M)$.

Define $\varepsilon : \mathrm{Ob}_{\mathbf{Mon}} \to \mathrm{Hom}_{\mathbf{Mon}}$ by $M \mapsto \varepsilon_M$ where $\varepsilon_M : \mathscr{F}\mathscr{U}(M) \to M$ is defined by $(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \cdots x_n$ for all $(x_1, x_2, \ldots, x_n) \in \mathscr{F}\mathscr{U}(M)$ and $()_M \mapsto 1_M$.

Now consider the diagram

$$\begin{tikzcd}
\mathscr{F}\mathscr{U}(M) \arrow[r, "\varepsilon_M"] \arrow[d, "\mathscr{F}\mathscr{U}(f)"'] & M \arrow[d, "f"] \\
\mathscr{F}\mathscr{U}(N) \arrow[r, "\varepsilon_N"'] & N
\end{tikzcd}$$

To prove commutativity, we need to show that $\varepsilon_M \circ \mathscr{F}\mathscr{U}(f) = f \circ \varepsilon_N$.

For each $(x_1, x_2, \ldots, x_n) \in \mathscr{F}\mathscr{U}(M)$, we have

1. $\varepsilon_N \circ \mathscr{F}\mathscr{U}(f) \circ \big((x_1, x_2, \ldots, x_n)\big) = \varepsilon_N\big((f(x_1), \ldots, f(x_n))\big) = f(x_1) *_N \cdots *_N f(x_n)$,

2. $f \circ \varepsilon_M\big((x_1, x_2, \ldots, x_n)\big) = f(x_1 *_M \cdots *_M x_n) = f(x_1) *_N \cdots *_N f(x_n)$,

3. $\varepsilon_N \circ \mathscr{F}\mathscr{U}(f)\big(()_M\big) = \varepsilon_N\big(()_N\big) = 1_N = f(1_M) = f \circ \varepsilon_M\big(()_M\big)$.

Thus, ε is a natural transformation.

Now consider $\mathscr{U}\mathscr{F}(A) = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in A\}$.

We see that a function $f : A \to B$ induces a function $\mathscr{U}\mathscr{F}(f) : \mathscr{U}\mathscr{F}(A) \to \mathscr{U}\mathscr{F}(B)$ defined by $(x_1, x_2, \ldots, x_n) \mapsto (f(x_1), f(x_2), \ldots, f(x_n))$.

Define $\eta : \mathrm{Ob}_{\mathbf{Set}} \to \mathrm{Hom}_{\mathbf{Set}}$ by $A \mapsto \eta_A$ where $\eta_A : A \to \mathscr{U}\mathscr{F}(A)$ is defined by $x \mapsto (x)$ for every $x \in A$.

Now consider the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathscr{U}\mathscr{F}(A) \arrow[d, "\mathscr{U}\mathscr{F}(f)"] \\ B \arrow[r, "\eta_B"'] & \mathscr{U}\mathscr{F}(B) \end{tikzcd}$$

To prove commutativity, we need to show that $\mathscr{U}\mathscr{F}(f) \circ \eta_A = \eta_B \circ f$.

For each $x \in A$, we get $\mathscr{U}\mathscr{F}(f) \circ \eta_A(x) = \mathscr{U}\mathscr{F}(f)\big((x)\big) = \big(f(x)\big) = \eta_B \circ f(x)$.

Hence, η is a natural transformation.

Ultimately, consider the following diagrams

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mathscr{F}\eta"] \arrow[dr, "1_{\mathscr{F}}"'] & \mathscr{F}\mathscr{U}\mathscr{F} \arrow[d, "\varepsilon\mathscr{F}"] \\ & \mathscr{F} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{U} \arrow[r, "\eta\mathscr{U}"] \arrow[dr, "1_{\mathscr{U}}"'] & \mathscr{U}\mathscr{F}\mathscr{U} \arrow[d, "\mathscr{U}\varepsilon"] \\ & \mathscr{U} \end{tikzcd}$$

To obtain an adjunction, we have to show that both triangles are commutative; that is, $\varepsilon\mathscr{F} \circ \mathscr{F}\eta = 1_{\mathscr{F}}$ and $\mathscr{U}\varepsilon \circ \eta\mathscr{U} = 1_{\mathscr{U}}$.

But verifying that such diagrams commute is equivalent to demonstrating that the following triangles commute:

$$
\begin{tikzcd}
\mathscr{F}(A) \arrow[r, "\mathscr{F}\eta_A"] \arrow[dr, "1_{\mathscr{F}(A)}"'] & \mathscr{F}\mathscr{U}\mathscr{F}(A) \arrow[d, "\varepsilon_A\mathscr{F}"] \\
& \mathscr{F}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{U}(M) \arrow[r, "\eta_M\mathscr{U}"] \arrow[dr, "1_{\mathscr{U}(M)}"'] & \mathscr{U}\mathscr{F}\mathscr{U}(M) \arrow[d, "\mathscr{U}\varepsilon_M"] \\
& \mathscr{U}(M)
\end{tikzcd}
$$

Recall that $\mathscr{F}(A) := \{(x_1,x_2,\dots,x_n) \mid x_i \in A, i = 1,2,\dots,n\} \cup \{()\}$.

This implies that $\mathscr{U}\mathscr{F}(A) = \{(x_1,x_2,\dots,x_n) \mid x_i \in A, i = 1,2,\dots,n\}$ and so $\mathscr{F}\mathscr{U}\mathscr{F}(A) = \{((x_{11},\dots,x_{1n}),\dots,(x_{m1},\dots,x_{mn})) \mid x_{ij} \in A, i = 1,\dots,m \ j = 1,\dots,n\} \cup \{(())\}$.

Define $\mathscr{F}\eta_A : \mathscr{F}(A) \to \mathscr{F}\mathscr{U}\mathscr{F}(A)$ by $(x_1,x_2,\dots,x_n) \mapsto ((x_1,x_2,\dots,x_n))$ for all $(x_1,x_2,\dots,x_n) \in \mathscr{F}(A)$ and $() \mapsto (())$.

Define $\varepsilon_A\mathscr{F} : \mathscr{F}\mathscr{U}\mathscr{F}(A) \to \mathscr{F}(A)$ by

$$((x_{11},\dots,x_{1n}),\dots,(x_{m1},\dots,x_{mn})) \mapsto (x_{11},\dots,x_{1n},\dots,x_{m1},\dots,x_{mn}).$$

It is easy to see that $\mathscr{F}\eta_A$ and $\varepsilon_A\mathscr{F}$ are homomorphisms of monoids.

Recall again that $\mathscr{U}(M) = M$ as a set.

It follows that $\mathscr{F}\mathscr{U}(M) = \{(x_1,x_2,\dots,x_n) \mid x_i \in \mathscr{U}(M), i = 1,\dots,n\} \cup \{()\}$ and so $\mathscr{U}\mathscr{F}\mathscr{U}(M) = \{(x_1,x_2,\dots,x_n) \mid x_i \in \mathscr{U}(M), i = 1,\dots,n\}$.

Define functions $\eta_M\mathscr{U} : \mathscr{U}(M) \to \mathscr{U}\mathscr{F}\mathscr{U}(M)$ and $\mathscr{U}\varepsilon_M : \mathscr{U}\mathscr{F}\mathscr{U}(M) \to \mathscr{U}(M)$ by $\eta_M\mathscr{U} : x \mapsto (x)$ and $\mathscr{U}\varepsilon_M : (x_1,x_2,\dots,x_n) \mapsto x_1x_2\cdots x_n \in M = U(M)$.

Next, we will check that $\varepsilon_A\mathscr{F} \circ \mathscr{F}\eta_A = 1_{\mathscr{F}(A)}$ and $\mathscr{U}\varepsilon_M \circ \eta_M\mathscr{U} = 1_{\mathscr{U}(M)}$.

For every $(x_1,x_2,\dots,x_n) \in \mathscr{F}(A)$ and $x \in \mathscr{U}(M)$, we obtain

1. $\varepsilon_A\mathscr{F} \circ \mathscr{F}\eta_A((x_1,x_2,\dots,x_n)) = \varepsilon_A\mathscr{F}\big(((x_1,x_2,\dots,x_n))\big) = (x_1,x_2,\dots,x_n)$
2. $\mathscr{U}\varepsilon_M \circ \eta_M\mathscr{U}(x) = \mathscr{U}\varepsilon_M((x)) = x$.

This yields that $\varepsilon_A\mathscr{F} \circ \mathscr{F}\eta_A = 1_{\mathscr{F}(A)}$ and $\mathscr{U}\varepsilon_M \circ \eta_M\mathscr{U} = 1_{\mathscr{U}(M)}$.

It follows that $\varepsilon\mathscr{F} \circ \mathscr{F}\eta = 1_{\mathscr{F}}$ and $\mathscr{U}\varepsilon \circ \eta\mathscr{U} = 1_{\mathscr{U}}$.

Therefore, $\mathscr{F}$ is left adjoint to $\mathscr{U}$. $\square$

2.3.7 Monads and Algebras

In this subsection, we discuss the notions of monads and algebras[2] for monads. The idea is that a monad gives us a way of describing a theory, such as the theory of groups, by encapsulating all the information about how structures in that theory are required to behave. In addition, an algebra is a set equipped with extra structure and the algebra action tells us how the operations are to be evaluated. For more details, see Cheng E. - Lauda A. [CL].

Any endofunctor T on a category $\mathscr{C}$ has composites $T^2 = T \circ T : \mathscr{C} \to \mathscr{C}$ and $T^3 = T^2 \circ T : \mathscr{C} \to \mathscr{C}$. If $\mu : T^2 \Rightarrow T$ is a natural transformation with components $\mu_X : T^2(X) \to T(X)$ for each $X \in \mathrm{Ob}_{\mathscr{C}}$, then $T\mu : T^3 \Rightarrow T^2$ gives the natural transformation with components $(T\mu)_X = T\mu_X : T^3(X) \to T^2(X)$ and also $\mu T : T^3 \Rightarrow T^2$ has components $(\mu T)_X = \mu_{T(X)}$.

Definition 2.3.7.1. A **monad** on a category $\mathscr{C}$ consists of a functor $T : \mathscr{C} \to \mathscr{C}$ and natural transformations $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow T$ (the **unit**) and $\mu : T^2 \Rightarrow T$ (the **multiplication**) such that the following diagrams commute:

$$\begin{tikzcd}
T(X) \arrow[r, "T\eta_X"] \arrow[dr, "1_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] & T(X) \arrow[l, "\eta_{T(X)}"'] \arrow[dl, "1_{T(X)}"] \\
& T(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3(X) \arrow[r, "\mu_X T"] \arrow[d, "T\mu_X"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2(X) \arrow[d, "\mu_X"] \\
T^2(X) \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}$$

that is, $\mu_X \circ T\eta_X = 1_{T(X)} = \mu_X \circ \eta_{T(X)}$ and $\mu_X \circ T\mu_X = \mu_X \circ \mu_X T$.

Theorem 2.3.7.2. *Every adjunction gives rise a monad.*

Remark 2.3.7.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. We see that an adjunction pair $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ has a composite $T = GF$ an endofunctor, the unit η of the adjunction is a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow T$, and the counit $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{C}}$ of the adjunction

[2] There are several definitions of algebras depending on the context. In universal algebra, an algebra over a ring R is an R-module with an R-bilinear multiplication together with some compatibility conditions.

produces a natural transformation $\mu = G\varepsilon F : GFGF \Rightarrow GF = T$. Now we consider the commutativity of the following diagrams:

$$\begin{tikzcd} GFGFGF \arrow[r, "GFG\varepsilon F"] \arrow[d, "G\varepsilon FGF"'] \arrow[dr, phantom, "\circlearrowleft"] & GFGF \arrow[d, "G\varepsilon F"] \\ GFGF \arrow[r, "G\varepsilon F"'] & GF \end{tikzcd} \qquad \begin{tikzcd} FGFG \arrow[r, "FG\varepsilon"] \arrow[d, "\varepsilon FG"'] \arrow[dr, phantom, "\circlearrowleft"] & FG \arrow[d, "\varepsilon"] \\ FG \arrow[r, "\varepsilon"'] & 1_{\mathscr{D}} \end{tikzcd}$$

Similarly, the left and right unit axioms reduce to the diagram:

$$\begin{tikzcd} 1_X GF \arrow[r, "\eta GF"] \arrow[dr, "="'] & GFGF \arrow[d, "G\varepsilon F"] & GF1_X \arrow[l, "GF\eta"'] \arrow[dl, "="] \\ & GF & \end{tikzcd}$$

which are simply two triangular identities $1_G = G\varepsilon \circ \eta G : G \Rightarrow G$ and $1_F = \varepsilon F \circ F\eta : F \Rightarrow F$ for the adjunction. Hence, $(GF, \eta, G\varepsilon F)$ is a monad on $\mathscr{C}$.

A standard example is the monad for monoids.

Example 2.3.7.4. There is a forgetful functor $\mathscr{U} : \mathbf{Mon} \to \mathbf{Set}$ from the category of monoids to the category of sets which simply forgets about the multiplication and identity. Moreover, there is a free functor $\mathscr{F} : \mathbf{Set} \to \mathbf{Mon}$ which sends every set to the free monoid on that set. We have shown in Example 2.3.6.8 that $\mathscr{F}$ is left adjoint to $\mathscr{U}$. Thus, $T = \mathscr{U}\mathscr{F}$ is a monad on **Set**.

Remark 2.3.7.5. Let's synthesize the data in Example 2.3.7.4: for each set A

1. $T(A) = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in A\}$, the set of strings of elements of A,

2. $T^2(A) = \{((x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn})) \mid x_{ij} \in A,\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$, the set of strings of strings of elements of A,

3. $T^3(A) = \{(((x_{111}, \ldots, x_{11n}), \ldots, (x_{1m1}, \ldots, x_{1mn})), \ldots, ((x_{l11}, \ldots, x_{l1n}), \ldots, (x_{lm1}, \ldots, x_{lmn}))) \mid x_{ijk} \in A,\ i = 1, \ldots, l,\ j = 1, \ldots, m,\ k = 1, \ldots, n\}$, the set of strings of strings of strings of elements of A,

4. a natural transformation $\eta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow T$ is defined by $A \mapsto \eta_A$, where $\eta_A : A \to T(A)$ is defined by $x \mapsto (x)$,

5. a natural transformation $\mu : T^2 \Rightarrow T$ is defined by $A \mapsto \mu_A$, where $\mu_A : T^2(A) \to T(A)$ is defined by $((x_{11}, \ldots, x_{1n}), \ldots, (x_{m1}, \ldots, x_{mn})) \mapsto (x_{11}, \ldots, x_{mn})$,

6. $T\eta_A : T(A) \to T^2(A)$ is defined by $(x_1, x_2, \ldots, x_n) \mapsto ((x_1), (x_2), \ldots, (x_n))$,

7. $\eta_{T(A)} : T(A) \to T^2(A)$ is defined by $(x_1, x_2, \ldots, x_n) \mapsto ((x_1, x_2, \ldots, x_n))$,

8. $\mu_A T : T^3(A) \to T^2(A)$ is defined by
$(((x_{111}, \ldots, x_{11n}), \ldots, (x_{1m1}, \ldots, x_{1mn})), \ldots, ((x_{l11}, \ldots, x_{l1n}), \ldots, (x_{lm1}, \ldots, x_{lmn})))$
$\mapsto ((x_{111}, \ldots, x_{11n}, \ldots, x_{1m1}, \ldots, x_{1mn}), \ldots, (x_{l11}, \ldots, x_{l1n}, \ldots, x_{lm1}, \ldots, x_{lmn}))$,

9. $T\mu_A : T^3(A) \to T^2(A)$ is defined by
$(((x_{111}, \ldots, x_{11n}), \ldots, (x_{1m1}, \ldots, x_{1mn})), \ldots, ((x_{l11}, \ldots, x_{l1n}), \ldots, (x_{lm1}, \ldots, x_{lmn})))$
$\mapsto ((x_{111}, \ldots, x_{11n}), \ldots, (x_{1m1}, \ldots, x_{1mn}), \ldots, (x_{l11}, \ldots, x_{l1n}), \ldots, (x_{lm1}, \ldots, x_{lmn}))$.

Definition 2.3.7.6. Let (T, η, μ) be a monad on a category $\mathscr{C}$. An **algebra** for a monad T consists of an object $A \in \mathrm{Ob}_{\mathscr{C}}$ together with a morphism $T(A) \xrightarrow{\theta} A$ such that the following diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & T(A) \arrow[d, "\theta"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2(A) \arrow[r, "\mu_A"] \arrow[d, "T\theta"'] \arrow[dr, phantom, "\circlearrowleft"] & T(A) \arrow[d, "\theta"] \\
T(A) \arrow[r, "\theta"'] & A
\end{tikzcd}
$$

that is, $\theta \circ \eta_A = 1_A$ and $\theta \circ T\theta = \theta \circ \mu_A$.

Example 2.3.7.7. An algebra for a monad for the category **Mon** is simply a monoid.

2.4 Weak ω-Categories: Penon's Definition

In this section we discuss mainly the Penon's definition of weak ω-categories. We start with the definitions of globular ω-magmas, strict ω-categories, and finally weak ω-categories. For further details, the readers are required to see Cheng E., Lauda A. [CL], Leinster T. [L1, L2], and Penon J. [P].

2.4.1 Globular ω-Magmas

As defined in the subsection 2.3.1, a quiver can be generalized to an n-quiver for any $n \in \mathbb{N}$ and ultimately an ω-quiver. In a similar way, since a category can be viewed to be based on a quiver, an n-category (ω-category, respectively) can be viewed to be based on an n-quiver (ω-quiver, respectively).

Definition 2.4.1.1. Let $n \in \mathbb{N}$. An n**-quiver** $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-2}}{\overset{s^{n-2}}{\leftleftarrows}} Q^{n-1} \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n$ is a finite family of sets Q^k for $k = 0,1,\ldots,n$ equipped with n pairs of source and target maps $s^k,t^k : Q^{k+1} \rightrightarrows Q^k$ for each $k = 0,1,\ldots,n-1$. Elements of Q^m are called m**-cells** of Q.

Remark 2.4.1.2. In this case, we also require that the sets Q^i and Q^j need to be disjoint for all $i \neq j$ so as to avoid the repetition of names of cells.

Definition 2.4.1.3. An ω**-quiver** $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-2}}{\overset{s^{n-2}}{\leftleftarrows}} Q^{n-1} \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^{n}}{\overset{s^{n}}{\leftleftarrows}} \cdots$ is an infinite family of sets Q^k, for any $k \in \mathbb{N}_0$, equipped with infinite pairs of source and target maps $s^k,t^k : Q^{k+1} \rightrightarrows Q^k$, for each $k \in \mathbb{N}_0$.

Definition 2.4.1.4. Let $n \in \mathbb{N}$. An n**-globular set** is an n-quiver which satisfies the globularity condition, i.e. $s^{k-1}s^k = s^{k-1}t^k$ and $t^{k-1}s^k = t^{k-1}t^k$ for all $k = 1,2,\ldots,n-1$.

Remark 2.4.1.5. For each n-globular set, there exists a colimit Q^n which can be considered as a bundle

$$\begin{tikzcd} Q^n \arrow[d] \\ Q^{n-1} \times Q^{n-1} \end{tikzcd}$$

with nontrivial fibers over $(f,g) \in Q^{n-1} \times Q^{n-1}$ that satisfies the globular condition.

Definition 2.4.1.6. An **ω-globular set** is an ω-quiver satisfying the globularity condition, i.e. $s^{k-1}s^k = s^{k-1}t^k$ and $t^{k-1}s^k = t^{k-1}t^k$ for all $k \in \mathbb{N}$.

Remark 2.4.1.7. For each ω-globular set, there exists a colimit Q^∞ which can be considered as a bundle

$$\begin{tikzcd} Q^\infty \arrow[d] \\ Q^\infty \end{tikzcd}$$

with trivial fibers .

Definition 2.4.1.8. A **globular n-magma** is an n-globular set equipped with a function $\circ_p^m : Q^m \times_p Q^m \to Q^m$ for each $0 \le p < m \le n$, where

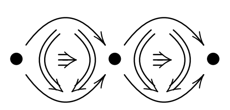
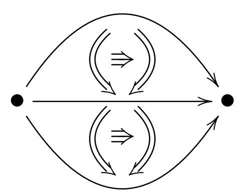
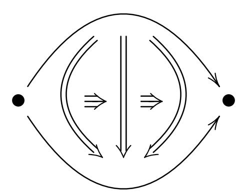
$$Q^m \times_p Q^m := \{(x',x) \in Q^m \times Q^m \mid t^p t^{p+1} \cdots t^{m-1}(x) = s^p s^{p+1} \cdots s^{m-1}(x')\},$$

such that the following conditions hold: if $0 \le p < m \le n$ and $(x',x) \in Q^m \times_p Q^m$,

- $s^q s^{q+1} \cdots s^{m-1}(x' \circ_p^m x) = \begin{cases} s^q s^{q+1} \cdots s^{m-1}(x') \circ_p^q s^q s^{q+1} \cdots s^{m-1}(x), & q > p; \\ s^q s^{q+1} \cdots s^{m-1}(x'), & q \le p. \end{cases}$,

- $t^q t^{q+1} \cdots t^{m-1}(x' \circ_p^m x) = \begin{cases} t^q t^{q+1} \cdots t^{m-1}(x') \circ_p^q t^q t^{q+1} \cdots t^{m-1}(x), & q > p; \\ t^q t^{q+1} \cdots t^{m-1}(x), & q \le p. \end{cases}$.

Remark 2.4.1.9. We can compose k-cells along boundary p-cells, for $0 \le p < k$. We call this **p-composition**. The case $k = 3$ is depicted below.

		
0-composition or composition along bounding 0-cells	1-composition or composition along bounding 1-cells	2-composition or composition along bounding 2-cells

Definition 2.4.1.10. A **globular ω-magma** is an ω-globular set equipped with a function $\circ^m_p : Q^m \times_p Q^m \to Q^m$ for each $0 \le p < m$, where

$$Q^m \times_p Q^m := \{(x',x) \in Q^m \times Q^m \mid t^p t^{p+1} \cdots t^{m-1}(x) = s^p s^{p+1} \cdots s^{m-1}(x')\},$$

such that the following conditions hold: if $0 \le p < m$ and $(x',x) \in Q^m \times_p Q^m$,

- $s^q s^{q+1} \cdots s^{m-1}(x' \circ^m_p x) = \begin{cases} s^q s^{q+1} \cdots s^{m-1}(x') \circ^q_p s^q s^{q+1} \cdots s^{m-1}(x), & q > p; \\ s^q s^{q+1} \cdots s^{m-1}(x'), & q \le p. \end{cases}$,

- $t^q t^{q+1} \cdots t^{m-1}(x' \circ^m_p x) = \begin{cases} t^q t^{q+1} \cdots t^{m-1}(x') \circ^q_p t^q t^{q+1} \cdots t^{m-1}(x), & q > p; \\ t^q t^{q+1} \cdots t^{m-1}(x), & q \le p. \end{cases}$.

Remark 2.4.1.11. In view of further investigation in the direction of ω-C*-categories, one might need to add linear structures and norms on all the fiber-blocks and in order to keep this feature also in the ω-case, it is convenient to adopt a slightly more general definition of globular ω-magmas that allows the possibility on nontrivial fiber-blocks on each cell in Q^∞.

2.4.2 Strict ω-Categories

Definition 2.4.2.1. Let $n \in \mathbb{N}$. A **strict** n**-category** is a globular n-magma $\mathscr{C}$ equipped with a function $\iota^p : \mathscr{C}^p \to \mathscr{C}^{p+1}$ for each $0 < p < n$; we call $\iota^p(x)$ the **identity** on x, satisfying the following axioms:

1. (sources and targets of identities) if $0 \le p < n$ and $x \in \mathscr{C}^p$, then
   $s^p(\iota^p(x)) = x = t^p(\iota^p(x))$,

2. (associativity) if $0 \le p < m \le n$ and $x,y,z \in \mathscr{C}^m$ with
   $(z,y),(y,x) \in \mathscr{C}^m \times_p \mathscr{C}^m$, then $(z \circ^m_p y) \circ^m_p x = z \circ^m_p (y \circ^m_p x)$,

3. (unitality) if $0 \le p < m \le n$ and $x \in \mathscr{C}^m$, then

$$\iota^{m-1} \cdots \iota^p t^p \cdots t^{m-1}(x) \circ^m_p x = x = x \circ^m_p \iota^{m-1} \cdots \iota^p s^p \cdots s^{m-1}(x),$$

4. (binary exchange) if $0 \leq q < p < m \leq n$ and $x, x', y, y' \in \mathscr{C}^m$ with $(y', y), (x', x) \in \mathscr{C}^m \times_p \mathscr{C}^m$ and $(y', x'), (y, x) \in \mathscr{C}^m \times_q \mathscr{C}^m$, then

$$(y' \circ_p^m y) \circ_q^m (x' \circ_p^m x) = (y' \circ_q^m x') \circ_p^m (y \circ_q^m x),$$

5. (functoriality of identities) if $0 \leq q < p < n$ and $(x', x) \in \mathscr{C}^p \times_q \mathscr{C}^p$, then

$$\iota^p(x') \circ_q^{p+1} \iota^p(x) = \iota^p(x' \circ_q^p x).$$

Definition 2.4.2.2. A **strict ω-category** is a globular ω-magma $\mathscr{C}$ equipped with a function $\iota^p : \mathscr{C}^p \to \mathscr{C}^{p+1}$ for each $p > 0$ satisfying the following axioms:

1. (sources and targets of identities) if $p \geq 0$ and $x \in \mathscr{C}^p$, then $s^p(\iota^p(x)) = x = t^p(\iota^p(x))$,

2. (associativity) if $0 \leq p < m$ and $x, y, z \in \mathscr{C}^m$ with $(z, y), (y, x) \in \mathscr{C}^m \times_p \mathscr{C}^m$, then $(z \circ_p^m y) \circ_p^m x = z \circ_p^m (y \circ_p^m x)$,

3. (unitality) if $0 \leq p < m$ and $x \in \mathscr{C}^m$, then

$$\iota^{m-1} \cdots \iota^p t^p \cdots t^{m-1}(x) \circ_p^m x = x = x \circ_p^m \iota^{m-1} \circ \iota^p s^p \cdots s^{m-1}(x),$$

4. (binary exchange) if $0 \leq q < p < m$ and $x, x', y, y' \in \mathscr{C}^m$ with $(y', y), (x', x) \in \mathscr{C}^m \times_p \mathscr{C}^m$ and $(y', x'), (y, x) \in \mathscr{C}^m \times_q \mathscr{C}^m$, then

$$(y' \circ_p^m y) \circ_q^m (x' \circ_p^m x) = (y' \circ_q^m x') \circ_p^m (y \circ_q^m x),$$

5. (functoriality of identities) if $0 \leq q < p$ and $(x', x) \in \mathscr{C}^p \times_q \mathscr{C}^p$, then

$$\iota^p(x') \circ_q^{p+1} \iota^p(x) = \iota^p(x' \circ_q^p x).$$

2.4.3 Weak ω-Category: Penon's Definition

Definition 2.4.3.1. We say that two k-cells are **parallel** if $k = 0$ or $k > 0$ and they have the same source and target as one another.

Definition 2.4.3.2. Let A be a globular ω-magma, B a strict ω-category, and $f : A \to B$ a morphism of globular ω-magmas. For each $\alpha, \beta \in A^k$ such that

1. α and β are parallel, and

2. $f(\alpha) = f(\beta)$,

a **contraction** $[\cdot,\cdot]$ on a map f gives a **contraction cell** $[\alpha,\beta] : \alpha \to \beta$ such that $f([\alpha,\beta]) = 1_{f(\alpha)} = 1_{f(\beta)} \in B$. Furthermore, if A is unital, then $[\alpha,\alpha] = 1_\alpha \in A$ for all α.

Remark 2.4.3.3. Informally, given any two parallel k-cells α and β with the same image under f, there is a given $(k+1)$-cell $\alpha \to \beta$ that maps to the identity under f.

Consider a category $\mathscr{Q}$ whose objects are of the form $\left(A \xrightarrow{f} B, [\cdot,\cdot]\right)$ where A is a globular ω-magma, B is a strict ω-category, f is a morphism of globular ω-magmas i.e. f preserves composition, equipped with a specified contraction $[\cdot,\cdot]$, and whose morphisms are of the form $\phi : \left(A \xrightarrow{f} B, [\cdot,\cdot]\right) \to \left(C \xrightarrow{g} D, [\cdot,\cdot]'\right)$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi_{\mathscr{M}}"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "g"] \\ B \arrow[r, "\phi_{\mathscr{C}}"'] & D \end{tikzcd}$$

that is, $g \circ \phi_{\mathscr{M}} = \phi_{\mathscr{C}} \circ f$ and they should also preserve contractions, i.e. $\phi_{\mathscr{M}}([x,y]) = [\phi_{\mathscr{M}}(x), \phi_{\mathscr{M}}(y)]'$.

We have a forgetful functor $G : \mathscr{Q} \to \mathbf{GSet}$ to the category of ω-globular sets which sends $\left(A \xrightarrow{f} B, [\cdot,\cdot]\right)$ to the underlying ω-globular set A. The important result is that G has a left adjoint F.

Theorem 2.4.3.4. *[P, P.69] A free ω-magma over an ω-quiver exists.*

Definition 2.4.3.5. A **weak ω-category** is an algebra for the monad $P = GF$.

2.5 Strict Involutive Globular Higher Categories

In this section we focus on one of the possible concepts of strict involution in the context of strict involutive globular ω-categories. In this work, involutions are defined as involutive functors together with certain covariance and contravariance properties with respect to compositions. For more detailed references, see Bertozzini P., et al [BCLS] and Jacobs B. [J].

2.5.1 Definition

Definition 2.5.1.1. Let $n \in \mathbb{N}$ and $\alpha \subseteq \{0,1,\dots,n-1\}$. An **$\alpha$-contravariant functor** between two strict globular n-categories $(\mathscr{C}, \{\circ^m_p\}_{0\le p<m\le n})$ and $(\mathscr{D}, \{\hat{\circ}^m_p\}_{0\le p<m\le n})$ is a map $\phi : \mathscr{C} \to \mathscr{D}$ such that: for all $x, y \in \mathscr{C}^m$ and for some integer $0 \le m \le n$,

1. for each $q \notin \alpha$, if $x \circ^m_q y$ exists, then $\phi(x \circ^m_q y) = \phi(x) \hat{\circ}^m_q \phi(y)$,
2. for each $q \in \alpha$, if $x \circ^m_q y$ exists, then $\phi(x \circ^m_q y) = \phi(y) \hat{\circ}^m_q \phi(x)$,
3. if $e \in \mathscr{C}^q$ is a $\circ^q_{q-1}$-identity, then $\phi(e) \in \mathscr{D}^q$ is a $\hat{\circ}^q_{q-1}$-identity.

Definition 2.5.1.2. Let $m, n \in \mathbb{N}$ and $\alpha \subseteq \{0,1,\dots,n-1\}$. An **$\alpha$-involution** $*^m_\alpha$ on a strict globular n-category $(\mathscr{C}, \{\circ^m_p\}_{0\le p<m\le n})$ is an α-contravariant endofunctor such that $(x^{*^m_\alpha})^{*^m_\alpha} = x$ for each $x \in \mathscr{C}^m$. Moreover, if $\{*^m_\alpha \mid \alpha \in \Lambda \subseteq P(\{0,1,\dots,n-1\})\}$ is a family of commuting α-involutions, the strict globular n-category is said to be **Λ-involutive**.

Remark 2.5.1.3. An α-involution is an involution that is a unital homomorphism for all $\circ^m_q$-compositions with $q \notin \alpha$ and is a unital anti-homomorphism for $\circ^m_q$-compositions with $q \in \alpha$.

Remark 2.5.1.4. Whenever the family $\alpha \subseteq \{0,1,\dots,n-1\}$ is a singleton $\alpha = \{q\}$, we will simply use the notation $*^m_q := *^m_{\{q\}}$ and in this particular case we will make use of the following terminology.

Definition 2.5.1.5. A strict ω-category $(\mathscr{C}, \{\circ_p^m\}_{0\leq p<m})$ is equipped with an **involution over q-arrows**, for some $q \in \mathbb{N}_0$, if there exists a map $*_q^m : \mathscr{C}^m \to \mathscr{C}^m$ for every $m \in \mathbb{N}$ such that for all $x, y \in \mathscr{C}^m$

1. for all $p \neq q$, if $(x \circ_p^m y)^{*_q^m}$ exists, then $(x \circ_p^m y)^{*_q^m} = x^{*_q^m} \circ_p^m y^{*_q^m}$,
2. for $p = q$, if $(x \circ_p^m y)^{*_q^m}$ exists, then $(x \circ_p^m y)^{*_q^m} = y^{*_q^m} \circ_p^m x^{*_q^m}$,
3. for all p, q, if x is a $\circ_p^m$-identity, $x^{*_q^m}$ is also a $\circ_p^m$-identity.
   That is, if $x \in \mathscr{C}^p$, then $\iota^p(x)^{*_q^{p+1}} = \iota^p(x^{*_q^p})$,
4. for all $q \in \mathbb{N}_0$, $(x^{*_q^m})^{*_q^m} = x$,
5. for all $p, q \in \mathbb{N}_0$, $(x^{*_p^m})^{*_q^m} = (x^{*_q^m})^{*_p^m}$.

The involution $*_q^m$ is **Hermitian** if, for $p = q$, if x is a $\circ_p^m$-identity, then $x^{*_q^m} = x$.

Definition 2.5.1.6. A **fully involutive strict n-category** is a strict n-category equipped with a q-involution for every $q = 0, 1, \ldots, n-1$. A strict n-category is **partially involutive** if it is endowed with a proper subset of the family of involutions $*_q$, for $q = 0, 1, \ldots, n-1$.

2.5.2 Examples

Here some examples of strict involutive categories are given.

Example 2.5.2.1. One of special cases of fully involutive strict globular n-categories is a strict globular n-groupoid, where a groupoid is a category in which every morphism is an isomorphism, when inverse maps take the role of involutions.

Example 2.5.2.2. Consider a *bipartite 1-quiver*, a pair of maps $A \xleftarrow{s} R \xrightarrow{t} B$, where each element $r \in R$ is interpreted as an arrow connecting its source $s(r) \in A$ to its target $t(r) \in B$. We see that bijective bipartite 1-quivers between sets (i.e. both maps are bijective) are an example of fully involutive strict globular n-groupoids, for all $n \in \mathbb{N}$.

The following example explains one of the possibilities of constructing an involution from a unital homomorphism of unital involutive monoids.

Example 2.5.2.3. Let $(M_1, \cdot_1, \dagger_1)$ and $(M_2, \cdot_2, \dagger_2)$ be two unital involutive monoids.

Suppose that $\phi : M_1 \to M_2$ a unital homomorphism of monoids.

Define $\phi^*(x) := \phi(x^{\dagger_1})^{\dagger_2}$, for all $x \in M_1$.

We see that $\phi^* : M_1 \to M_2$ is a unital homomorphism.

This is an example of $\varnothing$-involution (directions of arrows are not reversed) which is Hermitian.

CHAPTER 3

INVOLUTIVE WEAK GLOBULAR ω-CATEGORIES

Along the Penon's definition of weak globular ω-categories discussed in 2.4 and the definition of strict involutive higher categories discussed in 2.5 together with some modifications that generalize these ideas, we will combine these two notions to weak involutive globular ω-categories via the following guidelines.

Recall that the category $\mathcal{Q}$ has objects of the form $\left(A \xrightarrow{f} B, [\cdot,\cdot]\right)$, where A is a globular ω-magma, B is a strict globular ω-category, f is a morphism of globular ω-magmas, equipped with a certain contraction $[\cdot,\cdot]$, and whose morphisms preserve everything possible. Now we are going to define a category $\mathcal{Q}^*$ with a similar role to the category $\mathcal{Q}$. Its objects are of the form $\left(A^* \xrightarrow{f^*} B^*, [\cdot,\cdot]^*\right)$ where A^* is a self-dual globular ω-magma, B^* is a strict involutive globular ω-category, f^* is a morphism of self-dual globular ω-magmas, equipped with a certain contraction $[\cdot,\cdot]^*$. In addition, its morphisms should also preserve everything possible.

In order to define an involutive weak globular ω-category as an algebra for a certain monad, we must have an adjunction between the category $\mathcal{Q}^*$ and the category **GSet** of ω-globular sets in the sense of free-forgetful functors. Indeed, the main part of this work might be proving that this forgetful functor has a left adjoint that is a free one.

Remark 3.0.2.4. In this work, reflexivity is not assumed in ω-globular sets.

3.1 Free Reflexive Self-Dual Globular ω-Magmas

The concepts of *involution* and *self-duality* are different in that involution is a self-dual map that satisfies the condition of involutivity. In this section we will prove the existence of a free self-dual globular ω-magma over an ω-globular set via establishing a free self-dual ω-globular set and a free globular ω-magma over an ω-globular set.

3.1.1 Free Reflexive Self-Dual ω-Globular Sets

We begin this subsection with the definition of self-duality.

Definition 3.1.1.1. A (reflexive) ω-globular set Q is called **self-dual** if there exists a family of maps $*^n_\alpha : Q^n \to Q^n$, for every $n \in \mathbb{N}_0$ and $\alpha \subseteq \mathbb{N}_0$, such that

- $s^n(f^{*^{n+1}_\alpha}) = t^n(f)^{*^n_\alpha}$ and $t^n(f^{*^{n+1}_\alpha}) = s^n(f)^{*^n_\alpha}$ for every $n \in \alpha$ and $f \in Q^{n+1}$,
- $s^n(f^{*^{n+1}_\alpha}) = s^n(f)^{*^n_\alpha}$ and $t^n(f^{*^{n+1}_\alpha}) = t^n(f)^{*^n_\alpha}$ for every $n \notin \alpha$ and $f \in Q^{n+1}$.

Proposition 3.1.1.2. *A free reflexive self-dual* ω*-globular set over a reflexive* ω*-globular set exists.*

Proof. Let $\left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots, (\iota^n_Q)_{n\in\mathbb{N}_0} \right)$ be a reflexive ω-globular set.

For the further usage of notations, we set, for all $x \in Q^n$,

$$x^{\alpha^n} := (x, \alpha^n), \ldots, (\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m} := ((x, \alpha^n_1), \alpha^n_2), \ldots), \alpha^n_m), \ldots.$$

For each $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, let us set

$$\hat{Q}^n := \{(\cdots((y^{\beta^n_1})^{\beta^n_2})^{\cdots})^{\beta^n_m} \mid y \in Q^n,\ \beta^n_j \subseteq \mathbb{N}_0,\ j = 1, 2, \ldots, m\}.$$

We construct a new ω-globular set as follows: for any $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $(\cdots((y^{\beta^{n+1}_1})^{\beta^{n+1}_2})^{\cdots})^{\beta^{n+1}_m} \in \hat{Q}^{n+1}$, we define $s^n_{\hat{Q}}, t^n_{\hat{Q}} : \hat{Q}^{n+1} \to \hat{Q}^n$ by

- $s^n_{\hat{Q}}\big((\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big) := \begin{cases} (\cdots((s^n_Q(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m; \\ (\cdots((t^n_Q(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m. \end{cases}$
- $t^n_{\hat{Q}}\big((\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big) := \begin{cases} (\cdots((t^n_Q(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m; \\ (\cdots((s^n_Q(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m. \end{cases}$

Now we check that these constructions give us an ω-globular set.

Assume that $n \in \mathbb{N}$ and $(\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m} \in \hat{Q}^{n+1}$.

Consider $s^{n-1}_{\hat{Q}} s^n_{\hat{Q}}\big((\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big)$

$$= \begin{cases} (\cdots((s^{n-1}_Q s^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \notin \beta^n_1 \triangle \cdots \triangle \beta^n_m; \\ (\cdots((s^{n-1}_Q t^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \notin \beta^n_1 \triangle \cdots \triangle \beta^n_m; \\ (\cdots((t^{n-1}_Q s^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \in \beta^n_1 \triangle \cdots \triangle \beta^n_m; \\ (\cdots((t^{n-1}_Q t^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \in \beta^n_1 \triangle \cdots \triangle \beta^n_m. \end{cases}$$

and $s^{n-1}_{\hat{Q}} t^n_{\hat{Q}}\big((\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big)$

$$= \begin{cases} (\cdots((s^{n-1}_Q t^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \notin \beta^n_1 \triangle \cdots \triangle \beta^n_m; \\ (\cdots((s^{n-1}_Q s^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \notin \beta^n_1 \triangle \cdots \triangle \beta^n_m; \\ (\cdots((t^{n-1}_Q t^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \in \beta^n_1 \triangle \cdots \triangle \beta^n_m; \\ (\cdots((t^{n-1}_Q s^n_Q(y))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m,\ n-1 \in \beta^n_1 \triangle \cdots \triangle \beta^n_m. \end{cases}$$

By the globularity condition of the ω-globular set Q, we get $s^{n-1}_{\hat{Q}} s^n_{\hat{Q}} = s^{n-1}_{\hat{Q}} t^n_{\hat{Q}}$.

Using a similar argument, we also have $t^{n-1}_{\hat{Q}} s^n_{\hat{Q}} = t^{n-1}_{\hat{Q}} t^n_{\hat{Q}}$.

Thus, $\hat{Q}^0 \overset{s^0_{\hat{Q}}}{\underset{t^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \overset{s^1_{\hat{Q}}}{\underset{t^1_{\hat{Q}}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\hat{Q}}}{\underset{t^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \overset{s^n_{\hat{Q}}}{\underset{t^n_{\hat{Q}}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Moreover, define $\iota^n_{\hat{Q}} : \hat{Q}^n \to \hat{Q}^{n+1}$ by $(\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \mapsto (\cdots((\iota^n(y))^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}$.

In addition, for each $(\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \in \hat{Q}^n$, we have

$$\begin{aligned} s^n_{\hat{Q}} \circ \iota^n_{\hat{Q}}\big((\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m}\big) &= s^n_{\hat{Q}}\big((\cdots((\iota^n(y))^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big) \\ &= (\cdots((s^n \circ \iota^n(y))^{\beta^n_1})^{\cdots})^{\beta^n_m} \\ &= (\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \end{aligned}$$

and also

$$\begin{aligned} t^n_{\hat{Q}} \circ \iota^n_{\hat{Q}}\big((\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m}\big) &= t^n_{\hat{Q}}\big((\cdots((\iota^n(y))^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big) \\ &= (\cdots((t^n \circ \iota^n(y))^{\beta^n_1})^{\cdots})^{\beta^n_m} \\ &= (\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m}. \end{aligned}$$

This means that $s^n_{\hat{Q}} \circ \iota^n_{\hat{Q}} = \mathrm{Id}_{\hat{Q}^n} = t^n_{\hat{Q}} \circ \iota^n_{\hat{Q}}$.

Hence, $\left(\hat{Q}^0 \underset{t^0_{\hat{Q}}}{\overset{s^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \underset{t^1_{\hat{Q}}}{\overset{s^1_{\hat{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{Q}}}{\overset{s^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \underset{t^n_{\hat{Q}}}{\overset{s^n_{\hat{Q}}}{\leftleftarrows}} \cdots, (\iota^n_{\hat{Q}})_{n\in\mathbb{N}_0}\right)$ is a reflexive ω-globular set.

For each $n \in \mathbb{N}_0$ and $\alpha^n \subseteq \mathbb{N}_0$, we define $*^n_\alpha : \hat{Q}^n \to \hat{Q}^n$ by

$$(\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \mapsto ((\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m})^{\alpha^n}.$$

We see that $\left(\hat{Q}^0 \underset{t^0_{\hat{Q}}}{\overset{s^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \underset{t^1_{\hat{Q}}}{\overset{s^1_{\hat{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{Q}}}{\overset{s^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \underset{t^n_{\hat{Q}}}{\overset{s^n_{\hat{Q}}}{\leftleftarrows}} \cdots, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}, (\iota^n_{\hat{Q}})_{n\in\mathbb{N}_0}\right)$ is a reflexive self-dual ω-globular set.

Now consider a family of maps $i : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to$ $\left(\hat{Q}^0 \underset{t^0_{\hat{Q}}}{\overset{s^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \underset{t^1_{\hat{Q}}}{\overset{s^1_{\hat{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{Q}}}{\overset{s^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \underset{t^n_{\hat{Q}}}{\overset{s^n_{\hat{Q}}}{\leftleftarrows}} \cdots\right)$ defined by $x \mapsto x^{\varnothing^n}$ for every $x \in Q^n$ and $n \in \mathbb{N}_0$.

Assume that there exists a morphism $f : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to$ $\left(\hat{R}^0 \underset{t^0_{\hat{R}}}{\overset{s^0_{\hat{R}}}{\leftleftarrows}} \hat{R}^1 \underset{t^1_{\hat{R}}}{\overset{s^1_{\hat{R}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{R}}}{\overset{s^{n-1}_{\hat{R}}}{\leftleftarrows}} \hat{R}^n \underset{t^n_{\hat{R}}}{\overset{s^n_{\hat{R}}}{\leftleftarrows}} \cdots, (\hat{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}, (\iota^n_{\hat{R}})_{n\in\mathbb{N}_0}\right)$ into another reflexive self-dual ω-globular set.

The only choice of morphism of reflexive self-dual ω-globular sets satisfying the universal factorization property is given by the following.

Define $\phi : \left(\hat{Q}^0 \underset{t^0_{\hat{Q}}}{\overset{s^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \underset{t^1_{\hat{Q}}}{\overset{s^1_{\hat{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{Q}}}{\overset{s^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \underset{t^n_{\hat{Q}}}{\overset{s^n_{\hat{Q}}}{\leftleftarrows}} \cdots\right) \to \left(\hat{R}^0 \underset{t^0_{\hat{R}}}{\overset{s^0_{\hat{R}}}{\leftleftarrows}} \hat{R}^1 \underset{t^1_{\hat{R}}}{\overset{s^1_{\hat{R}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{R}}}{\overset{s^{n-1}_{\hat{R}}}{\leftleftarrows}} \hat{R}^n \underset{t^n_{\hat{R}}}{\overset{s^n_{\hat{R}}}{\leftleftarrows}} \cdots\right)$ by $(\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \to (\cdots((f(y))^{\hat{*}_{\beta_1}})^{\cdots})^{\hat{*}_{\beta_m}}$ for each $(\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \in \hat{Q}^n$.

For every $(\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m} \in \hat{Q}^n$, we have

$$\begin{aligned} \phi\big((\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m})^{\alpha^n}\big) &= ((\cdots(f(y)^{\hat{*}^n_{\beta_1}})^{\cdots})^{\hat{*}^n_{\beta_m}})^{\hat{*}^n_\alpha} \\ &= \big(\phi((\cdots(y^{\beta^n_1})^{\cdots})^{\beta^n_m}\big)^{\hat{*}^n_\alpha}. \end{aligned}$$

Hence, ϕ is a unique morphism of reflexive self-dual ω-globular sets satisfying $f = \phi \circ i$.

Therefore, $\left(\left(\hat{Q}^0 \underset{t^0_{\hat{Q}}}{\overset{s^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \underset{t^1_{\hat{Q}}}{\overset{s^1_{\hat{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{Q}}}{\overset{s^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \underset{t^n_{\hat{Q}}}{\overset{s^n_{\hat{Q}}}{\leftleftarrows}} \cdots, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}, (\iota^n_{\hat{Q}})_{n\in\mathbb{N}_0}\right), i\right)$ is a free reflexive self-dual ω-globular set over a reflexive ω-globular set. □

Similarly, we can construct a reflexive self-dual ω-globular set from a non-reflexive ω-globular set as described in the following proposition.

Proposition 3.1.1.3. *A free reflexive self-dual* ω*-globular set over an* ω*-globular set exists.*

Proof. Let $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$ be an ω-globular set.

For each $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we let

$$\hat{Q}^n := \{(\cdots((y^{\beta_1^n})^{\beta_2^n})^{\cdots})^{\beta_m^n} \mid y \in Q^n,\ \beta_j^n \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\}.$$

For all $m,n \in \mathbb{N}$ and $\alpha_1^n, \alpha_2^n, \ldots, \alpha_m^n \subseteq \mathbb{N}_0$, we construct the recursive families:

$$\begin{array}{rcl} \bar{\hat{Q}}^0 & := & \hat{Q}^0, \\ (\bar{\hat{Q}}^0)^1 & := & \{(\cdots((x,0)^{\alpha_1^1})^{\alpha_2^1})^{\cdots})^{\alpha_m^1} \mid x \in \bar{\hat{Q}}^0\}, \\ \bar{\hat{Q}}^1 & := & \hat{Q}^1 \cup\!\!\!\cdot\; (\bar{\hat{Q}}^0)^1, \\ (\bar{\hat{Q}}^1)^2 & := & \{(\cdots((y,1)^{\alpha_1^2})^{\alpha_2^2})^{\cdots})^{\alpha_m^2} \mid y \in \bar{\hat{Q}}^1\}, \\ \bar{\hat{Q}}^2 & := & \hat{Q}^2 \cup\!\!\!\cdot\; (\bar{\hat{Q}}^1)^2, \\ \vdots & & \vdots \\ (\bar{\hat{Q}}^n)^{n+1} & := & \{(\cdots((z,n)^{\alpha_1^{n+1}})^{\alpha_2^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \mid z \in \bar{\hat{Q}}^n\}, \\ \bar{\hat{Q}}^{n+1} & := & \hat{Q}^{n+1} \cup\!\!\!\cdot\; (\bar{\hat{Q}}^n)^{n+1}, \\ \vdots & & \vdots \end{array}$$

Next, we establish the new sources and targets in the new quiver as follows.

For every $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $(\cdots((y^{\alpha_1^{n+1}})^{\alpha_2^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \in \bar{\hat{Q}}^{n+1}$, we define $s^n_{\bar{\hat{Q}}}, t^n_{\bar{\hat{Q}}} : \bar{\hat{Q}}^{n+1} \to \bar{\hat{Q}}^n$ by

$$s^n_{\bar{\hat{Q}}}((\cdots((y^{\alpha_1^{n+1}})^{\alpha_2^{n+1}})^{\cdots})^{\alpha_m^{n+1}})$$

$$:= \begin{cases} (\cdots((s^n_Q(y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & y \in Q^{n+1},\ n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n_Q(y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & y \in Q^{n+1},\ n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((x^{\alpha_1^n})^{\alpha_2^n})^{\cdots})^{\alpha_m^n}, & y = (x,n), x \in \bar{\hat{Q}}^n. \end{cases}$$

$$t^n_{\bar{\tilde{Q}}}\big((\cdots((y^{\alpha_1^{n+1}})^{\alpha_2^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$$
$$:= \begin{cases} (\cdots((t^n_Q(y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & y \in Q^{n+1},\ n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((s^n_Q(y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & y \in Q^{n+1},\ n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((x^{\alpha_1^n})^{\alpha_2^n})^{\cdots})^{\alpha_m^n}, & y = (x,n), x \in \bar{\tilde{Q}}^n. \end{cases}$$

We now verify that these provide us an ω-globular set.

Assume that $n \in \mathbb{N}$ and $(\cdots((y^{\alpha_1^{n+1}})^{\alpha_2^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \in \bar{\tilde{Q}}^{n+1}$.

Consider $s^{n-1}_{\bar{\tilde{Q}}} s^n_{\bar{\tilde{Q}}}\big((\cdots((y^{\alpha_1^{n+1}})^{\alpha_2^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$

$$= \begin{cases} (\cdots(((s^{n-1}_Q s^n_Q(y))^{\alpha_1^{n-1}})^{\alpha_2^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y \in Q^{n+1}, n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & n-1 \notin \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots(((s^{n-1}_Q t^n_Q(y))^{\alpha_1^{n-1}})^{\alpha_2^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y \in Q^{n+1}, n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & n-1 \notin \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots(((t^{n-1}_Q s^n_Q(y))^{\alpha_1^{n-1}})^{\alpha_2^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y \in Q^{n+1}, n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & n-1 \in \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots(((t^{n-1}_Q t^n_Q(y))^{\alpha_1^{n-1}})^{\alpha_2^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y \in Q^{n+1}, n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & n-1 \in \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots(((\cdots((s^{n-1}_Q(z))^{\beta_1^{n-1}})^{\cdots})^{\beta_k^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y = ((\cdots(z^{\beta_1^n})^{\cdots})^{\beta_k^n}), n),\ z \in Q^n, \\ & n-1 \notin \beta_1^n \triangle \cdots \triangle \beta_k^n \triangle \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots(((\cdots((t^{n-1}_Q(z))^{\beta_1^{n-1}})^{\cdots})^{\beta_k^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y = ((\cdots(z^{\beta_1^n})^{\cdots})^{\beta_k^n}), n),\ z \in Q^n, \\ & n-1 \in \beta_1^n \triangle \cdots \triangle \beta_k^n \triangle \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots(((\cdots(w^{\beta_1^{n-1}})^{\cdots})^{\beta_k^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & y = \big((\cdots((w,n-1)^{\beta_1^n})^{\cdots})^{\beta_k^n}, n\big), w \in \bar{\tilde{Q}}^{n-1}. \end{cases}$$

and $s^{n-1}_{\bar{\hat{Q}}} t^{n}_{\bar{\hat{Q}}}\big((\cdots((y^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m}\big)$

$$= \begin{cases} (\cdots(((s^{n-1}_Q t^n_Q(y))^{\alpha^{n-1}_1})^{\alpha^{n-1}_2})^{\cdots})^{\alpha^{n-1}_m}, & y \in Q^{n+1}, n \notin \alpha^{n+1}_1 \triangle \cdots \triangle \alpha^{n+1}_m, \\ & n-1 \notin \alpha^n_1 \triangle \cdots \triangle \alpha^n_m; \\ (\cdots(((s^{n-1}_Q s^n_Q(y))^{\alpha^{n-1}_1})^{\alpha^{n-1}_2})^{\cdots})^{\alpha^{n-1}_m}, & y \in Q^{n+1}, n \in \alpha^{n+1}_1 \triangle \cdots \triangle \alpha^{n+1}_m, \\ & n-1 \notin \alpha^n_1 \triangle \cdots \triangle \alpha^n_m; \\ (\cdots(((t^{n-1}_Q t^n_Q(y))^{\alpha^{n-1}_1})^{\alpha^{n-1}_2})^{\cdots})^{\alpha^{n-1}_m}, & y \in Q^{n+1}, n \notin \alpha^{n+1}_1 \triangle \cdots \triangle \alpha^{n+1}_m, \\ & n-1 \in \alpha^n_1 \triangle \cdots \triangle \alpha^n_m; \\ (\cdots(((t^{n-1}_Q s^n_Q(y))^{\alpha^{n-1}_1})^{\alpha^{n-1}_2})^{\cdots})^{\alpha^{n-1}_m}, & y \in Q^{n+1}, n \in \alpha^{n+1}_1 \triangle \cdots \triangle \alpha^{n+1}_m, \\ & n-1 \in \alpha^n_1 \triangle \cdots \triangle \alpha^n_m; \\ (\cdots(((\cdots((s^{n-1}_Q(z))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_k})^{\alpha^{n-1}_1})^{\cdots})^{\alpha^{n-1}_m}, & y = ((\cdots(z^{\beta^n_1})^{\cdots})^{\beta^n_k}), n),\ z \in Q^n, \\ & n-1 \notin \beta^n_1 \triangle \cdots \triangle \beta^n_k \triangle \alpha^n_1 \triangle \cdots \triangle \alpha^n_m; \\ (\cdots(((\cdots((t^{n-1}_Q(z))^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_k})^{\alpha^{n-1}_1})^{\cdots})^{\alpha^{n-1}_m}, & y = ((\cdots(z^{\beta^n_1})^{\cdots})^{\beta^n_k}), n),\ z \in Q^n, \\ & n-1 \in \beta^n_1 \triangle \cdots \triangle \beta^n_k \triangle \alpha^n_1 \triangle \cdots \triangle \alpha^n_m; \\ (\cdots(((\cdots(w^{\beta^{n-1}_1})^{\cdots})^{\beta^{n-1}_k})^{\alpha^{n-1}_1})^{\cdots})^{\alpha^{n-1}_m}, & y = \big((\cdots((w,n-1)^{\beta^n_1})^{\cdots})^{\beta^n_k}, n\big), w \in \bar{\hat{Q}}^{n-1}. \end{cases}$$

By the globularity condition of Q, we get $s^{n-1}_{\bar{\hat{Q}}} s^n_{\bar{\hat{Q}}} = s^{n-1}_{\bar{\hat{Q}}} t^n_{\bar{\hat{Q}}}$.

Applying a similar method, we obtain $t^{n-1}_{\bar{\hat{Q}}} s^n_{\bar{\hat{Q}}} = t^{n-1}_{\bar{\hat{Q}}} t^n_{\bar{\hat{Q}}}$.

Thus, $\bar{\hat{Q}}^0 \underset{t^0_{\bar{\hat{Q}}}}{\overset{s^0_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^1 \underset{t^1_{\bar{\hat{Q}}}}{\overset{s^1_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{\hat{Q}}}}{\overset{s^{n-1}_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^n \underset{t^n_{\bar{\hat{Q}}}}{\overset{s^n_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Now define $\iota^n_{\bar{\hat{Q}}} : \bar{\hat{Q}}^n \to \bar{\hat{Q}}^{n+1}$ by $(\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m} \mapsto (\cdots(((x,n)^{\alpha^{n+1}_1})^{\alpha^{n+1}_2})^{\cdots})^{\alpha^{n+1}_m}$.

It is easy to see that $s^n_{\bar{\hat{Q}}} \circ \iota^n_{\bar{\hat{Q}}} = \mathrm{Id}_{\bar{\hat{Q}}^n} = t^n_{\bar{\hat{Q}}} \circ \iota^n_{\bar{\hat{Q}}}$ for every $n \in \mathbb{N}_0$.

For each $\alpha^n \subseteq \mathbb{N}_0$, define $\overline{*}^n_\alpha : \bar{\hat{Q}}^n \to \bar{\hat{Q}}^n$ by

$$(\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m} \mapsto ((\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m})^{\alpha^n}.$$

We see that $\left(\bar{\hat{Q}}^0 \underset{t^0_{\bar{\hat{Q}}}}{\overset{s^0_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^1 \underset{t^1_{\bar{\hat{Q}}}}{\overset{s^1_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{\hat{Q}}}}{\overset{s^{n-1}_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^n \underset{t^n_{\bar{\hat{Q}}}}{\overset{s^n_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots, (\overline{*}^n_\alpha)_{\alpha \subseteq \mathbb{N}_0, n \in \mathbb{N}_0}, (\iota^n_{\bar{\hat{Q}}})_{n \in \mathbb{N}_0}\right)$ is a reflexive self-dual ω-globular set.

Define $i : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to \left(\bar{\hat{Q}}^0 \underset{t^0_{\bar{\hat{Q}}}}{\overset{s^0_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^1 \underset{t^1_{\bar{\hat{Q}}}}{\overset{s^1_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{\hat{Q}}}}{\overset{s^{n-1}_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^n \underset{t^n_{\bar{\hat{Q}}}}{\overset{s^n_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots\right)$

by $x \mapsto x^{\varnothing^n}$ for every $x \in Q^n$ and $n \in \mathbb{N}_0$.

Assume that there exists a morphism $f : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots \right) \to$ $\left(\bar{R}^0 \underset{t^0_{\bar{R}}}{\overset{s^0_{\bar{R}}}{\leftleftarrows}} \bar{R}^1 \underset{t^1_{\bar{R}}}{\overset{s^1_{\bar{R}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{R}}}{\overset{s^{n-1}_{\bar{R}}}{\leftleftarrows}} \bar{R}^n \underset{t^n_{\bar{R}}}{\overset{s^n_{\bar{R}}}{\leftleftarrows}} \cdots, (*^n_\alpha)_{\alpha \subseteq \mathbb{N}_0, n \in \mathbb{N}_0}, (\iota^n_{\bar{R}})_{n \in \mathbb{N}_0} \right)$ into another reflexive self-dual ω-globular set.

The only choice of morphism of reflexive self-dual ω-globular sets satisfying the universal factorization property is given by the following.

Define $\phi : \left(\bar{Q}^0 \underset{t^0_{\bar{Q}}}{\overset{s^0_{\bar{Q}}}{\leftleftarrows}} \bar{Q}^1 \underset{t^1_{\bar{Q}}}{\overset{s^1_{\bar{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{Q}}}{\overset{s^{n-1}_{\bar{Q}}}{\leftleftarrows}} \bar{Q}^n \underset{t^n_{\bar{Q}}}{\overset{s^n_{\bar{Q}}}{\leftleftarrows}} \cdots \right) \to \left(\bar{R}^0 \underset{t^0_{\bar{R}}}{\overset{s^0_{\bar{R}}}{\leftleftarrows}} \bar{R}^1 \underset{t^1_{\bar{R}}}{\overset{s^1_{\bar{R}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{R}}}{\overset{s^{n-1}_{\bar{R}}}{\leftleftarrows}} \bar{R}^n \underset{t^n_{\bar{R}}}{\overset{s^n_{\bar{R}}}{\leftleftarrows}} \cdots \right)$ recursively by, for all $x \in Q^0$, $y \in Q^1$,

- $(\cdots((x^{\alpha^0_1})^{\alpha^0_2})^{\cdots})^{\alpha^0_m} \mapsto (\cdots(((f(x))^{*^0_{\alpha_1}})^{*^0_{\alpha_2}})^{\cdots})^{*^0_{\alpha_m}}$, and in particular, $x^{\varnothing^0} \mapsto f(x)$,
- $(\cdots(((\cdots(x^{\beta^0_1})^{\cdots})^{\beta^0_k}, 0)^{\alpha^1_1})^{\cdots})^{\alpha^1_m} \mapsto (\cdots((\iota^0_{\bar{R}}((\cdots((f(x))^{*^0_{\beta_1}})^{\cdots})^{*^0_{\beta_k}})^{*^1_{\alpha_1}})^{\cdots})^{*^1_{\alpha_m}}$,
- $(\cdots((y^{\alpha^1_1})^{\alpha^1_2})^{\cdots})^{\alpha^1_m} \mapsto (\cdots(((f(y))^{*^1_{\alpha_1}})^{*^1_{\alpha_2}})^{\cdots})^{*^1_{\alpha_m}}$, and in particular, $y^{\varnothing^1} \mapsto f(y)$,
- $(\cdots(((\cdots(y^{\beta^1_1})^{\cdots})^{\beta^1_k}, 1)^{\alpha^2_1})^{\cdots})^{\alpha^2_m} \mapsto (\cdots((\iota^1_{\bar{R}}((\cdots((f(y))^{*^1_{\beta_1}})^{\cdots})^{*^1_{\beta_k}}))^{*^2_{\alpha_1}})^{\cdots})^{*^2_{\alpha_m}}$,
- $(\cdots(((\cdots(((\cdots(x^{\gamma^0_1})^{\cdots})^{\gamma^0_t}, 0)^{\beta^1_1})^{\cdots})^{\beta^1_k}, 1)^{\alpha^2_1})^{\cdots})^{\alpha^2_m} \mapsto$ $(\cdots((\iota^1_{\bar{R}}((\cdots((\iota^0_{\bar{R}}((\cdots((f(x))^{*^0_{\gamma_1}})^{\cdots})^{*^0_{\gamma_t}}))^{*^1_{\beta_1}})^{\cdots})^{*^1_{\beta_k}}))^{*^2_{\alpha_1}})^{\cdots})^{*^2_{\alpha_m}}$,

$$\vdots$$

We will show that ϕ is a morphism of reflexive self-dual ω-globular set.

First, it is easy to see that, for any $x \in Q^n$,

$$\phi\big((\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m}\big) = (\cdots(((f(x))^{*^n_{\alpha_1}})^{*^n_{\alpha_2}})^{\cdots})^{*^n_{\alpha_m}} = (\cdots(((\phi(x))^{*^n_{\alpha_1}})^{*^n_{\alpha_2}})^{\cdots})^{*^n_{\alpha_m}}.$$

Then, for each $(\cdots(((\cdots(x^{\beta^0_1})^{\cdots})^{\beta^0_k}, 0)^{\alpha^1_1})^{\cdots})^{\alpha^1_m} \in \bar{Q}^1$, we have

$$\begin{aligned} (\cdots(((\cdots(x^{\beta^0_1})^{\cdots})^{\beta^0_k}, 0)^{\alpha^1_1})^{\cdots})^{\alpha^1_m} &= (\cdots((\iota^0_{\bar{R}}((\cdots((f(x))^{*^0_{\beta_1}})^{\cdots})^{*^0_{\beta_k}})^{*^1_{\alpha_1}})^{\cdots})^{*^1_{\alpha_m}} \\ &= (\cdots((\iota^0_{\bar{R}}(\phi((\cdots(x^{\beta^0_1})^{\cdots})^{\beta^0_k}))^{*^1_{\alpha_1}})^{\cdots})^{*^1_{\alpha_m}}. \end{aligned}$$

Suppose that $\phi\big((\cdots((z,h)^{\alpha_1^{h+1}})^{\cdots})^{\alpha_m^{h+1}})\big) = (\cdots((\iota^h_{\bar{\hat{R}}}(\phi(z)))^{*_{\alpha_1}^{h+1}})^{\cdots})^{*_{\alpha_m}^{h+1}}$ for all $h = 0,1,\ldots,n-1$ and $z \in \bar{\hat{Q}}^h$.

For every $z \in \bar{\hat{Q}}^n$, by the hypothesis, we obtain

$$\begin{aligned}
&\phi\big((\cdots(((\cdots(z^{\beta_1^n})^{\cdots})^{\beta_k^n},n)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) \\
= &\begin{cases} (\cdots((\iota^n_{\bar{\hat{R}}}((\cdots((f(z))^{*_{\beta_1}^n})^{\cdots})^{*_{\beta_k}^n}))^{*_{\alpha_1}^{n+1}})^{\cdots})^{*_{\alpha_1}^{n+1}}), & z \in Q^n; \\ (\cdots((\iota^n_{\bar{\hat{R}}}((\cdots((\phi(z))^{*_{\beta_1}^n})^{\cdots})^{*_{\beta_k}^n}))^{*_{\alpha_1}^{n+1}})^{\cdots})^{*_{\alpha_1}^{n+1}}), & z \in (\bar{\hat{Q}}^{n-1})^n. \end{cases} \\
= &(\cdots((\iota^n_{\bar{\hat{R}}}((\cdots((\phi(z))^{*_{\beta_1}^n})^{\cdots})^{*_{\beta_k}^n}))^{*_{\alpha_1}^{n+1}})^{\cdots})^{*_{\alpha_1}^{n+1}}).
\end{aligned}$$

Hence, ϕ is a unique morphism of reflexive self-dual ω-globular sets such that $f = \phi \circ i$.

Therefore, $\left(\left(\bar{\hat{Q}}^0 \underset{t^0_{\bar{\hat{Q}}}}{\overset{s^0_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^1 \underset{t^1_{\bar{\hat{Q}}}}{\overset{s^1_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{\hat{Q}}}}{\overset{s^{n-1}_{\bar{\hat{Q}}}}{\leftleftarrows}} \bar{\hat{Q}}^n \underset{t^n_{\bar{\hat{Q}}}}{\overset{s^n_{\bar{\hat{Q}}}}{\leftleftarrows}} \cdots, (\bar{*}^n_\alpha)_{\alpha \subseteq \mathbb{N}_0, n \in \mathbb{N}_0}, (\iota^n_{\bar{\hat{Q}}})_{n \in \mathbb{N}_0}\right), i\right)$ is a free reflexive self-dual ω-globular set over an ω-globular set. □

3.1.2 Free Reflexive Globular ω-Magmas

We give brief constructions of a free reflexive globular ω-magma over a reflexive ω-globular set first and then over an ω-globular set.

Proposition 3.1.2.1. *A free reflexive globular ω-magma over a reflexive ω-globular set exists.*

Proof. Let $\left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots, (\iota^n)_{n \in \mathbb{N}_0}\right)$ be a reflexive ω-globular set.

First of all, we construct a new ω-globular set that suits our situation.

Setting $k,n \in \mathbb{N}$ and $\langle Q^0 \rangle := Q^0$, we construct the following recursive family.

Consider $\langle Q^1[1] \rangle := Q^1$, $s^0[1] := s^0$, and $t^0[1] := t^0$.

Now let $\langle Q^1[2] \rangle := \{(x,0,y) \mid x,y \in \langle Q^1[1] \rangle,\ s^0[1](x) = t^0[1](y)\}$.

Define $s^0[2] : \langle Q^1[2] \rangle \to \langle Q^0 \rangle$ by $s^0[2]\big((x,0,y)\big) := s^0[1](y)$

and $t^0[2] : \langle Q^1[2] \rangle \to \langle Q^0 \rangle$ by $t^0[2]\big((x,0,y)\big) := t^0[1](x)$.

Suppose that we have $\langle Q^1[l]\rangle$, $s^0[l]$, and $t^0[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle Q^1[k]\rangle := \{(x,0,y) \mid x \in \langle Q^1[i]\rangle, y \in \langle Q^1[j]\rangle, i+j = k, s^0[i](x) = t^0[j](y)\}$.

If $(x,0,y) \in \langle Q^1[k]\rangle$, we define $s^0[k], t^0[k] : \langle Q^1[k]\rangle \to \langle Q^0\rangle$ by $s^0[k]\big((x,0,y)\big) := s^0[j](y)$ and $t^0[k]\big((x,0,y)\big) := t^0[i](x)$.

Set $\langle Q^1\rangle := \dot{\bigcup\limits_{k=1}^{\infty}} \langle Q^1[k]\rangle$, $s^0_{\langle Q\rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} s^0[k]$, and $t^0_{\langle Q\rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} t^0[k]$.

Assume that we have $\langle Q^m\rangle$, $s^{m-1}_{\langle Q\rangle}$, and $t^{m-1}_{\langle Q\rangle}$ for every $m = 1,2,\ldots,n-1$.

Consider $\langle Q^n[1]\rangle := Q^n$, $s^{n-1}[1] := s^{n-1}$, and $t^{n-1}[1] := t^{n-1}$.

Let $\langle Q^n[2]\rangle := \dot{\bigcup\limits_{p=0}^{n-1}} \{(x,p,y) \mid x,y \in \langle Q^n[1]\rangle, s^p \cdots s^{n-1}[1](x) = t^p \cdots t^{n-1}[1](y)\}$.

If $(x,p,y) \in \langle Q^n[2]\rangle$, then we define $s^{n-1}[2], t^{n-1}[2] : \langle Q^n[2]\rangle \to \langle Q^{n-1}\rangle$ by

- $s^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[1](x), p, s^{n-1}[1](y)), & n-1 > p; \\ s^{n-1}[1](y), & n-1 = p. \end{cases}$

- $t^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[1](x), p, t^{n-1}[1](y)), & n-1 > p; \\ t^{n-1}[1](x), & n-1 = p. \end{cases}$

Suppose that we have $\langle Q^n[l]\rangle$, $s^{n-1}[l]$, and $t^{n-1}[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle Q^n[k]\rangle := \dot{\bigcup\limits_{p=0}^{n-1}} \{(x,p,y) \mid x \in \langle Q^n[i]\rangle,\ y \in \langle Q^n[j]\rangle,\ i+j = k,$

$s^p s^{p+1} \cdots s^{n-1}[i](x) = t^p t^{p+1} \cdots t^{n-1}[j](y)\}$.

If $(x,p,y) \in \langle Q^n[k]\rangle$, then we define $s^{n-1}[k], t^{n-1}[k] : \langle Q^n[k]\rangle \to \langle Q^{n-1}\rangle$ by

- $s^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[i](x), p, s^{n-1}[j](y)), & n-1 > p; \\ s^{n-1}[j](y), & n-1 = p. \end{cases}$

- $t^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[i](x), p, t^{n-1}[j](y)), & n-1 > p; \\ t^{n-1}[i](x), & n-1 = p. \end{cases}$

Set $\langle Q^n\rangle := \dot{\bigcup\limits_{k=1}^{\infty}} \langle Q^n[k]\rangle$, $s^{n-1}_{\langle Q\rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} s^{n-1}[k]$, and $t^{n-1}_{\langle Q\rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} t^{n-1}[k]$.

We now check that these definitions give us an ω-globular set.

Let $n \in \mathbb{N}$ and $(x,p,y) \in \langle Q^{n+1}\rangle$.

By the globularity condition of the ω-globular set Q, we have

$$\begin{aligned} s^{n-1}_{\langle Q\rangle} s^n[2]\big((x,p,y)\big) &= \begin{cases} \big(s^{n-1}s^n(x), p, s^{n-1}s^n(y)\big), & n-1>p; \\ s^{n-1}s^n(y), & n-1=p. \end{cases} \\ &= \begin{cases} \big(s^{n-1}t^n(x), p, s^{n-1}t^n(y)\big), & n-1>p; \\ s^{n-1}t^n(y), & n-1=p. \end{cases} \\ &= s^{n-1}_{\langle Q\rangle} t^n[2]\big((x,p,y)\big). \end{aligned}$$

Suppose that $s^{n-1}_{\langle Q\rangle} s^n[m] = s^{n-1}_{\langle Q\rangle} t^n[m]$ for all $m = 1,2,\ldots,k-1$.

This implies that if $(x,p,y) \in \big\langle Q^{n+1}[k]\big\rangle$, then

$$s^{n-1}[i]s^n[i](x) = s^{n-1}[i]t^n[i](x)$$

and

$$s^{n-1}[j]s^n[j](y) = s^{n-1}[j]t^n[j](y).$$

Thus,

$$\begin{aligned} s^{n-1}_{\langle Q\rangle} s^n[k]\big((x,p,y)\big) &= \begin{cases} (s^{n-1}[i]s^n[i](x), p, s^{n-1}[j]s^n[j](y)), & n-1>p; \\ s^{n-1}[j]s^n[j](y), & n-1=p. \end{cases} \\ &= \begin{cases} (s^{n-1}[i]t^n[i](x), p, s^{n-1}[j]t^n[j](y)), & n-1>p; \\ s^{n-1}[j]t^n[j](y), & n-1=p. \end{cases} \\ &= s^{n-1}_{\langle Q\rangle} t^n[k]\big((x,p,y)\big). \end{aligned}$$

This means that $s^{n-1}_{\langle Q\rangle} s^n_{\langle Q\rangle} = s^{n-1}_{\langle Q\rangle} t^n_{\langle Q\rangle}$.

Similarly, we get $t^{n-1}_{\langle Q\rangle} s^n_{\langle Q\rangle} = t^{n-1}_{\langle Q\rangle} t^n_{\langle Q\rangle}$.

Additionally, we have to establish identity maps as follows.

First, we set $\iota^0_{\langle Q\rangle} := \iota^0$ and $\iota^n[1] := \iota^n$ for every $n \in \mathbb{N}_0$.

Next, define $\iota^1[2] : \big\langle Q^1[2]\big\rangle \to \big\langle Q^2[2]\big\rangle$ by $(x,0,y) \mapsto (\iota^1[1](x), 0, \iota^1[1](y))$.

Assume that we have $\iota^1[h]$ for all $h = 1,2,\ldots,k-1$.

If $(x,0,y) \in \big\langle Q^1[k]\big\rangle$, where $x \in \big\langle Q^1[i]\big\rangle$ and $y \in \big\langle Q^1[j]\big\rangle$, we define $\iota^1[k] : \big\langle Q^1[k]\big\rangle \to \big\langle Q^2[k]\big\rangle$ by $(x,0,y) \mapsto (\iota^1[i](x), 0, \iota^1[j](y))$.

Then we let $\iota^1_{\langle Q\rangle} := \dot{\bigcup_{k=1}^{\infty}} \iota^1[k]$.

Suppose that we have $\iota^l_{\langle Q\rangle}$ for all $l = 0,1,\ldots,n-1$.

Next, define $\iota^n[2] : \langle Q^n[2]\rangle \to \left\langle Q^{n+1}[2]\right\rangle$ by $(x,p,y) \mapsto (\iota^n[1](x), p, \iota^n[1](y))$.

Assume that we have $\iota^n[h]$ for all $h = 1,2,\ldots,k-1$.

If $(x,p,y) \in \langle Q^n[k]\rangle$, where $x \in \langle Q^n[i]\rangle$ and $y \in \langle Q^n[j]\rangle$, we define $\iota^n[k] : \langle Q^n[k]\rangle \to \left\langle Q^{n+1}[k]\right\rangle$ by $(x,p,y) \mapsto (\iota^n[i](x), p, \iota^n[j](y))$.

Then we let $\iota^n_{\langle Q\rangle} := \dot{\bigcup_{k=1}^{\infty}} \iota^n[k]$.

Consider, for each $(x,p,y) \in \langle Q^n[2]\rangle$,

$$s^n[2]\circ\iota^n[2]\big((x,p,y)\big) = s^n[2]\big((\iota^n(x),p,\iota^n(y))\big) = \big(s^n\circ\iota^n(x),p,s^n\circ\iota^n(y)\big) = (x,p,y).$$

Thus, $s^n[2]\circ\iota^n[2] = \mathrm{Id}_{\langle Q^n[2]\rangle}$.

Now assume that $s^n[m]\circ\iota^n[m] = \mathrm{Id}_{\langle Q^n[m]\rangle}$ for every $m = 1,2,\ldots,k-1$.

For any $(x,p,y) \in \langle Q^n[k]\rangle$, where $x \in \langle Q^n[i]\rangle$ and $y \in \langle Q^n[j]\rangle$,

$$\begin{aligned} s^n[k]\circ\iota^n[k]\big((x,p,y)\big) &= s^n[k]\big((\iota^n[i](x),p,\iota^n[j](y))\big) \\ &= \big(s^n[i]\circ\iota^n[i](x),p,s^n[j]\circ\iota^n[j](y)\big) \\ &= (x,p,y). \end{aligned}$$

It follows that $s^n_{\langle Q\rangle}\circ\iota^n_{\langle Q\rangle} = \mathrm{Id}_{\langle Q^n\rangle}$.

Similarly, we have $t^n_{\langle Q\rangle}\circ\iota^n_{\langle Q\rangle} = \mathrm{Id}_{\langle Q^n\rangle}$.

Hence, $\left(\left\langle Q^0\right\rangle \underset{t^0_{\langle Q\rangle}}{\overset{s^0_{\langle Q\rangle}}{\leftleftarrows}} \left\langle Q^1\right\rangle \underset{t^1_{\langle Q\rangle}}{\overset{s^1_{\langle Q\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle Q\rangle}}{\overset{s^{n-1}_{\langle Q\rangle}}{\leftleftarrows}} \langle Q^n\rangle \underset{t^n_{\langle Q\rangle}}{\overset{s^n_{\langle Q\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle Q\rangle})_{n\in\mathbb{N}_0}\right)$ is a reflexive ω-globular set.

For all $p,n \in \mathbb{N}_0$ such that $p < n$, we set

$$\langle Q^n\rangle \times_p \langle Q^n\rangle := \{(x,y) \in \langle Q^n\rangle \times \langle Q^n\rangle \mid s^p_{\langle Q\rangle}s^{p+1}_{\langle Q\rangle}\cdots s^{n-1}_{\langle Q\rangle}(x) = t^p_{\langle Q\rangle}t^{p+1}_{\langle Q\rangle}\cdots t^{n-1}_{\langle Q\rangle}(y)\}.$$

Then we define $\circ^n_p : \langle Q^n\rangle \times_p \langle Q^n\rangle \to \langle Q^n\rangle$ by $x \circ^n_p y \mapsto (x,p,y)$.

We see that $\left(\left\langle Q^0 \right\rangle \underset{t^0_{\langle Q \rangle}}{\overset{s^0_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^1 \right\rangle \underset{t^1_{\langle Q \rangle}}{\overset{s^1_{\langle Q \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle Q \rangle}}{\overset{s^{n-1}_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^n \right\rangle \underset{t^n_{\langle Q \rangle}}{\overset{s^n_{\langle Q \rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle Q \rangle})_{n \in \mathbb{N}_0}, (\circ^n_p)_{0 \leq p < n \in \mathbb{N}} \right)$ is a reflexive globular ω-magma.

Now consider a family of maps $i : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots \right) \to \left(\left\langle Q^0 \right\rangle \underset{t^0_{\langle Q \rangle}}{\overset{s^0_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^1 \right\rangle \underset{t^1_{\langle Q \rangle}}{\overset{s^1_{\langle Q \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle Q \rangle}}{\overset{s^{n-1}_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^n \right\rangle \underset{t^n_{\langle Q \rangle}}{\overset{s^n_{\langle Q \rangle}}{\leftleftarrows}} \cdots \right)$ defined by $x \mapsto x$ for $x \in Q^n$ and $n \in \mathbb{N}_0$.

Suppose that there exists a morphism $f : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots \right) \to \left(\left\langle R^0 \right\rangle \underset{t^0_{\langle R \rangle}}{\overset{s^0_{\langle R \rangle}}{\leftleftarrows}} \left\langle R^1 \right\rangle \underset{t^1_{\langle R \rangle}}{\overset{s^1_{\langle R \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle R \rangle}}{\overset{s^{n-1}_{\langle R \rangle}}{\leftleftarrows}} \left\langle R^n \right\rangle \underset{t^n_{\langle R \rangle}}{\overset{s^n_{\langle R \rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle R \rangle})_{n \in \mathbb{N}_0}, (\hat{\circ}^n_p)_{0 \leq p < n \in \mathbb{N}} \right)$ into another reflexive globular ω-magma.

Then we define a function $\phi : \left(\left\langle Q^0 \right\rangle \underset{t^0_{\langle Q \rangle}}{\overset{s^0_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^1 \right\rangle \underset{t^1_{\langle Q \rangle}}{\overset{s^1_{\langle Q \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle Q \rangle}}{\overset{s^{n-1}_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^n \right\rangle \underset{t^n_{\langle Q \rangle}}{\overset{s^n_{\langle Q \rangle}}{\leftleftarrows}} \cdots \right) \to \left(\left\langle R^0 \right\rangle \underset{t^0_{\langle R \rangle}}{\overset{s^0_{\langle R \rangle}}{\leftleftarrows}} \left\langle R^1 \right\rangle \underset{t^1_{\langle R \rangle}}{\overset{s^1_{\langle R \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle R \rangle}}{\overset{s^{n-1}_{\langle R \rangle}}{\leftleftarrows}} \left\langle R^n \right\rangle \underset{t^n_{\langle R \rangle}}{\overset{s^n_{\langle R \rangle}}{\leftleftarrows}} \cdots \right)$ recursively by

$$
\begin{array}{rcl}
\langle Q^n[1] \rangle \ni x & \mapsto & f(x) \\
\langle Q^n[2] \rangle \ni (x,p,y) & \mapsto & f(x) \hat{\circ}^n_p f(y) \\
\langle Q^n[3] \rangle \ni \big(x,p,(y,q,z)\big) & \mapsto & f(x) \hat{\circ}^n_p \big(f(y) \hat{\circ}^n_q f(z)\big) \\
\langle Q^n[3] \rangle \ni \big((x,p,y),q,z\big) & \mapsto & \big(f(x) \hat{\circ}^n_p f(y)\big) \hat{\circ}^n_q f(z) \\
\vdots & & \vdots
\end{array}
$$

Next, we will verify that $\phi\big((x,p,y)\big) = \phi(x) \hat{\circ}^n_p \phi(y)$ for every $(x,p,y) \in \langle Q^n \rangle$.

Notice that if $(x,p,y) \in \langle Q^n \rangle$, then $x \in \langle Q^n[i] \rangle$ and $y \in \langle Q^n[j] \rangle$, where $i + j = k$ for some $k \in \mathbb{N} \setminus \{1\}$.

If $i = j = 1$, we have $\phi\big((x,p,y)\big) = f(x) \hat{\circ}^n_p f(y) = \phi((x)) \hat{\circ}^n_p \phi((y))$.

Assume that this equation holds for $i = 1$ and $j = 1, 2, \ldots, n-1$.

We have $\phi\big((x,p,y)\big) = f(x) \hat{\circ}^n_p \phi(y) = \phi(x) \hat{\circ}^n_p \phi(y)$.

Now suppose that the equation holds for $i = 1,2,\ldots,m-1$ and $j \in \mathbb{N}$.

We have $\phi\big((x,p,y)\big) = \phi(x)\hat{\circ}^n_p\phi(y)$.

This means that $\phi\big((x,p,y)\big) = \phi(x)\hat{\circ}^n_p\phi(y)$ for each $(x,p,y) \in \langle Q^n \rangle$.

Thus, ϕ is a morphism of globular ω-magma satisfying $f = \phi \circ i$.

We see that ϕ is the only morphism of globular ω-magmas holding $f = \phi \circ i$.

Therefore, $\left(\left(\left\langle Q^0 \right\rangle \overset{s^0_{\langle Q \rangle}}{\underset{t^0_{\langle Q \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle Q \rangle}}{\underset{t^{n-1}_{\langle Q \rangle}}{\leftleftarrows}} \left\langle Q^n \right\rangle \overset{s^n_{\langle Q \rangle}}{\underset{t^n_{\langle Q \rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle Q \rangle})_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}\right), i\right)$
is a free reflexive globular ω-magma over a reflexive ω-globular set. $\square$

Without the reflexivity in the original ω-globular set, we can also guarantee the existence of a free reflexive globular ω-magma.

Proposition 3.1.2.2. *A free reflexive globular ω-magma over an ω-globular set exists.*

Proof. Let $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} \cdots$ be an ω-globular set.

First of all, we introduce a construction of a new ω-globular set.

Set $\left\langle \bar{Q}^0 \right\rangle := Q^0$, $\left(\left\langle \bar{Q}^0 \right\rangle\right)^1 := \{(x,0) \mid x \in \left\langle \bar{Q}^0 \right\rangle\}$, and $\left\langle \bar{Q}^1[1] \right\rangle := Q^1 \cup \left(\left\langle \bar{Q}^0 \right\rangle\right)^1$.

Define $s^0[1] : \left\langle \bar{Q}^1[1] \right\rangle \to \left\langle \bar{Q}^0 \right\rangle$ by

$$s^0[1](y) := \begin{cases} s^0(y), & y \in Q^1; \\ x, & y = (x,0),\ x \in \left\langle \bar{Q}^0 \right\rangle. \end{cases}$$

and also $t^0[1] : \left\langle \bar{Q}^1[1] \right\rangle \to \left\langle \bar{Q}^0 \right\rangle$ by

$$t^0[1](y) := \begin{cases} t^0(y), & y \in Q^1; \\ x, & y = (x,0),\ x \in \left\langle \bar{Q}^0 \right\rangle. \end{cases}$$

Let $\left\langle \bar{Q}^1[2] \right\rangle := \{(x,0,y) \mid x,y \in \left\langle \bar{Q}^1[1] \right\rangle, s^0[1](x) = t^0[1](y)\}$.

Define $s^0[2] : \left\langle \bar{Q}^1[2] \right\rangle \to \left\langle \bar{Q}^0 \right\rangle$ by $s^0[2]\big((x,0,y)\big) := s^0[1](y)$

and $t^0[2] : \left\langle \bar{Q}^1[2] \right\rangle \to \left\langle Q^0 \right\rangle$ by $t^0[2]\big((x,0,y)\big) := t^0[1](x)$.

Suppose that we have $\left\langle \bar{Q}^1[l] \right\rangle$, $s^0[l]$, and $t^0[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle \bar{Q}^1[k] \rangle := \{(x,0,y) \mid x \in \langle \bar{Q}^1[i] \rangle, y \in \langle \bar{Q}^1[j] \rangle, i+j=k, s^0[i](x) = t^0[j](y)\}$.

If $(x,0,y) \in \langle \bar{Q}^1[k] \rangle$, we define $s^0[k], t^0[k] : \langle \bar{Q}^1[k] \rangle \to \langle \bar{Q}^0 \rangle$ by $s^0[k]\big((x,0,y)\big) := s^0[j](y)$ and $t^0[k]\big((x,0,y)\big) := t^0[i](x)$.

Set $\langle \bar{Q}^1 \rangle := \bigcupdot_{k=1}^{\infty} \langle \bar{Q}^1[k] \rangle$, $s^0_{\langle \bar{Q} \rangle} := \bigcupdot_{k=1}^{\infty} s^0[k]$, and $t^0_{\langle \bar{Q} \rangle} := \bigcupdot_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle \bar{Q}^m \rangle$, $s^{m-1}_{\langle \bar{Q} \rangle}$, and $t^{m-1}_{\langle \bar{Q} \rangle}$ for every $m = 1,2,\ldots,n-1$.

Let $\langle \bar{Q}^n[1] \rangle := Q^n \cupdot \big(\langle \bar{Q}^{n-1} \rangle\big)^n$ and $s^{n-1}[1], t^{n-1}[1] : \langle \bar{Q}^n[1] \rangle \to \langle \bar{Q}^{n-1}[1] \rangle$ be defined by

$$s^{n-1}[1](y) := \begin{cases} s^{n-1}(y), & y \in Q^n; \\ x, & y = (x,n),\ x \in \langle \bar{Q}^{n-1} \rangle. \end{cases}$$

$$t^{n-1}[1](y) := \begin{cases} t^{n-1}(y), & y \in Q^n; \\ x, & y = (x,n),\ x \in \langle \bar{Q}^{n-1} \rangle. \end{cases}$$

Let $\langle \bar{Q}^n[2] \rangle := \bigcupdot_{p=0}^{n-1} \{(x,p,y) \mid x,y \in \langle \bar{Q}^n[1] \rangle, s^p \cdots s^{n-1}[1](x) = t^p \cdots t^{n-1}[1](y)\}$.

If $(x,p,y) \in \langle \bar{Q}^n[2] \rangle$, then we define $s^{n-1}[2], t^{n-1}[2] : \langle \bar{Q}^n[2] \rangle \to \langle \bar{Q}^{n-1} \rangle$ by

- $s^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[1](x), p, s^{n-1}[1](y)), & n-1 > p; \\ s^{n-1}[1](y), & n-1 = p. \end{cases}$

- $t^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[1](x), p, t^{n-1}[1](y)), & n-1 > p; \\ t^{n-1}[1](x), & n-1 = p. \end{cases}$

Suppose that we have $\langle \bar{Q}^n[l] \rangle$, $s^{n-1}[l]$, and $t^{n-1}[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle \bar{Q}^n[k] \rangle := \bigcupdot_{p=0}^{n-1} \{(x,p,y) \mid x \in \langle \bar{Q}^n[i] \rangle,\ y \in \langle \bar{Q}^n[j] \rangle,\ i+j=k,$

$$s^p s^{p+1} \cdots s^{n-1}[i](x) = t^p t^{p+1} \cdots t^{n-1}[j](y)\}.$$

If $(x,p,y) \in \langle \bar{Q}^n[k] \rangle$, then we define $s^{n-1}[k], t^{n-1}[k] : \langle \bar{Q}^n[k] \rangle \to \langle \bar{Q}^{n-1} \rangle$ by

- $s^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[i](x), p, s^{n-1}[j](y)), & n-1 > p; \\ s^{n-1}[j](y), & n-1 = p. \end{cases}$

- $t^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[i](x), p, t^{n-1}[j](y)), & n-1 > p; \\ t^{n-1}[i](x), & n-1 = p. \end{cases}$

Set $\langle \bar{Q}^n \rangle := \dot{\bigcup\limits_{k=1}^{\infty}} \langle \bar{Q}^n[k] \rangle$, $s^{n-1}_{\langle \bar{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} s^{n-1}[k]$, and $t^{n-1}_{\langle \bar{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} t^{n-1}[k]$.

In order to prove the globularity condition, we let $n \in \mathbb{N}$ and $(x,p,y) \in \langle \bar{Q}^{n+1} \rangle$.

By globularity condition of Q, we get $s^{n-1}[1]s^{n-1}[1] = s^{n-1}[1]t^{n-1}[1]$.

This means that

$$
\begin{aligned}
s^{n-1}_{\langle \bar{Q} \rangle} s^n[2]\big((x,p,y)\big) &= \begin{cases} \big(s^{n-1}[1]s^n[1](x), p, s^{n-1}[1]s^n[1](y)\big), & n-1 > p; \\ s^{n-1}[1]s^n[1](y), & n-1 = p. \end{cases} \\
&= \begin{cases} \big(s^{n-1}[1]t^n[1](x), p, s^{n-1}[1]t^n[1](y)\big), & n-1 > p; \\ s^{n-1}[1]t^n[1](y), & n-1 = p. \end{cases} \\
&= s^{n-1}_{\langle \bar{Q} \rangle} t^n[2]\big((x,p,y)\big).
\end{aligned}
$$

Suppose that $s^{n-1}_{\langle \bar{Q} \rangle} s^n[m] = s^{n-1}_{\langle \bar{Q} \rangle} t^n[m]$ for all $m = 1,2,\ldots,k-1$.

This implies that if $(x,p,y) \in \langle \bar{Q}^{n+1}[k] \rangle$, then

$$s^{n-1}[i]s^n[i](x) = s^{n-1}[i]t^n[i](x)$$

and

$$s^{n-1}[j]s^n[j](y) = s^{n-1}[j]t^n[j](y).$$

Thus,

$$
\begin{aligned}
s^{n-1}_{\langle \bar{Q} \rangle} s^n[k]\big((x,p,y)\big) &= \begin{cases} (s^{n-1}[i]s^n[i](x), p, s^{n-1}[j]s^n[j](y)), & n-1 > p; \\ s^{n-1}[j]s^n[j](y), & n-1 = p. \end{cases} \\
&= \begin{cases} (s^{n-1}[i]t^n[i](x), p, s^{n-1}[j]t^n[j](y)), & n-1 > p; \\ s^{n-1}[j]t^n[j](y), & n-1 = p. \end{cases} \\
&= s^{n-1}_{\langle \bar{Q} \rangle} t^n[k]\big((x,p,y)\big).
\end{aligned}
$$

This yields $s^{n-1}_{\langle \bar{Q} \rangle} s^n_{\langle \bar{Q} \rangle} = s^{n-1}_{\langle \bar{Q} \rangle} t^n_{\langle \bar{Q} \rangle}$.

Similarly, we get $t^{n-1}_{\langle \bar{Q} \rangle} s^n_{\langle \bar{Q} \rangle} = t^{n-1}_{\langle \bar{Q} \rangle} t^n_{\langle \bar{Q} \rangle}$.

Thus, $\langle \bar{Q}^0 \rangle \underset{t^0_{\langle \bar{Q} \rangle}}{\overset{s^0_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \langle \bar{Q}^1 \rangle \underset{t^1_{\langle \bar{Q} \rangle}}{\overset{s^1_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle \bar{Q} \rangle}}{\overset{s^{n-1}_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \langle \bar{Q}^n \rangle \underset{t^n_{\langle \bar{Q} \rangle}}{\overset{s^n_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Moreover, we need to establish identity maps as follows.

First, we define $\iota^0_{\langle \bar{Q} \rangle} : \langle \bar{Q}^0 \rangle \to \langle \bar{Q}^1 \rangle$ by $x \mapsto (x, 0)$ for all $x \in \langle \bar{Q}^0 \rangle$.

Next, define $\iota^1[1] : \langle \bar{Q}^1[1] \rangle \to \langle \bar{Q}^2[1] \rangle$ by $y \mapsto (y, 1)$ for any $y \in \langle \bar{Q}^1[1] \rangle$.

Assume that we have $\iota^1[h]$ for all $h = 1, 2, \ldots, k-1$.

If $(x, 0, y) \in \langle \bar{Q}^1[k] \rangle$, where $x \in \langle \bar{Q}^1[i] \rangle$ and $y \in \langle \bar{Q}^1[j] \rangle$, we define $\iota^1[k] : \langle \bar{Q}^1[k] \rangle \to \langle \bar{Q}^2[k] \rangle$ by $(x, 0, y) \mapsto (\iota^1[i](x), 0, \iota^1[j](y))$.

Then we let $\iota^1_{\langle \bar{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} \iota^1[k]$.

Suppose that we have $\iota^l_{\langle \bar{Q} \rangle}$ for all $l = 0, 1, \ldots, n-1$.

Next, define $\iota^n[1] : \langle \bar{Q}^n[1] \rangle \to \langle \bar{Q}^{n+1}[1] \rangle$ by $y \mapsto (y, n)$ for any $y \in \langle \bar{Q}^n[1] \rangle$.

Assume that we have $\iota^n[h]$ for all $h = 1, 2, \ldots, k-1$.

If $(x, p, y) \in \langle \bar{Q}^n[k] \rangle$, where $x \in \langle \bar{Q}^n[i] \rangle$ and $y \in \langle \bar{Q}^n[j] \rangle$, we define $\iota^n[k] : \langle \bar{Q}^n[k] \rangle \to \langle \bar{Q}^{n+1}[k] \rangle$ by $(x, p, y) \mapsto (\iota^n[i](x), p, \iota^n[j](y))$.

Then we let $\iota^n_{\langle \bar{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} \iota^n[k]$.

Consider, for each $(x, p, y) \in \langle \bar{Q}^n[2] \rangle$,

$$s^n[2] \circ \iota^n[2]\big((x, p, y)\big) = s^n[2]\big((\iota^n(x), p, \iota^n(y))\big) = \big(s^n \circ \iota^n(x), p, s^n \circ \iota^n(y)\big) = (x, p, y).$$

Thus, $s^n[2] \circ \iota^n[2] = \mathrm{Id}_{\langle \bar{Q}^n[2] \rangle}$.

Now assume that $s^n[m] \circ \iota^n[m] = \mathrm{Id}_{\langle \bar{Q}^n[m] \rangle}$ for every $m = 1, 2, \ldots, k-1$.

For any $(x, p, y) \in \langle \bar{Q}^n[k] \rangle$, where $x \in \langle \bar{Q}^n[i] \rangle$ and $y \in \langle \bar{Q}^n[j] \rangle$,

$$\begin{aligned} s^n[k] \circ \iota^n[k]\big((x, p, y)\big) &= s^n[k]\big((\iota^n[i](x), p, \iota^n[j](y))\big) \\ &= \big(s^n[i] \circ \iota^n[i](x), p, s^n[j] \circ \iota^n[j](y)\big) \\ &= (x, p, y). \end{aligned}$$

It follows that $s^n_{\langle \bar{Q} \rangle} \circ \iota^n_{\langle \bar{Q} \rangle} = \mathrm{Id}_{\langle \bar{Q}^n \rangle}$.

Similarly, we have $t^n_{\langle \bar{Q} \rangle} \circ \iota^n_{\langle \bar{Q} \rangle} = \mathrm{Id}_{\langle \bar{Q}^n \rangle}$.

Thus, $\left(\left\langle\bar{Q}^0\right\rangle \underset{t^0_{\langle\bar{Q}\rangle}}{\overset{s^0_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^1\right\rangle \underset{t^1_{\langle\bar{Q}\rangle}}{\overset{s^1_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\bar{Q}\rangle}}{\overset{s^{n-1}_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^n\right\rangle \underset{t^n_{\langle\bar{Q}\rangle}}{\overset{s^n_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\bar{Q}\rangle})_{n\in\mathbb{N}_0}\right)$ is a reflexive ω-globular set.

For $p \in \mathbb{N}_0$, set $\left\langle\bar{Q}^n\right\rangle \times_p \left\langle\bar{Q}^n\right\rangle := \{(x,y) \in \left\langle\bar{Q}^n\right\rangle \times \left\langle\bar{Q}^n\right\rangle \mid s^p_{\langle\bar{Q}\rangle}(x) = t^p_{\langle\bar{Q}\rangle}(y)\}$.

Define a family of operations $\circ^n_p : \left\langle\bar{Q}^n\right\rangle \times_p \left\langle\bar{Q}^n\right\rangle \to \left\langle\bar{Q}^n\right\rangle$ by $x \circ^n_p y \mapsto (x,p,y)$.

We see that $\left(\left\langle\bar{Q}^0\right\rangle \underset{t^0_{\langle\bar{Q}\rangle}}{\overset{s^0_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^1\right\rangle \underset{t^1_{\langle\bar{Q}\rangle}}{\overset{s^1_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\bar{Q}\rangle}}{\overset{s^{n-1}_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^n\right\rangle \underset{t^n_{\langle\bar{Q}\rangle}}{\overset{s^n_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\bar{Q}\rangle})_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}\right)$ is a reflexive globular ω-magma.

Now we define a map $i : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to$ $\left(\left\langle\bar{Q}^0\right\rangle \underset{t^0_{\langle\bar{Q}\rangle}}{\overset{s^0_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^1\right\rangle \underset{t^1_{\langle\bar{Q}\rangle}}{\overset{s^1_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\bar{Q}\rangle}}{\overset{s^{n-1}_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^n\right\rangle \underset{t^n_{\langle\bar{Q}\rangle}}{\overset{s^n_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots\right)$ by $x \mapsto x$ for $x \in Q^n$ and $n \in \mathbb{N}_0$.

Suppose that there exists a morphism $f : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to$ $\left(\left\langle\bar{R}^0\right\rangle \underset{t^0_{\langle\bar{R}\rangle}}{\overset{s^0_{\langle\bar{R}\rangle}}{\leftleftarrows}} \left\langle\bar{R}^1\right\rangle \underset{t^1_{\langle\bar{R}\rangle}}{\overset{s^1_{\langle\bar{R}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\bar{R}\rangle}}{\overset{s^{n-1}_{\langle\bar{R}\rangle}}{\leftleftarrows}} \left\langle\bar{R}^n\right\rangle \underset{t^n_{\langle\bar{R}\rangle}}{\overset{s^n_{\langle\bar{R}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\bar{R}\rangle})_{n\in\mathbb{N}_0}, (\hat{\circ}^n_p)_{0\le p<n\in\mathbb{N}}\right)$ into another reflexive globular ω-magma.

The only choice of morphism of reflexive globular ω-magmas holding its universal factorization property is given by the following.

We define a function $\phi : \left(\left\langle\bar{Q}^0\right\rangle \underset{t^0_{\langle\bar{Q}\rangle}}{\overset{s^0_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^1\right\rangle \underset{t^1_{\langle\bar{Q}\rangle}}{\overset{s^1_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\bar{Q}\rangle}}{\overset{s^{n-1}_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \left\langle\bar{Q}^n\right\rangle \underset{t^n_{\langle\bar{Q}\rangle}}{\overset{s^n_{\langle\bar{Q}\rangle}}{\leftleftarrows}} \cdots\right) \to$ $\left(\left\langle\bar{R}^0\right\rangle \underset{t^0_{\langle\bar{R}\rangle}}{\overset{s^0_{\langle\bar{R}\rangle}}{\leftleftarrows}} \left\langle\bar{R}^1\right\rangle \underset{t^1_{\langle\bar{R}\rangle}}{\overset{s^1_{\langle\bar{R}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\bar{R}\rangle}}{\overset{s^{n-1}_{\langle\bar{R}\rangle}}{\leftleftarrows}} \left\langle\bar{R}^n\right\rangle \underset{t^n_{\langle\bar{R}\rangle}}{\overset{s^n_{\langle\bar{R}\rangle}}{\leftleftarrows}} \cdots\right)$ by, for any $x \in Q^0$, $y \in Q^1$, $u,v,w \in Q^n$,

$$
\begin{array}{rcl}
x & \mapsto & f(x), \\
(x,0) & \mapsto & \iota^0_{\langle \bar{R} \rangle}(f(x)), \\
y & \mapsto & f(y), \\
(y,1) & \mapsto & \iota^1_{\langle \bar{R} \rangle}(f(y)), \\
((x,0),1) & \mapsto & \iota^1_{\langle \bar{R} \rangle}\iota^0_{\langle \bar{R} \rangle}(f(x)), \\
\vdots & & \vdots \\
((u,p,v),q,w) & \mapsto & (f(u)\hat{\circ}^n_p f(v))\hat{\circ}^n_q f(w), \\
(u,p,(v,q,w)) & \mapsto & f(u)\hat{\circ}^n_p (f(v)\hat{\circ}^n_q f(w)), \\
\vdots & & \vdots
\end{array}
$$

Next, we will verify that $\phi\big((x,p,y)\big) = \phi(x)\hat{\circ}^n_p\phi(y)$ for every $(x,p,y) \in \langle \bar{Q}^n \rangle$.

First, notice that $\phi\big((x,n)\big) = \iota^n_{\langle \bar{R} \rangle}\big(f(x)\big) = \iota^n_{\langle \bar{R} \rangle}\big(\phi(x)\big)$ for every $x \in Q^n$.

Notice that if $(x,p,y) \in \langle \bar{Q}^n \rangle$, then $x \in \langle \bar{Q}^n[i] \rangle$ and $y \in \langle \bar{Q}^n[j] \rangle$, where $i + j = k$ for some $k \in \mathbb{N} \setminus \{1\}$.

If $i = j = 1$, we have $\phi\big((x,p,y)\big) = f(x)\hat{\circ}^n_p f(y) = \phi(x)\hat{\circ}^n_p\phi(y)$.

Assume that this equation holds for $i = 1$ and $j = 1,2,\ldots,n-1$.

We have $\phi\big((x,p,y)\big) = f(x)\hat{\circ}^n_p\phi(y) = \phi(x)\hat{\circ}^n_p\phi(y)$.

Now suppose that the equation holds for $i = 1,2,\ldots,m-1$ and $j \in \mathbb{N}$.

We have $\phi\big((x,p,y)\big) = \phi(x)\hat{\circ}^n_p\phi(y)$.

This means that $\phi\big((x,p,y)\big) = \phi(x)\hat{\circ}^n_p\phi(y)$ for each $(x,p,y) \in \langle \bar{Q}^n \rangle$.

Thus, ϕ is a unique morphism of globular ω-magma satisfying $f = \phi \circ i$.

As a result,

$$
\left(\left(\langle \bar{Q}^0 \rangle \underset{t^0_{\langle \bar{Q} \rangle}}{\overset{s^0_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \langle \bar{Q}^1 \rangle \underset{t^1_{\langle \bar{Q} \rangle}}{\overset{s^1_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle \bar{Q} \rangle}}{\overset{s^{n-1}_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \langle \bar{Q}^n \rangle \underset{t^n_{\langle \bar{Q} \rangle}}{\overset{s^n_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle \bar{Q} \rangle})_{n \in \mathbb{N}_0}, (\circ^n_p)_{0 \le p < n \in \mathbb{N}}\right), i\right)
$$

is a free reflexive globular ω-magma over an ω-globular set. □

3.1.3 Free Reflexive Self-Dual Globular ω-Magmas

Applying the ideas of the proofs of Proposition 3.1.1.3 and Proposition 3.1.2.2, we can obtain a free reflexive self-dual globular ω-magma over a (reflexive) ω-globular set. Since the concept of self-duality does not introduce any equations among self-dual operations, we can talk about the notion of self-duality of a globular ω-magma regarding it as a self-dual ω-globular set equipped with a family of partially-defined compositions.

Proposition 3.1.3.1. *A free reflexive self-dual globular ω-magma over a reflexive ω-globular set exists.*

Proof. Let $\left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots, (\iota^n)_{n\in\mathbb{N}_0} \right)$ be a reflexive ω-globular set.

For any $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we first set

$$\hat{Q}^n := \{(\cdots((y^{\beta_1^n})^{\beta_2^n})^{\cdots})^{\beta_m^n} \mid y \in Q^n,\ \beta_j^n \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\}.$$

Let $k,m,n \in \mathbb{N}$, $\langle \hat{Q}^0 \rangle := \hat{Q}^0$, and $\langle \hat{Q}^1[1] \rangle := \hat{Q}^1$.

Define $s^0[1], t^0[1] : \langle \hat{Q}^1[1] \rangle \to \langle \hat{Q}^0 \rangle$ by

- $s^0[1]\big((\cdots(y^{\beta_1^1})^{\cdots})^{\beta_m^1}\big) := \begin{cases} (\cdots((s^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & 0 \notin \beta_1^1 \triangle \cdots \triangle \beta_m^1; \\ (\cdots((t^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & 0 \in \beta_1^1 \triangle \cdots \triangle \beta_m^1. \end{cases}$

- $t^0[1]\big((\cdots(y^{\beta_1^1})^{\cdots})^{\beta_m^1}\big) := \begin{cases} (\cdots((t^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & 0 \notin \beta_1^1 \triangle \cdots \triangle \beta_m^1; \\ (\cdots((s^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & 0 \in \beta_1^1 \triangle \cdots \triangle \beta_m^1. \end{cases}$

Now let $\langle \hat{Q}^1[2] \rangle := \{(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mid x,y \in \langle \hat{Q}^1[1] \rangle,\ \alpha_j^1 \subseteq \mathbb{N}_0,$

$$j = 1,2,\ldots,m,\ s^0[1](x) = t^0[1](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \in \langle \hat{Q}^1[2] \rangle$, we define $s^0[2], t^0[2] : \langle \hat{Q}^1[2] \rangle \to \langle \hat{Q}^0 \rangle$ by

- $s^0[2]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((s^0[1](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((t^0[1](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

- $t^0[2]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((t^0[1](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((s^0[1](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

Suppose that we have $\langle \hat{Q}^1[l] \rangle$, $s^0[l]$, and $t^0[l]$ for every $l = 1,2,\ldots,k-1$.

Let $\langle \hat{Q}^1[k] \rangle := \{(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mid x \in \langle \hat{Q}^1[i] \rangle,\ y \in \langle \hat{Q}^1[j] \rangle,\ i+j=k,$

$$\alpha_h^1 \subseteq \mathbb{N}_0,\ h = 1,2,\ldots,m,\ s^0[i](x) = t^0[j](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \in \langle \hat{Q}^1[k] \rangle$, we define $s^0[k], t^0[k] : \langle \hat{Q}^1[k] \rangle \to \langle \hat{Q}^0 \rangle$ by

- $s^0[k]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((s^0[j](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((t^0[j](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

- $t^0[k]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((t^0[i](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((s^0[i](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

Set $\langle \hat{Q}^1 \rangle := \bigcup_{k=1}^{\infty} \langle \hat{Q}^1[k] \rangle$, $s^0_{\langle \hat{Q} \rangle} := \bigcup_{k=1}^{\infty} s^0[k]$, and $t^0_{\langle \hat{Q} \rangle} := \bigcup_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle \hat{Q}^r \rangle$, $s^{r-1}_{\langle \hat{Q} \rangle}$, and $t^{r-1}_{\langle \hat{Q} \rangle}$ for every $r = 1,2,\ldots,n-1$.

Let $\langle \hat{Q}^n[1] \rangle := \hat{Q}^n$.

Define $s^{n-1}[1], t^{n-1}[1] : \langle \hat{Q}^n[1] \rangle \to \langle \hat{Q}^{n-1} \rangle$ by

- $s^{n-1}[1]\big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n}\big) := \begin{cases} (\cdots((s^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_m^{n-1}}, & n-1 \notin \beta_1^n \triangle \cdots \triangle \beta_m^n; \\ (\cdots((t^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_m^{n-1}}, & n-1 \in \beta_1^n \triangle \cdots \triangle \beta_m^n. \end{cases}$

- $t^{n-1}[1]\big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n}\big) := \begin{cases} (\cdots((t^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_m^{n-1}}, & n-1 \notin \beta_1^n \triangle \cdots \triangle \beta_m^n; \\ (\cdots((s^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_m^{n-1}}, & n-1 \in \beta_1^n \triangle \cdots \triangle \beta_m^n. \end{cases}$

Now let $\langle \hat{Q}^n[2] \rangle := \bigcup_{p=0}^{n-1} \{(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mid x,y \in \langle \hat{Q}^n[1] \rangle,\ \alpha_j^n \subseteq \mathbb{N}_0,$

$$j = 1,2,\ldots,m,\ s^p[1]\cdots s^{n-1}[1](x) = t^p[1]\cdots t^{n-1}[1](y)\}.$$

If $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \langle \hat{Q}^n[2] \rangle$, define $s^{n-1}[2], t^{n-1}[2] : \langle \hat{Q}^n[2] \rangle \to \langle \hat{Q}^{n-1} \rangle$ by

$$s^{n-1}[2]\big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big)$$
$$:= \begin{cases} (\cdots((s^{n-1}[1](x),p,s^{n-1}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((t^{n-1}[1](x),p,t^{n-1}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((s^{n-1}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((t^{n-1}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n. \end{cases}$$

$$t^{n-1}[2]\big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big)$$
$$:= \begin{cases} (\cdots((t^{n-1}[1](x),p,t^{n-1}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((s^{n-1}[1](x),p,s^{n-1}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((t^{n-1}[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((s^{n-1}[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n. \end{cases}$$

Suppose that we have $\langle \hat{Q}^n[l]\rangle$, $s^{n-1}[l]$, and $t^{n-1}[l]$ for every $l=1,2,\ldots,k-1$.

Let $\langle \hat{Q}^n[k]\rangle := \bigcup_{p=0}^{n-1}\{(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mid x\in\langle \hat{Q}^n[i]\rangle,\ y\in\langle \hat{Q}^n[j]\rangle,\ i+j=k,$

$\alpha_h^n\subseteq\mathbb{N}_0,\ h=1,2,\ldots,m,\ s^p[i]\cdots s^{n-1}[i](x)=t^p[j]\cdots t^{n-1}[j](y)\}.$

If $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\in\langle \hat{Q}^n[k]\rangle$, define $s^{n-1}[k],t^{n-1}[k]:\langle \hat{Q}^n[k]\rangle\to\langle \hat{Q}^{n-1}\rangle$ by

$$s^{n-1}[k]\big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big)$$
$$:= \begin{cases} (\cdots((s^{n-1}[i](x),p,s^{n-1}[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((t^{n-1}[i](x),p,t^{n-1}[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((s^{n-1}[i](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((t^{n-1}[i](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n. \end{cases}$$

$$t^{n-1}[k]\big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big)$$
$$:= \begin{cases} (\cdots((t^{n-1}[i](x),p,t^{n-1}[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((s^{n-1}[i](x),p,s^{n-1}[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p<n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((t^{n-1}[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\ (\cdots((s^{n-1}[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p=n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n. \end{cases}$$

Set $\langle \hat{Q}^n \rangle := \bigcup\limits_{k=1}^{\infty} \langle \hat{Q}^n[k] \rangle$, $s^{n-1}_{\langle \hat{Q} \rangle} := \bigcup\limits_{k=1}^{\infty} s^{n-1}[k]$, and $t^{n-1}_{\langle \hat{Q} \rangle} := \bigcup\limits_{k=1}^{\infty} t^{n-1}[k]$.

Notice that $s^{n-1}[1]s^n[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$

$$= \begin{cases} (\cdots((s^{n-1}s^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \notin \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots((t^{n-1}s^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \in \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots((s^{n-1}t^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \notin \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots((t^{n-1}t^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \in \alpha_1^n \triangle \cdots \triangle \alpha_m^n. \end{cases}$$

$$= \begin{cases} (\cdots((s^{n-1}t^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \notin \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots((t^{n-1}t^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \in \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots((s^{n-1}s^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \notin \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\ (\cdots((t^{n-1}s^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1},\ n-1 \in \alpha_1^n \triangle \cdots \triangle \alpha_m^n. \end{cases}$$

$= s^{n-1}[1]t^n[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$.

Similarly, $t^{n-1}[1]s^n[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) = t^{n-1}[1]t^n[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$.

To prove the globularity condition, we suppose $(\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \in \langle Q^{n+1} \rangle$.

Then $(\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \in \langle Q^{n+1}[k] \rangle$ for some $k \in \mathbb{N} \setminus \{1\}$.

This means that there exist $i, j \in \mathbb{N}$ such that $x \in \langle Q^{n+1}[i] \rangle$, $y \in \langle Q^{n+1}[j] \rangle$, $i + j = k$, and $s^p[i](x) = t^p[j](y)$.

For the case $k = 2$, we have $s^{n-1}[2]s^n[2]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$

$$
= \begin{cases}
(\cdots((s^{n-1}[1]s^{n}[1](x), p, s^{n-1}[1]s^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]s^{n}[1](x), p, t^{n-1}[1]s^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((s^{n-1}[1]s^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]s^{n}[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((s^{n-1}[1]t^{n}[1](x), p, s^{n-1}[1]t^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]t^{n}[1](x), p, t^{n-1}[1]t^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((s^{n-1}[1]t^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]t^{n}[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n},
\end{cases}
$$

$$
= \begin{cases}
(\cdots((s^{n-1}[1]s^{n}[1](x), p, s^{n-1}[1]s^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots((t^{n-1}[1]t^{n}[1](x), p, t^{n-1}[1]t^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots((s^{n-1}[1]s^{n}[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots((t^{n-1}[1]t^{n}[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}.
\end{cases}
$$

$$
= \begin{cases}
(\cdots((s^{n-1}[1]t^n[1](x), p, s^{n-1}[1]t^n[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]t^n[1](x), p, t^{n-1}[1]t^n[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((s^{n-1}[1]t^n[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]t^n[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((s^{n-1}[1]s^n[1](x), p, s^{n-1}[1]s^n[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]s^n[1](x), p, t^{n-1}[1]s^n[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((s^{n-1}[1]s^n[1](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\
(\cdots((t^{n-1}[1]s^n[1](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n},
\end{cases}
$$

$$= s^{n-1}[2]t^n[2]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big).$$

This implies that $s^{n-1}[2]s^n[2] = s^{n-1}[2]t^n[2]$.

Using a similar argument, we get $t^{n-1}[2]s^n[2] = t^{n-1}[2]t^n[2]$.

Assume that $s^{n-1}[h]s^n[h] = s^{n-1}[h]t^n[h]$ and $t^{n-1}[h]s^n[h] = t^{n-1}[h]t^n[h]$ for any $h = 1,2,\ldots,k-1$.

Consider $s^{n-1}[k]s^n[k]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$

$$= \begin{cases} (\cdots((s^{n-1}[i]s^n[i](x),p,s^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]s^n[i](x),p,t^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((s^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]s^n[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((s^{n-1}[i]t^n[i](x),p,s^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]t^n[i](x),p,t^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((s^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]t^n[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \end{cases}$$

$$= \begin{cases} (\cdots((s^{n-1}[i]s^n[i](x),p,s^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\ (\cdots((t^{n-1}[i]t^n[i](x),p,t^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\ (\cdots((s^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\ (\cdots((t^{n-1}[i]t^n[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}. \end{cases}$$

$$= \begin{cases} (\cdots((s^{n-1}[i]t^n[i](x), p, s^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]t^n[i](x), p, t^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((s^{n-1}[j]t^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]t^n[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((s^{n-1}[i]s^n[i](x), p, s^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]s^n[i](x), p, t^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p < n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((s^{n-1}[j]s^n[j](y))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \\ (\cdots((t^{n-1}[i]s^n[i](x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\ & p = n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}, \end{cases}$$

$$= s^{n-1}[k]t^n[k]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big).$$

It follows that $s^{n-1}[k]s^n[k] = s^{n-1}[k]t^n[k]$.

Applying a similar argument, we have $t^{n-1}[k]s^n[k] = t^{n-1}[k]t^n[k]$.

This yields $s^{n-1}_{\langle\hat{Q}\rangle}s^n_{\langle\hat{Q}\rangle} = s^{n-1}_{\langle\hat{Q}\rangle}t^n_{\langle\hat{Q}\rangle}$ and $t^{n-1}_{\langle\hat{Q}\rangle}s^n_{\langle\hat{Q}\rangle} = t^{n-1}_{\langle\hat{Q}\rangle}t^n_{\langle\hat{Q}\rangle}$.

So, $\langle\hat{Q}^0\rangle \overset{s^0_{\langle\hat{Q}\rangle}}{\underset{t^0_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \langle\hat{Q}^1\rangle \overset{s^1_{\langle\hat{Q}\rangle}}{\underset{t^1_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle\hat{Q}\rangle}}{\underset{t^{n-1}_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \langle\hat{Q}^n\rangle \overset{s^n_{\langle\hat{Q}\rangle}}{\underset{t^n_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots$ is an ω-globular set.

In addition, we need to construct identity maps as follows.

First of all, set $\iota^0_{\langle Q\rangle} := \iota^0$ and $\iota^n[1] : \langle\hat{Q}^n[1]\rangle \to \langle\hat{Q}^{n+1}[1]\rangle$ is defined by $(\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n} \mapsto (\cdots((\iota^n(y))^{\beta_1^{n+1}})^{\cdots})^{\beta_m^{n+1}}$ for all $n \in \mathbb{N}_0$.

Next, we define $\iota^1[2] : \langle\hat{Q}^1[2]\rangle \to \langle\hat{Q}^2[2]\rangle$ by

$$(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mapsto (\cdots((\iota^1[1](x), 0, \iota^1[1](y))^{\alpha_1^2})^{\cdots})^{\alpha_m^2}.$$

Assume that we have $\iota^1[h]$ for all $h = 1, 2, \ldots, k-1$.

If $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \in \langle \hat{Q}^1[k] \rangle$, where $x \in \langle \hat{Q}^1[i] \rangle$ and $y \in \langle \hat{Q}^1[j] \rangle$, we define $\iota^1[k] : \langle \hat{Q}^1[k] \rangle \to \langle \hat{Q}^2[k] \rangle$ by $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mapsto (\cdots((\iota^1[i](x), 0, \iota^1[j](y))^{\alpha_1^2})^{\cdots})^{\alpha_m^2}$.

Then we let $\iota^1_{\langle \hat{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} \iota^1[k]$.

Suppose that we have $\iota^l_{\langle \hat{Q} \rangle}$ for all $l = 0, 1, \ldots, n-1$.

Next, we define $\iota^n[2] : \langle \hat{Q}^n[2] \rangle \to \langle \hat{Q}^{n+1}[2] \rangle$ by

$$(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mapsto (\cdots((\iota^n[1](x), p, \iota^n[1](y))^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}.$$

Assume that we have $\iota^n[h]$ for every $h = 0, 1, \ldots, k-1$.

If $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \langle \hat{Q}^n[k] \rangle$, where $x \in \langle \hat{Q}^n[i] \rangle$ and $y \in \langle \hat{Q}^n[j] \rangle$, we define $\iota^n[k] : \langle \hat{Q}^n[k] \rangle \to \langle \hat{Q}^{n+1}[k] \rangle$ by

$$(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mapsto (\cdots((\iota^n[i](x), p, \iota^n[j](y))^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}.$$

Then we let $\iota^n_{\langle \hat{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} \iota^n[k]$.

Note that, for each $(\cdots(x)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \langle \hat{Q}^n[1] \rangle$,

$$\begin{aligned} s^n[1] \circ \iota^n[1]\big((\cdots(x)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big) &= s^n[1]\big((\cdots((\iota^n(x))^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) \\ &= (\cdots((s^n \circ \iota^n(x))^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \\ &= (\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n}. \end{aligned}$$

It follows that $s^n[1] \circ \iota^n[1] = \mathrm{Id}_{\langle \hat{Q}^n[1] \rangle}$.

Assume that $s^n[h] \circ \iota^n[h] = \mathrm{Id}_{\langle \hat{Q}^n[h] \rangle}$ for every $h = 1, 2, \ldots, k-1$.

For any $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \langle \hat{Q}^n[k] \rangle$, where $x \in \langle \hat{Q}^n[i] \rangle$ and $y \in \langle \hat{Q}^n[j] \rangle$,

$$\begin{aligned} s^n[k] \circ \iota^n[k]\big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big) &= s^n[k]\big((\cdots((\iota^n[i](x), p, \iota^n[j](y))^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) \\ &= (\cdots((s^n[i] \circ \iota^n[i](x), p, s^n[j] \circ \iota^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \\ &= (\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}. \end{aligned}$$

This means that $s^n[1] \circ \iota^n[1] = \mathrm{Id}_{\langle \hat{Q}^n[1] \rangle}$.

That is, $s^n_{\langle\hat{Q}\rangle} \circ \iota^n_{\langle\hat{Q}\rangle} = \mathrm{Id}_{\langle\hat{Q}^n\rangle}$.

Similarly, we have $t^n_{\langle\hat{Q}\rangle} \circ \iota^n_{\langle\hat{Q}\rangle} = \mathrm{Id}_{\langle\hat{Q}^n\rangle}$.

Hence, $\left(\left\langle\hat{Q}^0\right\rangle \underset{t^0_{\langle\hat{Q}\rangle}}{\overset{s^0_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^1\right\rangle \underset{t^1_{\langle\hat{Q}\rangle}}{\overset{s^1_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\hat{Q}\rangle}}{\overset{s^{n-1}_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^n\right\rangle \underset{t^n_{\langle\hat{Q}\rangle}}{\overset{s^n_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\hat{Q}\rangle})_{n\in\mathbb{N}_0}\right)$ is a reflexive ω-globular set.

For all $p,n \in \mathbb{N}_0$ such that $p < n$, we set

$$\left\langle\hat{Q}^n\right\rangle \times_p \left\langle\hat{Q}^n\right\rangle := \{(x,y) \in \left\langle\hat{Q}^n\right\rangle \times \left\langle\hat{Q}^n\right\rangle \mid s^p_{\langle\hat{Q}\rangle} s^{p+1}_{\langle\hat{Q}\rangle} \cdots s^{n-1}_{\langle\hat{Q}\rangle}(x) = t^p_{\langle\hat{Q}\rangle} t^{p+1}_{\langle\hat{Q}\rangle} \cdots t^{n-1}_{\langle\hat{Q}\rangle}(y)\}.$$

Then we define $\circ^n_p : \left\langle\hat{Q}^n\right\rangle \times_p \left\langle\hat{Q}^n\right\rangle \to \left\langle\hat{Q}^n\right\rangle$ by $x \circ^n_p y \mapsto (x,p,y)$.

For each $\alpha^n \subseteq \mathbb{N}_0$, we define $*^n_\alpha : \left\langle\hat{Q}^n\right\rangle \to \left\langle\hat{Q}^n\right\rangle$ by

$$(\cdots((x,p,y)^{\alpha^n_1})^{\cdots})^{\alpha^n_m} \mapsto ((\cdots((x,p,y)^{\alpha^n_1})^{\cdots})^{\alpha^n_m})^{\alpha^n}.$$

So, $\left(\left\langle\hat{Q}^0\right\rangle \underset{t^0_{\langle\hat{Q}\rangle}}{\overset{s^0_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\hat{Q}\rangle}}{\overset{s^{n-1}_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^n\right\rangle \underset{t^n_{\langle\hat{Q}\rangle}}{\overset{s^n_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\hat{Q}\rangle})_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right)$ is a reflexive self-dual globular ω-magma.

Now consider a family of maps $i : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to \left(\left\langle\hat{Q}^0\right\rangle \underset{t^0_{\langle\hat{Q}\rangle}}{\overset{s^0_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^1\right\rangle \underset{t^1_{\langle\hat{Q}\rangle}}{\overset{s^1_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\hat{Q}\rangle}}{\overset{s^{n-1}_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^n\right\rangle \underset{t^n_{\langle\hat{Q}\rangle}}{\overset{s^n_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots\right)$ defined by $x \mapsto x^{\varnothing^n}$ for $x \in Q^n$ and $n \in \mathbb{N}_0$.

Assume that there exists a morphism $f : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots\right) \to \left(\left\langle\hat{R}^0\right\rangle \underset{t^0_{\langle\hat{R}\rangle}}{\overset{s^0_{\langle\hat{R}\rangle}}{\leftleftarrows}} \left\langle\hat{R}^1\right\rangle \underset{t^1_{\langle\hat{R}\rangle}}{\overset{s^1_{\langle\hat{R}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\hat{R}\rangle}}{\overset{s^{n-1}_{\langle\hat{R}\rangle}}{\leftleftarrows}} \left\langle\hat{R}^n\right\rangle \underset{t^n_{\langle\hat{R}\rangle}}{\overset{s^n_{\langle\hat{R}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\hat{R}\rangle})_{n\in\mathbb{N}_0}, (\hat{\circ}^n_p)_{0\le p<n\in\mathbb{N}}, (\hat{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right)$ of reflexive ω-globular sets into another reflexive self-dual globular ω-magma.

The only choice of morphism of self-dual globular ω-magmas satisfying the universal factorization property is given by the following.

Then we define a function $\phi : \left(\left\langle\hat{Q}^0\right\rangle \underset{t^0_{\langle\hat{Q}\rangle}}{\overset{s^0_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^1\right\rangle \underset{t^1_{\langle\hat{Q}\rangle}}{\overset{s^1_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\hat{Q}\rangle}}{\overset{s^{n-1}_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \left\langle\hat{Q}^n\right\rangle \underset{t^n_{\langle\hat{Q}\rangle}}{\overset{s^n_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots\right) \to$

$$\left(\langle \hat{R}^0\rangle \underset{t^0_{\langle \hat{R}\rangle}}{\overset{s^0_{\langle \hat{R}\rangle}}{\leftleftarrows}} \langle \hat{R}^1\rangle \underset{t^1_{\langle \hat{R}\rangle}}{\overset{s^1_{\langle \hat{R}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle \hat{R}\rangle}}{\overset{s^{n-1}_{\langle \hat{R}\rangle}}{\leftleftarrows}} \langle \hat{R}^n\rangle \underset{t^n_{\langle \hat{R}\rangle}}{\overset{s^n_{\langle \hat{R}\rangle}}{\leftleftarrows}} \cdots \right) \text{ recursively by}$$

$$\langle \hat{Q}^n[1]\rangle \ni \big((\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big) \mapsto (\cdots(f(x)^{\hat{*}^n_{\alpha_1}})^{\cdots})^{\hat{*}^n_{\alpha_m}},$$

$$\langle \hat{Q}^n[2]\rangle \ni \Big(\cdots\Big(\Big(\big((\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big), p, \big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_l^n}\big)\Big)^{\delta_1^n}\Big)^{\cdots}\Big)^{\delta_t^n}$$

$$\mapsto \Big(\cdots\Big(\Big(\Big((\cdots(f(x)^{\hat{*}^n_{\alpha_1}})^{\cdots})^{\hat{*}^n_{\alpha_m}}\,\hat{\circ}^n_p(\cdots(f(y)^{\hat{*}^n_{\beta_1}})^{\cdots})^{\hat{*}^n_{\beta_l}}\Big)^{\hat{*}^n_{\delta_1}}\Big)^{\cdots}\Big)^{\hat{*}^n_{\delta_t}},$$

$$\vdots$$

Next, we will show that, for each $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \langle \hat{Q}^n\rangle$,

$$\phi\Big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\Big) = \Big(\cdots\Big(\Big(\phi(x)\hat{\circ}^n_p\phi(y)\Big)^{\hat{*}^n_{\alpha_1}}\Big)^{\cdots}\Big)^{\hat{*}^n_{\alpha_m}}.$$

Note that if $(\cdots((x,p,y)^{\delta_1^n})^{\cdots})^{\delta_t^n} \in \langle \hat{Q}^n\rangle$, then $x \in \langle \hat{Q}^n[i]\rangle$ and $y \in \langle \hat{Q}^n[j]\rangle$, where $i+j=k$ for some $k \in \mathbb{N}\setminus\{1\}$.

Indeed, for every $(\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n} \in \langle \hat{Q}^n[1]\rangle$, we have

$$\begin{aligned} \phi\big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n})^{\alpha^n}\big) &= ((\cdots(f(y)^{\hat{*}^n_{\beta_1}})^{\cdots})^{\hat{*}^n_{\beta_m}})^{\hat{*}^n_{\alpha}} \\ &= \big(\phi((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n}\big)^{\hat{*}^n_{\alpha}}. \end{aligned}$$

If $i=j=1$, then there exist $a,b \in \hat{Q}^n$ and $\alpha_1^n,\ldots,\alpha_m^n,\beta_1^n,\ldots,\beta_l^n \subseteq \mathbb{N}_0$ such that $x = (\cdots(a^{\alpha_1^n})^{\cdots})^{\alpha_m^n}$ and $y = (\cdots(b^{\beta_1^n})^{\cdots})^{\beta_l^n}$.

We see that $\phi\Big(\big(\cdots\big(\big(\big((\cdots(a^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big), p, \big((\cdots(b^{\beta_1^n})^{\cdots})^{\beta_l^n}\big)\big)^{\delta_1^n}\big)^{\cdots}\big)^{\delta_t^n}\Big)$

$$= \Big(\cdots\Big(\Big(\Big((\cdots(f(a)^{\hat{*}^n_{\alpha_1}})^{\cdots})^{\hat{*}^n_{\alpha_m}}\,\hat{\circ}^n_p(\cdots(f(b)^{\hat{*}^n_{\beta_1}})^{\cdots})^{\hat{*}^n_{\beta_l}}\Big)^{\hat{*}^n_{\delta_1}}\Big)^{\cdots}\Big)^{\hat{*}^n_{\delta_t}}$$

$$= \Big(\cdots\Big(\Big(\Big(\phi\big(((\cdots(a^{\alpha_1^n})^{\cdots})^{\alpha_m^n})\big)\hat{\circ}^n_p\phi\big(((\cdots(b^{\beta_1^n})^{\cdots})^{\beta_l^n})\big)\Big)^{\hat{*}^n_{\delta_1}}\Big)^{\cdots}\Big)^{\hat{*}^n_{\delta_t}}.$$

Suppose that this equation holds for $i=1$ and $j=1,2,\ldots,n-1$.

We have $\phi\Big(\big(\cdots\big(\big(\big((\cdots(a^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big), p, y\big)^{\delta_1^n}\big)^{\cdots}\big)^{\delta_t^n}\Big)$

$$= \Big(\cdots\Big(\Big(\Big((\cdots(f(a)^{\hat{*}^n_{\alpha_1}})^{\cdots})^{\hat{*}^n_{\alpha_m}}\,\hat{\circ}^n_p\phi(y)\Big)^{\hat{*}^n_{\delta_1}}\Big)^{\cdots}\Big)^{\hat{*}^n_{\delta_t}}$$

$$= \Big(\cdots\Big(\Big(\Big(\phi\big(((\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n})\big)\hat{\circ}^n_p\phi(y)\Big)^{\hat{*}^n_{\delta_1}}\Big)^{\cdots}\Big)^{\hat{*}^n_{\delta_t}}.$$

Now assume that the equation holds for $i = 1,2,\dots,l-1$ and $j \in \mathbb{N}$.

We have $\phi\Big(\big(\cdots((x,p,y)^{\delta_1^n})^{\cdots}\big)^{\delta_t^n}\Big) = \Big(\cdots\Big(\Big(\phi(x)\hat{\circ}_p^n\phi(y)\Big)^{\hat{*}_{\delta_1}^n}\Big)^{\cdots}\Big)^{\hat{*}_{\delta_t}^n}$.

This yields that ϕ is a unique morphism of reflexive self-dual globular ω-magmas such that $f = \phi \circ i$.

Therefore,

$$\left(\left(\langle\hat{Q}^0\rangle \underset{t^0_{\langle\hat{Q}\rangle}}{\overset{s^0_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle\hat{Q}\rangle}}{\overset{s^{n-1}_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \langle\hat{Q}^n\rangle \underset{t^n_{\langle\hat{Q}\rangle}}{\overset{s^n_{\langle\hat{Q}\rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle\hat{Q}\rangle})_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right), i\right)$$

is a free reflexive self-dual globular ω-magma over a reflexive ω-globular set. $\square$

Proposition 3.1.3.2. *A free reflexive self-dual globular* ω*-magma over an* ω*-globular set exists.*

Proof. Let $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$ be an ω-globular set.

First, we introduce the following notation, for every $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$,

$$\hat{Q}^n := \{(\cdots(y^{\beta_1^n})^{\cdots})^{\beta_m^n} \mid y \in Q^n,\ \beta_j^n \subseteq \mathbb{N}_0,\ j = 1,2,\dots,m\}.$$

Then we establish a new ω-globular set as follows: $\left\langle\bar{\hat{Q}}^0\right\rangle := \hat{Q}^0$,

$$\left(\widehat{\left\langle\bar{\hat{Q}}^0\right\rangle}\right)^1 := \{(\cdots(x,0)^{\beta_1^1})^{\cdots})^{\beta_m^1} \mid x \in \left\langle\bar{\hat{Q}}^0\right\rangle,\ \beta_j^1 \subseteq \mathbb{N}_0,\ j = 1,2,\dots,m\}.$$

Set $\left\langle\bar{\hat{Q}}^1[1]\right\rangle := \hat{Q}^1 \cup \left(\widehat{\left\langle\bar{\hat{Q}}^0\right\rangle}\right)^1$ and define $s^0[1], t^0[1] : \left\langle\bar{\hat{Q}}^1[1]\right\rangle \to \left\langle\bar{\hat{Q}}^0\right\rangle$ by

- $$s^0[1]\big((\cdots(y^{\beta_1^1})^{\cdots})^{\beta_m^1}\big) := \begin{cases} (\cdots((s^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & y \in Q^1,\ 0 \notin \beta_1^1 \triangle \cdots \triangle \beta_m^1; \\ (\cdots((t^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & y \in Q^1,\ 0 \in \beta_1^1 \triangle \cdots \triangle \beta_m^1; \\ (\cdots(x^{\beta_1^0})^{\cdots})^{\beta_m^0}, & y = (x,0),\ x \in \left\langle\bar{\hat{Q}}^0\right\rangle. \end{cases}$$

- $$t^0[1]\big((\cdots(y^{\beta_1^1})^{\cdots})^{\beta_m^1}\big) := \begin{cases} (\cdots((t^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & y \in Q^1,\ 0 \notin \beta_1^1 \triangle \cdots \triangle \beta_m^1; \\ (\cdots((s^0(y))^{\beta_1^0})^{\cdots})^{\beta_m^0}, & y \in Q^1,\ 0 \in \beta_1^1 \triangle \cdots \triangle \beta_m^1; \\ (\cdots(x^{\beta_1^0})^{\cdots})^{\beta_m^0}, & y = (x,0),\ x \in \left\langle\bar{\hat{Q}}^0\right\rangle. \end{cases}$$

Now let $\left\langle \bar{\hat{Q}}^1[2] \right\rangle := \{(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mid x,y \in \left\langle \bar{\hat{Q}}^1[1] \right\rangle,\ \alpha_j^1 \subseteq \mathbb{N}_0,$

$$j = 1,2,\ldots,m,\ s^0[1](x) = t^0[1](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \in \left\langle \bar{\hat{Q}}^1[2] \right\rangle$, we define $s^0[2], t^0[2] : \left\langle \bar{\hat{Q}}^1[2] \right\rangle \to \left\langle \bar{\hat{Q}}^0 \right\rangle$ by

- $s^0[2]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((s^0[1](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((t^0[1](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

- $t^0[2]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((t^0[1](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((s^0[1](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

Suppose that we have $\left\langle \bar{\hat{Q}}^1[l] \right\rangle$, $s^0[l]$, and $t^0[l]$ for every $l = 1,2,\ldots,k-1$.

Let $\left\langle \bar{\hat{Q}}^1[k] \right\rangle := \{(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mid x \in \left\langle \bar{\hat{Q}}^1[i] \right\rangle,\ y \in \left\langle \bar{\hat{Q}}^1[j] \right\rangle,\ i+j=k,$

$$\alpha_h^1 \subseteq \mathbb{N}_0,\ h = 1,2,\ldots,m,\ s^0[i](x) = t^0[j](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \in \left\langle \bar{\hat{Q}}^1[k] \right\rangle$, we define $s^0[k], t^0[k] : \left\langle \bar{\hat{Q}}^1[k] \right\rangle \to \left\langle \bar{\hat{Q}}^0 \right\rangle$ by

- $s^0[k]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((s^0[j](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((t^0[j](y))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

- $t^0[k]\big((\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}\big) := \begin{cases} (\cdots((t^0[i](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \notin \alpha_1^1 \triangle \cdots \triangle \alpha_m^1; \\ (\cdots((s^0[i](x))^{\alpha_1^0})^{\cdots})^{\alpha_m^0}, & 0 \in \alpha_1^1 \triangle \cdots \triangle \alpha_m^1. \end{cases}$

Set $\left\langle \bar{\hat{Q}}^1 \right\rangle := \dot{\bigcup}_{k=1}^{\infty} \left\langle \bar{\hat{Q}}^1[k] \right\rangle$, $s^0_{\langle \bar{\hat{Q}} \rangle} := \dot{\bigcup}_{k=1}^{\infty} s^0[k]$, and $t^0_{\langle \bar{\hat{Q}} \rangle} := \dot{\bigcup}_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle \hat{Q}^r \rangle$, $s^{r-1}_{\langle \bar{\hat{Q}} \rangle}$, and $t^{r-1}_{\langle \bar{\hat{Q}} \rangle}$ for every $r = 1,2,\ldots,n$.

Let $\left\langle \bar{\hat{Q}}^{n+1}[1] \right\rangle := \hat{Q}^{n+1} \,\dot{\cup}\, \widehat{\left(\left\langle \bar{\hat{Q}}^n \right\rangle\right)}^{n+1}$.

Define $s^n[1], t^n[1] : \left\langle \bar{\hat{Q}}^{n+1}[1] \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by

$$s^n[1]\big((\cdots(y^{\beta_1^{n+1}})^{\cdots})^{\beta_m^{n+1}}\big) := \begin{cases} (\cdots((s^n(y))^{\beta_1^n})^{\cdots})^{\beta_m^n}, & y \in Q^{n+1},\ n \notin \beta_1^{n+1} \triangle \cdots \triangle \beta_m^{n+1}; \\ (\cdots((t^n(y))^{\beta_1^n})^{\cdots})^{\beta_m^n}, & y \in Q^{n+1},\ n \in \beta_1^{n+1} \triangle \cdots \triangle \beta_m^{n+1}; \\ (\cdots(x^{\beta_1^n})^{\cdots})^{\beta_m^n}, & y = (x,n),\ x \in \left\langle \bar{\hat{Q}}^n \right\rangle. \end{cases}$$

$$t^n[1]\big((\cdots(y^{\beta_1^{n+1}})^{\cdots})^{\beta_m^{n+1}}\big) := \begin{cases} (\cdots((t^n(y))^{\beta_1^n})^{\cdots})^{\beta_m^n}, & y \in Q^{n+1},\ n \notin \beta_1^{n+1} \triangle \cdots \triangle \beta_m^{n+1}; \\ (\cdots((s^n(y))^{\beta_1^n})^{\cdots})^{\beta_m^n}, & y \in Q^{n+1},\ n \in \beta_1^{n+1} \triangle \cdots \triangle \beta_m^{n+1}; \\ (\cdots(x^{\beta_1^n})^{\cdots})^{\beta_m^n}, & y = (x,n),\ x \in \left\langle \bar{\hat{Q}}^n \right\rangle. \end{cases}$$

Now let $\left\langle \bar{\hat{Q}}^{n+1}[2] \right\rangle := \dot{\bigcup}_{p=0}^{n} \{(\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \mid x,y \in \langle \hat{Q}^{n+1}[1] \rangle,$

$\alpha_j^{n+1} \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m,\ s^p[1]\cdots s^n[1](x) = t^p[1]\cdots t^n[1](y)\}.$

Define $s^n[2], t^n[2] : \left\langle \bar{\hat{Q}}^{n+1}[2] \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by

$$s^n[2]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$$
$$:= \begin{cases} (\cdots((s^n[1](x),p,s^n[1](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n[1](x),p,t^n[1](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((s^n[1](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p = n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n[1](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p = n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}. \end{cases}$$

$$t^n[2]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$$
$$:= \begin{cases} (\cdots((t^n[1](x),p,t^n[1](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((s^n[1](x),p,s^n[1](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n[1](x))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p = n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((s^n[1](x))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p = n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}. \end{cases}$$

Suppose that we have $\left\langle \bar{\hat{Q}}^{n+1}[l] \right\rangle$, $s^n[l]$, and $t^n[l]$ for every $l = 1,2,\ldots,k-1$.

Let $\left\langle \bar{\hat{Q}}^{n+1}[k] \right\rangle := \dot{\bigcup}_{p=0}^{n} \{(\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}} \mid x \in \langle \hat{Q}^{n+1}[i] \rangle,\ y \in \langle \hat{Q}^{n+1}[j] \rangle,$

$i+j=k,\ \alpha_h^{n+1} \subseteq \mathbb{N}_0,\ h = 1,2,\ldots,m,\ s^p[i]\cdots s^n[i](x) = t^p[j]\cdots t^n[j](y)\}.$

Define $s^n[k], t^n[k] : \left\langle \bar{\hat{Q}}^{n+1}[k] \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by

$$s^n[k]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$$
$$:= \begin{cases} (\cdots((s^n[i](x),p,s^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p < n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n[i](x),p,t^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p < n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((s^n[i](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p = n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n[i](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p = n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}. \end{cases}$$

$$
\begin{aligned}
& t^n[k]\big((\cdots((x,p,y)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) \\
:= & \begin{cases}
(\cdots((t^n[i](x),p,t^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p<n\notin \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}; \\
(\cdots((s^n[i](x),p,s^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p<n\in \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}; \\
(\cdots((t^n[i](x))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p=n\notin \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}; \\
(\cdots((s^n[i](x))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & p=n\in \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}.
\end{cases}
\end{aligned}
$$

Set $\left\langle \bar{\hat{Q}}^{n+1}\right\rangle := \dot{\bigcup}_{k=1}^{\infty}\left\langle \bar{\hat{Q}}^{n+1}[k]\right\rangle$, $s^n_{\left\langle \bar{\hat{Q}}\right\rangle} := \dot{\bigcup}_{k=1}^{\infty} s^n[k]$, and $t^n_{\left\langle \bar{\hat{Q}}\right\rangle} := \dot{\bigcup}_{k=1}^{\infty} t^n[k]$.

We see that $s^{n-1}[1]s^n[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$

$$
= \begin{cases}
(\cdots((s^{n-1}s^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x\in Q^{n+1},\ n\notin \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}, \\
 & n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\
(\cdots((t^{n-1}s^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x\in Q^{n+1},\ n\notin \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}, \\
 & n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\
(\cdots((s^{n-1}t^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x\in Q^{n+1},\ n\in \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}, \\
 & n-1\notin \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\
(\cdots((t^{n-1}t^n(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x\in Q^{n+1},\ n\in \alpha_1^{n+1}\triangle\cdots\triangle\alpha_m^{n+1}, \\
 & n-1\in \alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\
(\cdots(((\cdots((s^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_t^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x=\big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_t^n},n\big),\ y\in Q^n, \\
 & n-1\in \beta_1^n\triangle\cdots\triangle\beta_t^n\triangle\alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\
(\cdots(((\cdots((t^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_t^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x=\big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_t^n},n\big),\ y\in Q^n, \\
 & n-1\notin \beta_1^n\triangle\cdots\triangle\beta_t^n\triangle\alpha_1^n\triangle\cdots\triangle\alpha_m^n; \\
(\cdots(((\cdots(z^{\beta_1^{n-1}})^{\cdots})^{\beta_t^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x=\big((\cdots((z,n-1)^{\beta_1^n})^{\cdots})^{\beta_t^n},n\big), \\
 & z\in\left\langle \bar{\hat{Q}}^{n-1}\right\rangle.
\end{cases}
$$

$$
= \begin{cases}
(\cdots((s^{n-1}t^{n}(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x \in Q^{n+1},\ n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots((t^{n-1}t^{n}(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x \in Q^{n+1},\ n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots((s^{n-1}s^{n}(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x \in Q^{n+1},\ n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & n-1 \notin \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots((t^{n-1}s^{n}(x))^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x \in Q^{n+1},\ n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}, \\
 & n-1 \in \alpha_1^{n} \triangle \cdots \triangle \alpha_m^{n}; \\
(\cdots(((\cdots((s^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_t^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x = \big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_t^n}, n\big),\ y \in Q^n, \\
 & n-1 \in \beta_1^n \triangle \cdots \triangle \beta_t^n \triangle \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\
(\cdots(((\cdots((t^{n-1}(y))^{\beta_1^{n-1}})^{\cdots})^{\beta_t^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x = \big((\cdots(y^{\beta_1^n})^{\cdots})^{\beta_t^n}, n\big),\ y \in Q^n, \\
 & n-1 \notin \beta_1^n \triangle \cdots \triangle \beta_t^n \triangle \alpha_1^n \triangle \cdots \triangle \alpha_m^n; \\
(\cdots(((\cdots(z^{\beta_1^{n-1}})^{\cdots})^{\beta_t^{n-1}})^{\alpha_1^{n-1}})^{\cdots})^{\alpha_m^{n-1}}, & x = \big((\cdots((z, n-1)^{\beta_1^n})^{\cdots})^{\beta_t^n}, n\big), \\
 & z \in \left\langle \bar{\hat{Q}}^{n-1} \right\rangle.
\end{cases}
$$

$$
= s^{n-1}[1]s^{n}[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big).
$$

Similarly, $t^{n-1}[1]s^{n}[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) = t^{n-1}[1]t^{n}[1]\big((\cdots(x^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big)$.

For the induction step, we refer to the previous proposition.

Thus, $\left\langle \bar{\hat{Q}}^0 \right\rangle \overset{s^0_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^0_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^1 \right\rangle \overset{s^1_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^1_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^n \right\rangle \overset{s^n_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^n_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots$ is an ω-globular set.

For the reflexivity, we proceed as follows.

First, we define $\iota^0_{\langle \bar{\hat{Q}} \rangle} : \left\langle \bar{\hat{Q}}^0 \right\rangle \to \left\langle \bar{\hat{Q}}^1 \right\rangle$ by $(\cdots(x^{\alpha_1^0})^{\cdots})^{\alpha_m^0} \mapsto (\cdots((x,0)^{\alpha_1^1})^{\cdots})^{\alpha_m^1}$.

Then define $\iota^1[1] : \left\langle \bar{\hat{Q}}^1[1] \right\rangle \to \left\langle \bar{\hat{Q}}^2[1] \right\rangle$ by $(\cdots(x^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mapsto (\cdots((x,1)^{\alpha_1^2})^{\cdots})^{\alpha_m^2}$.

Assume that we have $\iota^1[h]$ for all $h = 1, 2, \ldots, k-1$.

If $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \in \left\langle \bar{\hat{Q}}^1[k] \right\rangle$, where $x \in \left\langle \bar{\hat{Q}}^1[i] \right\rangle$ and $y \in \left\langle \bar{\hat{Q}}^1[j] \right\rangle$, define $\iota^1[k] : \left\langle \bar{\hat{Q}}^1[k] \right\rangle \to \left\langle \bar{\hat{Q}}^2[k] \right\rangle$ by $(\cdots((x,0,y)^{\alpha_1^1})^{\cdots})^{\alpha_m^1} \mapsto (\cdots((\iota^1[i](x), 0, \iota^1[j](y))^{\alpha_1^2})^{\cdots})^{\alpha_m^2}$.

Now we consider $\iota^1_{\langle \bar{\hat{Q}} \rangle} := \dot{\bigcup_{k=1}^{\infty}} \iota^1[k]$.

Suppose that we have $\iota^l_{\langle \bar{\hat{Q}} \rangle}$ for every $l = 0, 1, \dots, n-1$.

Define $\iota^n[1] : \left\langle \bar{\hat{Q}}^n[1] \right\rangle \to \left\langle \bar{\hat{Q}}^{n+1}[1] \right\rangle$ by $(\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mapsto (\cdots((x,n)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}$.

If $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \left\langle \bar{\hat{Q}}^n[k] \right\rangle$, where $x \in \left\langle \bar{\hat{Q}}^n[i] \right\rangle$ and $y \in \left\langle \bar{\hat{Q}}^n[j] \right\rangle$, we define $\iota^n[k] : \left\langle \bar{\hat{Q}}^n[k] \right\rangle \to \left\langle \bar{\hat{Q}}^{n+1}[k] \right\rangle$ by

$$(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mapsto (\cdots((\iota^n[i](x), p, \iota^n[j](y))^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}.$$

For each $(\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \left\langle \bar{\hat{Q}}^n[1] \right\rangle$, we have

$$s^n[1] \circ \iota^1[1]\big((\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big) = s^n[1]\big((\cdots((x,n)^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}} = (\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n}.$$

This means that $s^n[1] \circ \iota^n[1] = \mathrm{Id}_{\langle \bar{\hat{Q}}^n[1] \rangle}$.

Similarly, we have $t^n[1] \circ \iota^n[1] = \mathrm{Id}_{\langle \bar{\hat{Q}}^n[1] \rangle}$.

Assume that $s^n[h] \circ \iota^n[h] = \mathrm{Id}_{\langle \bar{\hat{Q}}^n[h] \rangle} = t^n[h] \circ \iota^n[h]$ for all $h = 1, 2, \dots, k-1$.

For every $(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \in \left\langle \bar{\hat{Q}}^n[k] \right\rangle$ with $x \in \left\langle \bar{\hat{Q}}^n[i] \right\rangle$ and $y \in \left\langle \bar{\hat{Q}}^n[j] \right\rangle$,

$$\begin{aligned}
& s^n[k] \circ \iota^n[k]\big((\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}\big) \\
= \ & s^n[k]\big((\cdots((\iota^n[i](x), p, \iota^n[j](y))^{\alpha_1^{n+1}})^{\cdots})^{\alpha_m^{n+1}}\big) \\
= \ & \begin{cases} (\cdots((s^n[i] \circ \iota^n[i](x), p, s^n[j] \circ \iota^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & n \notin \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}; \\ (\cdots((t^n[i] \circ \iota^n[i](x), p, t^n[j] \circ \iota^n[j](y))^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, & n \in \alpha_1^{n+1} \triangle \cdots \triangle \alpha_m^{n+1}. \end{cases} \\
= \ & (\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n}.
\end{aligned}$$

It follows that $s^n[k] \circ \iota^n[k] = \mathrm{Id}_{\langle \bar{\hat{Q}}^n[k] \rangle}$.

Similarly, $t^n[k] \circ \iota^n[k] = \mathrm{Id}_{\langle \bar{\hat{Q}}^n[k] \rangle}$.

This means that $s^n_{\langle \bar{\hat{Q}} \rangle} \circ \iota^n_{\langle \bar{\hat{Q}} \rangle} = \mathrm{Id}_{\langle \bar{\hat{Q}}^n \rangle} = t^n_{\langle \bar{\hat{Q}} \rangle} \circ \iota^n_{\langle \bar{\hat{Q}} \rangle}$.

That is, $\left(\left\langle \bar{\hat{Q}}^0 \right\rangle \underset{t^0_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^0_{\langle \bar{\hat{Q}} \rangle}}{\Leftarrow}} \left\langle \bar{\hat{Q}}^1 \right\rangle \underset{t^1_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^1_{\langle \bar{\hat{Q}} \rangle}}{\Leftarrow}} \cdots \underset{t^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\Leftarrow}} \left\langle \bar{\hat{Q}}^n \right\rangle \underset{t^n_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^n_{\langle \bar{\hat{Q}} \rangle}}{\Leftarrow}} \cdots, (\iota^n_{\langle \bar{\hat{Q}} \rangle})_{n \in \mathbb{N}_0} \right)$ becomes a reflexive ω-globular set.

For $p \in \mathbb{N}_0$, set $\left\langle \bar{\hat{Q}}^n \right\rangle \times_p \left\langle \bar{\hat{Q}}^n \right\rangle := \{(x,y) \in \left\langle \bar{\hat{Q}}^n \right\rangle \times \left\langle \bar{\hat{Q}}^n \right\rangle \mid s^p_{\langle \bar{\hat{Q}} \rangle}(x) = t^p_{\langle \bar{\hat{Q}} \rangle}(y)\}$.

Define a family of operations $\circ^n_p : \left\langle \bar{\hat{Q}}^n \right\rangle \times_p \left\langle \bar{\hat{Q}}^n \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by $x \circ^n_p y \mapsto (x, p, y)$. For each $\alpha^n \subseteq \mathbb{N}_0$, define $\bar{*}^n_\alpha : \left\langle \bar{\hat{Q}}^n \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by

$$(\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m} \mapsto ((\cdots((x^{\alpha^n_1})^{\alpha^n_2})^{\cdots})^{\alpha^n_m})^{\alpha^n}.$$

This yields that we have a reflexive self-dual globular ω-magma

$$\left(\left\langle \bar{\hat{Q}}^0 \right\rangle \overset{s^0_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^0_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^n \right\rangle \overset{s^n_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^n_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle \bar{\hat{Q}} \rangle})_{n \in \mathbb{N}_0}, (\circ^n_p)_{0 \le p < n \in \mathbb{N}}, (*^n_\alpha)_{\alpha \subseteq \mathbb{N}_0, n \in \mathbb{N}_0} \right).$$

Now we define a map $i : \left(Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} \cdots \right) \to$ $\left(\left\langle \bar{\hat{Q}}^0 \right\rangle \overset{s^0_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^0_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^1 \right\rangle \overset{s^1_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^1_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^n \right\rangle \overset{s^n_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^n_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \right)$ by $x \mapsto x$ for $x \in Q^n$ and $n \in \mathbb{N}_0$.

Suppose that there exists a morphism $f : \left(Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} \cdots \right) \to$ $\left(\left\langle \bar{\hat{R}}^0 \right\rangle \overset{s^0_{\langle \bar{\hat{R}} \rangle}}{\underset{t^0_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{R}}^1 \right\rangle \overset{s^1_{\langle \bar{\hat{R}} \rangle}}{\underset{t^1_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle \bar{\hat{R}} \rangle}}{\underset{t^{n-1}_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{R}}^n \right\rangle \overset{s^n_{\langle \bar{\hat{R}} \rangle}}{\underset{t^n_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \cdots, (\iota^n_{\langle \bar{\hat{R}} \rangle})_{n \in \mathbb{N}_0}, (\hat{\circ}^n_p)_{0 \le p < n \in \mathbb{N}}, (\hat{*}^n_\alpha)_{\alpha \subseteq \mathbb{N}_0, n \in \mathbb{N}_0} \right)$ into another reflexive self-dual globular ω-magma.

The only choice of morphism of reflexive self-dual globular ω-magmas holding its universal factorization property is given by the following.

We define a function $\phi : \left(\left\langle \bar{\hat{Q}}^0 \right\rangle \overset{s^0_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^0_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^1 \right\rangle \overset{s^1_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^1_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^n \right\rangle \overset{s^n_{\langle \bar{\hat{Q}} \rangle}}{\underset{t^n_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \right) \to$ $\left(\left\langle \bar{\hat{R}}^0 \right\rangle \overset{s^0_{\langle \bar{\hat{R}} \rangle}}{\underset{t^0_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{R}}^1 \right\rangle \overset{s^1_{\langle \bar{\hat{R}} \rangle}}{\underset{t^1_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\langle \bar{\hat{R}} \rangle}}{\underset{t^{n-1}_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{R}}^n \right\rangle \overset{s^n_{\langle \bar{\hat{R}} \rangle}}{\underset{t^n_{\langle \bar{\hat{R}} \rangle}}{\leftleftarrows}} \cdots \right)$ by, for any $x \in Q^0$, $y \in Q^1$, and $u, v \in Q^n$,

- $(\cdots((x^{\alpha^0_1})^{\alpha^0_2})^{\cdots})^{\alpha^0_m} \mapsto (\cdots(((f(x))^{\hat{*}^0_{\alpha_1}})^{\hat{*}^0_{\alpha_2}})^{\cdots})^{\hat{*}^0_{\alpha_m}}$, and in particular, $x^{\varnothing^0} \mapsto f(x)$,
- $(\cdots(((\cdots(x^{\beta^0_1})^{\cdots})^{\beta^0_k}, 0)^{\alpha^1_1})^{\cdots})^{\alpha^1_m} \mapsto (\cdots((\iota^0_{\langle \bar{\hat{R}} \rangle}((\cdots((f(x))^{\hat{*}^0_{\beta_1}})^{\cdots})^{\hat{*}^0_{\beta_k}})^{\hat{*}^1_{\alpha_1}})^{\cdots})^{\hat{*}^1_{\alpha_m}}$,
- $(\cdots((y^{\alpha^1_1})^{\alpha^1_2})^{\cdots})^{\alpha^1_m} \mapsto (\cdots(((f(y))^{\hat{*}^1_{\alpha_1}})^{\hat{*}^1_{\alpha_2}})^{\cdots})^{\hat{*}^1_{\alpha_m}}$, and in particular, $y^{\varnothing^1} \mapsto f(y)$,

- $(\cdots(((\cdots(y^{\beta_1^1})^{\cdots})^{\beta_k^1}, 1)^{\alpha_1^2})^{\cdots})^{\alpha_m^2} \mapsto (\cdots((\iota^1_{\langle\hat{\bar{R}}\rangle}((\cdots((f(y))^{\hat{*}_{\beta_1}^1})^{\cdots})^{\hat{*}_{\beta_k}^1}))^{\hat{*}_{\alpha_1}^2})^{\cdots})^{\hat{*}_{\alpha_m}^2}$,
- $(\cdots(((\cdots(((\cdots(x^{\gamma_1^0})^{\cdots})^{\gamma_t^0}, 0)^{\beta_1^1})^{\cdots})^{\beta_k^1}, 1)^{\alpha_1^2})^{\cdots})^{\alpha_m^2} \mapsto$
  $(\cdots((\iota^1_{\langle\hat{\bar{R}}\rangle}((\cdots((\iota^0_{\langle\hat{\bar{R}}\rangle}((\cdots((f(x))^{\hat{*}_{\gamma_1}^0})^{\cdots})^{\hat{*}_{\gamma_t}^0}))^{\hat{*}_{\beta_1}^1})^{\cdots})^{\hat{*}_{\beta_k}^1}))^{\hat{*}_{\alpha_1}^2})^{\cdots})^{\hat{*}_{\alpha_m}^2}$,

$$\vdots$$

- $\left(\cdots\left(\left((\cdots(x^{\alpha_1^n})^{\cdots})^{\alpha_m^n}, p, (\cdots(y^{\beta_1^n})^{\cdots})^{\beta_k^n}\right)^{\delta_1^n}\right)^{\cdots}\right)^{\delta_t^n} \mapsto$
  $\left(\cdots\left(\left(\left((\cdots(f(x)^{\hat{*}_{\alpha_1^n}})^{\cdots})^{\hat{*}_{\alpha_m^n}} \hat{\circ}_p (\cdots(f(y)^{\hat{*}_{\beta_1^n}})^{\cdots})^{\hat{*}_{\beta_k^n}}\right)^{\hat{*}_{\delta_1^n}}\right)^{\cdots}\right)^{\hat{*}_{\delta_t^n}}$,

$$\vdots$$

Combining the proofs of Proposition 3.1.2.2 and Proposition 3.1.1.3, we get that ϕ is a unique morphism of reflexive self-dual globular ω-magmas such that $f = \phi \circ i$.

Therefore,

$$\left(\left(\left\langle \hat{\bar{Q}}^0 \right\rangle \underset{t^0_{\langle\hat{\bar{Q}}\rangle}}{\overset{s^0_{\langle\hat{\bar{Q}}\rangle}}{\Leftarrow}} \cdots \underset{t^{n-1}_{\langle\hat{\bar{Q}}\rangle}}{\overset{s^{n-1}_{\langle\hat{\bar{Q}}\rangle}}{\Leftarrow}} \left\langle \hat{\bar{Q}}^n \right\rangle \underset{t^n_{\langle\hat{\bar{Q}}\rangle}}{\overset{s^n_{\langle\hat{\bar{Q}}\rangle}}{\Leftarrow}} \cdots, (\iota^n_{\langle\hat{\bar{Q}}\rangle})_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right), i\right)$$

becomes a free reflexive self-dual globular ω-magma over an ω-globular set. □

3.2 Free Involutive Penon Contractions

In this section we will examine the notion of a *free involutive Penon contraction* over an ω-globular set applying the previous results. We begin with an investigation of a free strict involutive globular ω-category over an ω-globular set.

3.2.1 Free Strict Involutive Globular ω-Categories

Since the existence of a free reflexive self-dual globular ω-magma over an ω-globular set has been discussed, we will use this result to obtain a free strict globular ω-category via a specific *congruence*. First, we give a definition of congruence relation on a reflexive self-dual globular ω-magma as follows.

Lemma 3.2.1.1. *A product of reflexive self-dual globular* ω*-magmas is a reflexive self-dual globular* ω*-magma with pointwise operations.*

Proof. Let $\left(M^0 \underset{t_M^0}{\overset{s_M^0}{\leftleftarrows}} \cdots \underset{t_M^{n-1}}{\overset{s_M^{n-1}}{\leftleftarrows}} M^n \underset{t_M^n}{\overset{s_M^n}{\leftleftarrows}} \cdots, (\iota_M^n)_{n\in\mathbb{N}_0}, (\circ_p^n)_{0\le p<n\in\mathbb{N}}, (*_\alpha^n)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0} \right)$ and $\left(N^0 \underset{t_N^0}{\overset{s_N^0}{\leftleftarrows}} \cdots \underset{t_N^{n-1}}{\overset{s_N^{n-1}}{\leftleftarrows}} N^n \underset{t_N^n}{\overset{s_N^n}{\leftleftarrows}} \cdots, (\iota_N^n)_{n\in\mathbb{N}_0}, (\hat{\circ}_p^n)_{0\le p<n\in\mathbb{N}}, (\hat{*}_\alpha^n)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0} \right)$ be two reflexive self-dual globular ω-magmas.

For each M^n and N^n, we consider a product $M^n \times N^n$.

Define $s_{M\times N}^n, t_{M\times N}^n : M^{n+1}\times N^{n+1} \to M^n \times N^n$, for all $(x,y)\in M^{n+1}\times N^{n+1}$, by $s_{M\times N}^n\big((x,y)\big) := \big(s_M^n(x), s_N^n(y)\big)$ and $t_{M\times N}^n\big((x,y)\big) := \big(t_M^n(x), t_N^n(y)\big)$.

For each $(x,y)\in M^{n+1}\times N^{n+1}$, we have

$$\begin{aligned} s_{M\times N}^{n-1}s_{M\times N}^n\big((x,y)\big) &= \big(s_M^{n-1}s_M^n(x), s_N^{n-1}s_N^n(y)\big) \\ &= \big(s_M^{n-1}t_M^n(x), s_N^{n-1}t_N^n(y)\big) \\ &= s_{M\times N}^{n-1}t_{M\times N}^n\big((x,y)\big). \end{aligned}$$

This means that $s_{M\times N}^{n-1}s_{M\times N}^n = s_{M\times N}^{n-1}t_{M\times N}^n$.

Using a similar deduction, we get $t_{M\times N}^{n-1}s_{M\times N}^n = t_{M\times N}^{n-1}t_{M\times N}^n$.

Thus, $M^0\times N^0 \underset{t_{M\times N}^0}{\overset{s_{M\times N}^0}{\leftleftarrows}} \cdots \underset{t_{M\times N}^{n-1}}{\overset{s_{M\times N}^{n-1}}{\leftleftarrows}} M^n\times N^n \underset{t_{M\times N}^n}{\overset{s_{M\times N}^n}{\leftleftarrows}} \cdots$ is an ω-globular set.

In addition, define $\iota_{M\times N}^n : M^n\times N^n \to M^{n+1}\times N^{n+1}$ by $(x,y)\mapsto \big(\iota_M^n(x), \iota_N^n(y)\big)$.

It is easy to see that $s_{M\times N}^n \circ \iota_{M\times N}^n = \mathrm{Id}_{M^n\times N^n} = t_{M\times N}^n \circ \iota_{M\times N}^n$.

This implies that $\left(M^0\times N^0 \underset{t_{M\times N}^0}{\overset{s_{M\times N}^0}{\leftleftarrows}} \cdots \underset{t_{M\times N}^{n-1}}{\overset{s_{M\times N}^{n-1}}{\leftleftarrows}} M^n\times N^n \underset{t_{M\times N}^n}{\overset{s_{M\times N}^n}{\leftleftarrows}} \cdots, (\iota_{M\times N}^n)_{n\in\mathbb{N}_0} \right)$ becomes a reflexive ω-globular set.

Set $(M^n\times N^n)\times_p(M^n\times N^n) := \{\big((x,y),(a,b)\big) \mid (x,a),(y,b)\in M^n\times_p N^n\}$.

Defining $\bar{\circ}_p^n : (M^n\times N^n)\times_p(M^n\times N^n) \to M^n\times N^n$ by $\big((x,y),(a,b)\big)\mapsto (x\circ_p^n a, y\hat{\circ}_p^n b)$ and $\bar{*}_\alpha^n : M^n\times N^n \to M^n\times N^n$ by $(x,y)\mapsto (x^{*_\alpha^n}, y^{\hat{*}_\alpha^n})$ gives us compositions and self-dual operations in the new quiver.

That is,

$$\left(M^0\times N^0 \underset{t^0_{M\times N}}{\overset{s^0_{M\times N}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{M\times N}}{\overset{s^{n-1}_{M\times N}}{\leftleftarrows}} M^n\times N^n \underset{t^n_{M\times N}}{\overset{s^n_{M\times N}}{\leftleftarrows}} \cdots, (\iota^n_{M\times N})_{n\in\mathbb{N}_0}, (\bar{\circ}^n_p)_{0\le p<n\in\mathbb{N}}, (\bar{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right)$$

becomes a reflexive self-dual globular ω-magma. □

Definition 3.2.1.2. Given a reflexive self-dual globular ω-magma

$$\left(M^0 \underset{t^0_M}{\overset{s^0_M}{\leftleftarrows}} \cdots \underset{t^{n-1}_M}{\overset{s^{n-1}_M}{\leftleftarrows}} M^n \underset{t^n_M}{\overset{s^n_M}{\leftleftarrows}} \cdots, (\iota^n_M)_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right),$$

a **congruence relation** on M is a subquiver R of $M\times M$ such that $R^n\subseteq M^n\times M^n$ is an equivalence relation, for all $n\in\mathbb{N}_0$, such that

1. if $(x,y),(a,b)\in R^n$ and $(x,a),(y,b)\in M^n\times_p M^n$, then $\left(x\circ^n_p a, y\circ^n_p b\right)\in R^n$,
2. if $(x,y)\in R^n$, then $\left(x^{*^n_\alpha}, y^{*^n_\alpha}\right)\in R^n$,
3. if $(x,y)\in R^n$, then $(\iota^n_M(x),\iota^n_M(y))\in R^{n+1}$.

Proposition 3.2.1.3. *A free strict involutive globular ω-category over an ω-globular set exists.*

Proof. Recall that we have a free reflexive self-dual globular ω-magma

$$\left(\left(M^0 \underset{t^0_M}{\overset{s^0_M}{\leftleftarrows}} M^1 \underset{t^1_M}{\overset{s^1_M}{\leftleftarrows}} \cdots \underset{t^{n-1}_M}{\overset{s^{n-1}_M}{\leftleftarrows}} M^n \underset{t^n_M}{\overset{s^n_M}{\leftleftarrows}} \cdots, (\iota^n_M)_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\right), i\right)$$

over an ω-globular set $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$.

Next, we construct a congruence in M containing axioms that need to hold.

Since there are no axioms for $M^0\times M^0$, we define $\mathrm{Ax}^0 := \varnothing\subseteq M^0\times M^0$.

For every $n\in\mathbb{N}$, we define

$$\begin{aligned}\mathrm{Ax}^n := &\left\{\left((x\circ^n_p y)\circ^n_p z, x\circ^n_p(y\circ^n_p z)\right) \mid (x,y),(y,z)\in M^n\times_p M^n, p=0,1,\dots,n-1\right\}\\ &\cup\{(\iota^{n-1}\cdots\iota^p t^p\cdots t^{n-1}(x)\circ^n_p x, x) \mid x\in M^n,\ p=0,1,\dots,n-1\}\\ &\cup\{(x\circ^n_p \iota^{n-1}\cdots\iota^p s^p\cdots s^{n-1}(x), x) \mid x\in M^n,\ p=0,1,\dots,n-1\}\end{aligned}$$

$$\cup\{\big(\iota^n(x\circ_p^n y),\iota^n(x)\circ_p^{n+1}\iota^n(y)\big)\mid (x,y)\in M^n\times M^n, p=0,1,\dots,n-1\}$$

$$\cup\{\big((y'\circ_p^n y)\circ_q^n(x'\circ_p^n x),(y'\circ_q^n x')\circ_p^n(y\circ_q^n x)\big)\mid (y',y),(x',x)\in M^n\times_p M^n,$$

$$(y',x'),(y,x)\in M^n\times_q M^n\}\cup\{\big((x^{*^n_\alpha})^{*^n_\beta},(x^{*^n_\beta})^{*^n_\alpha}\big)\mid x\in M^n,\ \alpha,\beta\subseteq\mathbb{N}_0\}$$

$$\cup\{\big((x\circ_p^n y)^{*^n_\alpha},x^{*^n_\alpha}\circ_p^n y^{*^n_\alpha}\big)\mid (x,y)\in M^n\times_p M^n,\ \mathbb{N}_0\supseteq\alpha\not\ni p=0,1,\dots,n-1\}$$

$$\cup\{\big((x\circ_p^n y)^{*^n_\alpha},y^{*^n_\alpha}\circ_p^n x^{*^n_\alpha}\big)\mid (x,y)\in M^n\times_p M^n,\ \mathbb{N}_0\supseteq\alpha\ni p=0,1,\dots,n-1\}$$

$$\cup\{\iota^{n-1}(w^{*^{n-1}_\alpha}),(\iota^{n-1}(w))^{*^n_\alpha}\mid w\in M^{n-1},\alpha\subseteq\mathbb{N}_0\}$$

$$\cup\{((x^{*^n_\alpha})^{*^n_\alpha},x)\mid x\in M^n,\ \alpha\subseteq\mathbb{N}_0\}\subseteq M^n\times M^n.$$

Let R be the smallest congruence such that $\mathrm{Ax}^n\subseteq R^n$ for any $n\in\mathbb{N}_0$.

Restricting domains of compositions to $R^n\times_p R^n$ and of self-dual operations to R^n provides us a reflexive self-dual globular ω-magma $R^0\underset{t_R^0}{\overset{s_R^0}{\leftleftarrows}}R^1\underset{t_R^1}{\overset{s_R^1}{\leftleftarrows}}\cdots\underset{t_R^{n-1}}{\overset{s_R^{n-1}}{\leftleftarrows}}R^n\underset{t_R^n}{\overset{s_R^n}{\leftleftarrows}}\cdots$.

Set $M^n/R^n:=\{[x]_n\mid x\in M^n\}$, where $[x]_n:=\{y\in M^n\mid (x,y)\in R^n\}$, and define a family of maps $\pi^n:M^n\to M^n/R^n$ by $x\mapsto[x]_n$.

Define $s^n_{M/R},t^n_{M/R}:M^{n+1}/R^{n+1}\to M^n/R^n$ by $s^n_{M/R}\big([x]_{n+1}\big):=\big[s^n_M(x)\big]_n$ and $t^n_{M/R}\big([x]_{n+1}\big):=\big[t^n_M(x)\big]_n$.

Since $(x,y)\in R^{n+1}$ implies both $\big(s^n_M(x),s^n_M(y)\big)\in R^n$ and $\big(t^n_M(x),t^n_M(y)\big)\in R^n$, $s^n_{M/R}$ and $t^n_{M/R}$ are well-defined.

So, $M^0/R^0\underset{t^0_{M/R}}{\overset{s^0_{M/R}}{\leftleftarrows}}M^1/R^1\underset{t^1_{M/R}}{\overset{s^1_{M/R}}{\leftleftarrows}}\cdots\underset{t^{n-1}_{M/R}}{\overset{s^{n-1}_{M/R}}{\leftleftarrows}}M^n/R^n\underset{t^n_{M/R}}{\overset{s^n_{M/R}}{\leftleftarrows}}\cdots$ is an ω-globular set.

Moreover, we define $\iota^n_{M/R}:M^n/R^n\to M^{n+1}/R^{n+1}$ by $[x]_n\mapsto[\iota^n_M(x)]_{n+1}$.

It is easy to check that $s^{n-1}_{M/R}\circ s^n_{M/R}=s^{n-1}_{M/R}\circ t^n_{M/R}$, $t^{n-1}_{M/R}\circ s^n_{M/R}=t^{n-1}_{M/R}\circ t^n_{M/R}$, and $s^n_{M/R}\circ\iota^n_{M/R}=\mathrm{Id}_{M^n/R^n}=t^n_{M/R}\circ\iota^n_{M/R}$.

This yields that $\left(M^0/R^0\underset{t^0_{M/R}}{\overset{s^0_{M/R}}{\leftleftarrows}}M^1/R^1\underset{t^1_{M/R}}{\overset{s^1_{M/R}}{\leftleftarrows}}\cdots\underset{t^{n-1}_{M/R}}{\overset{s^{n-1}_{M/R}}{\leftleftarrows}}M^n/R^n\underset{t^n_{M/R}}{\overset{s^n_{M/R}}{\leftleftarrows}}\cdots,(\iota^n_{M/R})_{n\in\mathbb{N}_0}\right)$ is a reflexive ω-globular set.

For every $0\leq p<n\in\mathbb{N}$, we set as usual

$$M^n/R^n\times_p M^n/R^n:=\left\{\big([x]_n,[y]_n\big)\mid s^p_{M/R}\cdots s^{n-1}_{M/R}\big([x]_n\big)=t^p_{M/R}\cdots t^{n-1}_{M/R}\big([y]_n\big)\right\}.$$

Define $\hat{\circ}^n_p : M^n/R^n \times_p M^n/R^n \to M^n/R^n$ by $[x]_n \hat{\circ}^n_p [y]_n := [x \circ^n_p y]_n$.

Also, define $\hat{*}^n_\alpha : M^n/R^n \to M^n/R^n$ by $[x]_n^{\hat{*}^n_\alpha} := [x^{*^n_\alpha}]_n$.

These definitions are well-defined thanks to the definition of congruence.

This implies that we obtain a strict involutive globular ω-category

$$\left(M^0/R^0 \underset{t^0_{M/R}}{\overset{s^0_{M/R}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{M/R}}{\overset{s^{n-1}_{M/R}}{\leftleftarrows}} M^n/R^n \underset{t^n_{M/R}}{\overset{s^n_{M/R}}{\leftleftarrows}} \cdots, (\iota^n_{M/R})_{n\in\mathbb{N}_0}, (\hat{\circ}^n_p)_{0\le p<n\in\mathbb{N}}, (\hat{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0} \right).$$

We now check the universal factorization property.

Assume that there exists a morphism $f : \left(Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots \right) \to$ $\left(C^0 \underset{t^0_C}{\overset{s^0_C}{\leftleftarrows}} C^1 \underset{t^1_C}{\overset{s^1_C}{\leftleftarrows}} \cdots \underset{t^{n-1}_C}{\overset{s^{n-1}_C}{\leftleftarrows}} C^n \underset{t^n_C}{\overset{s^n_C}{\leftleftarrows}} \cdots, (\iota^n_C)_{n\in\mathbb{N}_0}, (\tilde{\circ}^n_p)_{0\le p<n\in\mathbb{N}}, (\tilde{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0} \right)$ into another strict involutive globular ω-category.

Since (M,i) is a free reflexive self-dual globular ω-magma over an ω-globular set, there exists a unique morphism of reflexive self-dual globular ω−magmas $\phi : \left(M^0 \underset{t^0_M}{\overset{s^0_M}{\leftleftarrows}} M^1 \underset{t^1_M}{\overset{s^1_M}{\leftleftarrows}} \cdots \underset{t^{n-1}_M}{\overset{s^{n-1}_M}{\leftleftarrows}} M^n \underset{t^n_M}{\overset{s^n_M}{\leftleftarrows}} \cdots \right) \to \left(C^0 \underset{t^0_C}{\overset{s^0_C}{\leftleftarrows}} C^1 \underset{t^1_C}{\overset{s^1_C}{\leftleftarrows}} \cdots \underset{t^{n-1}_C}{\overset{s^{n-1}_C}{\leftleftarrows}} C^n \underset{t^n_C}{\overset{s^n_C}{\leftleftarrows}} \cdots \right)$ such that $f = \phi \circ i$.

Consider, for all $n \in \mathbb{N}_0$, $R^n_\phi := \{(x,y) \in M^n \times M^n \mid \phi_n(x) = \phi_n(y)\}$.

We first claim that R_ϕ becomes a congruence in M due to the fact that ϕ is a morphism of reflexive self-dual globular ω-magmas.

To see that R_ϕ is an ω-globular set, suppose that $(x,y) \in R^{n+1}_\phi$.

This means that $\phi_{n+1}(x) = \phi_{n+1}(y)$ and so $\phi_n(s^n_M(x)) = \phi_n(s^n_M(y))$.

Thus, $(s^n_M(x), s^n_M(y)) \in R^n_\phi$ as desired.

With a similar argument, we also get $(t^n_M(x), t^n_M(y)) \in R^n_\phi$.

So R_ϕ with restriction of sources and targets to $M \times M$ is an ω-globular set.

If we assume $(x,y),(a,b) \in R^n_\phi$ with $(x,a),(y,b) \in M^n \times_p M^n$, then $\phi_n(x) = \phi_n(y)$, $\phi_n(a) = \phi_n(b)$, $x \circ^n_p a$ and $y \circ^n_p b$ exist.

The fact that ϕ is such morphism yields the following implications:

- $\phi_n\left(x^{*^n_\alpha}\right) = \phi_n\left(y^{*^n_\alpha}\right)$ and then $\left(x^{*^n_\alpha}, y^{*^n_\alpha}\right) \in R^n_\phi$,

- $\phi_{n+1}(\iota^n_M(x)) = \phi_{n+1}(\iota^n_M(y))$ and hence $\left(\iota^n_M(x), \iota^n_M(y)\right) \in R^{n+1}_\phi$,

- $\phi_n(x \circ^n_p a) = \phi_n(x) \hat{\circ}^n_p \phi_n(a) = \phi_n(y) \hat{\circ}^n_p \phi_n(b) = \phi_n(y \circ^n_p b)$ and so $(x \circ^n_p a, y \circ^n_p b) \in R^n_\phi$.

Since C is a strict involutive globular ω-category and all axioms in Ax^n need to be satisfied in C, $\mathrm{Ax}^n \subseteq R^n_\phi$ for every $n \in \mathbb{N}_0$.

This also implies that M/R_ϕ is a strict involutive globular ω-category.

Note that $[x]^\phi = [y]^\phi$ implies $(x,y) \in R_\phi$ and so $\phi(x) = \phi(y)$ for any $x, y \in M$.

If we define $\tilde{\phi} : M/R_\phi \to C$ by $\tilde{\phi}\left([x]^\phi\right) := \phi(x)$, it becomes a unique map such that $\tilde{\phi} \circ \pi_\phi = \phi$, where $\pi_\phi : M \to M/R_\phi$ is defined by $x \mapsto [x]^\phi$.

As R is the smallest congruence containing Ax, $R \subseteq R_\phi$ and so $\theta : M/R \to M/R_\phi$, defined by $[x] \mapsto [x]^\phi$, is a unique map such that $\pi_\phi = \theta \circ \pi$.

Combining all the previous maps, we get that $\hat{\phi} := \tilde{\phi} \circ \theta : M/R \to C$ is a unique morphism of strict involutive globular ω-categories satisfying the equation

$$f = \phi \circ i = \tilde{\phi} \circ \pi_\phi \circ i = \tilde{\phi} \circ \theta \circ \pi \circ i = \hat{\phi} \circ (\pi \circ i).$$

Therefore,

$$\left(\left(M^0/R^0 \underset{t^0_{M/R}}{\overset{s^0_{M/R}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{M/R}}{\overset{s^{n-1}_{M/R}}{\leftleftarrows}} M^n/R^n \underset{t^n_{M/R}}{\overset{s^n_{M/R}}{\leftleftarrows}} \cdots, (\iota^n_{M/R})_{n \in \mathbb{N}_0}, (\hat{\circ}^n_p)_{0 \le p < n}, (\hat{*}^n_\alpha)_{\alpha \subseteq \mathbb{N}_0 \ni n}\right), \pi \circ i\right)$$

is a free strict involutive globular ω-category over the ω-globular set Q. $\square$

Remark 3.2.1.4. We denote $[Q]$ a free strict involutive globular ω-category over an ω-globular set Q.

3.2.2 Free Involutive Penon Contractions

Let us establish a category $\mathscr{Q}^*$ whose objects are of the form $\left(M \xrightarrow{f} C, [\cdot,\cdot]\right)$, where M is a self-dual globular ω-magma, C is a strict involutive globular ω-category, and f is a morphism of self-dual globular ω-magmas, equipped with a Penon contraction $[\cdot,\cdot]$. We simply call these objects *involutive Penon contractions*.

Theorem 3.2.2.1. *A free involutive Penon contraction over an* ω*-globular set exists.*

Proof. Let $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$ be an ω-globular set.

Using the same terminology as in Proposition 3.1.3.2 and Proposition 3.2.1.3, we set $M^0 := \left\langle \bar{\tilde{Q}}^0 \right\rangle =: C^0$ and $\pi^0 : M^0 \to C^0$ as the identity map.

As there is no contraction induced by π^0, the domain of $[\cdot,\cdot]_0$ is $\varnothing$.

Define $M^1 := \left\langle \bar{\tilde{Q}}^1 \right\rangle$, $C^1 := M^1/R^1$, and $\pi^1 : M^1 \to C^1$ as the quotient map by the smallest congruence $R^1 \subseteq M^1 \times M^1$ generated by all algebraic axioms Ax^1.

Notice that the domain of $[\cdot,\cdot]_1$ is exactly Ax^1.

Let $M^2[1] := \left\langle \bar{\tilde{Q}}^2[1] \right\rangle \cup \widehat{\mathrm{Ax}^1}$, where $\widehat{\mathrm{Ax}^1} := \{(\cdots(y^{\beta_1^2})^{\cdots})^{\beta_m^2} \mid y \in \mathrm{Ax}^1,\ \beta_j^2 \subseteq \mathbb{N}_0\}$.

Define $s_M^1[1], t_M^1[1] : M^2[1] \to M^1$ by

- $s_M^1[1]\big((\cdots(y^{\beta_1^2})^{\cdots})^{\beta_m^2}\big) := \begin{cases} (\cdots((s^1(y))^{\beta_1^1})^{\cdots})^{\beta_m^1}, & y \in Q^2,\ 1 \notin \beta_1^2 \triangle \cdots \triangle \beta_m^2; \\ (\cdots((t^1(y))^{\beta_1^1})^{\cdots})^{\beta_m^1}, & y \in Q^2,\ 1 \in \beta_1^2 \triangle \cdots \triangle \beta_m^2; \\ (\cdots(x^{\beta_1^1})^{\cdots})^{\beta_m^1}, & y = (x,z)_1 \in \mathrm{Ax}^1. \end{cases}$

- $t_M^1[1]\big((\cdots(y^{\beta_1^2})^{\cdots})^{\beta_m^2}\big) := \begin{cases} (\cdots((t^1(y))^{\beta_1^1})^{\cdots})^{\beta_m^1}, & y \in Q^2,\ 1 \notin \beta_1^2 \triangle \cdots \triangle \beta_m^2; \\ (\cdots((s^1(y))^{\beta_1^1})^{\cdots})^{\beta_m^1}, & y \in Q^2,\ 1 \in \beta_1^2 \triangle \cdots \triangle \beta_m^2; \\ (\cdots(z^{\beta_1^1})^{\cdots})^{\beta_m^1}, & y = (x,z)_1 \in \mathrm{Ax}^1. \end{cases}$

Then we follow exactly as in the proof of Proposition 3.1.3.2 to inductively define $M^2[k]$, $s_M^1[k]$, and $t_M^1[k]$, for each $k \in \mathbb{N}$.

Set $M^2 := \bigcup_{k=1}^{\infty} M^2[k]$, $s_M^1 := \bigcup_{k=1}^{\infty} s_M^1[k]$, and $t_M^1 := \bigcup_{k=1}^{\infty} t_M^1[k]$.

Consider $[\mathrm{Ax}^{k+1}] := \mathrm{Ax}^{k+1} \cup \{(s^k((x,y)_k),x) \mid (x,y) \in \mathrm{Ax}^k\}$

$$\cup\{(t^k((x,y)_k),y) \mid (x,y) \in \mathrm{Ax}^k\} \cup \{((x,x)_k,(x,k)) \mid x \in M^k\}$$

and $[R^{k+1}]$ is the smallest congruence generated by $[\mathrm{Ax}^{k+1}]$ for any $k \in \mathbb{N}$.

Assume that we have M^r, s_M^{r-1}, t_M^{r-1}, and $\pi^r : M^r \to C^r$ for every $r = 1,2,\ldots,n$.

Let $M^{n+1}[1] := \left\langle \bar{\hat{Q}}^{n+1} \right\rangle \mathbin{\dot\cup} [\widehat{\mathrm{Ax}^{n+1}}]$.

Define $s^n_M[1], t^n_M[1] : M^{n+1}[1] \to M^n$ by

- $$s^n_M[1]\big((\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big) := \begin{cases} (\cdots((s^n(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m; \\ (\cdots((t^n(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m; \\ (\cdots(x^{\beta^n_1})^{\cdots})^{\beta^n_m}, & y = (x,z)_n \in [\mathrm{Ax}^n]. \end{cases}$$

- $$t^n_M[1]\big((\cdots(y^{\beta^{n+1}_1})^{\cdots})^{\beta^{n+1}_m}\big) := \begin{cases} (\cdots((t^n(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \notin \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m; \\ (\cdots((s^n(y))^{\beta^n_1})^{\cdots})^{\beta^n_m}, & n \in \beta^{n+1}_1 \triangle \cdots \triangle \beta^{n+1}_m; \\ (\cdots(z^{\beta^n_1})^{\cdots})^{\beta^n_m}, & y = (x,z)_n \in [\mathrm{Ax}^n]. \end{cases}$$

By Proposition 3.1.3.2, we can set $M^{n+1} := \dot{\bigcup\limits_{k=1}^{\infty}} M^{n+1}[k]$, $s^n_M := \dot{\bigcup\limits_{k=1}^{\infty}} s^n_M[k]$, and $t^n_M := \dot{\bigcup\limits_{k=1}^{\infty}} t^n_M[k]$ and also $\pi^{n+1} : M^{n+1} \to C^{n+1} := M^{n+1}/[R^{n+1}]$.

Arguing in a similar fashion as in Proposition 3.1.3.2, we get a reflexive ω-globular set $\left(M^0 \underset{t^0_M}{\overset{s^0_M}{\leftleftarrows}} M^1 \underset{t^1_M}{\overset{s^1_M}{\leftleftarrows}} \cdots \underset{t^{n-1}_M}{\overset{s^{n-1}_M}{\leftleftarrows}} M^n \underset{t^n_M}{\overset{s^n_M}{\leftleftarrows}} \cdots, (\iota^n_M)_{n\in\mathbb{N}_0} \right)$ with a similarly-defined family of identity maps $(\iota^n_M)_{n\in\mathbb{N}_0}$.

For all $p, n \in \mathbb{N}_0$ such that $p < n$, we set

$$M^n \times_p M^n := \{(x,y) \in M^n \times M^n \mid s^p_M s^{p+1}_M \cdots s^{n-1}_M(x) = t^p_M t^{p+1}_M \cdots t^{n-1}_M(y)\}.$$

Then we define $\circ^n_p : M^n \times_p M^n \to M^n$ by $x \circ^n_p y \mapsto (x,p,y)$.

For each $\alpha^n \subseteq \mathbb{N}_0$, we define $*^n_\alpha : M^n \to M^n$ by

$$(\cdots((x,p,y)^{\alpha^n_1})^{\cdots})^{\alpha^n_m} \mapsto ((\cdots((x,p,y)^{\alpha^n_1})^{\cdots})^{\alpha^n_m})^{\alpha^n}.$$

So, $\left(M^0 \underset{t^0_M}{\overset{s^0_M}{\leftleftarrows}} \cdots \underset{t^{n-1}_M}{\overset{s^{n-1}_M}{\leftleftarrows}} M^n \underset{t^n_M}{\overset{s^n_M}{\leftleftarrows}} \cdots, (\iota^n_M)_{n\in\mathbb{N}_0}, (\circ^n_p)_{0\le p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0} \right)$ is a reflexive self-dual globular ω-magma.

By Proposition 3.1.3.2, this quadruple becomes a free reflexive self-dual globular ω-magma over an ω-globular set.

Define $[\cdot,\cdot]_n : [\mathrm{Ax}^n] \to M^{n+1}$ by $[x,y]_n := (x,y)_n$.

We see that the following equations are satisfied: $s^n_M\big([x,y]_n\big) = x$, $t^n_M\big([x,y]_n\big) = y$, $\pi^{n+1}\big([x,y]_n\big) = \iota^n_C\big(\pi^n(x)\big) = \iota^n_C\big(\pi^n(y)\big)$, and $[x,x]_n = \iota^n_M(x)$.

Hence, $\Big(M \overset{\pi}{\to} C, \big([\cdot,\cdot]_n\big)_{n\in\mathbb{N}}\Big)$ becomes an involutive Penon contraction.

To prove the universal factorization property of this category, we define a map $g : Q \to \Big(M \overset{\pi}{\to} C, \big([\cdot,\cdot]_n\big)_{n\in\mathbb{N}}\Big)$ by $x \mapsto x^{\varnothing}$.

Assume that $\phi : Q \to \Big(\hat{M} \overset{\hat{\pi}}{\to} \hat{C}, \big(\widehat{[\cdot,\cdot]}_n\big)_{n\in\mathbb{N}}\Big)$ is a morphism into another involutive Penon contraction.

Since we want such morphism to preserve contractions as well, we have to add the following assignment to these extra arrows: for any $n \in \mathbb{N}$,

$$\phi^{n+1}_M\Big((\cdots(([x,y]_n)^{\alpha^{n+1}_1})^{\cdots})^{\alpha^{n+1}_m}\Big) \mapsto (\cdots(\widehat{[\phi^n_M(x),\phi^n_M(y)]}_n)^{\hat{*}^{n+1}_1})^{\cdots})^{\hat{*}^{n+1}_m}.$$

By Propositions 3.1.3.2 and 3.2.1.3, there exist a unique morphism of self-dual globular ω-magmas $\hat{\phi}_M : M \to M'$ such that $\phi_M = \hat{\phi}_M \circ g_M$ and a unique morphism of strict involutive globular ω-categories $\hat{\phi}_C : C \to C'$ such that $\phi_C = \hat{\phi}_C \circ g_C$.

As a consequence, $\Big(\Big(M \overset{f}{\to} C, \big([\cdot,\cdot]_n\big)_{n\in\mathbb{N}}\Big), g\Big)$ is a free involutive Penon contraction over an ω-globular set. $\square$

3.3 Involutive Weak Globular ω-Categories

We know from Proposition 3.2.2.1 that there exists a free involutive Penon contraction over an ω-globular set. So we obtain a pair of free-forgetful functors $\mathbf{GSet} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{Q}^*$, where **GSet** is the category of ω-globular sets and $\mathscr{Q}^*$ is the category of involutive Penon contractions. This leads us to a significant adjunction.

3.3.1 Adjunction between Free-Forgetful Functors

This subsection is devoted to the proof of existence of a desired adjunction and so a desired monad immediately.

Theorem 3.3.1.1. *The free functor $F : \textbf{\textit{GSet}} \to \mathscr{Q}^*$ is left adjoint to the forgetful functor $U : \mathscr{Q}^* \to \textbf{\textit{GSet}}$.*

Proof. Let $Q \in \mathrm{Ob}_{\mathbf{GSet}}$ and $\left(M \xrightarrow{f} C, [\cdot,\cdot]\right) \in \mathrm{Ob}_{\mathscr{Q}^*}$.

First of all, let us separate the functor F into the following components: $F_{\mathfrak{M}} : Q \mapsto \left\langle \bar{\hat{Q}} \right\rangle$ and $F_{\mathfrak{C}} : Q \mapsto [Q]$.

Consider $U : M \mapsto U(M)$ forgetting the reflexivity, self-duality, and compositions and $U : C \mapsto U(C)$ remaining the original ω-globular set.

We see that $(M \to F_{\mathfrak{M}}U(M))$ is a free reflexive self-dual globular ω-magma over the underlying ω-globular set of a reflexive self-dual globular ω-magma M.

Suppose that $\theta : M \to N$ is a morphism of reflexive self-dual globular ω-magmas.

We get that $F_{\mathfrak{M}}U(\theta) : F_{\mathfrak{M}}U(M) \to F_{\mathfrak{M}}U(N)$, defined similarly as in Proposition 3.1.3.2, becomes a morphism of reflexive self-dual globular ω-magmas.

Now consider a free strict involutive globular ω-category $(C \to F_{\mathfrak{C}}U(C))$ over the underlying ω-globular set of a strict involutive globular ω-category C.

Assume that $\nu : C \to D$ is a morphism of strict involutive globular ω-categories.

We obtain a map $F_{\mathfrak{C}}U(\nu) : F_{\mathfrak{C}}U(C) \to F_{\mathfrak{C}}U(D)$ defined by $[x]_n \mapsto [\nu(x)]_n$.

It follows from Proposition 3.2.1.3 that $F_{\mathfrak{C}}U(\nu)$ is a morphism of strict involutive globular ω-categories.

Define $\varepsilon : \mathrm{Ob}_{\mathscr{Q}^*} \to \mathrm{Hom}_{\mathscr{Q}^*}$ by $\left(M \xrightarrow{f} C, [\cdot,\cdot]\right) \mapsto \varepsilon_{\left(M \xrightarrow{f} C, [\cdot,\cdot]\right)}$, where $\varepsilon_{\left(M \xrightarrow{f} C, [\cdot,\cdot]\right)} : FU\left(\left(M \xrightarrow{f} C, [\cdot,\cdot]\right)\right) \to \left(M \xrightarrow{f} C, [\cdot,\cdot]\right)$ is divided into two as follows.

For each $\left((\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots})^{\bar{*}^n_{\alpha_m}}, (\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots})^{\bar{*}^n_{\beta_k}}\right) \in \left\langle \hat{M}^n[i] \right\rangle \times_p \left\langle \hat{M}^n[j] \right\rangle$, we define $\varepsilon^{\mathfrak{M}}_M : F_{\mathfrak{M}}U(M) \to M$ by $\left(\cdots\left(\left((\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots})^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p (\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots})^{\bar{*}^n_{\beta_k}}\right)^{\bar{*}^n_{\gamma_1}}\right)^{\cdots}\right)^{\bar{*}^n_{\gamma_t}}$
$\mapsto \left(\cdots\left(\left((\cdots(x^{*^n_{\alpha_1}})^{\cdots})^{*^n_{\alpha_m}} \circ^n_p (\cdots(y^{*^n_{\beta_1}})^{\cdots})^{*^n_{\beta_k}}\right)^{*^n_{\gamma_1}}\right)^{\cdots}\right)^{*^n_{\gamma_t}}$.

For every $\left(\left[(\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots})^{\bar{*}^n_{\alpha_m}}\right]_n, \left[(\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots})^{\bar{*}^n_{\beta_k}}\right]_n\right) \in [C[i]] \times_p [C[j]]$, define $\varepsilon^{\mathfrak{C}}_C : F_{\mathfrak{C}}U(C) \to C$ by $\left[\left(\cdots\left(\left((\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots})^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p (\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots})^{\bar{*}^n_{\beta_k}}\right)^{\bar{*}^n_{\gamma_1}}\right)^{\cdots}\right)^{\bar{*}^n_{\gamma_t}}\right]_{n+1}$

$$\mapsto \Big(\cdots\big(\big(\big(\cdots(x^{*^n_{\alpha_1}})^{\cdots}\big)^{*^n_{\alpha_m}} \circ^n_p \big(\cdots(y^{*^n_{\beta_1}})^{\cdots}\big)^{*^n_{\beta_k}}\big)^{*^n_{\gamma_1}}\big)^{\cdots}\Big)^{*^n_{\gamma_t}}.$$

First, consider the diagram

$$\begin{tikzcd}
\big(F_{\mathfrak{M}}U(M), (\bar{\circ}^n_p)_{0\leq p<n\in\mathbb{N}}, (\bar{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big) \arrow[r, "\varepsilon^{\mathfrak{M}}_M"] \arrow[d, "F_{\mathfrak{M}}U(\theta)"'] & \big(M, (\circ^n_p)_{0\leq p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big) \arrow[d, "\theta"] \\
\big(F_{\mathfrak{M}}U(N), (\tilde{\circ}^n_p)_{0\leq p<n\in\mathbb{N}}, (\tilde{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big) \arrow[r, "\varepsilon^{\mathfrak{M}}_N"'] & \big(N, (\hat{\circ}^n_p)_{0\leq p<n\in\mathbb{N}}, (\hat{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big)
\end{tikzcd}$$

To prove commutativity, we need to show that $\varepsilon^{\mathfrak{M}}_N \circ F_{\mathfrak{M}}U(\theta) = \theta \circ \varepsilon^{\mathfrak{M}}_M$.

We see that

$$\begin{aligned}
& \varepsilon^{\mathfrak{M}}_N \circ F_{\mathfrak{M}}U(\theta)\Big(\Big(\cdots\big(\big(\big(\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots}\big)^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p \big(\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots}\big)^{\bar{*}^n_{\beta_k}}\big)^{\bar{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\bar{*}^n_{\gamma_t}}\Big) \\
= \; & \varepsilon^{\mathfrak{M}}_N\Big(\Big(\cdots\big(\big(\big(\cdots((\theta(x))^{\tilde{*}^n_{\alpha_1}})^{\cdots}\big)^{\tilde{*}^n_{\alpha_m}} \tilde{\circ}^n_p \big(\cdots((\theta(y))^{\tilde{*}^n_{\beta_1}})^{\cdots}\big)^{\tilde{*}^n_{\beta_k}}\big)^{\tilde{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\tilde{*}^n_{\gamma_t}}\Big) \\
= \; & \Big(\cdots\big(\big(\big(\cdots((\theta(x))^{\tilde{*}^n_{\alpha_1}})^{\cdots}\big)^{\hat{*}^n_{\alpha_m}} \hat{\circ}^n_p \big(\cdots((\theta(y))^{\hat{*}^n_{\beta_1}})^{\cdots}\big)^{\hat{*}^n_{\beta_k}}\big)^{\hat{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\hat{*}^n_{\gamma_t}} \\
= \; & \theta\Big(\Big(\cdots\big(\big(\big(\cdots(x^{*^n_{\alpha_1}})^{\cdots}\big)^{*^n_{\alpha_m}} \circ^n_p \big(\cdots(y^{*^n_{\beta_1}})^{\cdots}\big)^{*^n_{\beta_k}}\big)^{*^n_{\gamma_1}}\big)^{\cdots}\Big)^{*^n_{\gamma_t}}\Big) \\
= \; & \theta \circ \varepsilon^{\mathfrak{M}}_M\Big(\Big(\cdots\big(\big(\big(\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots}\big)^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p \big(\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots}\big)^{\bar{*}^n_{\beta_k}}\big)^{\bar{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\bar{*}^n_{\gamma_t}}\Big).
\end{aligned}$$

Then we will show that this diagram commutes:

$$\begin{tikzcd}
\big(F_{\mathfrak{C}}U(C), (\bar{\circ}^n_p)_{0\leq p<n\in\mathbb{N}}, (\bar{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big) \arrow[r, "\varepsilon^{\mathfrak{C}}_C"] \arrow[d, "F_{\mathfrak{C}}U(\nu)"'] & \big(C, (\circ^n_p)_{0\leq p<n\in\mathbb{N}}, (*^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big) \arrow[d, "\nu"] \\
\big(F_{\mathfrak{C}}U(D), (\tilde{\circ}^n_p)_{0\leq p<n\in\mathbb{N}}, (\tilde{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big) \arrow[r, "\varepsilon^{\mathfrak{C}}_D"'] & \big(D, (\hat{\circ}^n_p)_{0\leq p<n\in\mathbb{N}}, (\hat{*}^n_\alpha)_{\alpha\subseteq\mathbb{N}_0, n\in\mathbb{N}_0}\big)
\end{tikzcd}$$

We also see that

$$\begin{aligned}
& \varepsilon^{\mathfrak{C}}_D \circ F_{\mathfrak{C}}U(\nu)\Big(\Big[\Big(\cdots\big(\big(\big(\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots}\big)^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p \big(\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots}\big)^{\bar{*}^n_{\beta_k}}\big)^{\bar{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\bar{*}^n_{\gamma_t}}\Big]_{n+1}\Big) \\
= \; & \varepsilon^{\mathfrak{C}}_D\left(\Big[\Big(\cdots\big(\big(\big(\cdots((\nu(x))^{\bar{*}^n_{\alpha_1}})^{\cdots}\big)^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p \big(\cdots((\nu(y))^{\bar{*}^n_{\beta_1}})^{\cdots}\big)^{\bar{*}^n_{\beta_k}}\big)^{\bar{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\bar{*}^n_{\gamma_t}}\Big]'_{n+1}\right) \\
= \; & \Big(\cdots\big(\big(\big(\cdots((\nu(x))^{\tilde{*}^n_{\alpha_1}})^{\cdots}\big)^{\hat{*}^n_{\alpha_m}} \hat{\circ}^n_p \big(\cdots((\nu(y))^{\hat{*}^n_{\beta_1}})^{\cdots}\big)^{\hat{*}^n_{\beta_k}}\big)^{\hat{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\hat{*}^n_{\gamma_t}} \\
= \; & \nu\Big(\Big(\cdots\big(\big(\big(\cdots(x^{*^n_{\alpha_1}})^{\cdots}\big)^{*^n_{\alpha_m}} \circ^n_p \big(\cdots(y^{*^n_{\beta_1}})^{\cdots}\big)^{*^n_{\beta_k}}\big)^{*^n_{\gamma_1}}\big)^{\cdots}\Big)^{*^n_{\gamma_t}}\Big) \\
= \; & \nu \circ \varepsilon^{\mathfrak{C}}_C\Big(\Big[\Big(\cdots\big(\big(\big(\cdots(x^{\bar{*}^n_{\alpha_1}})^{\cdots}\big)^{\bar{*}^n_{\alpha_m}} \bar{\circ}^n_p \big(\cdots(y^{\bar{*}^n_{\beta_1}})^{\cdots}\big)^{\bar{*}^n_{\beta_k}}\big)^{\bar{*}^n_{\gamma_1}}\big)^{\cdots}\Big)^{\bar{*}^n_{\gamma_t}}\Big]_{n+1}\Big).
\end{aligned}$$

Thus, ε is a natural transformation.

Define $\eta : \mathrm{Ob}_{\mathbf{GSet}} \to \mathrm{Hom}_{\mathbf{GSet}}$ by $Q \mapsto \eta_Q$, where $\eta_Q : Q \to UF(Q)$ is defined by $x \mapsto x^{*^n_\varnothing}$ for every $x \in Q^n$.

Now consider the diagram

$$\begin{tikzcd} Q \arrow[r, "\eta_Q"] \arrow[d, "\lambda"'] & UF(Q) \arrow[d, "UF(\lambda)"] \\ R \arrow[r, "\eta_R"'] & UF(R) \end{tikzcd}$$

To prove commutativity, we need to show that $\eta_R \circ \lambda = UF(\lambda) \circ \eta_Q$.

For any $x \in Q^n$, we see that

$$UF(\lambda) \circ \eta_Q(x) = UF(\lambda)\left(x^{*^n_\varnothing}\right) = (\lambda(x))^{\hat{*}^n_\varnothing} = \eta_R(\lambda(x)) = \eta_R \circ \lambda(x).$$

Hence, η is a natural transformation.

Finally, consider the following diagrams

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FUF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, "1_U"'] & UFU \arrow[d, "U\varepsilon"] \\ & U \end{tikzcd}$$

To obtain an adjunction, we have to show that both triangles are commutative; that is, $\varepsilon F \circ F\eta = 1_F$ and $U\varepsilon \circ \eta U = 1_U$.

But verifying that the diagrams commute is equivalent to demonstrating that the following triangles commute:

$$\begin{tikzcd} F_{\mathfrak{M}}(Q) \arrow[r, "F_{\mathfrak{M}}\eta^{\mathfrak{M}}_Q"] \arrow[dr, "1_{F_{\mathfrak{M}}(Q)}"'] & F_{\mathfrak{M}}UF_{\mathfrak{M}}(Q) \arrow[d, "\varepsilon^{\mathfrak{M}}_Q F_{\mathfrak{M}}"] \\ & F_{\mathfrak{M}}(Q) \end{tikzcd} \qquad \begin{tikzcd} F_{\mathfrak{C}}(Q) \arrow[r, "F_{\mathfrak{C}}\eta^{\mathfrak{C}}_Q"] \arrow[dr, "1_{F_{\mathfrak{C}}(Q)}"'] & F_{\mathfrak{C}}UF_{\mathfrak{C}}(Q) \arrow[d, "\varepsilon^{\mathfrak{C}}_Q F_{\mathfrak{C}}"] \\ & F_{\mathfrak{C}}(Q) \end{tikzcd}$$

$$\begin{tikzcd} U(M) \arrow[r, "\eta^{\mathfrak{M}}_M U"] \arrow[dr, "1_{U(M)}"'] & UF_{\mathfrak{M}}U(M) \arrow[d, "U\varepsilon^{\mathfrak{M}}_M"] \\ & U(M) \end{tikzcd} \qquad \begin{tikzcd} U(C) \arrow[r, "\eta^{\mathfrak{C}}_C U"] \arrow[dr, "1_{U(C)}"'] & UF_{\mathfrak{C}}U(C) \arrow[d, "U\varepsilon^{\mathfrak{C}}_C"] \\ & U(C) \end{tikzcd}$$

First, using the same notation as before, we get $F_{\mathfrak{M}}(Q) = \left\langle \bar{\hat{Q}} \right\rangle$.

Thus, $UF_{\mathfrak{M}}(Q) = \left\langle \hat{\bar{Q}} \right\rangle$ as an ω-globular set and so $F_{\mathscr{M}}UF_{\mathscr{M}}(Q) = \left\langle \widehat{\overline{\left\langle \hat{\bar{Q}} \right\rangle}} \right\rangle$.

Define $F_{\mathfrak{M}}\eta_Q^{\mathfrak{M}} : F_{\mathfrak{M}}(Q) \to F_{\mathfrak{M}}UF_{\mathfrak{M}}(Q)$ by $x \mapsto (x)^{\hat{\bar{*}}^n_{\varnothing}}$ for every $x \in \left\langle \hat{\bar{Q}}^n \right\rangle$.

For each $(x,y) \in \left\langle \hat{\bar{Q}}^n[i] \right\rangle \times_p \left\langle \hat{\bar{Q}}^n[j] \right\rangle$ and $z \in \left\langle \hat{\bar{Q}}^n[1] \right\rangle$, we define $\varepsilon_Q^{\mathfrak{M}}F_{\mathfrak{M}} : F_{\mathfrak{M}}UF_{\mathfrak{M}}(Q) \to F_{\mathfrak{M}}(Q)$ by

$$\begin{aligned}
(\cdots((x\hat{\circ}^n_p y)^{\hat{*}^n_{\gamma_1}})^{\cdots})^{\hat{*}^n_{\gamma_t}}) &\mapsto (\cdots((x \circ^n_p y)^{*^n_{\gamma_1}})^{\cdots})^{*^n_{\gamma_t}}, \\
((\cdots((z)^{\hat{*}^n_{\gamma_1}})^{\cdots})^{\hat{*}^n_{\gamma_t}})^{\hat{*}^n_{\varnothing}} &\mapsto (\cdots(z^{*^n_{\gamma_1}})^{\cdots})^{*^n_{\gamma_t}}.
\end{aligned}$$

It follows that $\varepsilon_Q^{\mathfrak{M}}F_{\mathfrak{M}} \circ F_{\mathfrak{M}}\eta_Q^{\mathfrak{M}}(x) = \varepsilon_Q^{\mathfrak{M}}F_{\mathfrak{M}}\left((x)^{\hat{\bar{*}}^n_{\varnothing}}\right) = x$.

This implies that $\varepsilon_Q^{\mathfrak{M}}F_{\mathfrak{M}} \circ F_{\mathfrak{M}}\eta_Q^{\mathfrak{M}} = 1_{F_{\mathfrak{M}}(Q)}$.

Second, since $F_{\mathfrak{C}}(Q) = [Q]$, $UF_{\mathfrak{C}}(Q) = [Q]$ as an ω-globular set and so $F_{\mathfrak{C}}UF_{\mathfrak{C}}(Q) = [[Q]]$.

Define $F_{\mathfrak{C}}\eta_Q^{\mathfrak{C}} : F_{\mathfrak{C}}(Q) \to F_{\mathfrak{C}}UF_{\mathfrak{C}}(Q)$ by $[y]_n \mapsto [[y]_n]'_n$.

Define $\varepsilon_Q^{\mathfrak{C}}F_{\mathfrak{C}} : F_{\mathfrak{C}}UF_{\mathfrak{C}}(Q) \to F_{\mathfrak{C}}(Q)$ by

$$\begin{aligned}
\Big[(\cdots(([x]_n\tilde{\circ}^n_p[y]_n)^{\tilde{*}^n_{\gamma_1}})^{\cdots})^{\tilde{*}^n_{\gamma_t}}\Big]'_{n+1} &\mapsto (\cdots(([x]_n \circ^n_p [y]_n)^{*^n_{\gamma_1}})^{\cdots})^{*^n_{\gamma_t}}, \\
[[z]_n]'_n &\mapsto [z]_n.
\end{aligned}$$

for any $(x,y) \in [C^n[i]] \times_p [C^n[j]]$ and $z \in [C^n[1]]$.

It is obvious that $\varepsilon_Q^{\mathfrak{C}}F_{\mathfrak{C}} \circ F_{\mathfrak{C}}\eta_Q^{\mathfrak{C}} = 1_{F_{\mathfrak{C}}(Q)}$.

Third, consider $U(M) = M$ as an ω-globular set.

So $F_{\mathfrak{M}}U(M) = \left\langle \bar{M} \right\rangle$ constructed from elements of M.

Thus, $UF_{\mathfrak{M}}U(M) = \left\langle \bar{M} \right\rangle$ as an ω-globular set.

Define $\eta_M^{\mathfrak{M}}U : U(M) \to UF_{\mathfrak{M}}U(M)$ by $x \mapsto (x)^{\bar{*}^n_{\varnothing}}$ for every $x \in M^n$.

Define $U\varepsilon_M^{\mathfrak{M}} : UF_{\mathfrak{M}}U(M) \to U(M)$ by

$$\begin{aligned}
(\cdots((x\bar{\circ}^n_p y)^{\bar{*}^n_{\gamma_1}})^{\cdots})^{\bar{*}^n_{\gamma_t}} &\mapsto (\cdots((x \circ^n_p y)^{*^n_{\gamma_1}})^{\cdots})^{*^n_{\gamma_t}}, \\
((\cdots((z)^{\bar{*}^n_{\gamma_1}})^{\cdots})^{\bar{*}^n_{\gamma_t}})^{\bar{*}^n_{\varnothing}} &\mapsto (\cdots(z^{*^n_{\gamma_1}})^{\cdots})^{*^n_{\gamma_t}}.
\end{aligned}$$

for any $(x,y) \in \left\langle \bar{\hat{M}}^n[i] \right\rangle \times_p \left\langle \bar{\hat{M}}^n[j] \right\rangle$ and $z \in \left\langle \bar{\hat{M}}^n[1] \right\rangle$.

We see that $U\varepsilon_M^{\mathfrak{M}} \circ \eta_M^{\mathfrak{M}} U(x) = U\varepsilon_M^{\mathfrak{M}} \left(x^{\bar{*}_{\varnothing}^n} \right) = x$.

This means that $U\varepsilon_M^{\mathfrak{M}} \circ \eta_M^{\mathscr{M}} U = 1_{U(M)}$.

Fourth, consider $U(C) = C$ as an ω-globular set.

Then $F_{\mathfrak{C}}U(C) = [C]$ established from elements of C and so $UF_{\mathfrak{C}}U(C) = [C]$ as an ω-globular set.

Define $\eta_C^{\mathfrak{C}} U : U(C) \to UF_{\mathfrak{C}}U(C)$ by $x \mapsto [x]_n$.

Define $U\varepsilon_C^{\mathfrak{C}} : UF_{\mathfrak{C}}U(C) \to U(C)$ by

$$\begin{array}{rcl} \left[(\cdots(([x]_n \tilde{\circ}_p^n [y]_n)^{\tilde{*}_{\gamma_1}^n})^{\cdots})^{\tilde{*}_{\gamma_t}^n} \right]'_{n+1} & \mapsto & (\cdots(([x]_n \circ_p^n [y]_n)^{*_{\gamma_1}^n})^{\cdots})^{*_{\gamma_t}^n}, \\ [[z]_n]'_n & \mapsto & [z]_n. \end{array}$$

for any $(x,y) \in [C^n[i]] \times_p [C^n[j]]$ and $z \in [C^n[1]]$.

It is easy to see that $U\varepsilon_C^{\mathfrak{C}} \circ \eta_C^{\mathfrak{C}} U = 1_{U(C)}$.

This means that $\varepsilon F \circ F\eta = 1_F$ and $U\varepsilon \circ \eta U = 1_U$.

Therefore, F is left adjoint to U. $\square$

3.3.2 Involutive Weak Globular ω-Categories

Now we can provide our main definition as follows.

Definition 3.3.2.1. A **Penon involutive weak globular ω-category** is an algebra for the monad $(UF, U\varepsilon F, \eta)$.

Finally, we list here some examples of Penon involutive weak globular ω-categories but discuss some of them in detail.

Example 3.3.2.2. Weak ω-groupoids are just special cases of weak involutive globular ω-categories with involutions given by (suitable composition of) the inverses. In particular the most elementary and well-known examples fitting our definition of weak involutive globular ω-category are the **fundamental ω-groupoids** $\Pi_\omega(X)$ of a topological space X.

Proof. First of all, let X be a topological space and $\Pi_\omega(X)^0 := X$.

Consider $\Pi_\omega(X)^1 := C([0,1];X) := \{f^{(1)} : [0,1] \to X \mid f^{(1)} \text{ is continuous}\}$ the set of continuous paths in X.

Define $s^0, t^0 : \Pi_\omega(X)^1 \to \Pi_\omega(X)^0$ by $s^0(f^{(1)}) := f^{(1)}(0)$ and $t^0(f^{(1)}) := f^{(1)}(1)$.

Define $\circ_0^1 : \Pi_\omega(X)^1 \times_0 \Pi_\omega(X)^1 \to \Pi_\omega(X)^1$ by $\left(f^{(1)}, g^{(1)}\right) \mapsto f^{(1)} \circ_0^1 g^{(1)}$, where

$$f^{(1)} \circ_0^1 g^{(1)}(u) := \begin{cases} g^{(1)}(2u), & 0 \le u \le \frac{1}{2}; \\ f^{(1)}(2u-1), & \frac{1}{2} \le u \le 1. \end{cases}$$

and $\Pi_\omega(X)^1 \times_0 \Pi_\omega(X)^1 := \left\{\left(f^{(1)}, g^{(1)}\right) \in \Pi_\omega(X)^1 \times \Pi_\omega(X)^1 \mid s^0(f^{(1)}) = t^0(g^{(1)})\right\}$.

Suppose that we have already defined $\Pi_\omega(X)^k$, s^{k-1}, t^{k-1}, and $\circ_p^k$ for every $p < k = 1, 2, \ldots, n-1$.

Then we let

$$\begin{aligned}\Pi_\omega(X)^n := \Big\{ f^{(n)} \in C([0,1]^n;X) \mid\ & f^{(n)}\big|_{[0,1]^{n-1}\times\{0\}}, f^{(n)}\big|_{[0,1]^{n-1}\times\{1\}} \in \Pi_\omega(X)^{n-1} \\ & \forall k = 0, \ldots, n-1,\ \forall u'_1, \ldots, u'_{k-1}, u''_1, \ldots, u''_{k-1}, u_{k+1}, \ldots, u_n, \bar{u}_{k+1}, \ldots, \bar{u}_n, \\ & f^{(n)}(u'_1, \ldots, u'_{k-1}, 0, u_{k+1}, \ldots, u_n) = f^{(n)}(u'_1, \ldots, u'_{k-1}, 0, \bar{u}_{k+1}, \ldots, \bar{u}_n), \\ & f^{(n)}(u''_1, \ldots, u''_{k-1}, 0, u_{k+1}, \ldots, u_n) = f^{(n)}(u''_1, \ldots, u''_{k-1}, 0, \bar{u}_{k+1}, \ldots, \bar{u}_n) \Big\}.\end{aligned}$$

Define $s^{n-1}, t^{n-1} : \Pi_\omega(X)^n \to \Pi_\omega(X)^{n-1}$ by $s^{n-1}(f^{(n)}) := f^{(n)}\big|_{[0,1]^{n-1}\times\{0\}}$ and $t^{n-1}(f^{(n)}) := f^{(n)}\big|_{[0,1]^{n-1}\times\{1\}}$.

We see that $\Pi_\omega(X)^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} \Pi_\omega(X)^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} \Pi_\omega(X)^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$ is an ω-globular set.

Define $\circ_p^n : \Pi_\omega(X)^n \times_p \Pi_\omega(X)^n \to \Pi_\omega(X)^n$ by $\left(f^{(n)}, g^{(n)}\right) \mapsto f^{(n)} \circ_p^n g^{(n)}$, where

$$f^{(n)} \circ_p^n g^{(n)}(x_1, \ldots, x_{p+1}, \ldots, x_n) := \begin{cases} g^{(n)}(x_1, \ldots, 2x_{p+1}, \ldots, x_n), & 0 \le x_{p+1} \le \frac{1}{2}; \\ f^{(n)}(x_1, \ldots, 2x_{p+1}-1, \ldots, x_n), & \frac{1}{2} \le x_{p+1} \le 1. \end{cases}$$

and $\Pi_\omega(X)^n \times_p \Pi_\omega(X)^n := \left\{\left(f^{(n)}, g^{(n)}\right) \in \Pi_\omega(X)^n \times \Pi_\omega(X)^n \mid s^p(f^{(n)}) = t^p(g^{(n)})\right\}$.

Moreover, for each $n \in \mathbb{N}_0$, we define $\iota^n : \Pi_\omega(X)^n \to \Pi_\omega(X)^{n+1}$ by $\iota^n(f^{(n)})(x_1,\ldots,x_n,x_{n+1}) := f^{(n)}(x_1,\ldots,x_n)$ for any $f^{(n)} \in \Pi_\omega(X)^n$ and $x_{n+1} \in [0,1]$.

It is obvious that $s^n \circ \iota^n = \mathrm{Id}_{\Pi_\omega(X)^n} = t^n \circ \iota^n$ for every $n \in \mathbb{N}_0$.

Next, for all $\alpha \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}_0$, we define $*^n_\alpha : \Pi_\omega(X)^n \to \Pi_\omega(X)^n$ by

$$(f^{(n)})^{*^n_\alpha}(x_1,x_2,\ldots,x_n,\ldots) := f^{(n)}(\gamma_1(x_1),\gamma_2(x_2),\ldots,\gamma_n(x_n))$$

where $\gamma_k(z) := \begin{cases} z, & k \notin \alpha; \\ 1-z, & k \in \alpha. \end{cases}$ for all $k = 1,2,\ldots,n$.

Let $\mathbf{GSet} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{Q}^*$ be the pair of free-forgetful functors and $T = UF$.

Then $T(\Pi_\omega(X))$ is the family of all possible concatenated elements of $\Pi_\omega(X)$ attached by compositions, involutions, and identities discussed before.

It remains for us to define a suitable evaluation map $\theta : T(\Pi_\omega(X)) \to \Pi_\omega(X)$; however, we give here a brief construction on simple elements as follows:

$$\begin{aligned} (x) &\mapsto x, \\ (x,p,y) &\mapsto x \circ^n_p y, \\ x^{\alpha^n} &\mapsto x^{*^n_\alpha}, \\ (x,n) &\mapsto \iota^n(x). \end{aligned}$$

It is easy to check that $(\Pi_\omega(X),\theta)$ is an algebra for the monad $(UF,\eta,U\varepsilon F)$. $\square$

Example 3.3.2.3. Every strict involutive globular ω-category is a very particular trivial case of involutive weak globular ω-category; in particular, strict globular ω-groupoids.

Example 3.3.2.4. Globular ω-quivers are an example of strict involutive globular ω-category. Globular propagators of globular ω-quivers give an example of involutive weak globular ω-categories (see [BJ]).

Example 3.3.2.5. Let $\mathscr{M}^0$ be a family of involutive monoids $A,B,C,\ldots$ and $\mathscr{M}^1$ the family of the bimodules ${}_AM_B$, with $A,B \in \mathscr{M}^0$. Composition $\circ^1_0$ of bimodules is given by the Rieffel tensor product ${}_AM_B \otimes_B {}_BN_C$ and involution $*^1_0$ of bimodules is provided by the Rieffel dual ${}_B\overline{M}_A$ where $\overline{M} := \{\overline{x} \mid x \in M\}$ is just a (specific) disjoint copy of M and the bimodule actions are $b \cdot \overline{x} \cdot a := \overline{a^* x b^*}$, for all $a \in A$, $b \in B$ and $x \in M$. Similarly starting from a class $\mathscr{M}^0$ of strict involutive 1-categories, the family $\mathscr{M}^1$ of "bimodules" between them is an involutive weak 1-category. Introducing a suitable notion of "bimodule" between strict involutive globular n-categories, we obtain an involutive weak globular n-category. If $\mathscr{M}^0$ is a family of strict globular ω-categories, the family $\mathscr{M}^1$ of "bimodules" between them is an involutive weak globular ω-category.

CHAPTER 4

INVOLUTIVE WEAK GLOBULAR-CONE ω-CATEGORIES

In this chapter we slightly modify the Penon's approach to obtain a generalized notion of involutive weak globular higher categories as discussed in Chapter 3.

In order to be able to capture the notion of weak C*-categories, which Penon's weak globular higher categories constructed from an algebra for a certain monad does not, we are forced to generalize this concept. A special class of infinite cells will be added and simultaneously dominate its own ω-globular set. We call this new structure a *globular cone*. In the first section we study some of its basic properties so as to understand how to deal with further operations involved.

Consider the category $\hat{\mathscr{Q}}$ whose objects are of the form $\left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right)$ where $\mathscr{M}$ is a globular-cone ω-magma, $\mathscr{C}$ is a strict globular-cone ω-category, f is a morphism of globular-cone globular ω-magmas, together with a specific contraction $[\cdot,\cdot]$, and whose morphisms preserve everything possible. Then we will define a category $\hat{\mathscr{Q}}^*$ with a similar role to the category $\hat{\mathscr{Q}}$. Its objects are of the form $\left(\mathscr{M}^* \xrightarrow{f^*} \mathscr{C}^*, \widehat{[\cdot,\cdot]}\right)$ where $\mathscr{M}^*$ is a self-dual globular-cone ω-magma, $\mathscr{M}^*$ is a strict involutive globular-cone ω-category, f^* is a morphism of self-dual globular-cone ω-magmas, equipped with a certain contraction $\widehat{[\cdot,\cdot]}$.

In order to define an involutive weak globular-cone ω-category as an algebra for a certain monad, we must have an adjunction between the category $\hat{\mathscr{Q}}^*$ and the category **GCone** of globular cones in the sense of free-forgetful functors.

So, in this chapter, we will mainly give constructions of a free self-dual globular-cone ω-magma and a free strict involutive globular-cone ω-category over a globular cone. Then we adopt some modification to the original contraction suiting our situation. The monadic definition and examples of free involutive weak globular-cone ω-categories are also discussed.

4.1 Globular Cones

This first section deserves to investigate basic behaviors of the so-called globular cones.

Let $Q^0 \underset{t_Q^0}{\overset{s_Q^0}{\leftleftarrows}} Q^1 \underset{t_Q^1}{\overset{s_Q^1}{\leftleftarrows}} \cdots \underset{t_Q^{n-1}}{\overset{s_Q^{n-1}}{\leftleftarrows}} Q^n \underset{t_Q^n}{\overset{s_Q^n}{\leftleftarrows}} \cdots$ be an ω-quiver with globularity condition: $s_Q^{n-1}s_Q^n = s_Q^{n-1}t_Q^n$ and $t_Q^{n-1}s_Q^n = t_Q^{n-1}t_Q^n$, for any natural number n.

Consider a *globular cone* $\mathscr{E}$ over an ω-globular set

$$\begin{tikzcd}
 & & & & \mathscr{E} \arrow[llll, shift left, "s_{\mathscr{E}}^0"'] \arrow[llll, shift right, "t_{\mathscr{E}}^0"] \arrow[lll, shift left, "s_{\mathscr{E}}^1"'] \arrow[lll, shift right, "t_{\mathscr{E}}^1"] \arrow[d, shift right, "s_{\mathscr{E}}^n"'] \arrow[d, shift left, "t_{\mathscr{E}}^n"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s_Q^0"'] \arrow[l, shift left, "t_Q^0"] & \cdots \arrow[l, shift right, "s_Q^1"'] \arrow[l, shift left, "t_Q^1"] & & Q^n \arrow[ll, shift right, "s_Q^{n-1}"'] \arrow[ll, shift left, "t_Q^{n-1}"] & \cdots \arrow[l, shift right, "s_Q^n"'] \arrow[l, shift left, "t_Q^n"]
\end{tikzcd}$$

that is, $(\mathscr{E}, s_{\mathscr{E}}^n)_{n\in\mathbb{N}_0}$ is a cone over $Q^0 \overset{s_Q^0}{\leftarrow} Q^1 \overset{s_Q^1}{\leftarrow} \cdots \overset{s_Q^{n-1}}{\leftarrow} Q^n \overset{s_Q^n}{\leftarrow} \cdots$ and $(\mathscr{E}, t_{\mathscr{E}}^n)_{n\in\mathbb{N}_0}$ is a cone over $Q^0 \underset{t_Q^0}{\leftarrow} Q^1 \underset{t_Q^1}{\leftarrow} \cdots \underset{t_Q^{n-1}}{\leftarrow} Q^n \underset{t_Q^n}{\leftarrow} \cdots$. Moreover, we also require the following natural compatibility: $s_Q^k t_{\mathscr{E}}^{k+1} = s_{\mathscr{E}}^k$ and $t_Q^k s_{\mathscr{E}}^{k+1} = t_{\mathscr{E}}^k$, for each $k \in \mathbb{N}_0$.

4.1.1 Truncated Globular Cones

Definition 4.1.1.1. Let $n \in \mathbb{N}$. An n-**truncated globular cone** over an ω-globular set is a diagram

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[lll, shift left, "s_{\mathscr{E}}^0"'] \arrow[lll, shift right, "t_{\mathscr{E}}^0"] \arrow[ll, shift left, "s_{\mathscr{E}}^1"'] \arrow[ll, shift right, "t_{\mathscr{E}}^1"] \arrow[d, shift right, "s_{\mathscr{E}}^n"'] \arrow[d, shift left, "t_{\mathscr{E}}^n"] \\
Q^0 & Q^1 \arrow[l, shift right, "s_Q^0"'] \arrow[l, shift left, "t_Q^0"] & \cdots \arrow[l, shift right, "s_Q^1"'] \arrow[l, shift left, "t_Q^1"] & Q^n \arrow[l, shift right, "s_Q^{n-1}"'] \arrow[l, shift left, "t_Q^{n-1}"]
\end{tikzcd}$$

such that $s_Q^k s_{\mathscr{E}}^{k+1} = s_{\mathscr{E}}^k = s_Q^k t_{\mathscr{E}}^{k+1}$ and $t_Q^k s_{\mathscr{E}}^{k+1} = t_{\mathscr{E}}^k = t_Q^k t_{\mathscr{E}}^{k+1}$ for each $k = 0, 1, \ldots, n-1$.

We observe that, for any $n \in \mathbb{N}$, *n-truncated* globular cones can be constructed from $(n+1)$-globular sets and vice versa.

Lemma 4.1.1.2. *$Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} Q^{n+1}$ is an $(n+1)$-globular set iff*

$$\begin{tikzcd}
 & & & & \mathscr{E} \arrow[dllll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dllll, shift left, "t^0_{\mathscr{E}}"] \arrow[dlll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^1_{\mathscr{E}}"] \arrow[dl, phantom, "\cdots"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & & Q^n \arrow[ll, shift right, "s^{n-1}_Q"'] \arrow[ll, shift left, "t^{n-1}_Q"]
\end{tikzcd}$$

is an n-truncated globular cone, where $\mathscr{E} := Q^{n+1}$, $s^k_Q := s^k$, $t^k_Q := t^k$,
$s^m_{\mathscr{E}} := s^m s^{m+1} \cdots s^{n-1} s^n$, and $t^m_{\mathscr{E}} := t^m t^{m+1} \cdots t^{n-1} t^n$
for each $k = 0, 1, \ldots, n-1$ and $m = 0, 1, \ldots, n$.

Proof. Suppose that

$$\begin{tikzcd}
 & & & & \mathscr{E} \arrow[dllll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dllll, shift left, "t^0_{\mathscr{E}}"] \arrow[dlll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^1_{\mathscr{E}}"] \arrow[dl, phantom, "\cdots"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & & Q^n \arrow[ll, shift right, "s^{n-1}_Q"'] \arrow[ll, shift left, "t^{n-1}_Q"]
\end{tikzcd}$$

is an n-truncated globular cone.

First, set $s^m := s^m_Q$ and $t^m := t^m_Q$ for all $m = 1, 2, \ldots, n$.

Then, let $Q^{n+1} := \mathscr{E}$, $s^n := s^n_{\mathscr{E}}$ and $t^n := t^n_{\mathscr{E}}$.

Since $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n$ is already an n-globular set by hypothesis, it remains to prove that $s^{n-1}_Q s^n_{\mathscr{E}} = s^{n-1}_{\mathscr{E}} = s^{n-1}_Q t^n_{\mathscr{E}}$ and $t^{n-1}_Q s^n_{\mathscr{E}} = t^{n-1}_{\mathscr{E}} = t^{n-1}_Q t^n_{\mathscr{E}}$.

By the property of the n-truncated globular cone, we have
$s^{n-1}_Q s^n_{\mathscr{E}} = s^{n-1}_{\mathscr{E}} = s^{n-1}_Q t^n_{\mathscr{E}}$ and $t^{n-1}_Q s^n_{\mathscr{E}} = t^{n-1}_{\mathscr{E}} = t^{n-1}_Q t^n_{\mathscr{E}}$.

Thus, $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} Q^{n+1}$ is an $(n+1)$-globular set.

Conversely, let $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} Q^{n+1}$ be an $(n+1)$-globular set. First, let $\mathscr{E} := Q^{n+1}$, $s^m_Q := s^m$ and $t^m_Q := t^m$ for all $m = 0, 1, \ldots, n-1$.

Set $s^m_{\mathscr{E}} := s^m_Q s^{m+1}_Q \cdots s^{n-1}_Q s^n_{\mathscr{E}}$ and $t^m_{\mathscr{E}} := t^m_Q t^{m+1}_Q \cdots t^{n-1}_Q t^n_{\mathscr{E}}$ for $m = 0, 1, \ldots, n$.

Next we need to prove the following compatibility conditions:

$s^k_{\mathscr{E}} = s^k_Q s^{k+1}_{\mathscr{E}}$, $t^k_{\mathscr{E}} = t^k_Q t^{k+1}_{\mathscr{E}}$, $s^k_{\mathscr{E}} = s^k_Q t^{k+1}_{\mathscr{E}}$, and $t^k_{\mathscr{E}} = t^k_Q s^{k+1}_{\mathscr{E}}$ for each $k = 0, 1, \ldots, n-1$.

For each $k = 0, 1, \ldots, n-1$, we have

1. $s^k_{\mathscr{E}} = s^k_Q s^{k+1}_Q \cdots s^{n-1}_Q s^n_{\mathscr{E}} = s^k_Q (s^{k+1}_Q \cdots s^{n-1}_Q s^n_{\mathscr{E}}) = s^k_Q s^{k+1}_{\mathscr{E}}$,

2. $t^k_{\mathscr{E}} = t^k_Q t^{k+1}_Q \cdots t^{n-1}_Q t^n_{\mathscr{E}} = t^k_Q (t^{k+1}_Q \cdots t^{n-1}_Q t^n_{\mathscr{E}}) = t^k_Q t^{k+1}_{\mathscr{E}}$,

3. $s^k_{\mathscr{E}} = s^k_Q s^{k+1}_Q \cdots s^{n-1}_Q s^n_{\mathscr{E}} = s^k_Q s^{k+1}_Q \cdots s^{n-1}_Q t^n_{\mathscr{E}} = s^k_Q s^{k+1}_Q \cdots t^{n-1}_Q t^n_{\mathscr{E}}$

   $= \cdots = s^k_Q t^{k+1}_Q \cdots t^{n-1}_Q t^n_{\mathscr{E}} = s^k_Q t^{k+1}_{\mathscr{E}}$,

4. $t^k_{\mathscr{E}} = t^k_Q t^{k+1}_Q \cdots t^{n-1}_Q t^n_{\mathscr{E}} = t^k_Q t^{k+1}_Q \cdots t^{n-1}_Q s^n_{\mathscr{E}} = t^k_Q t^{k+1}_Q \cdots s^{n-1}_Q s^n_{\mathscr{E}}$

   $= \cdots = t^k_Q s^{k+1}_Q \cdots s^{n-1}_Q s^n_{\mathscr{E}} = t^k_Q s^{k+1}_{\mathscr{E}}$.

Hence, we obtain an n-truncated globular cone

$$
\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{E}}"] \arrow[dll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] \arrow[dl, phantom, "\cdots"] \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"]
\end{tikzcd}
$$

as desired. $\square$

Lemma 4.1.1.3. *Let $n \in \mathbb{N}$. Given an n-globular set $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n$, we can construct an ω-globular set.*

Proof. Let $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n$ be an n-globular set.

To construct an $(n+1)$-globular set, consider $Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} Q^{n+1}$ with $Q^{n+1} := Q^n$, $s^n := i$ and $t^n := i$, where $i(x) = x$ for all $x \in Q^{n+1}$.

We see that, for all $x \in Q^{n+1}$,

1. $s^{n-1}s^n(x) = s^{n-1}(s^n(x)) = s^{n-1}(i(x)) = s^{n-1}(t^n(x)) = s^{n-1}t^n(x)$,

2. $t^{n-1}s^n(x) = t^{n-1}(s^n(x)) = t^{n-1}(i(x)) = t^{n-1}(t^n(x)) = t^{n-1}t^n(x)$.

Thus, $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} Q^{n+1}$ is an $(n+1)$-globular set.

Suppose that $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} \cdots \overset{s^{m-1}}{\underset{t^{m-1}}{\leftleftarrows}} Q^m$ is an m-globular set.

Consider $Q^m \overset{s^m}{\underset{t^m}{\leftleftarrows}} Q^{m+1}$ with $Q^{m+1} := Q^m$, $s^m := i$ and $t^m := i$, where $i(x) = x$ for all $x \in Q^{m+1}$.

We see that, for all $x \in Q^{m+1}$,

1. $s^{m-1}s^m(x) = s^{m-1}(s^m(x)) = s^{m-1}(i(x)) = s^{m-1}(t^m(x)) = s^{m-1}t^m(x)$,

2. $t^{m-1}s^m(x) = t^{m-1}(s^m(x)) = t^{m-1}(i(x)) = t^{m-1}(t^m(x)) = t^{m-1}t^m(x)$.

By induction, $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} \cdots \overset{s^{m-1}}{\underset{t^{m-1}}{\leftleftarrows}} Q^m \overset{s^m}{\underset{t^m}{\leftleftarrows}} Q^{m+1}$ is an $(m+1)$-globular set for each $m > n$.

Furthermore, every n-globular set $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n$ can be embedded into an ω-globular set $Q^0 \overset{s^0}{\underset{t^0}{\leftleftarrows}} Q^1 \overset{s^1}{\underset{t^1}{\leftleftarrows}} \cdots \overset{s^{n-1}}{\underset{t^{n-1}}{\leftleftarrows}} Q^n \overset{s^n}{\underset{t^n}{\leftleftarrows}} Q^{n+1} \overset{s^{n+1}}{\underset{t^{n+1}}{\leftleftarrows}} \cdots$ by setting, for all $m > n$, $Q^m := Q^n$, $s^{m-1} := i$ and $t^{m-1} := i$, where $i(x) = x$ for all $x \in Q^n$. $\square$

4.1.2 Cones over Globular Products

Theorem 4.1.2.1. *Let* $Q^0 \overset{s_Q^0}{\underset{t_Q^0}{\leftleftarrows}} Q^1 \overset{s_Q^1}{\underset{t_Q^1}{\leftleftarrows}} \cdots \overset{s_Q^{n-1}}{\underset{t_Q^{n-1}}{\leftleftarrows}} Q^n \overset{s_Q^n}{\underset{t_Q^n}{\leftleftarrows}} \cdots$ *be an* ω*-globular set. Then*

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[llld, shift left, "s_{\mathscr{E}}^0"'] \arrow[llld, shift right, "t_{\mathscr{E}}^0"] \arrow[lld, shift left, "s_{\mathscr{E}}^1"'] \arrow[lld, shift right, "t_{\mathscr{E}}^1"] \arrow[d, shift right, "s_{\mathscr{E}}^n"'] \arrow[d, shift left, "t_{\mathscr{E}}^n"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s_Q^0"'] \arrow[l, shift left, "t_Q^0"] & \cdots \arrow[l, shift right, "s_Q^1"'] \arrow[l, shift left, "t_Q^1"] & Q^n \arrow[l, shift right, "s_Q^{n-1}"'] \arrow[l, shift left, "t_Q^{n-1}"] & \cdots \arrow[l, shift right, "s_Q^n"'] \arrow[l, shift left, "t_Q^n"]
\end{tikzcd}$$

is a globular cone if and only if

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[llld, "\pi_{\mathscr{E}}^0"'] \arrow[lld, "\pi_{\mathscr{E}}^1"'] \arrow[d, "\pi_{\mathscr{E}}^n"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi_Q^0"'] & \cdots \arrow[l, "\pi_Q^1"'] & Q^n \times_G Q^n \arrow[l, "\pi_Q^{n-1}"'] & \cdots \arrow[l, "\pi_Q^n"']
\end{tikzcd}$$

is a cone, where

$$Q^n \times_G Q^n := \{(x,y) \in Q^n \times Q^n \mid s_Q^n(x) = s_Q^n(y) \text{ and } t_Q^n(x) = t_Q^n(y)\}$$

for all $n = 1, 2, \ldots$ *and* $Q^0 \times_G Q^0 := Q^0 \times Q^0$.

Proof. Given $k = 0, 1, \ldots$ and a globular cone

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[llld, shift left, "s_{\mathscr{E}}^0"'] \arrow[llld, shift right, "t_{\mathscr{E}}^0"] \arrow[lld, shift left, "s_{\mathscr{E}}^1"'] \arrow[lld, shift right, "t_{\mathscr{E}}^1"] \arrow[d, shift right, "s_{\mathscr{E}}^n"'] \arrow[d, shift left, "t_{\mathscr{E}}^n"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s_Q^0"'] \arrow[l, shift left, "t_Q^0"] & \cdots \arrow[l, shift right, "s_Q^1"'] \arrow[l, shift left, "t_Q^1"] & Q^n \arrow[l, shift right, "s_Q^{n-1}"'] \arrow[l, shift left, "t_Q^{n-1}"] & \cdots \arrow[l, shift right, "s_Q^n"'] \arrow[l, shift left, "t_Q^n"]
\end{tikzcd}$$

Define $\pi_Q^k : Q^{k+1} \times_G Q^{k+1} \to Q^k \times_G Q^k$ by $\pi_Q^k(x,y) := (s_Q^k(x), t_Q^k(y))$ for all $(x,y) \in Q^{k+1} \times_G Q^{k+1}$.

Note that $s_Q^k(x) = s_Q^k(y)$ and $t_Q^k(x) = t_Q^k(y)$ by the globularity condition.

Define $\pi_{\mathscr{E}}^k : \mathscr{E} \to Q^k \times_G Q^k$ by $\pi_{\mathscr{E}}^k(x) := (s_{\mathscr{E}}^k(x), t_{\mathscr{E}}^k(x))$ for all $x \in \mathscr{E}$.

Indeed, the maps π_Q^k and $\pi_{\mathscr{E}}^k$ are well-defined thanks to the globularity condition and the compatibility in the globular cone, respectively.

Next we check the commutativity of the diagrams.

By compatibilities of sources and targets, we have, for all $x \in \mathscr{E}$,

$$\begin{aligned}
\pi_{\mathscr{E}}^k(x) &= (s_{\mathscr{E}}^k(x), t_{\mathscr{E}}^k(x)) \\
&= (s_Q^k s_{\mathscr{E}}^{k+1}(x), t_Q^k t_{\mathscr{E}}^{k+1}(x)) \\
&= \left(s_Q^k(s_{\mathscr{E}}^{k+1}(x)), t_Q^k(t_{\mathscr{E}}^{k+1}(x))\right) \\
&= \pi_Q^k(s_{\mathscr{E}}^{k+1}(x), t_{\mathscr{E}}^{k+1}(x)) \\
&= \pi_Q^k \pi_{\mathscr{E}}^{k+1}(x).
\end{aligned}$$

Thus, $\pi_{\mathscr{E}}^k = \pi_Q^k \pi_{\mathscr{E}}^{k+1}$.

Hence, we obtain a cone of the form

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[lll, "\pi_{\mathscr{E}}^0"'] \arrow[dll, "\pi_{\mathscr{E}}^1"'] \arrow[d, "\pi_{\mathscr{E}}^n"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi_Q^0"'] & \cdots \arrow[l, "\pi_Q^1"'] & Q^n \times_G Q^n \arrow[l, "\pi_Q^{n-1}"'] & \cdots \arrow[l, "\pi_Q^n"']
\end{tikzcd}$$

Now we assume that

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[lll, "\pi_{\mathscr{E}}^0"'] \arrow[dll, "\pi_{\mathscr{E}}^1"'] \arrow[d, "\pi_{\mathscr{E}}^n"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi_Q^0"'] & \cdots \arrow[l, "\pi_Q^1"'] & Q^n \times_G Q^n \arrow[l, "\pi_Q^{n-1}"'] & \cdots \arrow[l, "\pi_Q^n"']
\end{tikzcd}$$

is a cone.

Define $p_1^k : Q^k \times Q^k \to Q^k$ and $p_2^k : Q^k \times Q^k \to Q^k$ by $p_1^k(x,y) := x$ and $p_2^k(x,y) := y$ for all $(x,y) \in Q^k \times Q^k$.

Then we define $s_{\mathscr{E}}^k := p_1^k \pi_{\mathscr{E}}^k$ and $t_{\mathscr{E}}^k := p_2^k \pi_{\mathscr{E}}^k$.

Now we check that $s_{\mathscr{E}}^k = s_Q^k s_{\mathscr{E}}^{k+1}$, $t_{\mathscr{E}}^k = t_Q^k t_{\mathscr{E}}^{k+1}$, $s_{\mathscr{E}}^k = s_Q^k t_{\mathscr{E}}^{k+1}$, and $t_{\mathscr{E}}^k = t_Q^k s_{\mathscr{E}}^{k+1}$.

Let $x \in \mathscr{E}$.

Thus, $\pi_{\mathscr{E}}^k(x) = (p_1^k \pi_{\mathscr{E}}^k(x), p_2^k \pi_{\mathscr{E}}^k(x))$ and $\pi_{\mathscr{E}}^{k+1}(x) = (p_1^{k+1}\pi_{\mathscr{E}}^{k+1}(x), p_2^{k+1}\pi_{\mathscr{E}}^{k+1}(x))$.

So, $\pi_Q^k \pi_{\mathscr{E}}^{k+1}(x) = \pi_Q^k(p_1^{k+1}\pi_{\mathscr{E}}^{k+1}(x), p_2^{k+1}\pi_{\mathscr{E}}^{k+1}(x)) = (s_Q^k p_1^{k+1}\pi_{\mathscr{E}}^{k+1}(x), t_Q^k p_2^{k+1}\pi_{\mathscr{E}}^{k+1}(x))$.

By the commutativity of the diagrams, we have $\pi_{\mathscr{E}}^k = \pi_Q^k \pi_{\mathscr{E}}^{k+1}$.

This implies that $(p_1^k \pi_{\mathscr{E}}^k(x), p_2^k \pi_{\mathscr{E}}^k(x)) = (s_Q^k p_1^{k+1}\pi_{\mathscr{E}}^{k+1}(x), t_Q^k p_2^{k+1}\pi_{\mathscr{E}}^{k+1}(x))$.

It follows that $p_1^k \pi_{\mathscr{E}}^k(x) = s_Q^k p_1^{k+1}\pi_{\mathscr{E}}^{k+1}(x)$ and $p_2^k \pi_{\mathscr{E}}^k(x) = t_Q^k p_2^{k+1}\pi_{\mathscr{E}}^{k+1}(x)$.

Hence, $s_{\mathscr{E}}^k = p_1^k \pi_{\mathscr{E}}^k = s_Q^k p_1^{k+1}\pi_{\mathscr{E}}^{k+1} = s_Q^k s_{\mathscr{E}}^{k+1}$

and $t_{\mathscr{E}}^k = p_2^k \pi_{\mathscr{E}}^k = t_Q^k p_2^{k+1}\pi_{\mathscr{E}}^{k+1} = t_Q^k s_{\mathscr{E}}^{k+1}$.

So, $(\mathscr{E}, s_{\mathscr{E}}^n)_{n\in\mathbb{N}_0}$ and $(\mathscr{E}, t_{\mathscr{E}}^n)_{n\in\mathbb{N}_0}$ are cones over an ω-globular set.

Furthermore, we get $s_{\mathscr{E}}^k = s_Q^k s_{\mathscr{E}}^{k+1} = s_Q^k t_{\mathscr{E}}^{k+1}$ and $t_{\mathscr{E}}^k = t_Q^k t_{\mathscr{E}}^{k+1} = t_Q^k s_{\mathscr{E}}^{k+1}$ by the globularity condition of the ω-globular set.

Therefore, we obtain a globular cone

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift right, "s_{\mathscr{E}}^0"'] \arrow[dlll, shift left, "t_{\mathscr{E}}^0"] \arrow[dll, shift right, "s_{\mathscr{E}}^1"'] \arrow[dll, shift left, "t_{\mathscr{E}}^1"] \arrow[d, shift right, "s_{\mathscr{E}}^n"'] \arrow[d, shift left, "t_{\mathscr{E}}^n"] & \\
Q^0 & Q^1 \arrow[l, shift right, "s_Q^0"'] \arrow[l, shift left, "t_Q^0"] & \cdots \arrow[l, shift right, "s_Q^1"'] \arrow[l, shift left, "t_Q^1"] & Q^n \arrow[l, shift right, "s_Q^{n-1}"'] \arrow[l, shift left, "t_Q^{n-1}"] & \cdots \arrow[l, shift right, "s_Q^n"'] \arrow[l, shift left, "t_Q^n"]
\end{tikzcd}$$

as required. $\square$

Remark 4.1.1. To make the more precise terminology, we will call the former cone a *globular cone over an* ω*-globular set* and the latter cone a *cone over globular products*.

From Lemma 4.1.2.1, we have the following important theorem from the categorical point of view.

Theorem 4.1.2. *There is an isomorphism between the category of globular cones over an* ω*-globular set and the category of cones over globular products.*

Proof. Let $Q^0 \underset{t_Q^0}{\overset{s_Q^0}{\leftleftarrows}} Q^1 \underset{t_Q^1}{\overset{s_Q^1}{\leftleftarrows}} \cdots \underset{t_Q^{n-1}}{\overset{s_Q^{n-1}}{\leftleftarrows}} Q^n \underset{t_Q^n}{\overset{s_Q^n}{\leftleftarrows}} \cdots$ be an ω-globular set.

First of all, we need to synthesize objects and morphisms of both categories.

Let $\mathscr{A}$ be the category of globular cones over an ω-globular set.

Objects of $\mathscr{A}$ are globular cones over an ω-globular set of the form

$$
\begin{tikzcd}
 & & \cdots & \mathscr{E} \arrow[dlll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{E}}"] \arrow[dll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}
$$

Suppose that

$$
\begin{tikzcd}
 & & \cdots & \mathscr{F} \arrow[dlll, shift right, "s^0_{\mathscr{F}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{F}}"] \arrow[dll, shift right, "s^1_{\mathscr{F}}"'] \arrow[dll, shift left, "t^1_{\mathscr{F}}"] \arrow[d, shift right, "s^n_{\mathscr{F}}"'] \arrow[d, shift left, "t^n_{\mathscr{F}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}
$$

is another globular cone over the same ω-globular set.

Morphisms from a globular cone $\mathscr{E}$ to a globular cone $\mathscr{F}$ are functions of the form $\psi : \mathscr{E} \to \mathscr{F}$

$$
\begin{tikzcd}[column sep=large, row sep=huge]
& \mathscr{E} \arrow[rr, "\psi"] \arrow[dl, shift right, "s^0_{\mathscr{E}}"'] \arrow[dl, shift left, "t^0_{\mathscr{E}}"] \arrow[d, shift right, "s^1_{\mathscr{E}}"'] \arrow[d, shift left, "t^1_{\mathscr{E}}"] \arrow[drr, shift right, "s^n_{\mathscr{E}}"'] \arrow[drr, shift left, "t^n_{\mathscr{E}}"] & \cdots & \mathscr{F} \arrow[dlll, shift right, "s^0_{\mathscr{F}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{F}}"] \arrow[dll, shift right, "s^1_{\mathscr{F}}"'] \arrow[dll, shift left, "t^1_{\mathscr{F}}"] \arrow[d, shift right, "s^n_{\mathscr{F}}"'] \arrow[d, shift left, "t^n_{\mathscr{F}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}
$$

making all triangles commutative: that is, $s^k_{\mathscr{E}} = s^k_{\mathscr{F}}\psi$ and $t^k_{\mathscr{E}} = t^k_{\mathscr{F}}\psi$ for every $k = 0, 1, \ldots$.

Let $\mathscr{B}$ be the category of cones over globular products of the form

$$
\begin{tikzcd}
& & & \mathscr{E} \arrow[dlll, "\pi^0_{\mathscr{E}}"'] \arrow[dll, "\pi^1_{\mathscr{E}}"] \arrow[d, "\pi^n_{\mathscr{E}}"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi^0_Q"'] & \cdots \arrow[l, "\pi^1_Q"'] & Q^n \times_G Q^n \arrow[l, "\pi^{n-1}_Q"'] & \cdots \arrow[l, "\pi^n_Q"']
\end{tikzcd}
$$

Suppose that

$$
\begin{tikzcd}
& & & \mathscr{F} \arrow[dlll, "\pi^0_{\mathscr{F}}"'] \arrow[dll, "\pi^1_{\mathscr{F}}"] \arrow[d, "\pi^n_{\mathscr{F}}"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi^0_Q"'] & \cdots \arrow[l, "\pi^1_Q"'] & Q^n \times_G Q^n \arrow[l, "\pi^{n-1}_Q"'] & \cdots \arrow[l, "\pi^n_Q"']
\end{tikzcd}
$$

is another cone over globular products.

Morphisms from a cone $\mathscr{E}$ to a cone $\mathscr{F}$ are functions of the form $\phi : \mathscr{E} \to \mathscr{F}$

$$
\begin{tikzcd}
& \mathscr{E} \arrow[rr, "\phi"] \arrow[dl, "\pi^0_{\mathscr{E}}"'] \arrow[d, "\pi^1_{\mathscr{E}}"] \arrow[drr, "\pi^n_{\mathscr{E}}"] & \cdots & \mathscr{F} \arrow[dlll, "\pi^0_{\mathscr{F}}"'] \arrow[dll, "\pi^1_{\mathscr{F}}"] \arrow[d, "\pi^n_{\mathscr{F}}"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi^0_Q"'] & \cdots \arrow[l, "\pi^1_Q"'] & Q^n \times_G Q^n \arrow[l, "\pi^{n-1}_Q"'] & \cdots \arrow[l, "\pi^n_Q"']
\end{tikzcd}
$$

making all triangles commutative: that is, $\pi^k_{\mathscr{E}} = \pi^k_{\mathscr{F}}\psi$ for every $k = 0, 1, \ldots$.

For our convenience, we let $n, k = 0, 1, \ldots$ from now on.

Now we construct a functor $F : \mathscr{A} \to \mathscr{B}$ which is defined as follows:

$$\begin{aligned} \mathscr{E} &\mapsto \mathscr{E} \\ Q^n &\mapsto Q^n \times_G Q^n \\ (s^k_Q, t^k_Q) &\mapsto \pi^k_Q \\ (s^k_{\mathscr{E}}, t^k_{\mathscr{E}}) &\mapsto \pi^k_{\mathscr{E}} \\ \psi &\mapsto \psi \end{aligned}$$

Indeed, F is a functor since it acts as an identity on morphisms, i.e.

1. $F(\iota_{\mathscr{E}}) = \iota_{\mathscr{E}} = \iota_{F(\mathscr{E})}$ for every globular cone $\mathscr{E} \in \mathrm{Ob}_{\mathscr{A}}$,
2. $F(\psi \circ_{\mathscr{A}} \phi) = \psi \circ_{\mathscr{B}} \phi = F(\psi) \circ_{\mathscr{B}} F(\phi)$ for every globular cone $\mathscr{E}, \mathscr{F}, \mathscr{G} \in \mathrm{Ob}_{\mathscr{A}}$, $\phi \in \mathrm{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{F})$ and $\psi \in \mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$.

Then we construct a functor $G : \mathscr{B} \to \mathscr{A}$ which is defined by

$$\begin{aligned} \mathscr{E} &\mapsto \mathscr{E} \\ Q^n \times_G Q^n &\mapsto Q^n \\ \pi^k_Q &\mapsto (s^k_Q, t^k_Q) \\ \pi^k_{\mathscr{E}} &\mapsto (s^k_{\mathscr{E}}, t^k_{\mathscr{E}}) \\ \phi &\mapsto \phi \end{aligned}$$

Indeed, G is a functor because it maps morphisms onto the same ones, i.e.

1. $G(\iota_{\mathscr{E}}) = \iota_{\mathscr{E}} = \iota_{G(\mathscr{E})}$ for every cone $\mathscr{E} \in \mathrm{Ob}_{\mathscr{B}}$,
2. $G(\psi \circ_{\mathscr{B}} \phi) = \psi \circ_{\mathscr{A}} \phi = G(\psi) \circ_{\mathscr{A}} G(\phi)$ for every globular cone $\mathscr{E}, \mathscr{F}, \mathscr{G} \in \mathrm{Ob}_{\mathscr{B}}$, $\phi \in \mathrm{Hom}_{\mathscr{B}}(\mathscr{E}, \mathscr{F})$ and $\psi \in \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}, \mathscr{G})$.

To obtain an isomorphism, we need to show that $GF = \mathrm{Id}_{\mathscr{A}}$ and $FG = \mathrm{Id}_{\mathscr{B}}$.

For the first equality, we have

1. $GF(\mathscr{E}) = G\big(F(\mathscr{E})\big) = G(\mathscr{E}) = \mathscr{E} = \mathrm{Id}_{\mathscr{A}}(\mathscr{E})$,
2. $GF(Q^n) = G\big(F(Q^n)\big) = G(Q^n \times_G Q^n) = Q^n = \mathrm{Id}_{\mathscr{A}}(Q^n)$,
3. $GF(s^k_Q, t^k_Q) = G\big(F(s^k_Q, t^k_Q)\big) = G(\pi^k_Q) = (s^k_Q, t^k_Q) = \mathrm{Id}_{\mathscr{A}}(s^k_Q, t^k_Q)$,
4. $GF(s^k_{\mathscr{E}}, t^k_{\mathscr{E}}) = G\big(F(s^k_{\mathscr{E}}, t^k_{\mathscr{E}})\big) = G(\pi^k_{\mathscr{E}}) = (s^k_{\mathscr{E}}, t^k_{\mathscr{E}}) = \mathrm{Id}_{\mathscr{A}}(s^k_{\mathscr{E}}, t^k_{\mathscr{E}})$,
5. $GF(\psi) = G\big(F(\psi)\big) = G(\psi) = \psi = \mathrm{Id}_{\mathscr{A}}(\psi)$.

For the second equality, we also get

1. $FG(\mathscr{E}) = F\big(G(\mathscr{E})\big) = F(\mathscr{E}) = \mathscr{E} = \mathrm{Id}_{\mathscr{B}}(\mathscr{E})$,
2. $FG(Q^n \times_G Q^n) = F\big(G(Q^n \times_G Q^n)\big) = F(Q^n) = Q^n \times_G Q^n = \mathrm{Id}_{\mathscr{B}}(Q^n \times_G Q^n)$,
3. $FG(\pi^k_Q) = F\big(G(\pi^k_Q)\big) = F(s^k_Q, t^k_Q) = (\pi^k_Q) = \mathrm{Id}_{\mathscr{B}}(\pi^k_Q)$,
4. $FG(\pi^k_{\mathscr{E}}) = F\big(G(\pi^k_{\mathscr{E}})\big) = F(s^k_{\mathscr{E}}, t^k_{\mathscr{E}}) = (\pi^k_{\mathscr{E}}) = \mathrm{Id}_{\mathscr{B}}(\pi^k_{\mathscr{E}})$,
5. $FG(\phi) = F\big(G(\phi)\big) = F(\phi) = \phi = \mathrm{Id}_{\mathscr{B}}(\phi)$.

Therefore, the category of globular cones over an ω-globular set and the category of cones over globular products are isomorphic. □

4.1.3 Reflexive Globular Cones

In this subsection we investigate how reflexive ω-globular sets affect the notion of identities in our globular cone. The result is that if the ω-globular set is not reflexive, we can recover the reflexive one via identities in the globular cone. But before having this result the notions of both reflexive ω-globular sets and identities in a globular cone need to be described first.

Definition 4.1.3.1. We say that an ω-globular set $Q^0 \underset{t^0}{\overset{s^0}{\leftleftarrows}} Q^1 \underset{t^1}{\overset{s^1}{\leftleftarrows}} \cdots \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} Q^n \underset{t^n}{\overset{s^n}{\leftleftarrows}} \cdots$ is **reflexive** if there exists a family of maps $Q^0 \overset{\iota^0}{\to} Q^1 \overset{\iota^1}{\to} \cdots \overset{\iota^{n-1}}{\to} Q^n \overset{\iota^n}{\to} \cdots$ such that $s^k \circ \iota^k = \mathrm{Id}_{Q^k} = t^k \circ \iota^k$ for every $k = 0, 1, \ldots$ and we call these maps **identities** in the ω-globular set.

Remark 4.1.3.2. The definition of identities tells us that they map elements in certain levels to be loops in any higher levels, i.e. $s^m \circ s^{m-1} \circ \cdots s^{k+1} \circ \iota^k = t^m \circ t^{m-1} \circ \cdots t^{k+1} \circ \iota^k$ for each $0 \leq k < m$.

Now we can define identities in our globular cone over a reflexive ω-globular set in a similar fashion as follows.

First of all, we need to glue a globular cone over a reflexive ω-globular set and a cone over globular products together by defining **diagonal maps**, for all $k = 0, 1, \ldots$, $\Delta^k : Q^k \to Q^k \times_G Q^k$ by $x \mapsto (x, x)$ for each $x \in Q^k$.

Constructing the diagonal maps gives the following connection:

$$
\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, "\pi^0_{\mathscr{E}}"'] \arrow[dll, "\pi^1_{\mathscr{E}}"] \arrow[d, "\pi^n_{\mathscr{E}}"] & \cdots \\
Q^0 \times_G Q^0 & Q^1 \times_G Q^1 \arrow[l, "\pi^0_Q"'] & \cdots \arrow[l, "\pi^1_Q"'] & Q^n \times_G Q^n \arrow[l, "\pi^{n-1}_Q"'] & \cdots \arrow[l, "\pi^n_Q"'] \\
Q^0 \arrow[u, "\Delta^0"] & Q^1 \arrow[u, "\Delta^1"] \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[u, "\Delta^n"] \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}
$$

Now we can give one of the possible definitions of identities in a globular cone over a reflexive ω-globular set.

Definition 4.1.3.3. Using the same terminology as in the context, a globular cone $\mathscr{E}$ is said to be **reflexive** if there exists a family of maps $\iota^n_{\mathscr{E}} : Q^n \to \mathscr{E}$ such that $\pi^{n+k}_{\mathscr{E}} \circ \iota^n_{\mathscr{E}} =$

$\Delta^{n+k} \circ (\iota_Q^{n+k-1} \circ \cdots \circ \iota_Q^{n+1} \circ \iota_Q^{n})$ for each $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$ and we call these maps **identities** in a globular cone.

The following lemma says that we can recover a reflexive ω-globular set in the process of constructing identities in a globular cone over an ω-globular set.

Lemma 4.1.3.4. *Using the same terminology as in the previous definition, a family of maps* $\iota_{\mathscr{E}}^n : Q^n \to \mathscr{E}$ *such that* $\pi_{\mathscr{E}}^n \circ \iota_{\mathscr{E}}^n = \Delta^n$ *and* $\pi_{\mathscr{E}}^{n+k} \circ \iota_{\mathscr{E}}^n(Q^n) \subseteq \Delta^{n+k}(Q^{n+k})$ *for each* $n = 0, 1, \ldots$ *and* $k = 1, 2, \ldots$ *gives rise to identities in a globular cone over a reflexive* ω*-globular set.*

Proof. Let $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$ be given.

Consider a family of maps $\iota_{\mathscr{E}}^n : Q^n \to \mathscr{E}$ such that $\pi_{\mathscr{E}}^n \circ \iota_{\mathscr{E}}^n = \Delta^n$ and $\pi_{\mathscr{E}}^{n+k} \circ \iota_{\mathscr{E}}^n(Q^n) \subseteq \Delta^{n+k}(Q^{n+k})$.

If $x \in Q^n$ for some n, then $\pi_{\mathscr{E}}^{n+k} \circ \iota_{\mathscr{E}}^n(x) = (y, y)$ for some $y \in Q^{n+k}$.

So we can define $p^n : \Delta^n(Q^n) \to Q^n$ by $p^n(x, x) := x$ for each $(x, x) \in \Delta^n(Q^n)$.

Obviously, $\Delta^n \circ p^n = \mathrm{Id}_{\Delta^n(Q^n)}$ and $p^n \circ \Delta^n = \mathrm{Id}_{Q^n}$.

Consider the following diagram:

$$\begin{tikzcd}
 & \mathscr{E} \arrow[dl, "\pi_{\mathscr{E}}^n"'] \arrow[dr, "\pi_{\mathscr{E}}^{n+1}"] & \\
Q^n \times_G Q^n & & Q^{n+1} \times_G Q^{n+1} \arrow[ll, "\pi_Q^n"] \\
Q^n \arrow[u, "\Delta^n"] \arrow[uur, bend left=60, "\iota_{\mathscr{E}}^n"] \arrow[rr, dotted, bend right, "\iota_Q^n"'] & & Q^{n+1} \arrow[u, "\Delta^{n+1}"'] \arrow[ll, shift right, "s_Q^n"'] \arrow[ll, shift left, "t_Q^n"]
\end{tikzcd}$$

Define $\iota_Q^n : Q^n \to Q^{n+1}$ such that $\iota_Q^n := p^{n+1} \circ \pi_{\mathscr{E}}^{n+1} \circ \iota_{\mathscr{E}}^n$ satisfying $\iota_{\mathscr{E}}^n = \iota_{\mathscr{E}}^{n+k} \circ \iota_Q^{n+k-1} \circ \cdots \circ \iota_Q^{n+1} \circ \iota_Q^n$.

Thus, $\pi^{n+k} \circ \iota_{\mathscr{E}}^n = \pi^{n+k} \circ \iota_{\mathscr{E}}^{n+k} \circ \iota_Q^{n+k-1} \circ \cdots \circ \iota_Q^{n+1} \circ \iota_Q^n = \Delta^{n+k} \circ (\iota_Q^{n+k-1} \circ \cdots \circ \iota_Q^{n+1} \circ \iota_Q^n)$.

This implies that $\Delta^{n+1} \circ \iota^n_Q = \Delta^{n+1} \circ p^{n+1} \circ \pi^{n+1}_{\mathscr{E}} \circ \iota^n_{\mathscr{E}} = \pi^{n+1}_{\mathscr{E}} \circ \iota^n_{\mathscr{E}}$.

It follows that $\Delta^n = \pi^n_{\mathscr{E}} \circ \iota^n_{\mathscr{E}} = \pi^n_Q \circ \pi^{n+1}_{\mathscr{E}} \circ \iota^n_{\mathscr{E}} = \pi^n_Q \circ \Delta^{n+1} \circ \iota^n_Q$.

Thus, for every $x \in Q^n$,

$$\begin{aligned}
(s^n_Q \circ \iota^n_Q(x), t^n_Q \circ \iota^n_Q(x)) &= \pi^n_Q\big((\iota^n_Q(x), \iota^n_Q(x))\big) \\
&= \pi^n_Q \circ \Delta^{n+1} \circ \iota^n_Q(x) \\
&= \Delta^n(x) \\
&= (x, x).
\end{aligned}$$

Hence, $s^n_Q \circ \iota^n_Q(x) = x = t^n_Q \circ \iota^n_Q(x)$ and so $s^n_Q \circ \iota^n_Q = \mathrm{Id}_{Q^n} = t^n_Q \circ \iota^n_Q$.

Therefore, the ω-globular set is reflexive. $\square$

4.2 Free Self-Dual Globular-Cone ω-Magmas

The concepts of *involution* and *self-duality* are different in that involution is a self-dual map that satisfies the condition of involutivity. In this section we will prove the existence of a free self-dual globular-cone ω-magma over a globular cone via the following diagram

$$\begin{tikzcd}
\boxed{\text{Globular Cones}} \arrow[r, "{}_M F_C"'] \arrow[d, "{}_D F_C"] & \boxed{\text{Globular-Cone } \omega\text{-Magmas}} \arrow[d, "{}_{DM} F_M"'] \\
\boxed{\text{Self-Dual Globular Cones}} \arrow[r, "{}_{DM} F_D"] & \boxed{\text{Self-Dual Globular-Cone } \omega\text{-Magmas}}
\end{tikzcd}$$

that is, we will construct these four free functors and verify that this diagram commutes up to isomorphism.

As a result, we divide this section into three parts. The existence of a free self-dual globular cone and a free globular-cone ω-magma over a globular cone is considered in the first two subsections. The last subsection we investigate the notion of free self-dual globular-cone ω-magma over a self-dual globular cone, a globular-cone ω-magma, and a globular cone, respectively.

4.2.1 Free Self-Dual Globular Cones

We begin this subsection with the definition of *self-duality* of a globular cone.

Definition 4.2.1.1. A globular cone $\mathscr{E}$ is called **self-dual** if there exists $*_\alpha : \mathscr{E} \to \mathscr{E}$, for all $\alpha \subseteq \mathbb{N}_0$, such that

- $s^q_{\mathscr{E}}(f^{*_\alpha}) = t^q_{\mathscr{E}}(f)^{*^q_\alpha}$ and $t^q_{\mathscr{E}}(f^{*_\alpha}) = s^q_{\mathscr{E}}(f)^{*^q_\alpha}$ for every $q \in \alpha$ and $f \in \mathscr{E}$,
- $s^q_{\mathscr{E}}(f^{*_\alpha}) = s^q_{\mathscr{E}}(f)^{*^q_\alpha}$ and $t^q_{\mathscr{E}}(f^{*_\alpha}) = t^q_{\mathscr{E}}(f)^{*^q_\alpha}$ for every $q \notin \alpha$ and $f \in \mathscr{E}$.

Remark 4.2.1.2. Being $\mathscr{E}$ a self-dual globular cone implies partial self-duality in the ω-globular set.

Proposition 4.2.1.3. *A free self-dual globular cone over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{E}}"] \arrow[dll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

First of all, we construct a new ω-globular set that suits our situation.

For each $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we let

$$\hat{Q}^n := \{(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \mid y \in Q^n,\ \beta_j \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\}.$$

We also establish the new sources and targets in the new quiver as follows: for each $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \in \hat{Q}^{n+1}$, define $s^n_{\hat{Q}}, t^n_{\hat{Q}} : \hat{Q}^{n+1} \to \hat{Q}^n$ by

- $s^n_{\hat{Q}}\big((\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots(((s^n_Q(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & n \notin \beta_1 \triangle \beta_2 \triangle \cdots \triangle \beta_m; \\ (\cdots(((t^n_Q(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & n \in \beta_1 \triangle \beta_2 \triangle \cdots \triangle \beta_m. \end{cases}$

- $t^n_{\hat{Q}}\big((\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots(((t^n_Q(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & n \notin \beta_1 \triangle \beta_2 \triangle \cdots \triangle \beta_m; \\ (\cdots(((s^n_Q(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & n \in \beta_1 \triangle \beta_2 \triangle \cdots \triangle \beta_m. \end{cases}$

By Proposition 3.1.1.2, $\hat{Q}^0 \underset{t^0_{\hat{Q}}}{\overset{s^0_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^1 \underset{t^1_{\hat{Q}}}{\overset{s^1_{\hat{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\hat{Q}}}{\overset{s^{n-1}_{\hat{Q}}}{\leftleftarrows}} \hat{Q}^n \underset{t^n_{\hat{Q}}}{\overset{s^n_{\hat{Q}}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Next, we establish the following recursive family:

$$\begin{aligned} \hat{\mathscr{E}}^1 &= \{x^\alpha \mid x \in \mathscr{E}, \alpha \subseteq \mathbb{N}_0\}, \\ \hat{\mathscr{E}}^2 &= \{(x^\alpha)^\beta \mid x \in \mathscr{E}, \alpha, \beta \subseteq \mathbb{N}_0\}, \\ &\vdots \\ \hat{\mathscr{E}}^n &= \{(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_n} \mid x \in \mathscr{E},\ \alpha_i \subseteq \mathbb{N}_0,\ i = 1, \ldots, n\}, \\ &\vdots \end{aligned}$$

Then we define $\hat{\mathscr{E}} := \dot{\bigcup}_{n=1}^{\infty} \hat{\mathscr{E}}^n$.

To obtain a globular cone, we first define $s^n_{\hat{\mathscr{E}}}, t^n_{\hat{\mathscr{E}}} : \hat{\mathscr{E}} \to \hat{Q}^n$ by

- $s^n_{\hat{\mathscr{E}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$

- $t^n_{\hat{\mathscr{E}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$

for every $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \hat{\mathscr{E}}$.

It remains to show $s^n_{\hat{Q}} s^{n+1}_{\hat{\mathscr{E}}} = s^n_{\hat{\mathscr{E}}} = s^n_{\hat{Q}} t^{n+1}_{\hat{\mathscr{E}}}$ and $t^n_{\hat{Q}} s^{n+1}_{\hat{\mathscr{E}}} = t^n_{\hat{\mathscr{E}}} = t^n_{\hat{Q}} t^{n+1}_{\hat{\mathscr{E}}}$ for $n \in \mathbb{N}_0$.

Let $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \hat{\mathscr{E}}$.

By the compatibility of the globular cone $\mathscr{E}$, we have

$$
\begin{aligned}
& s^n_{\hat{Q}} s^{n+1}_{\hat{\mathscr{E}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) \\
= & \begin{cases}
(\cdots((s^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n, n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n+1; \\
(\cdots((t^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\
(\cdots((t^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n, n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m.
\end{cases} \\
= & \begin{cases}
(\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m.
\end{cases} \\
= & s^n_{\hat{\mathscr{E}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
& s^n_{\hat{Q}} t^{n+1}_{\hat{\mathscr{E}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) \\
= & \begin{cases}
(\cdots((s^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n, n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n+1; \\
(\cdots((t^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\
(\cdots((t^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\cdots})^{\alpha_m}, & n, n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m.
\end{cases} \\
= & \begin{cases}
(\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m.
\end{cases} \\
= & s^n_{\hat{\mathscr{E}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).
\end{aligned}
$$

Hence, $s^n_{\hat{Q}} s^{n+1}_{\hat{\mathscr{E}}} = s^n_{\hat{\mathscr{E}}} = s^n_{\hat{Q}} t^{n+1}_{\hat{\mathscr{E}}}$.

Similarly, we also get $t^n_{\hat{Q}} s^{n+1}_{\hat{\mathscr{E}}} = t^n_{\hat{\mathscr{E}}} = t^n_{\hat{Q}} t^{n+1}_{\hat{\mathscr{E}}}$.

As a result, we obtain a globular cone

$$
\begin{tikzcd}
 & & & & \hat{\mathscr{E}} \arrow[dllll, shift right, "s^0_{\hat{\mathscr{E}}}"'] \arrow[dllll, shift left, "t^0_{\hat{\mathscr{E}}}"] \arrow[dlll, shift right, "s^1_{\hat{\mathscr{E}}}"'] \arrow[dlll, shift left, "t^1_{\hat{\mathscr{E}}}"] \arrow[d, shift right, "s^n_{\hat{\mathscr{E}}}"'] \arrow[d, shift left, "t^n_{\hat{\mathscr{E}}}"] & \cdots \\
\hat{Q}^0 & \hat{Q}^1 \arrow[l, shift right, "s^0_{\hat{Q}}"'] \arrow[l, shift left, "t^0_{\hat{Q}}"] & \cdots \arrow[l, shift right, "s^1_{\hat{Q}}"'] \arrow[l, shift left, "t^1_{\hat{Q}}"] & & \hat{Q}^n \arrow[ll, shift right, "s^{n-1}_{\hat{Q}}"'] \arrow[ll, shift left, "t^{n-1}_{\hat{Q}}"] & \cdots \arrow[l, shift right, "s^n_{\hat{Q}}"'] \arrow[l, shift left, "t^n_{\hat{Q}}"]
\end{tikzcd}
$$

For $\alpha \subseteq \mathbb{N}_0$, define $*_\alpha : \hat{\mathscr{E}} \to \hat{\mathscr{E}}$ by $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}$.

We see that $(\hat{\mathscr{E}}, (*_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ is a self-dual globular cone.

Now consider a map $i : \mathscr{E} \to \hat{\mathscr{E}}$ by $x \mapsto x^{\varnothing}$ for each $x \in \mathscr{E}$.

Suppose $f : \mathscr{E} \to (\mathscr{F}, (\hat{*}_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ is a morphism into a self-dual globular cone.

Define $\psi : \hat{\mathscr{E}} \to \mathscr{F}$ by $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\cdots((f(x)^{\hat{*}\alpha_1})^{\hat{*}\alpha_2})^{\cdots})^{\hat{*}\alpha_m}$.

For every $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \hat{\mathscr{E}}$, we have

$$\begin{aligned} \psi\big(((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}\big) &= ((\cdots((f(x)^{\hat{*}\alpha_1})^{\hat{*}\alpha_2})^{\cdots})^{\hat{*}\alpha_m})^{\hat{*}\alpha} \\ &= \big(\psi((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)^{\hat{*}\alpha}. \end{aligned}$$

Hence, ψ is a morphism of self-dual globular cones satisfying $f = \psi \circ i$.

Assume that $\phi : \hat{\mathscr{E}} \to \mathscr{F}$ is another morphism of self-dual globular cones satisfying $f = \phi \circ i$.

So, $\psi\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) = (\cdots((f(x)^{\hat{*}\alpha_1})^{\hat{*}\alpha_2})^{\cdots})^{\hat{*}\alpha_m} = \phi\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$.

Hence, ψ is a unique morphism of self-dual globular cones satisfying $f = \psi \circ i$.

Thus, $\Big((\hat{\mathscr{E}}, (*_\alpha)_{\alpha \subseteq \mathbb{N}_0}), i\Big)$ is a free self-dual globular cone over a globular cone.

□

4.2.2 Free Globular-Cone ω-Magmas

Definition 4.2.2.1. A globular cone $\mathscr{E}$ is called a **globular-cone ω-magma** if there exists a family of partially-defined operations $\circ_p : \mathscr{E} \times_p \mathscr{E} \to \mathscr{E}$, for $p = 0, 1, \ldots$, where

$$\mathscr{E} \times_p \mathscr{E} := \{(x,y) \in \mathscr{E} \times \mathscr{E} \mid s^p_{\mathscr{E}}(x) = t^p_{\mathscr{E}}(y)\},$$

by $(x,y) \mapsto x \circ_p y$ for all $(x,y) \in \mathscr{E} \times_p \mathscr{E}$ satisfying the compatibility between compositions and sources and targets as follows:

- $s^q_{\mathscr{E}}(x \circ_p y) = s^q_{\mathscr{E}}(x) \circ^q_p s^q_{\mathscr{E}}(y)$ and $t^q_{\mathscr{E}}(x \circ_p y) = t^q_{\mathscr{E}}(x) \circ^q_p t^q_{\mathscr{E}}(y)$ for all $q > p$,
- $s^q_{\mathscr{E}}(x \circ_p y) = s^q_{\mathscr{E}}(y)$ and $t^q_{\mathscr{E}}(x \circ_p y) = t^q_{\mathscr{E}}(x)$ for all $q \leq p$.

Remark 4.2.2.2. Being $\mathscr{E}$ a globular-cone ω-magma implies partially-defined compositions in the ω-globular set (the first condition).

Proposition 4.2.2.3. *A free globular-cone ω-magma over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$
\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{E}}"] \arrow[dll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}
$$

First of all, we construct a new ω-globular set that suits our situation.

Setting $k, n \in \mathbb{N}$ and $\langle Q^0 \rangle := Q^0$, we construct the following recursive family.

Consider $\langle Q^1[1] \rangle := Q^1$, $s^0[1] := s^0_Q$, and $t^0[1] := t^0_Q$.

Now let $\langle Q^1[2] \rangle := \{(x,0,y) \mid x,y \in \langle Q^1[1] \rangle,\ s^0[1](x) = t^0[1](y)\}$.

Define $s^0[2] : \langle Q^1[2] \rangle \to \langle Q^0 \rangle$ by $s^0[2]\big((x,0,y)\big) := s^0[1](y)$

and $t^0[2] : \langle Q^1[2] \rangle \to \langle Q^0 \rangle$ by $t^0[2]\big((x,0,y)\big) := t^0[1](x)$.

Suppose that we have $\langle Q^1[l] \rangle$, $s^0[l]$, and $t^0[l]$ for each $l = 1, 2, \ldots, k-1$

Let $\langle Q^1[k] \rangle := \{(x,0,y) \mid x \in \langle Q^1[i] \rangle, y \in \langle Q^1[j] \rangle, i+j = k, s^0[i](x) = t^0[j](y)\}$.

If $(x,0,y) \in \langle Q^1[k] \rangle$, we define $s^0[k], t^0[k] : \langle Q^1[k] \rangle \to \langle Q^0 \rangle$ by $s^0[k]\big((x,0,y)\big) := s^0[j](y)$ and $t^0[k]\big((x,0,y)\big) := t^0[i](x)$.

Set $\langle Q^1 \rangle := \dot{\bigcup}_{k=1}^{\infty} \langle Q^1[k] \rangle$, $s^0_{\langle Q \rangle} := \dot{\bigcup}_{k=1}^{\infty} s^0[k]$, and $t^0_{\langle Q \rangle} := \dot{\bigcup}_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle Q^m \rangle$, $s^{m-1}_{\langle Q \rangle}$, and $t^{m-1}_{\langle Q \rangle}$ for every $m = 1, 2, \ldots, n-1$.

Consider $\langle Q^n[1] \rangle := Q^n$, $s^{n-1}[1] := s^{n-1}_Q$, and $t^{n-1}[1] := t^{n-1}_Q$.

Let $\langle Q^n[2] \rangle := \dot{\bigcup}_{p=0}^{n-1} \{(x,p,y) \mid x,y \in \langle Q^n[1] \rangle, s^p_Q \cdots s^{n-1}[1](x) = t^p_Q \cdots t^{n-1}[1](y)\}$.

If $(x,p,y) \in \langle Q^n[2] \rangle$, then we define $s^{n-1}[2], t^{n-1}[2] : \langle Q^n[2] \rangle \to \langle Q^{n-1} \rangle$ by

- $s^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[1](x), p, s^{n-1}[1](y)), & n-1 > p; \\ s^{n-1}[1](y), & n-1 = p. \end{cases}$

- $t^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[1](x), p, t^{n-1}[1](y)), & n-1 > p; \\ t^{n-1}[1](x), & n-1 = p. \end{cases}$

Suppose that we have $\langle Q^n[l]\rangle$, $s^{n-1}[l]$, and $t^{n-1}[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle Q^n[k]\rangle := \dot{\bigcup\limits_{p=0}^{n-1}} \{(x,p,y) \mid x \in \langle Q^n[i]\rangle\,,\ y \in \langle Q^n[j]\rangle\,,\ i+j=k,$

$$s^p s^{p+1} \cdots s^{n-1}[i](x) = t^p t^{p+1} \cdots t^{n-1}[j](y)\}.$$

If $(x,p,y) \in \langle Q^n[k]\rangle$, then we define $s^{n-1}[k], t^{n-1}[k] : \langle Q^n[k]\rangle \to \left\langle Q^{n-1}\right\rangle$ by

- $s^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[i](x), p, s^{n-1}[j](y)), & n-1 > p; \\ s^{n-1}[j](y), & n-1 = p. \end{cases}$

- $t^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[i](x), p, t^{n-1}[j](y)), & n-1 > p; \\ t^{n-1}[i](x), & n-1 = p. \end{cases}$

Set $\langle Q^n\rangle := \dot{\bigcup\limits_{k=1}^{\infty}} \langle Q^n[k]\rangle$, $s^{n-1}_{\langle Q\rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} s^{n-1}[k]$, and $t^{n-1}_{\langle Q\rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} t^{n-1}[k]$.

By Proposition 3.1.2.1, $\left\langle Q^0\right\rangle \underset{t^0_{\langle Q\rangle}}{\overset{s^0_{\langle Q\rangle}}{\leftleftarrows}} \left\langle Q^1\right\rangle \underset{t^1_{\langle Q\rangle}}{\overset{s^1_{\langle Q\rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle Q\rangle}}{\overset{s^{n-1}_{\langle Q\rangle}}{\leftleftarrows}} \left\langle Q^n\right\rangle \underset{t^n_{\langle Q\rangle}}{\overset{s^n_{\langle Q\rangle}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Next, we establish the following recursive family.

For our convenience, we assume $q \in \mathbb{N}_0$.

Let $\langle \mathscr{E}[1]\rangle := \{(z) \mid z \in \mathscr{E}\}$, $s^q[1]((z)) := s^q_{\mathscr{E}}(z)$, and $t^q[1]((z)) := t^q_{\mathscr{E}}(z)$ for $z \in \mathscr{E}$.

Now set $\langle \mathscr{E}[2]\rangle := \dot{\bigcup\limits_{p=0}^{\infty}} \big\{(x,p,y) \mid x,y \in \langle \mathscr{E}[1]\rangle\,,\ s^p[1](x) = t^p[1](y)\big\}$.

If $(x,p,y) \in \langle \mathscr{E}[2]\rangle$, then we define $s^q[2], t^q[2] : \langle \mathscr{E}[2]\rangle \to \langle Q^q\rangle$ by

- $s^q[2]\big((x,p,y)\big) := \begin{cases} (s^q[1](x), p, s^q[1](y)), & q > p; \\ s^q[1](y), & q \le p. \end{cases}$

- $t^q[2]\big((x,p,y)\big) := \begin{cases} (t^q[1](x), p, t^q[1](y)), & q > p; \\ t^q[1](x), & q \le p. \end{cases}$

Suppose that we have $\langle\mathscr{E}[k]\rangle$, $s^q[k]$, and $t^q[k]$ for all $k = 1,2,\ldots,n-1$.

Let $\langle\mathscr{E}[n]\rangle := \dot{\bigcup\limits_{p=0}^{\infty}} \big\{(x,p,y) \mid x \in \langle\mathscr{E}[i]\rangle\,,\, y \in \langle\mathscr{E}[j]\rangle\,,\, i+j=n,\, s^p[i](x) = t^p[j](y)\big\}$.

If $(x,p,y) \in \langle\mathscr{E}[n]\rangle$, then we define $s^q[n], t^q[n] : \langle\mathscr{E}[n]\rangle \to \langle Q^q\rangle$ by

- $s^q[n]\big((x,p,y)\big) := \begin{cases} (s^q[i](x), p, s^q[j](y)), & q > p; \\ s^q[j](y), & q \leq p. \end{cases}$

- $t^q[n]\big((x,p,y)\big) := \begin{cases} (t^q[i](x), p, t^q[j](y)), & q > p; \\ t^q[i](x), & q \leq p. \end{cases}$

Let $\langle\mathscr{E}\rangle := \dot{\bigcup\limits_{n=1}^{\infty}} \langle\mathscr{E}[n]\rangle$, $s^q_{\langle\mathscr{E}\rangle} := \dot{\bigcup\limits_{n=1}^{\infty}} s^q[n]$, and $t^q_{\langle\mathscr{E}\rangle} := \dot{\bigcup\limits_{n=1}^{\infty}} t^q[n]$.

It remains to show that $s^n_{\langle Q\rangle} s^{n+1}_{\langle\mathscr{E}\rangle} = s^n_{\langle\mathscr{E}\rangle} = s^n_{\langle Q\rangle} t^{n+1}_{\langle\mathscr{E}\rangle}$ and $t^n_{\langle Q\rangle} s^{n+1}_{\langle\mathscr{E}\rangle} = t^n_{\langle\mathscr{E}\rangle} = t^n_{\langle Q\rangle} t^{n+1}_{\langle\mathscr{E}\rangle}$ for all $n \in \mathbb{N}_0$.

Suppose that $n \in \mathbb{N}_0$ and $(x,p,y) \in \langle\mathscr{E}\rangle$.

Then $(x,p,y) \in \langle\mathscr{E}[k]\rangle$ for some $k \in \mathbb{N}\setminus\{1\}$.

Note that the case $k=1$ holds immediately owing to the globularity of the cone.

We will only prove $s^n_{\langle Q\rangle} s^{n+1}[k]\big((x,p,y)\big) = s^n[k]\big((x,p,y)\big) = s^n_{\langle Q\rangle} t^{n+1}[k]\big((x,p,y)\big)$ while the remaining part can be similarly discussed.

If $k=2$, then $x, y \in \mathscr{E}$ and

$$\begin{aligned} s^{n+1}[2]\big((x,p,y)\big) &= \begin{cases} (s^{n+1}[1](x), p, s^{n+1}[1](y)), & n+1 > p; \\ s^{n+1}[1](y), & n+1 \leq p. \end{cases} \\ &= \begin{cases} (s^{n+1}_{\mathscr{E}}(x), p, s^{n+1}_{\mathscr{E}}(y)), & n+1 > p; \\ s^{n+1}_{\mathscr{E}}(y), & n+1 \leq p. \end{cases} \\ t^{n+1}[2]\big((x,p,y)\big) &= \begin{cases} (t^{n+1}_{\mathscr{E}}(x), p, t^{n+1}_{\mathscr{E}}(y)), & n+1 > p; \\ t^{n+1}_{\mathscr{E}}(x), & n+1 \leq p. \end{cases} \end{aligned}$$

It follows from the globularity of the cone that

$$
\begin{aligned}
s^n_{\langle Q\rangle} s^{n+1}[2]\big((x,p,y)\big) &= \begin{cases} (s^n_Q s^{n+1}_{\mathscr{E}}(x), p, s^n_Q s^{n+1}_{\mathscr{E}}(y)), & n+1>p \wedge n>p; \\ s^n_Q s^{n+1}_{\mathscr{E}}(y), & (n+1>p \wedge n \le p) \vee n+1 \le p. \end{cases} \\
&= \begin{cases} (s^n_{\mathscr{E}}(x), p, s^n_{\mathscr{E}}(y)), & n>p; \\ s^n_{\mathscr{E}}(y), & n \le p. \end{cases} \\
&= s^n[2]\big((x,p,y)\big)
\end{aligned}
$$

Note that $n<p$ and $k=2$ imply $s^p_{\mathscr{E}}(x) = t^p_{\mathscr{E}}(y)$ and

$$
s^n_{\mathscr{E}}(x) = s^n_Q s^{n+1}_Q \cdots s^{p-1}_Q s^p_{\mathscr{E}}(x) = s^n_Q s^{n+1}_Q \cdots s^{p-1}_Q t^p_{\mathscr{E}}(y) = s^n_{\mathscr{E}}(y).
$$

We also have the other part of equalities

$$
\begin{aligned}
s^n_{\langle Q\rangle} t^{n+1}[2]\big((x,p,y)\big) &= \begin{cases} (s^n_Q t^{n+1}_{\mathscr{E}}(x), p, s^n_Q t^{n+1}_{\mathscr{E}}(y)), & n+1>p \wedge n>p; \\ s^n_Q t^{n+1}_{\mathscr{E}}(y), & (n+1>p \wedge n \le p) \vee n+1 \le p. \\ s^n_Q t^{n+1}_{\mathscr{E}}(x), & n+1 \le p. \end{cases} \\
&= \begin{cases} (s^n_{\mathscr{E}}(x), p, s^n_{\mathscr{E}}(y)), & n>p; \\ s^n_{\mathscr{E}}(y), & n=p; \\ s^n_{\mathscr{E}}(x), & n<p. \end{cases} \\
&= \begin{cases} (s^n_{\mathscr{E}}(x), p, s^n_{\mathscr{E}}(y)), & n>p; \\ s^n_{\mathscr{E}}(y), & n \le p. \end{cases} \\
&= s^n[2]\big((x,p,y)\big).
\end{aligned}
$$

Thus, $s^n_{\langle Q\rangle} s^{n+1}[2]\big((x,p,y)\big) = s^n[2]\big((x,p,y)\big) = s^n_{\langle Q\rangle} t^{n+1}[2]\big((x,p,y)\big)$.

Assume that $s^n_{\langle Q\rangle} s^{n+1}[m]\big((x,p,y)\big) = s^n[m]\big((x,p,y)\big) = s^n_{\langle Q\rangle} t^{n+1}[m]\big((x,p,y)\big)$ holds for every $m = 1,2,\ldots,k-1$.

Recall that

$$
\begin{aligned}
s^{n+1}[k]\big((x,p,y)\big) &= \begin{cases} (s^{n+1}[i](x), p, s^{n+1}[j](y)), & n+1>p; \\ s^{n+1}[j](y), & n+1 \le p. \end{cases} \\
t^{n+1}[k]\big((x,p,y)\big) &= \begin{cases} (t^{n+1}[i](x), p, t^{n+1}[j](y)), & n+1>p; \\ t^{n+1}[i](x), & n+1 \le p. \end{cases}
\end{aligned}
$$

It follows from assumption that

$$
\begin{aligned}
s^n_{\langle Q\rangle} s^{n+1}[k]\big((x,p,y)\big) &= \begin{cases} (s^n_Q s^{n+1}[i](x), p, s^n_Q s^{n+1}[j](y)), & n > p; \\ s^n_Q s^{n+1}[j](y), & n \leq p. \end{cases} \\
&= \begin{cases} (s^n[i](x), p, s^n[j](y)), & n > p; \\ s^n[j](y), & n \leq p. \end{cases} \\
&= s^n[k]\big((x,p,y)\big).
\end{aligned}
$$

Note that $n < p$ and $s^p[i](x) = t^p[j](y)$ imply

$$
s^n[i](x) = s^n_{\langle Q\rangle} s^{n+1}_{\langle Q\rangle} \cdots s^{p-1}_{\langle Q\rangle} s^p[i](x) = s^n_{\langle Q\rangle} s^{n+1}_{\langle Q\rangle} \cdots s^{p-1}_{\langle Q\rangle} t^p[j](y) = s^n[j](y).
$$

We also have the other part of equalities

$$
\begin{aligned}
s^n_{\langle Q\rangle} t^{n+1}[k]\big((x,p,y)\big) &= \begin{cases} (s^n_Q t^{n+1}[i](x), p, s^n_Q t^{n+1}[j](y)), & n > p; \\ s^n_Q t^{n+1}[j](y), & n \leq p. \\ s^n_Q t^{n+1}[i](x), & n < p. \end{cases} \\
&= \begin{cases} (s^n[i](x), p, s^n[j](y)), & n > p; \\ s^n[j](y), & n = p; \\ s^n[i](x), & n < p. \end{cases} \\
&= \begin{cases} (s^n[i](x), p, s^n[j](y)), & n > p; \\ s^n[j](y), & n \leq p. \end{cases} \\
&= s^n[k]\big((x,p,y)\big).
\end{aligned}
$$

This means that $s^n_{\langle Q\rangle} s^{n+1}[k]\big((x,p,y)\big) = s^n[k]\big((x,p,y)\big) = s^n_{\langle Q\rangle} t^{n+1}[k]\big((x,p,y)\big)$.

Applying similar argument, we get

$t^n_{\langle Q\rangle} s^{n+1}[k]\big((x,p,y)\big) = t^n[k]\big((x,p,y)\big) = t^n_{\langle Q\rangle} t^{n+1}[k]\big((x,p,y)\big)$.

Thus, $s^n_{\langle Q\rangle} s^{n+1}_{\langle \mathscr{E}\rangle} = s^n_{\langle \mathscr{E}\rangle} = s^n_{\langle Q\rangle} t^{n+1}_{\langle \mathscr{E}\rangle}$ and $t^n_{\langle Q\rangle} s^{n+1}_{\langle \mathscr{E}\rangle} = t^n_{\langle \mathscr{E}\rangle} = t^n_{\langle Q\rangle} t^{n+1}_{\langle \mathscr{E}\rangle}$ for all $n \in \mathbb{N}_0$.

As a consequence,

$$
\begin{tikzcd}[column sep=large]
 & & & \langle\mathscr{E}\rangle \arrow[dlll, shift right, "s^0_{\langle\mathscr{E}\rangle}"'] \arrow[dlll, shift left, "t^0_{\langle\mathscr{E}\rangle}"] \arrow[dll, shift right, "s^1_{\langle\mathscr{E}\rangle}"'] \arrow[dll, shift left, "t^1_{\langle\mathscr{E}\rangle}"] \arrow[d, shift right, "s^n_{\langle\mathscr{E}\rangle}"'] \arrow[d, shift left, "t^n_{\langle\mathscr{E}\rangle}"] & \cdots \\
\langle Q^0\rangle & \langle Q^1\rangle \arrow[l, shift right, "s^0_{\langle Q\rangle}"'] \arrow[l, shift left, "t^0_{\langle Q\rangle}"] & \cdots \arrow[l, shift right, "s^1_{\langle Q\rangle}"'] \arrow[l, shift left, "t^1_{\langle Q\rangle}"] & \langle Q^n\rangle \arrow[l, shift right, "s^{n-1}_{\langle Q\rangle}"'] \arrow[l, shift left, "t^{n-1}_{\langle Q\rangle}"] & \cdots \arrow[l, shift right, "s^n_{\langle Q\rangle}"'] \arrow[l, shift left, "t^n_{\langle Q\rangle}"]
\end{tikzcd}
$$

is a globular cone.

For all $p \in \mathbb{N}_0$, set

$$\langle\mathscr{E}\rangle \times_p \langle\mathscr{E}\rangle := \{(x,y) \in \langle\mathscr{E}\rangle \times \langle\mathscr{E}\rangle \mid s^p_{\langle\mathscr{E}\rangle}(x) = t^p_{\langle\mathscr{E}\rangle}(y)\},$$

Define a family of partial operations $\circ_p : \langle\mathscr{E}\rangle \times_p \langle\mathscr{E}\rangle \to \langle\mathscr{E}\rangle$ by $x \circ_p y \mapsto (x,p,y)$.

We see that $(\langle\mathscr{E}\rangle, (\circ_p)_{p\in\mathbb{N}_0})$ is a globular-cone ω-magma.

Define a map $i : \mathscr{E} \to \langle\mathscr{E}\rangle$ by $x \mapsto (x)$.

Suppose that $f : \mathscr{E} \to (\mathscr{M}, (\hat{\circ}_p)_{p\in\mathbb{N}_0})$ is a morphism from the original globular cone into another globular-cone ω-magma.

Define $\psi : \langle\mathscr{E}\rangle \to \mathscr{M}$ recursively by

$$
\begin{array}{rcl}
\langle\mathscr{E}[1]\rangle \ni (x) & \mapsto & f(x) \\
\langle\mathscr{E}[2]\rangle \ni ((x),p,(y)) & \mapsto & f(x)\hat{\circ}_p f(y) \\
\langle\mathscr{E}[3]\rangle \ni \big((x),p,((y),q,(z))\big) & \mapsto & f(x)\hat{\circ}_p\big(f(y)\hat{\circ}_q f(z)\big) \\
\langle\mathscr{E}[3]\rangle \ni \big(((x),p,(y)),q,(z)\big) & \mapsto & \big(f(x)\hat{\circ}_p f(y)\big)\hat{\circ}_q f(z) \\
\vdots & & \vdots
\end{array}
$$

Next, we will verify that $\psi\big((x,p,y)\big) = \psi(x)\hat{\circ}_p\psi(y)$ for every $(x,p,y) \in \langle\mathscr{E}\rangle$.

Notice that if $(x,p,y) \in \langle\mathscr{E}\rangle$, then $x \in \langle\mathscr{E}[i]\rangle$ and $y \in \langle\mathscr{E}[j]\rangle$, where $i+j=k$ for some $k \in \mathbb{N}\setminus\{1\}$.

If $i=j=1$, we have $\psi\big(((x),p,(y))\big) = f(x)\hat{\circ}_p f(y) = \psi((x))\hat{\circ}_p\psi((y))$.

Assume that this equation holds for $i=1$ and $j=1,2,\ldots,n-1$.

We have $\psi\big(((x),p,y)\big) = f(x)\hat{\circ}_p\psi(y) = \psi((x))\hat{\circ}_p\psi(y)$.

Now suppose that the equation holds for $i = 1,2,\ldots,m-1$ and $j \in \mathbb{N}$.

We have $\psi\big((x,p,y)\big) = \psi(x)\hat{\circ}_p\psi(y)$.

This means that $\psi\big((x,p,y)\big) = \psi(x)\hat{\circ}_p\psi(y)$ for each $(x,p,y) \in \langle\mathscr{E}\rangle$.

Thus, ψ is a morphism of globular-cone ω-magmas satisfying $f = \psi \circ i$.

We see that ψ is the only morphism of globular-cone ω-magmas holding $f = \psi \circ i$.

As a result, $\big((\langle\mathscr{E}\rangle, (\circ_p)_{p\in\mathbb{N}_0}), i\big)$ is a free globular-cone ω-magma over a globular cone. □

4.2.3 Free Self-Dual Globular-Cone ω-Magmas

In the previous subsections we have already proved the existence of a free self-dual globular cone and a free globular-cone ω-magma over a globular cone. Applying these two propositions, we obtain a free *self-dual globular-cone* ω*-magma*. But the notion of self-dual globular-cone ω-magmas need to be defined first.

Definition 4.2.3.1. A globular cone $\mathscr{E}$ is said to be a **self-dual globular-cone ω-magma** if there exist $\circ_p : \mathscr{E} \times_p \mathscr{E} \to \mathscr{E}$ and $*_\alpha : \mathscr{E} \to \mathscr{E}$, for all $p \in \mathbb{N}_0$ and $\alpha \subseteq \mathbb{N}_0$, such that $(\mathscr{E}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0})$ is a self-dual globular cone and also $(\mathscr{E}, (\circ_p)_{p\in\mathbb{N}_0})$ is a globular-cone ω-magma

We will prove first that a free self-dual globular-cone ω-cone over both a self-dual globular cone and a globular-cone ω-magma exists. Later we will show that both of them satisfy the universal factorization property of free self-dual globular-cone ω-magmas.

Proposition 4.2.3.2. *A free self-dual globular-cone* ω*-magma over a self-dual globular cone exists.*

Proof. Let a self-dual globular cone $(\hat{\mathscr{E}}, (\bar{*}_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ over an ω-globular set be given

$$\begin{tikzcd}
 & & & \hat{\mathscr{E}} \arrow[dlll, shift left, "s^0_{\hat{\mathscr{E}}}"'] \arrow[dlll, shift right, "t^0_{\hat{\mathscr{E}}}"] \arrow[dll, shift left, "s^1_{\hat{\mathscr{E}}}"'] \arrow[dll, shift right, "t^1_{\hat{\mathscr{E}}}"] \arrow[d, shift right, "s^n_{\hat{\mathscr{E}}}"'] \arrow[d, shift left, "t^n_{\hat{\mathscr{E}}}"] & \\
\hat{Q}^0 & \hat{Q}^1 \arrow[l, shift right, "s^0_{\hat{Q}}"'] \arrow[l, shift left, "t^0_{\hat{Q}}"] & \cdots \arrow[l, shift right, "s^1_{\hat{Q}}"'] \arrow[l, shift left, "t^1_{\hat{Q}}"] & \hat{Q}^n \arrow[l, shift right, "s^{n-1}_{\hat{Q}}"'] \arrow[l, shift left, "t^{n-1}_{\hat{Q}}"] & \cdots \arrow[l, shift right, "s^n_{\hat{Q}}"'] \arrow[l, shift left, "t^n_{\hat{Q}}"]
\end{tikzcd}$$

First, we establish a new ω-globular set that fits our situation.

For any $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we first set

$$\hat{Q}^n := \{(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \mid y \in Q^n,\ \beta_j \subseteq \mathbb{N}_0,\ j = 1, 2, \ldots, m\}.$$

Let $k, m, n \in \mathbb{N}$, $\langle \hat{Q}^0 \rangle := \hat{Q}^0$, and $\langle \hat{Q}^1[1] \rangle := \hat{Q}^1$.

Define $s^0[1], t^0[1] : \langle \hat{Q}^1[1] \rangle \to \langle \hat{Q}^0 \rangle$ by

- $s^0[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((s^0_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & 0 \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((t^0_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & 0 \in \beta_1 \triangle \cdots \triangle \beta_m. \end{cases}$

- $t^0[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((t^0_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & 0 \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((s^0_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & 0 \in \beta_1 \triangle \cdots \triangle \beta_m. \end{cases}$

Now let $\langle \hat{Q}^1[2] \rangle := \{(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x, y \in \langle \hat{Q}^1[1] \rangle,\ \alpha_j \subseteq \mathbb{N}_0,$

$$j = 1, 2, \ldots, m,\ s^0[1](x) = t^0[1](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \in \langle \hat{Q}^1[2] \rangle$, we define $s^0[2], t^0[2] : \langle \hat{Q}^1[2] \rangle \to \langle \hat{Q}^0 \rangle$ by

- $s^0[2]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((s^0[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^0[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

- $t^0[2]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((t^0[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^0[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

Suppose that we have $\langle \hat{Q}^1[l] \rangle$, $s^0[l]$, and $t^0[l]$ for every $l = 1,2,\ldots,k-1$.

Let $\langle \hat{Q}^1[k] \rangle := \{(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x \in \langle \hat{Q}^1[i] \rangle,\ y \in \langle \hat{Q}^1[j] \rangle,\ i+j=k,$

$$\alpha_h \subseteq \mathbb{N}_0,\ h=1,2,\ldots,m,\ s^0[i](x) = t^0[j](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \in \langle \hat{Q}^1[k] \rangle$, we define $s^0[k], t^0[k] : \langle \hat{Q}^1[k] \rangle \to \langle \hat{Q}^0 \rangle$ by

- $s^0[k]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((s^0[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^0[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

- $t^0[k]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((t^0[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^0[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

Set $\langle \hat{Q}^1 \rangle := \dot{\bigcup}_{k=1}^{\infty} \langle \hat{Q}^1[k] \rangle$, $s^0_{\langle \hat{Q} \rangle} := \dot{\bigcup}_{k=1}^{\infty} s^0[k]$, and $t^0_{\langle \hat{Q} \rangle} := \dot{\bigcup}_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle \hat{Q}^r \rangle$, $s^{r-1}_{\langle \hat{Q} \rangle}$, and $t^{r-1}_{\langle \hat{Q} \rangle}$ for every $r = 1,2,\ldots,n-1$.

Let $\langle \hat{Q}^n[1] \rangle := \hat{Q}^n$.

Define $s^{n-1}[1], t^{n-1}[1] : \langle \hat{Q}^n[1] \rangle \to \langle \hat{Q}^{n-1} \rangle$ by

- $s^{n-1}[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((s^{n-1}_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & n-1 \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((t^{n-1}_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & n-1 \in \beta_1 \triangle \cdots \triangle \beta_m. \end{cases}$

- $t^{n-1}[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((t^{n-1}_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & n-1 \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((s^{n-1}_{\hat{Q}}(y))^{\beta_1})^{\cdots})^{\beta_m}, & n-1 \in \beta_1 \triangle \cdots \triangle \beta_m. \end{cases}$

Now let $\langle \hat{Q}^n[2] \rangle := \dot{\bigcup}_{p=0}^{n-1} \{(\cdots((x,p,y)^{\alpha_1^n})^{\cdots})^{\alpha_m^n} \mid x,y \in \langle \hat{Q}^n[1] \rangle,\ \alpha_j \subseteq \mathbb{N}_0,$

$$j=1,2,\ldots,m,\ s^p[1]\cdots s^{n-1}[1](x) = t^p[1]\cdots t^{n-1}[1](y)\}.$$

If $(\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m} \in \langle \hat{Q}^n[2] \rangle$, define $s^{n-1}[2], t^{n-1}[2] : \langle \hat{Q}^n[2] \rangle \to \langle \hat{Q}^{n-1} \rangle$ by

$$s^{n-1}[2]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((s^{n-1}[1](x),p,s^{n-1}[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((t^{n-1}[1](x),p,t^{n-1}[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\in \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((s^{n-1}[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((t^{n-1}[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\in \alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

$$t^{n-1}[2]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((t^{n-1}[1](x),p,t^{n-1}[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((s^{n-1}[1](x),p,s^{n-1}[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\in \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((t^{n-1}[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((s^{n-1}[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\in \alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

Suppose that we have $\langle\hat{Q}^n[l]\rangle$, $s^{n-1}[l]$, and $t^{n-1}[l]$ for every $l=1,2,\ldots,k-1$.

Let $\langle\hat{Q}^n[k]\rangle := \bigcup_{p=0}^{n-1}\{(\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x\in\langle\hat{Q}^n[i]\rangle,\ y\in\langle\hat{Q}^n[j]\rangle,\ i+j=k,$

$\alpha_h\subseteq\mathbb{N}_0,\ h=1,2,\ldots,m,\ s^p[i]\cdots s^{n-1}[i](x)=t^p[j]\cdots t^{n-1}[j](y)\}.$

If $(\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\in\langle\hat{Q}^n[k]\rangle$, define $s^{n-1}[k],t^{n-1}[k]:\langle\hat{Q}^n[k]\rangle\to\langle\hat{Q}^{n-1}\rangle$ by

$$s^{n-1}[k]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((s^{n-1}[i](x),p,s^{n-1}[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((t^{n-1}[i](x),p,t^{n-1}[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\in \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((s^{n-1}[i](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((t^{n-1}[i](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\in \alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

$$t^{n-1}[k]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((t^{n-1}[i](x),p,t^{n-1}[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((s^{n-1}[i](x),p,s^{n-1}[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n-1\in \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((t^{n-1}[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\notin \alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots((s^{n-1}[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n-1\in \alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

Set $\langle \hat{Q}^n \rangle := \dot{\bigcup}_{k=1}^{\infty} \langle \hat{Q}^n[k] \rangle$, $s^{n-1}_{\langle \hat{Q} \rangle} := \dot{\bigcup}_{k=1}^{\infty} s^{n-1}[k]$, and $t^{n-1}_{\langle \hat{Q} \rangle} := \dot{\bigcup}_{k=1}^{\infty} t^{n-1}[k]$.

By Proposition 3.1.3.1, we have that $\langle \hat{Q}^0 \rangle \underset{t^0_{\langle \hat{Q} \rangle}}{\overset{s^0_{\langle \hat{Q} \rangle}}{\leftleftarrows}} \langle \hat{Q}^1 \rangle \underset{t^1_{\langle \hat{Q} \rangle}}{\overset{s^1_{\langle \hat{Q} \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle \hat{Q} \rangle}}{\overset{s^{n-1}_{\langle \hat{Q} \rangle}}{\leftleftarrows}} \langle \hat{Q}^n \rangle \underset{t^n_{\langle \hat{Q} \rangle}}{\overset{s^n_{\langle \hat{Q} \rangle}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Then we construct the following recursive family.

For our convenience, we suppose $q \in \mathbb{N}_0$.

Consider $\left\langle \hat{\mathscr{E}}[1] \right\rangle := \{(z) \mid z \in \hat{\mathscr{E}}\}$ and $s^q[1], t^q[1] : \left\langle \hat{\mathscr{E}}[1] \right\rangle \to \langle \hat{Q}^1 \rangle$ which are defined by $s^q[1]((z)) := s^q_{\hat{\mathscr{E}}}(z)$, and $t^q[1]((z)) := t^q_{\hat{\mathscr{E}}}(z)$ for $z \in \hat{\mathscr{E}}$.

Now let $\left\langle \hat{\mathscr{E}}[2] \right\rangle := \dot{\bigcup}_{p=0}^{\infty} \big\{ (\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x, y \in \left\langle \hat{\mathscr{E}}[1] \right\rangle,$

$s^p[1](x) = t^p[1](y),\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0 \big\}.$

If $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \left\langle \hat{\mathscr{E}}[2] \right\rangle$, then we define

- $s^q[2] : \left\langle \hat{\mathscr{E}}[2] \right\rangle \to \langle \hat{Q}^q \rangle$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((s^q[1](x), p, s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[1](x), p, t^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[1](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m, \end{cases}$$

- $t^q[2] : \left\langle \hat{\mathscr{E}}[2] \right\rangle \to \langle \hat{Q}^q \rangle$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((t^q[1](x), p, t^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[1](x), p, s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[1](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$$

Suppose that we have $\left\langle \hat{\mathscr{E}}[k] \right\rangle$, $s^q[k]$, and $t^q[k]$ for all $k = 1, 2, \ldots, n-1$.

Let $\left\langle \hat{\mathscr{E}}[n] \right\rangle := \dot{\bigcup}_{p=0}^{\infty} \big\{ (\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \left\langle \hat{\mathscr{E}}[i] \right\rangle,\ y \in \left\langle \hat{\mathscr{E}}[j] \right\rangle,$
$i + j = n,\ s^p[i](x) = t^p[j](y),\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0 \big\}.$

If $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \left\langle \hat{\mathscr{E}}[n] \right\rangle$, then we define

- $s^q[n] : \left\langle \hat{\mathscr{E}}[n] \right\rangle \to \langle \hat{Q}^q \rangle$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$
$$\mapsto \begin{cases} (\cdots(((s^q[i](x),p,s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[i](x),p,t^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m, \end{cases}$$

- $t^q[n] : \left\langle \hat{\mathscr{E}}[n] \right\rangle \to \langle \hat{Q}^q \rangle$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$
$$\mapsto \begin{cases} (\cdots(((t^q[i](x),p,t^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[i](x),p,s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$$

Let $\left\langle \hat{\mathscr{E}} \right\rangle := \dot{\bigcup}_{n=1}^{\infty} \left\langle \hat{\mathscr{E}}[n] \right\rangle$, $s^q_{\langle \hat{\mathscr{E}} \rangle} := \dot{\bigcup}_{n=1}^{\infty} s^q[n]$, and $t^q_{\langle \hat{\mathscr{E}} \rangle} := \dot{\bigcup}_{n=1}^{\infty} t^q[n]$.

It remains to show that $s^n_{\langle \hat{Q} \rangle} s^{n+1}_{\langle \hat{\mathscr{E}} \rangle} = s^n_{\langle \hat{\mathscr{E}} \rangle} = s^n_{\langle \hat{Q} \rangle} t^{n+1}_{\langle \hat{\mathscr{E}} \rangle}$ and $t^n_{\langle \hat{Q} \rangle} s^{n+1}_{\langle \hat{\mathscr{E}} \rangle} = t^n_{\langle \hat{\mathscr{E}} \rangle} = t^n_{\langle \hat{Q} \rangle} t^{n+1}_{\langle \hat{\mathscr{E}} \rangle}$ for all $n \in \mathbb{N}_0$.

Assume that $n \in \mathbb{N}_0$ and $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \left\langle \hat{\mathscr{E}} \right\rangle$.

Then $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \left\langle \hat{\mathscr{E}}[k] \right\rangle$ for some $k \in \mathbb{N} \setminus \{1\}$.

Note that the case $k=1$ is satisfied thanks to the globularity and self-duality conditions of the cone.

We will only show $s^n_{\langle \hat{Q} \rangle} s^{n+1}[k] = s^n[k] = s^n_{\langle \hat{Q} \rangle} t^{n+1}[k]$ while the other part is similarly argued.

Note that $n<p$ and $k=2$ imply $s^p_{\hat{\mathscr{E}}}(x) = t^p_{\hat{\mathscr{E}}}(y)$,

$$s^n_{\hat{\mathscr{E}}}(x) = s^n_{\hat{Q}} s^{n+1}_{\hat{Q}} \cdots s^{p-1}_{\hat{Q}} s^p_{\hat{\mathscr{E}}}(x) = s^n_{\hat{Q}} s^{n+1}_{\hat{Q}} \cdots s^{p-1}_{\hat{Q}} t^p_{\hat{\mathscr{E}}}(y) = s^n_{\hat{\mathscr{E}}}(y),$$

and

$$t^n_{\hat{\mathscr{E}}}(x) = t^n_{\hat{Q}} t^{n+1}_{\hat{Q}} \cdots t^{p-1}_{\hat{Q}} s^p_{\hat{\mathscr{E}}}(x) = t^n_{\hat{Q}} t^{n+1}_{\hat{Q}} \cdots t^{p-1}_{\hat{Q}} t^p_{\hat{\mathscr{E}}}(y) = t^n_{\hat{\mathscr{E}}}(y).$$

For $k=2$, we see that $s^n_{\langle \hat{Q} \rangle} s^{n+1}[2]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$= \begin{cases} (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a), p, t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n > p; \\ (\cdots(((s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a), p, s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n > p; \\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots(((s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\ (\cdots(((t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(a), p, t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n > p; \\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(a), p, s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n > p; \\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots(((s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\ (\cdots(((t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p. \end{cases}$$

$$= \begin{cases} (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a), p, t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(a), p, s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \notin \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$$

$$= \begin{cases} (\cdots(((t^n_{\hat{\mathscr{E}}}(a), p, t^n_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\hat{\mathscr{E}}}(a), p, s^n_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^n_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \notin \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$$

$$= s^n[2]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).$$

Also, $s^n_{\langle\hat{Q}\rangle}t^{n+1}[2]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$= \begin{cases} (\cdots(((t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(a),p,t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\in\alpha_1\triangle\cdots\triangle\alpha_m\ni n>p;\\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(a),p,s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\in\alpha_1\triangle\cdots\triangle\alpha_m\not\ni n>p;\\ (\cdots(((t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\in\alpha_1\triangle\cdots\triangle\alpha_m\ni n\le p;\\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\in\alpha_1\triangle\cdots\triangle\alpha_m\not\ni n\le p;\\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a),p,t^n_{Q}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\notin\alpha_1\triangle\cdots\triangle\alpha_m\ni n>p;\\ (\cdots(((s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a),p,s^n_{Q}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\notin\alpha_1\triangle\cdots\triangle\alpha_m\not\ni n>p;\\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\notin\alpha_1\triangle\cdots\triangle\alpha_m\ni n\le p;\\ (\cdots(((s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n+1\notin\alpha_1\triangle\cdots\triangle\alpha_m\not\ni n\le p;\\ (\cdots(((t^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\ge n+1\in\alpha_1\triangle\cdots\triangle\alpha_m\ni n\le p;\\ (\cdots(((s^n_{\hat{Q}}s^{n+1}_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\ge n+1\in\alpha_1\triangle\cdots\triangle\alpha_m\not\ni n\le p;\\ (\cdots(((t^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\ge n+1\notin\alpha_1\triangle\cdots\triangle\alpha_m\ni n\le p;\\ (\cdots(((s^n_{\hat{Q}}t^{n+1}_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\ge n+1\notin\alpha_1\triangle\cdots\triangle\alpha_m\not\ni n\le p. \end{cases}$$

$$= \begin{cases} (\cdots(((t^n_{\hat{\mathscr{E}}}(a),p,t^n_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n\in\alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots(((s^n_{\hat{\mathscr{E}}}(a),p,s^n_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<n\notin\alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots(((t^n_{\hat{\mathscr{E}}}(a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\ge n\in\alpha_1\triangle\cdots\triangle\alpha_m;\\ (\cdots(((s^n_{\hat{\mathscr{E}}}(b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\ge n\notin\alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

$$= s^n[2]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).$$

It follows that $s^n_{\langle\hat{Q}\rangle}s^{n+1}[2]=s^n[2]=s^n_{\langle\hat{Q}\rangle}t^{n+1}[2]$.

Suppose that $s^n_{\langle\hat{Q}\rangle}s^{n+1}[m]=s^n[m]=s^n_{\langle\hat{Q}\rangle}t^{n+1}[m]$ holds for all $m=1,\ldots,k-1$.

Note that $n<p$ and $s^p[i](x)=t^p[j](y)$ imply

$$s^n[i](x)=s^n_{\langle\hat{Q}\rangle}s^{n+1}_{\langle\hat{Q}\rangle}\cdots s^{p-1}_{\langle\hat{Q}\rangle}s^p[i](x)=s^n_{\langle\hat{Q}\rangle}s^{n+1}_{\langle\hat{Q}\rangle}\cdots s^{p-1}_{\langle\hat{Q}\rangle}t^p[j](y)=s^n[j](y).$$

We see that $s^n_{\langle\hat{Q}\rangle}s^{n+1}[k]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$
= \begin{cases}
(\cdots((t^n_{\hat{Q}}t^{n+1}[i](a), p, t^n_{\hat{Q}}t^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n > p; \\
(\cdots((s^n_{\hat{Q}}t^{n+1}[i](a), p, s^n_{\hat{Q}}t^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n > p; \\
(\cdots((t^n_{\hat{Q}}t^{n+1}[i](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\
(\cdots((s^n_{\hat{Q}}t^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\
(\cdots((t^n_{\hat{Q}}s^{n+1}[i](a), p, t^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n > p; \\
(\cdots((s^n_{\hat{Q}}s^{n+1}[i](a), p, s^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n > p; \\
(\cdots((t^n_{\hat{Q}}s^{n+1}[i](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\
(\cdots((s^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\
(\cdots((t^n_{\hat{Q}}t^{n+1}[i](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\
(\cdots((s^n_{\hat{Q}}t^{n+1}[i](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\
(\cdots((t^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\
(\cdots((s^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p.
\end{cases}
$$

$$
= \begin{cases}
(\cdots(((t^n_{\hat{Q}}t^{n+1}[i](a), p, t^n_{\hat{Q}}t^{n+1}[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((s^n_{\hat{Q}}s^{n+1}[i](a), p, s^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((t^n_{\hat{Q}}t^{n+1}[i](a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((s^n_{\hat{Q}}s^{n+1}[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \notin \alpha_1 \triangle \cdots \triangle \alpha_m.
\end{cases}
$$

$$
= \begin{cases}
(\cdots(((t^n[i](a), p, t^n[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((s^n[i](a), p, s^n[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((t^n[i](a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((s^n[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \notin \alpha_1 \triangle \cdots \triangle \alpha_m.
\end{cases}
$$

$$
= s^n[k]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).
$$

Also, $s^n_{\langle\hat{Q}\rangle} t^{n+1}[k]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$= \begin{cases} (\cdots((t^n_{\hat{Q}} s^{n+1}[i](a), p, t^n_{\hat{Q}} s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n > p; \\ (\cdots((s^n_{\hat{Q}} s^{n+1}[i](a), p, s^n_{\hat{Q}} s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n > p; \\ (\cdots((t^n_{\hat{Q}} s^{n+1}[i](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots((s^n_{\hat{Q}} s^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\ (\cdots((t^n_{\hat{Q}} t^{n+1}[i](a), p, t^n_{\hat{Q}} t^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n > p; \\ (\cdots((s^n_{\hat{Q}} t^{n+1}[i](a), p, s^n_{\hat{Q}} t^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n > p; \\ (\cdots((t^n_{\hat{Q}} t^{n+1}[i](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots((s^n_{\hat{Q}} t^{n+1}[j](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\ (\cdots((t^n_{\hat{Q}} s^{n+1}[i](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots((s^n_{\hat{Q}} s^{n+1}[i](b))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p; \\ (\cdots((t^n_{\hat{Q}} t^{n+1}[j](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n \leq p; \\ (\cdots((s^n_{\hat{Q}} t^{n+1}[j](a))^{\alpha_1})^{\cdots})^{\alpha_m}, & p \geq n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n \leq p. \end{cases}$$

$$= \begin{cases} (\cdots(((t^n_{\hat{Q}} t^{n+1}[i](a), p, t^n_{\hat{Q}} t^{n+1}[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\hat{Q}} s^{n+1}[i](a), p, s^n_{\hat{Q}} s^{n+1}[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^n_{\hat{Q}} t^{n+1}[i](a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n_{\hat{Q}} s^{n+1}[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \notin \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$$

$$= \begin{cases} (\cdots(((t^n[i](a), p, t^n[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n[i](a), p, s^n[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^n[i](a))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^n[j](b))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq n \notin \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$$

$$= s^n[k]\big((\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).$$

This means that $s^n_{\langle\hat{Q}\rangle} s^{n+1}[k] = s^n[k] = s^n_{\langle\hat{Q}\rangle} t^{n+1}[k]$.

Applying similar argument, we get $t^n_{\langle\hat{Q}\rangle} s^{n+1}[k] = t^n[k] = t^n_{\langle\hat{Q}\rangle} t^{n+1}[k]$.

Thus, $s^n_{\langle\hat{Q}\rangle} s^{n+1}_{\langle\hat{\mathscr{E}}\rangle} = s^n_{\langle\hat{\mathscr{E}}\rangle} = s^n_{\langle\hat{Q}\rangle} t^{n+1}_{\langle\hat{\mathscr{E}}\rangle}$ and $t^n_{\langle\hat{Q}\rangle} s^{n+1}_{\langle\hat{\mathscr{E}}\rangle} = t^n_{\langle\hat{\mathscr{E}}\rangle} = t^n_{\langle\hat{Q}\rangle} t^{n+1}_{\langle\hat{\mathscr{E}}\rangle}$ for all $n \in \mathbb{N}_0$.

As a consequence,

$$\begin{tikzcd}
 & & \cdots & \left\langle \hat{\mathscr{E}} \right\rangle \arrow[llld, shift right, "s^{0}_{\langle\hat{\mathscr{E}}\rangle}"'] \arrow[llld, shift left, "t^{0}_{\langle\hat{\mathscr{E}}\rangle}"] \arrow[lld, shift right, "s^{1}_{\langle\hat{\mathscr{E}}\rangle}"'] \arrow[lld, shift left, "t^{1}_{\langle\hat{\mathscr{E}}\rangle}"] \arrow[d, shift right, "s^{n}_{\langle\hat{\mathscr{E}}\rangle}"'] \arrow[d, shift left, "t^{n}_{\langle\hat{\mathscr{E}}\rangle}"] & \cdots \\
\left\langle \hat{Q}^0 \right\rangle & \left\langle \hat{Q}^1 \right\rangle \arrow[l, shift right, "s^{0}_{\langle\hat{Q}\rangle}"'] \arrow[l, shift left, "t^{0}_{\langle\hat{Q}\rangle}"] & \cdots \arrow[l, shift right, "s^{1}_{\langle\hat{Q}\rangle}"'] \arrow[l, shift left, "t^{1}_{\langle\hat{Q}\rangle}"] & \left\langle \hat{Q}^n \right\rangle \arrow[l, shift right, "s^{n-1}_{\langle\hat{Q}\rangle}"'] \arrow[l, shift left, "t^{n-1}_{\langle\hat{Q}\rangle}"] & \cdots \arrow[l, shift right, "s^{n}_{\langle\hat{Q}\rangle}"'] \arrow[l, shift left, "t^{n}_{\langle\hat{Q}\rangle}"]
\end{tikzcd}$$

is a globular cone.

For all $p \in \mathbb{N}_0$, set $\left\langle \hat{\mathscr{E}} \right\rangle \times_p \left\langle \hat{\mathscr{E}} \right\rangle := \{(x,y) \in \left\langle \hat{\mathscr{E}} \right\rangle \times \left\langle \hat{\mathscr{E}} \right\rangle \mid s^p_{\langle\hat{\mathscr{E}}\rangle}(x) = t^p_{\langle\hat{\mathscr{E}}\rangle}(y)\}$.

Define a family of partial operations $\circ_p : \left\langle \hat{\mathscr{E}} \right\rangle \times_p \left\langle \hat{\mathscr{E}} \right\rangle \to \left\langle \hat{\mathscr{E}} \right\rangle$ by $x \circ_p y \mapsto (x,p,y)$.

For each $\alpha \subseteq \mathbb{N}_0$, define $*_\alpha : \left\langle \hat{\mathscr{E}} \right\rangle \to \left\langle \hat{\mathscr{E}} \right\rangle$ by

$$(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}.$$

In particular, $\left\langle \hat{\mathscr{E}}[1] \right\rangle \ni (x) \mapsto (x^{\bar{*}\alpha})$.

Then $(\left\langle \hat{\mathscr{E}} \right\rangle, (\circ_p)_{p\in\mathbb{N}_0}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0})$ is a self-dual globular-cone ω-magma.

Define a map $i : \hat{\mathscr{E}} \to \left\langle \hat{\mathscr{E}} \right\rangle$ by $x \mapsto (x)$.

Suppose that $f : \hat{\mathscr{E}} \to (\mathscr{M}, (\hat{\circ}_p)_{p\in\mathbb{N}_0}, (\hat{*}_\alpha)_{\alpha\subseteq\mathbb{N}_0})$ is a morphism from the original self-dual globular cone into another self-dual globular-cone ω-magma.

Define $\psi : \left\langle \hat{\mathscr{E}} \right\rangle \to \mathscr{M}$ recursively by

$$\left\langle \hat{\mathscr{E}}[1] \right\rangle \ni (\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m} \mapsto (\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m},$$

$$\left\langle \hat{\mathscr{E}}[2] \right\rangle \ni \left(\cdots\left(\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}, p, (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_n}\right)^{\delta_1}\right)^{\cdots}\right)^{\delta_t}$$

$$\mapsto \left(\cdots\left(\left(\left((\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m} \hat{\circ}_p (\cdots(f(y)^{\hat{*}\beta_1})^{\cdots})^{\hat{*}\beta_n}\right)^{\hat{*}\delta_1}\right)^{\cdots}\right)^{\hat{*}\delta_t},$$

$$\vdots$$

Next, we will show that, for each $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \left\langle \hat{\mathscr{E}} \right\rangle$,

$$\psi\Big((\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\Big) = \left(\cdots\left(\left(\psi(x)\hat{\circ}_p\psi(y)\right)^{\hat{*}\alpha_1}\right)^{\cdots}\right)^{\hat{*}\alpha_m}.$$

Note that if $(\cdots(((x,p,y)^{\delta_1})^{\delta_2})^{\cdots})^{\delta_t} \in \left\langle \hat{\mathscr{E}} \right\rangle$, then $x \in \left\langle \hat{\mathscr{E}}[i] \right\rangle$ and $y \in \left\langle \hat{\mathscr{E}}[j] \right\rangle$, where $i + j = k$ for some $k \in \mathbb{N} \setminus \{1\}$.

If $i = j = 1$, then there exist $a, b \in \hat{\mathscr{E}}$ and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \subseteq \mathbb{N}_0$ such that $x = (\cdots(a^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}$ and $y = (\cdots(b^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_n}$.

We see that $\psi\Big((\cdots((((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}), p, ((\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_n}))^{\delta_1})^{\cdots})^{\delta_t} \Big)$

$$= \Big(\cdots\Big(\Big(\Big((\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m} \hat{\circ}_p (\cdots(f(y)^{\hat{*}\beta_1})^{\cdots})^{\hat{*}\beta_n}\Big)^{\hat{*}\delta_1}\Big)^{\cdots}\Big)^{\hat{*}\delta_t}$$

$$= \Big(\cdots\Big(\Big(\Big(\psi(((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}))\hat{\circ}_p\psi(((\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_n}))\Big)^{\hat{*}\delta_1}\Big)^{\cdots}\Big)^{\hat{*}\delta_t}.$$

Suppose that this equation holds for $i = 1$ and $j = 1, 2, \ldots, n-1$.

We have $\psi\Big((\cdots((((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}), p, y)^{\delta_1})^{\cdots})^{\delta_t} \Big)$

$$= \Big(\cdots\Big(\Big(\Big((\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m} \hat{\circ}_p \psi(y)\Big)^{\hat{*}\delta_1}\Big)^{\cdots}\Big)^{\hat{*}\delta_t}$$

$$= \Big(\cdots\Big(\Big(\Big(\psi(((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}))\hat{\circ}_p\psi(y)\Big)^{\hat{*}\delta_1}\Big)^{\cdots}\Big)^{\hat{*}\delta_t}.$$

Now assume that the equation holds for $i = 1, 2, \ldots, l-1$ and $j = 1, 2, \ldots, n$.

We have $\psi\Big((\cdots((x,p,y)^{\delta_1})^{\cdots})^{\delta_t} \Big) = \Big(\cdots\Big(\Big(\psi(x)\hat{\circ}_p\psi(y)\Big)^{\hat{*}\delta_1}\Big)^{\cdots}\Big)^{\hat{*}\delta_t}$.

Thus, ψ is a morphism of self-dual globular-cone ω-magmas satisfying $f = \psi \circ i$.

We see that ψ is the only such morphism.

Therefore, $((\left\langle \hat{\mathscr{E}} \right\rangle, (\circ_p)_{p\in\mathbb{N}_0}, (*\alpha)_{\alpha\subseteq\mathbb{N}_0}), i)$ is a free self-dual globular-cone ω-magma over a self-dual globular cone. □

Starting from a globular-cone ω-magma, we can also get a free self-dual globular-cone ω-magma as in the following proposition.

Proposition 4.2.3.3. *A free self-dual globular-cone ω-magma over a globular-cone ω-magma exists.*

Proof. Let a globular-cone ω-magma $(\langle\mathscr{E}\rangle, (\bar{\circ}_p)_{p\in\mathbb{N}_0})$ over an ω-globular set be given

$$\begin{tikzcd}[column sep=large, row sep=huge]
 & & & & \langle \mathscr{E} \rangle \arrow[dllll, shift left, "s^0_{\langle \mathscr{E} \rangle}"'] \arrow[dllll, shift right, "t^0_{\langle \mathscr{E} \rangle}"] \arrow[dlll, shift left, "s^1_{\langle \mathscr{E} \rangle}"'] \arrow[dlll, shift right, "t^1_{\langle \mathscr{E} \rangle}"] \arrow[d, shift right, "s^n_{\langle \mathscr{E} \rangle}"'] \arrow[d, shift left, "t^n_{\langle \mathscr{E} \rangle}"] & \cdots \\
\langle Q^0 \rangle & \langle Q^1 \rangle \arrow[l, shift right, "s^0_{\langle Q \rangle}"'] \arrow[l, shift left, "t^0_{\langle Q \rangle}"'] & \cdots \arrow[l, shift right, "s^1_{\langle Q \rangle}"'] \arrow[l, shift left, "t^1_{\langle Q \rangle}"'] & & \langle Q^n \rangle \arrow[ll, shift right, "s^{n-1}_{\langle Q \rangle}"'] \arrow[ll, shift left, "t^{n-1}_{\langle Q \rangle}"'] & \cdots \arrow[l, shift right, "s^n_{\langle Q \rangle}"'] \arrow[l, shift left, "t^n_{\langle Q \rangle}"']
\end{tikzcd}$$

Setting $\widehat{\langle Q^n \rangle} := \langle \hat{Q}^n \rangle$, $s^n_{\widehat{\langle Q \rangle}} := s^n_{\langle Q \rangle}$, and $t^n_{\widehat{\langle Q \rangle}} := t^n_{\langle Q \rangle}$, for every $n \in \mathbb{N}_0$, we obtain an ω-globular set $\widehat{\langle Q^0 \rangle} \underset{t^0_{\widehat{\langle Q \rangle}}}{\overset{s^0_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \widehat{\langle Q^1 \rangle} \underset{t^1_{\widehat{\langle Q \rangle}}}{\overset{s^1_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\widehat{\langle Q \rangle}}}{\overset{s^{n-1}_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \widehat{\langle Q^n \rangle} \underset{t^n_{\widehat{\langle Q \rangle}}}{\overset{s^n_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \cdots$.

Consider $\widehat{\langle \mathscr{E}[1] \rangle} := \{(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \mid y \in \langle \mathscr{E} \rangle,\ \beta_j \subseteq \mathbb{N}_0,\ j = 1, 2, \ldots, m\}$.

Letting $q \in \mathbb{N}_0$, we first define $s^q[1], t^q[1] : \widehat{\langle \mathscr{E}[1] \rangle} \to \widehat{\langle Q^q \rangle}$ by

- $s^q[1]\big((\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots(((s^q_{\langle \mathscr{E} \rangle}(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & q \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots(((t^q_{\langle \mathscr{E} \rangle}(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & q \in \beta_1 \triangle \cdots \triangle \beta_m. \end{cases}$

- $t^q[1]\big((\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots(((t^q_{\langle \mathscr{E} \rangle}(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & q \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots(((s^q_{\langle \mathscr{E} \rangle}(y))^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}, & q \in \beta_1 \triangle \cdots \triangle \beta_m. \end{cases}$

Set $\widehat{\langle \mathscr{E}[2] \rangle} := \bigcup\limits_{p=0}^{\infty} \big\{(\cdots(((x, p, y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x, y \in \widehat{\langle \mathscr{E}[1] \rangle},\ s^p[1](x) = t^p[1](y),$ $\alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0\big\}$.

If $(\cdots(((x, p, y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \widehat{\langle \mathscr{E}[2] \rangle}$, then we define

- $s^q[2] : \widehat{\langle \mathscr{E}[2] \rangle} \to \widehat{\langle Q^q \rangle}$ by $(\cdots(((a, p, b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$
$$\mapsto \begin{cases} (\cdots(((s^q[1](x), p, s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[1](x), p, t^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[1](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m, \end{cases}$$

- $t^q[2] : \widehat{\langle \mathscr{E}[2] \rangle} \to \widehat{\langle Q^q \rangle}$ by $(\cdots(((a, p, b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((t^q[1](x),p,t^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[1](x),p,s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[1](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[1](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$$

Suppose that we have $\widehat{\langle \mathscr{E}[k] \rangle}$, $s^q[k]$, and $t^q[k]$ for all $k = 1, 2, \ldots, n-1$.

Let $\widehat{\langle \mathscr{E}[n] \rangle} := \dot{\bigcup\limits_{p=0}^{\infty}} \big\{ (\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \widehat{\langle \mathscr{E}[i] \rangle},\ y \in \widehat{\langle \mathscr{E}[j] \rangle},$ $i + j = n,\ s^p[i](x) = t^p[j](y),\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0 \big\}$.

If $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \widehat{\langle \mathscr{E}[n] \rangle}$, then we define

- $s^q[n] : \widehat{\langle \mathscr{E}[n] \rangle} \to \widehat{\langle Q^q \rangle}$ by $(\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$
$$\mapsto \begin{cases} (\cdots(((s^q[i](x),p,s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[i](x),p,t^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m, \end{cases}$$
- $t^q[n] : \widehat{\langle \mathscr{E}[n] \rangle} \to \widehat{\langle Q^q \rangle}$ by $(\cdots(((a,p,b)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$
$$\mapsto \begin{cases} (\cdots(((t^q[i](x),p,t^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[i](x),p,s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$$

Let $\widehat{\langle \mathscr{E} \rangle} := \dot{\bigcup\limits_{n=1}^{\infty}} \widehat{\langle \mathscr{E}[n] \rangle}$, $s^q_{\widehat{\langle \mathscr{E} \rangle}} := \dot{\bigcup\limits_{n=1}^{\infty}} s^q[n]$, and $t^q_{\widehat{\langle \mathscr{E} \rangle}} := \dot{\bigcup\limits_{n=1}^{\infty}} t^q[n]$.

By Proposition 4.2.3.2,

$$\begin{tikzcd}
 & & & & \widehat{\langle \mathscr{E} \rangle} \arrow[dllll, shift right, "s^0_{\widehat{\langle \mathscr{E} \rangle}}"'] \arrow[dllll, shift left, "t^0_{\widehat{\langle \mathscr{E} \rangle}}"] \arrow[dlll, shift right, "s^1_{\widehat{\langle \mathscr{E} \rangle}}"'] \arrow[dlll, shift left, "t^1_{\widehat{\langle \mathscr{E} \rangle}}"] \arrow[d, shift right, "s^n_{\widehat{\langle \mathscr{E} \rangle}}"'] \arrow[d, shift left, "t^n_{\widehat{\langle \mathscr{E} \rangle}}"] & \cdots \\
\widehat{\langle Q^0 \rangle} & \widehat{\langle Q^1 \rangle} \arrow[l, shift right, "s^0_{\widehat{\langle Q \rangle}}"'] \arrow[l, shift left, "t^0_{\widehat{\langle Q \rangle}}"] & \cdots \arrow[l, shift right, "s^1_{\widehat{\langle Q \rangle}}"'] \arrow[l, shift left, "t^1_{\widehat{\langle Q \rangle}}"] & & \widehat{\langle Q^n \rangle} \arrow[ll, shift right, "s^{n-1}_{\widehat{\langle Q \rangle}}"'] \arrow[ll, shift left, "t^{n-1}_{\widehat{\langle Q \rangle}}"] & \cdots \arrow[l, shift right, "s^n_{\widehat{\langle Q \rangle}}"'] \arrow[l, shift left, "t^n_{\widehat{\langle Q \rangle}}"]
\end{tikzcd}$$

is a globular cone.

For all $p \in \mathbb{N}_0$, set $\widehat{\langle \mathscr{E} \rangle} \times_p \widehat{\langle \mathscr{E} \rangle} := \{(x,y) \in \widehat{\langle \mathscr{E} \rangle} \times \widehat{\langle \mathscr{E} \rangle} \mid s^p_{\widehat{\langle \mathscr{E} \rangle}}(x) = t^p_{\widehat{\langle \mathscr{E} \rangle}}(y)\}$.

Define a family of partial operations $\circ_p : \widehat{\langle \mathscr{E} \rangle} \times_p \widehat{\langle \mathscr{E} \rangle} \to \widehat{\langle \mathscr{E} \rangle}$ by $x \circ_p y \mapsto (x,p,y)$.

In particular, for $\widehat{\langle \mathscr{E}[1] \rangle} \times_p \widehat{\langle \mathscr{E}[1] \rangle} \ni (x^{\varnothing}, y^{\varnothing})$, define $x^{\varnothing} \circ_p y^{\varnothing} \mapsto x^{\varnothing} \bar{\circ}_p y^{\varnothing}$.

For each $\alpha \subseteq \mathbb{N}_0$, define $*_\alpha : \widehat{\langle \mathscr{E} \rangle} \to \widehat{\langle \mathscr{E} \rangle}$ by

$$(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}.$$

Then $(\widehat{\langle \mathscr{E} \rangle}, (\circ_p)_{p \in \mathbb{N}_0}, (*_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ is a self-dual globular-cone ω-magma.

Define a map $i : \hat{\mathscr{E}} \to \widehat{\langle \mathscr{E} \rangle}$ by $x \mapsto x^{\varnothing}$.

Suppose that $f : \hat{\mathscr{E}} \to (\mathscr{M}, (\hat{\circ}_p)_{p \in \mathbb{N}_0}, (\hat{*}_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ is a morphism from the original globular-cone ω-magma into another self-dual globular-cone ω-magma.

Define $\psi : \widehat{\langle \mathscr{E} \rangle} \to \mathscr{M}$ recursively by

$$\widehat{\langle \mathscr{E}[1] \rangle} \ni (\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m} \mapsto (\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m},$$

$$\widehat{\langle \mathscr{E}[2] \rangle} \ni \left(\cdots\left(\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}, p, (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_n}\right)^{\delta_1}\right)^{\cdots}\right)^{\delta_t}$$

$$\mapsto \left(\cdots\left(\left(\left((\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m} \hat{\circ}_p (\cdots(f(y)^{\hat{*}\beta_1})^{\cdots})^{\hat{*}\beta_n}\right)^{\hat{*}\delta_1}\right)^{\cdots}\right)^{\hat{*}\delta_t},$$

$$\vdots$$

By Proposition 4.2.3.2 again, ψ is a unique morphism of self-dual globular-cone ω-magmas satisfying $f = \psi \circ i$.

Therefore, $\left((\widehat{\langle \mathscr{E} \rangle}, (\circ_p)_{p \in \mathbb{N}_0}, (*_\alpha)_{\alpha \subseteq \mathbb{N}_0}), i\right)$ is a free self-dual globular-cone ω-magma over a globular-cone ω-magma. □

Theorem 4.2.3.4. *A free self-dual globular-cone* ω*-magma over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
 & & & & \mathscr{E} \arrow[llll, shift left, "s^0_{\mathscr{E}}"'] \arrow[llll, shift right, "t^0_{\mathscr{E}}"] \arrow[lll, shift left, "s^1_{\mathscr{E}}"'] \arrow[lll, shift right, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & & Q^n \arrow[ll, shift right, "s^{n-1}_Q"'] \arrow[ll, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

Defining $\widehat{\langle Q^n \rangle}$, $s^n_{\widehat{\langle Q \rangle}}$, and $t^n_{\widehat{\langle Q \rangle}}$ similarly as in Proposition 4.2.3.2, we obtain an ω-globular set $\widehat{\langle Q^0 \rangle} \overset{s^0_{\widehat{\langle Q \rangle}}}{\underset{t^0_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \widehat{\langle Q^1 \rangle} \overset{s^1_{\widehat{\langle Q \rangle}}}{\underset{t^1_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \cdots \overset{s^{n-1}_{\widehat{\langle Q \rangle}}}{\underset{t^{n-1}_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \widehat{\langle Q^n \rangle} \overset{s^n_{\widehat{\langle Q \rangle}}}{\underset{t^n_{\widehat{\langle Q \rangle}}}{\leftleftarrows}} \cdots$.

Combining Proposition 4.2.1.3 with Proposition 4.2.3.2, we have

$$\begin{aligned}
\left\langle \hat{\mathscr{E}} \right\rangle := {} & \{\left((\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\right) \mid y \in \mathscr{E},\ \beta_j \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\} \,\dot{\cup} \\
& \dot{\bigcup_{n=2}^{\infty}} \dot{\bigcup_{p=0}^{\infty}} \big\{(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \left\langle \hat{\mathscr{E}}[i] \right\rangle,\ y \in \left\langle \hat{\mathscr{E}}[j] \right\rangle, \\
& i + j = n,\ s^p[i](x) = t^p[j](y),\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0 \big\}.
\end{aligned}$$

As before, we define a family of partial operations $\circ_p : \left\langle \hat{\mathscr{E}} \right\rangle \times_p \left\langle \hat{\mathscr{E}} \right\rangle \to \left\langle \hat{\mathscr{E}} \right\rangle$ by $x \circ_p y \mapsto (x,p,y)$.

Moreover, for any $\alpha \subseteq \mathbb{N}_0$, define $*_\alpha : \left\langle \hat{\mathscr{E}} \right\rangle \to \left\langle \hat{\mathscr{E}} \right\rangle$ by

$$(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}.$$

Thus, $(\left\langle \hat{\mathscr{E}} \right\rangle, (\circ_p)_{p \in \mathbb{N}_0}, (*_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ is a self-dual globular-cone ω-magma.

It remains to check the universal factorization property.

Define a map $i : \mathscr{E} \to \left\langle \hat{\mathscr{E}} \right\rangle$ by $x \mapsto (x^{\varnothing})$.

Suppose that $f : \mathscr{E} \to (\mathscr{M}, (\hat{\circ}_p)_{p \in \mathbb{N}_0}, (\hat{*}_\alpha)_{\alpha \subseteq \mathbb{N}_0})$ is a morphism from the original globular cone into another self-dual globular-cone ω-magma.

Define $\psi : \left\langle \hat{\mathscr{E}} \right\rangle \to \mathscr{M}$ recursively by

$$\left\langle \hat{\mathscr{E}}[1] \right\rangle \ni \left((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}\right) \mapsto (\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m},$$

$$\left\langle \hat{\mathscr{E}}[2] \right\rangle \ni \left(\cdots\left(\left(\left((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}\right), p, \left((\cdots(y^{\beta_1})^{\cdots})^{\beta_n}\right)\right)^{\delta_1}\right)^{\cdots}\right)^{\delta_t}$$

$$\mapsto \left(\cdots\left(\left(\left((\cdots(f(x)^{\hat{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m} \,\hat{\circ}_p (\cdots(f(y)^{\hat{*}\beta_1})^{\cdots})^{\hat{*}\beta_n}\right)^{\hat{*}\delta_1}\right)^{\cdots}\right)^{\hat{*}\delta_t},$$

$$\vdots$$

By Proposition 4.2.1.3 and Proposition 4.2.3.2, ψ is the unique morphism of self-dual globular-cone ω-magma which satisfies the property $f = \psi \circ i$.

As a consequence, $\left((\left\langle \hat{\mathscr{E}} \right\rangle, (\circ_p)_{p\in\mathbb{N}_0}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0}), i\right)$ is a free self-dual globular-cone ω-magma over a globular cone. $\square$

As promised, we will prove that there exists an isomorphism between free self-dual globular-cone ω-magma in both ways.

Theorem 4.2.3.5. *A free self-dual globular-cone ω-magma over a free self-dual globular cone over a globular cone and a free self-dual globular-cone ω-magma over a free globular-cone ω-magma over a globular cone are isomorphic.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{E}}"] \arrow[dll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

Combining Proposition 4.2.2.3 and Proposition 4.2.3.3, we get

$$\widehat{\langle\mathscr{E}\rangle} := \{(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \mid y \in \mathscr{E},\ \beta_j \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\} \,\dot\cup$$

$$\dot\bigcup_{n=2}^{\infty} \dot\bigcup_{p=0}^{\infty} \{(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \widehat{\langle\mathscr{E}[i]\rangle},\ y \in \widehat{\langle\mathscr{E}[j]\rangle},$$

$$i+j=n,\ s^p[i](x)=t^p[j](y),\ \alpha_1,\alpha_2,\ldots,\alpha_m\subseteq\mathbb{N}_0\big\}.$$

It suffices to show that $\widehat{\langle\mathscr{E}\rangle}\simeq\left\langle\hat{\mathscr{E}}\right\rangle$.

First of all, we define $\gamma:\widehat{\langle\mathscr{E}\rangle}\to\left\langle\hat{\mathscr{E}}\right\rangle$ by

$$(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\mapsto\big((\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m}\big),$$

$$\left(\cdots\left(\left((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m},p,(\cdots(y^{\beta_1})^{\cdots})^{\beta_n}\right)^{\delta_1}\right)^{\cdots}\right)^{\delta_t}\mapsto$$

$$\left(\cdots\left(\left(((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}),p,((\cdots(y^{\beta_1})^{\cdots})^{\beta_n})\right)^{\delta_1}\right)^{\cdots}\right)^{\delta_t}$$

$$\vdots$$

It is easy to see that γ is an isomorphism.

Therefore, free self-dual globular-cone ω-magmas over both a self-dual globular cone and a globular-cone ω-magma are isomorphic. □

4.3 Free Reflexive Self-Dual Globular-Cone ω-Magmas

The next section we will prove the existence of a free involutive globular ω-category. Since a globular ω-category is a globular ω-magma which satisfies associativity and unitality, we do not construct it directly; yet we apply the existence of a free self-dual globular-cone ω-magma and a free reflexive globular cone over a globular cone to guarantee the existence of a free strict involutive globular-cone ω-category over a globular cone.

Indeed, as we have already proved the existence of a free self-dual globular-cone ω-magma over both a self-dual globular cone and a globular-cone ω-magma and finally over a globular cone, we remain to add the final ingredient, which is reflexivity, to our globular cone. So, instead of having a commutative (up to isomorphism) square, we will obtain a commutative (up to isomorphism) cube:

$$\begin{tikzcd}
& \boxed{\text{Reflexive Self-Dual Globular Cones}} \arrow[rr] & & \boxed{\text{Reflexive Self-Dual Globular-Cone } \omega\text{-Magmas}} \\
\boxed{\text{Self-Dual Globular Cones}} \arrow[ur] \arrow[rr] & & \boxed{\text{Self-Dual Globular-Cone } \omega\text{-Magmas}} \arrow[ur] & \\
& \boxed{\text{Reflexive Globular Cones}} \arrow[uu] \arrow[rr] & & \boxed{\text{Reflexive Globular-Cone } \omega\text{-Magmas}} \arrow[uu] \\
\boxed{\text{Globular Cones}} \arrow[uu] \arrow[ur] \arrow[rr] & & \boxed{\text{Globular-Cone } \omega\text{-Magmas}} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

In the final subsection, we show that there exist certain equivalence relations making a reflexive self-dual globular-cone ω-magma a strict involutive globular-cone ω-category.

4.3.1 Free Reflexive Globular Cones

We begin this section by proving that a free reflexive globular cone over a globular cone exists as shown in the following proposition.

Proposition 4.3.1.1. *A free reflexive globular cone over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
& & & & & & \mathscr{E} \arrow[dllllll, shift left, "s^0_{\mathscr{E}}"'] \arrow[dllllll, shift right, "t^0_{\mathscr{E}}"] \arrow[dllll, shift left, "s^1_{\mathscr{E}}"'] \arrow[dllll, shift right, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & & Q^1 \arrow[ll, shift right, "s^0_Q"'] \arrow[ll, shift left, "t^0_Q"] & & \cdots \arrow[ll, shift right, "s^1_Q"'] \arrow[ll, shift left, "t^1_Q"] & & Q^n \arrow[ll, shift right, "s^{n-1}_Q"'] \arrow[ll, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

First of all, we need to form a new appropriate ω-globular set.

For each $n \in \mathbb{N}$, we construct the following recursive families:

$$\begin{array}{rcl} \bar{Q}^0 &:=& Q^0, \\ (\bar{Q}^0)^1 &:=& \{(x,0) \mid x \in \bar{Q}^0\}, \\ \bar{Q}^1 &:=& Q^1 \dot{\cup} (\bar{Q}^0)^1, \\ (\bar{Q}^1)^2 &:=& \{(y,1) \mid y \in \bar{Q}^1\}, \\ \bar{Q}^2 &:=& Q^2 \dot{\cup} (\bar{Q}^1)^2, \\ \vdots && \vdots \\ (\bar{Q}^n)^{n+1} &:=& \{(z,n) \mid z \in \bar{Q}^n\}, \\ \bar{Q}^{n+1} &:=& Q^{n+1} \dot{\cup} (\bar{Q}^n)^{n+1}, \\ \vdots && \vdots \end{array}$$

Then we define new sources and targets as follows: for all $n \in \mathbb{N}_0$,

- $s^n_{\bar{Q}} : \bar{Q}^{n+1} \to \bar{Q}^n$ is defined by $s^n_{\bar{Q}}(x) := \begin{cases} s^n_Q(x), & x \in Q^{n+1}; \\ y, & x = (y,n),\ y \in \bar{Q}^n. \end{cases}$

- $t^n_{\bar{Q}} : \bar{Q}^{n+1} \to \bar{Q}^n$ is defined by $t^n_{\bar{Q}}(x) := \begin{cases} t^n_Q(x), & x \in Q^{n+1}; \\ y, & x = (y,n),\ y \in \bar{Q}^n. \end{cases}$

With these definitions, we obtain an ω-globular set $\bar{Q}^0 \underset{t^0_{\bar{Q}}}{\overset{s^0_{\bar{Q}}}{\leftleftarrows}} \bar{Q}^1 \underset{t^1_{\bar{Q}}}{\overset{s^1_{\bar{Q}}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\bar{Q}}}{\overset{s^{n-1}_{\bar{Q}}}{\leftleftarrows}} \bar{Q}^n \underset{t^n_{\bar{Q}}}{\overset{s^n_{\bar{Q}}}{\leftleftarrows}} \cdots$.

Indeed, for every $n \in \mathbb{N}$ and $c \in \bar{Q}^{n+1}$,

$$\begin{aligned} s^{n-1}_{\bar{Q}} s^n_{\bar{Q}}(c) &= \begin{cases} s^{n-1}_Q s^n_Q(c), & c \in Q^{n+1}; \\ s^{n-1}_Q(b), & c = (b,n),\ b \in Q^n; \\ a, & c = ((a,n-1),n),\ a \in \bar{Q}^{n-1}. \end{cases} \\ &= \begin{cases} s^{n-1}_Q t^n_Q(c), & c \in Q^{n+1}; \\ s^{n-1}_Q(b), & c = (b,n),\ b \in Q^n; \\ a, & c = ((a,n-1),n),\ a \in \bar{Q}^{n-1}. \end{cases} \\ &= s^{n-1}_{\bar{Q}} t^n_{\bar{Q}}(c). \end{aligned}$$

Next, we establish the following recursive family:

$$\begin{aligned}
\mathscr{E}^0 &:= \{(\ldots,(1,(0,x))\cdots) \mid x \in \bar{Q}^0\},\\
\mathscr{E}^1 &:= \{(\ldots,(2,(1,y))\cdots) \mid y \in \bar{Q}^1\},\\
\vdots\; & \;\;\vdots\\
\mathscr{E}^n &= \{(\ldots,(n+1,(n,z))\cdots) \mid z \in \bar{Q}^n\},\\
\vdots\; & \;\;\vdots\\
\bar{\mathscr{E}} &:= \mathscr{E} \cup \dot{\bigcup_{n=0}^{\infty}} \mathscr{E}^n.
\end{aligned}$$

For each $n \in \mathbb{N}_0$, define $s^n_{\bar{\mathscr{E}}}, t^n_{\bar{\mathscr{E}}} : \bar{\mathscr{E}} \to \bar{Q}^n$ by

$$s^n_{\bar{\mathscr{E}}}(x) := \begin{cases} s^n_{\mathscr{E}}(x), & x \in \mathscr{E};\\ s^n_{\bar{Q}} \cdots s^{k-1}_{\bar{Q}}(y), & x = (\ldots,(k,y)\cdots),\ y \in \bar{Q}^k,\ n<k;\\ z, & x = (\ldots,(n,z)\cdots),\ z \in \bar{Q}^n;\\ ((\cdots(w,k),\ldots),n-1), & x = (\ldots,(k,w)\cdots),\ w \in \bar{Q}^k,\ 0 \le k < n. \end{cases}$$

$$t^n_{\bar{\mathscr{E}}}(x) := \begin{cases} t^n_{\mathscr{E}}(x), & x \in \mathscr{E};\\ t^n_{\bar{Q}} \cdots t^{k-1}_{\bar{Q}}(y), & x = (\ldots,(k,y)\cdots),\ y \in \bar{Q}^k,\ n<k;\\ z, & x = (\ldots,(n,z)\cdots),\ z \in \bar{Q}^n;\\ ((\cdots(w,k),\ldots),n-1), & x = (\ldots,(k,w)\cdots),\ w \in \bar{Q}^k,\ 0 \le k < n. \end{cases}$$

For each $m,n \in \mathbb{N}_0$, we have the following equalities:

$$\begin{aligned}
s^n_{\bar{Q}} s^{n+1}_{\bar{\mathscr{E}}}(x) &= \begin{cases} s^n_Q s^{n+1}_{\mathscr{E}}(x), & x \in \mathscr{E};\\ s^n_{\bar{Q}} s^{n+1}_{\bar{Q}} \cdots s^{k-1}_{\bar{Q}}(y), & x = (\ldots,(k,y)\cdots),\ y \in \bar{Q}^k,\ n+1<k;\\ s^n_{\bar{Q}}(z), & x = (\ldots,(n+1,z)\cdots),\ z \in \bar{Q}^{n+1};\\ v, & x = (\ldots,(n,v)\cdots),\ v \in \bar{Q}^n;\\ ((\cdots(w,l),\ldots),n-1), & x = (\ldots,(k,w)\cdots),\ w \in \bar{Q}^k,\ 0 \le k < n. \end{cases}\\
&= \begin{cases} s^n_{\mathscr{E}}(x), & x \in \mathscr{E};\\ s^n_{\bar{Q}} \cdots s^{k-1}_{\bar{Q}}(y), & x = (\ldots,(k,y)\cdots),\ y \in \bar{Q}^k,\ n<k;\\ v, & x = (\ldots,(n,v)\cdots),\ v \in \bar{Q}^n;\\ ((\cdots(w,l),\ldots),n-1), & x = (\ldots,(k,w)\cdots),\ w \in \bar{Q}^k,\ 0 \le k < n. \end{cases}\\
&= s^n_{\bar{\mathscr{E}}}(x)
\end{aligned}$$

and

$$s^n_{\bar{Q}}t^{n+1}_{\bar{\mathscr{E}}}(x) \quad = \quad \begin{cases} s^n_Q t^{n+1}_{\mathscr{E}}(x), & x \in \mathscr{E}; \\ s^n_{\bar{Q}} t^{n+1}_{\bar{Q}} \cdots t^{k-1}_{\bar{Q}}(y), & x = (\ldots,(k,y)\cdots),\ y \in \bar{Q}^k,\ n+1 < k; \\ s^n_{\bar{Q}}(z), & x = (\ldots,(n+1,z)\cdots),\ z \in \bar{Q}^{n+1}; \\ v, & x = (\ldots,(n,v)\cdots),\ v \in \bar{Q}^n; \\ ((\cdots(w,l),\ldots),n-1), & x = (\ldots,(k,w)\cdots),\ w \in \bar{Q}^k,\ 0 \le k < n. \end{cases}$$

$$= \begin{cases} s^n_{\mathscr{E}}(x), & x \in \mathscr{E}; \\ s^n_{\bar{Q}} \cdots s^{k-1}_{\bar{Q}}(y), & x = (\ldots,(k,y)\cdots),\ y \in \bar{Q}^k,\ n < k; \\ v, & x = (\ldots,(v,n)\cdots),\ v \in \bar{Q}^n; \\ ((\cdots(w,l),\ldots),n-1), & x = (\ldots(k,w)\cdots),\ w \in \bar{Q}^k,\ 0 \le k < n. \end{cases}$$

$$= \quad s^n_{\bar{\mathscr{E}}}(x).$$

It follows that $s^n_{\bar{Q}}s^{n+1}_{\bar{\mathscr{E}}} = s^n_{\bar{\mathscr{E}}}(x) = s^n_{\bar{Q}}t^{n+1}_{\bar{\mathscr{E}}}$.

Arguing in a similar way, we get $t^n_{\bar{Q}}s^{n+1}_{\bar{\mathscr{E}}} = t^n_{\bar{\mathscr{E}}}(x) = t^n_{\bar{Q}}t^{n+1}_{\bar{\mathscr{E}}}$.

This means that we obtain a globular cone

$$\begin{tikzcd}
 & & & \bar{\mathscr{E}} \arrow[dlll, shift right, "s^0_{\bar{\mathscr{E}}}"'] \arrow[dlll, shift left, "t^0_{\bar{\mathscr{E}}}"] \arrow[dll, shift right, "s^1_{\bar{\mathscr{E}}}"'] \arrow[dll, shift left, "t^1_{\bar{\mathscr{E}}}"] \arrow[d, shift right, "s^n_{\bar{\mathscr{E}}}"'] \arrow[d, shift left, "t^n_{\bar{\mathscr{E}}}"] & \cdots \\
\bar{Q}^0 & \bar{Q}^1 \arrow[l, shift right, "s^0_{\bar{Q}}"'] \arrow[l, shift left, "t^0_{\bar{Q}}"] & \cdots \arrow[l, shift right, "s^1_{\bar{Q}}"'] \arrow[l, shift left, "t^1_{\bar{Q}}"] & \bar{Q}^n \arrow[l, shift right, "s^{n-1}_{\bar{Q}}"'] \arrow[l, shift left, "t^{n-1}_{\bar{Q}}"] & \cdots \arrow[l, shift right, "s^n_{\bar{Q}}"'] \arrow[l, shift left, "t^n_{\bar{Q}}"]
\end{tikzcd}$$

Defining $\iota^n_{\bar{Q}} : \bar{Q}^n \to \bar{Q}^{n+1}$ by $x \mapsto (x,n)$ provides $s^n_{\bar{Q}} \circ \iota^n_{\bar{Q}} = \mathrm{Id}_{\bar{Q}} = t^n_{\bar{Q}} \circ \iota^n_{\bar{Q}}$.

Now we define $\iota^n_{\bar{\mathscr{E}}} : \bar{Q}^n \to \bar{\mathscr{E}}$ by $x \mapsto (\ldots,(n+1,(n,x))\cdots)$.

According to the reflexivity, we need to establish the following components:

1. $\bar{Q}^n \times_G \bar{Q}^n := \{(x,y) \in \bar{Q}^n \times \bar{Q}^n \mid s^n_{\bar{Q}}(x) = s^n_{\bar{Q}}(y) \text{ and } t^n_{\bar{Q}}(x) = t^n_{\bar{Q}}(y)\}$,

2. $\pi^n_{\bar{\mathscr{E}}} : \bar{\mathscr{E}} \to \bar{Q}^n \times_G \bar{Q}^n$ is defined by $\pi^n_{\bar{\mathscr{E}}}(x) := \big(s^n_{\bar{\mathscr{E}}}(x), t^n_{\bar{\mathscr{E}}}(x)\big)$,

3. $\Delta^n : \bar{Q}^n \to \bar{Q}^n \times_G \bar{Q}^n$ is defined by $\Delta^n(x) := (x,x)$.

We need to check that $\pi^{n+k}_{\bar{\mathscr{E}}} \circ \iota^{n}_{\bar{\mathscr{E}}} = \Delta^{n+k} \circ \iota^{n+k-1}_{\bar{Q}} \circ \cdots \circ \iota^{n}_{\bar{Q}}$.

These components induce the following equations:

$$\begin{aligned}
\pi^{n+k}_{\bar{\mathscr{E}}} \circ \iota^{n}_{\bar{\mathscr{E}}}(x) &= \pi^{n+k}_{\bar{\mathscr{E}}}\big((\ldots,(n+1,(n,x))\cdots)\big) \\
&= \Big(s^{n+k}_{\bar{\mathscr{E}}}\big((\ldots,(n+1,(n,x))\cdots)\big), t^{n+k}_{\bar{\mathscr{E}}}\big((\ldots,(n+1,(n,x))\cdots)\big)\Big) \\
&= \Big((\cdots((x,n),\ldots),n+k-1),(\cdots((x,n),\ldots),n+k-1)\Big) \\
&= \Delta^{n+k}\Big(((\cdots((x,n),n+1),\ldots),n+k-1)\Big) \\
&= \Delta^{n+k} \circ \iota^{n+k-1}_{\bar{Q}}\Big(((\cdots((x,n),n+1),\ldots),n+k-2)\Big) \\
&\vdots \quad \vdots \\
&= \Delta^{n+k} \circ \iota^{n+k-1}_{\bar{Q}} \circ \cdots \circ \iota^{n}_{\bar{Q}}(x).
\end{aligned}$$

This yields $\pi^{n+k}_{\bar{\mathscr{E}}} \circ \iota^{n}_{\bar{\mathscr{E}}} = \Delta^{n+k} \circ \iota^{n+k-1}_{\bar{Q}} \circ \cdots \circ \iota^{n}_{\bar{Q}}$.

That is, $(\bar{\mathscr{E}}, (\iota^{n}_{\bar{\mathscr{E}}})_{n\in\mathbb{N}_0})$ is a reflexive globular cone.

Next, we define $i : \mathscr{E} \to \bar{\mathscr{E}}$ by $x \mapsto x$.

Assume that $f : \mathscr{E} \to (\mathscr{R}, (\iota^{n}_{\mathscr{R}})_{n\in\mathbb{N}_0}), (\iota^{n}_{Q_{\mathscr{R}}})_{n\in\mathbb{N}_0})$ is a morphism from the original globular cone into another reflexive globular cone.

The only choice of morphism of reflexive globular cones holding its universal factorization property is given by the following.

Define $\psi : \bar{\mathscr{E}} \to \mathscr{R}$ recursively by, for each $w \in \mathscr{E}$, $x \in Q^0$, $y \in Q^1$, and $z \in Q^2$,

$$\begin{array}{rcl}
x & \mapsto & f(x), \\
(x,0) & \mapsto & \iota^0_{Q_{\mathscr{R}}}(f(x)), \\
y & \mapsto & f(y), \\
(y,1) & \mapsto & \iota^1_{Q_{\mathscr{R}}}(f(y)), \\
((x,0),1) & \mapsto & \iota^1_{Q_{\mathscr{R}}}\iota^0_{Q_{\mathscr{R}}}(f(x)), \\
\vdots & & \vdots \\
w & \mapsto & f(w), \\
(\ldots,(1,(0,x))\cdots) & \mapsto & \iota^0_{\mathscr{R}}(f(x)), \\
(\ldots,(2,(1,y))\cdots) & \mapsto & \iota^1_{\mathscr{R}}(f(y)), \\
(\ldots,(1,(x,0))\cdots) & \mapsto & \iota^1_{\mathscr{R}}\iota^0_{Q_{\mathscr{R}}}(f(x)), \\
(\ldots,(3,(2,z))\cdots) & \mapsto & \iota^2_{\mathscr{R}}(f(z)), \\
(\ldots,(2,(y,1))\cdots) & \mapsto & \iota^2_{\mathscr{R}}\iota^1_{Q_{\mathscr{R}}}(f(y)), \\
(\ldots,(2,((x,0),1))\cdots) & \mapsto & \iota^2_{\mathscr{R}}\iota^1_{Q_{\mathscr{R}}}\iota^0_{Q_{\mathscr{R}}}(f(x)), \\
\vdots & & \vdots
\end{array}$$

We will verify that ψ is a morphism of reflexive globular cones by induction.

First, it is obvious that, for any $x \in \bar{Q}^0$,

$$\psi\big((\ldots,(1,(0,x))\cdots)\big) = \iota^0_{\mathscr{R}}(f(x)) = \iota^0_{\mathscr{R}}(\psi(x)).$$

Next, for every $x \in \bar{Q}^1$,

$$\begin{array}{rcl}
\psi\big((\ldots,(2,(1,x))\cdots)\big) & = & \begin{cases} \iota^1_{\mathscr{R}}(f(x)), & x \in Q^1; \\ \iota^1_{\mathscr{R}}\iota^0_{Q_{\mathscr{R}}}(f(y)), & x = (y,0),\ y \in Q^0. \end{cases} \\
& = & \begin{cases} \iota^1_{\mathscr{R}}(\psi(x)), & x \in Q^1; \\ \iota^1_{\mathscr{R}}(\psi((y,0))), & x = (y,0),\ y \in Q^0. \end{cases} \\
& = & \iota^1_{\mathscr{R}}(\psi(x)).
\end{array}$$

Then, for each $x \in \bar{Q}^2$,

$$\psi\big((\ldots,(3,(2,x))\cdots)\big) = \begin{cases} \iota^2_{\mathscr{R}}(f(x)), & x \in Q^2; \\ \iota^2_{\mathscr{R}}\iota^1_{Q_{\mathscr{R}}}(f(y)), & x = (y,1),\ y \in Q^1; \\ \iota^2_{\mathscr{R}}\iota^1_{Q_{\mathscr{R}}}\iota^0_{Q_{\mathscr{R}}}(f(z)), & x = ((z,0),1),\ z \in Q^0. \end{cases}$$

$$= \begin{cases} \iota^2_{\mathscr{R}}(\psi(x)), & x \in Q^2; \\ \iota^2_{\mathscr{R}}(\psi((y,1))), & x = (y,1),\ y \in Q^1; \\ \iota^2_{\mathscr{R}}(\psi(((z,0),1)), & x = ((z,0),1),\ z \in Q^0. \end{cases}$$

$$= \iota^2_{\mathscr{R}}(\psi(x)).$$

Suppose that $\psi\big((\ldots,(k+1,(k,x))\cdots)\big) = \iota^k_{\mathscr{R}}(\psi(x))$ for all $k = 0,1,\ldots,n-1$ and $x \in \bar{Q}^k$.

For $z \in \bar{Q}^n$, we have $\psi\big((\ldots,(n+1,(n,z))\cdots)\big)$

$$= \begin{cases} \iota^n_{\mathscr{R}}(f(z)), & z \in Q^n; \\ \iota^n_{\mathscr{R}}\iota^{n-1}_{Q_{\mathscr{R}}}(f(z_{n-1})), & z = (z_{n-1},n-1),\ z_{n-1} \in Q^{n-1}; \\ \vdots & \vdots \\ \iota^n_{\mathscr{R}}\cdots\iota^1_{Q_{\mathscr{R}}}(f(z_1)), & z = ((\cdots(z_1,1),\ldots),n-1),\ z_1 \in Q^1; \\ \iota^n_{\mathscr{R}}\cdots\iota^0_{Q_{\mathscr{R}}}(f(z_0)), & z = ((\cdots(z_0,0),\ldots),n-1),\ z_0 \in Q^0. \end{cases}$$

$$= \iota^n_{\mathscr{R}}(\psi(z)).$$

Thus, ψ is a unique morphism of reflexive globular cones such that $f = \psi \circ i$.

Hence, $\big((\bar{\mathscr{E}},(\iota^n_{\bar{\mathscr{E}}})_{n\in\mathbb{N}_0}),i\big)$ is a free reflexive globular cone over a globular cone.

□

4.3.2 Free Reflexive Self-Dual Globular Cones

This subsection is devoted to the proof the existence of a free reflexive self-dual globular cone over a globular cone. We begin by its definition as follows.

Definition 4.3.2.1. A globular cone $\mathscr{E}$ is called a **reflexive self-dual globular cone** if it is a self-dual globular cone equipped with a family of identities $\iota^n_{\mathscr{E}} : Q^n \to \mathscr{E}$, for all $n \in \mathbb{N}_0$, such that $(\mathscr{E}, (\iota^n_{\mathscr{E}})_{n\in\mathbb{N}_0})$ is a reflexive globular cone.

Proposition 4.3.2.2. *A free reflexive self-dual globular cone over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[llld, shift right, "s^0_{\mathscr{E}}"'] \arrow[llld, shift left, "t^0_{\mathscr{E}}"] \arrow[lld, shift right, "s^1_{\mathscr{E}}"'] \arrow[lld, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

First of all, we construct a new ω-globular set that suits our situation.

For each $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we let

$$\hat{Q}^n := \{(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \mid y \in Q^n,\ \beta_j \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\}.$$

For all $m, n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0$, we construct the recursive families:

$$\begin{aligned}
\bar{\hat{Q}}^0 &:= \hat{Q}^0, \\
(\bar{\hat{Q}}^0)^1 &:= \{(\cdots((x,0)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \bar{\hat{Q}}^0\}, \\
\bar{\hat{Q}}^1 &:= \hat{Q}^1 \cup\!\!\!\cdot\ (\bar{\hat{Q}}^0)^1, \\
(\bar{\hat{Q}}^1)^2 &:= \{(\cdots((y,1)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid y \in \bar{\hat{Q}}^1\}, \\
\bar{\hat{Q}}^2 &:= \hat{Q}^2 \cup\!\!\!\cdot\ (\bar{\hat{Q}}^1)^2, \\
&\ \ \vdots \qquad\qquad \vdots \\
(\bar{\hat{Q}}^n)^{n+1} &:= \{(\cdots((z,n)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid z \in \bar{\hat{Q}}^n\}, \\
\bar{\hat{Q}}^{n+1} &:= \hat{Q}^{n+1} \cup\!\!\!\cdot\ (\bar{\hat{Q}}^n)^{n+1}, \\
&\ \ \vdots \qquad\qquad \vdots
\end{aligned}$$

Next, we establish the new sources and targets in the new quiver as follows.

For every $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $(\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \bar{\hat{Q}}^{n+1}$, we define $s^n_{\bar{\hat{Q}}}, t^n_{\bar{\hat{Q}}} : \bar{\hat{Q}}^{n+1} \to \bar{\hat{Q}}^n$ by

$$s^n_{\bar{\hat{Q}}}\big((\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((s^n_Q(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & y \in Q^{n+1},\ n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^n_Q(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & y \in Q^{n+1},\ n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y = (x,n), x \in \bar{\hat{Q}}^n. \end{cases}$$

$$t^n_{\bar{\hat{Q}}}\big((\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((t^n_Q(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & y \in Q^{n+1},\ n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^n_Q(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & y \in Q^{n+1},\ n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y = (x,n), x \in \bar{\hat{Q}}^n. \end{cases}$$

We now verify that these provide us an ω-globular set.

Assume that $n \in \mathbb{N}$ and $(\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \bar{\hat{Q}}^{n+1}$.

Consider $s^{n-1}_{\bar{\hat{Q}}} s^n_{\bar{\hat{Q}}}\big((\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$= \begin{cases} (\cdots(((s^{n-1}_Q s^n_Q(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n-1, n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^{n-1}_Q t^n_Q(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m \ni n-1; \\ (\cdots(((t^{n-1}_Q s^n_Q(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n-1 \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m \ni n; \\ (\cdots(((t^{n-1}_Q t^n_Q(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n-1, n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots((s^{n-1}_Q(z))^{\beta_1})^{\cdots})^{\beta_k})^{\alpha_1})^{\cdots})^{\alpha_m}, & y = ((\cdots(z^{\beta_1})^{\cdots})^{\beta_k}), n), z \in Q^n, \\ & n-1 \notin \beta_1 \triangle \cdots \triangle \beta_k \triangle \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots((t^{n-1}_Q(z))^{\beta_1})^{\cdots})^{\beta_k})^{\alpha_1})^{\cdots})^{\alpha_m}, & y = ((\cdots(z^{\beta_1})^{\cdots})^{\beta_k}), n), z \in Q^n, \\ & n-1 \in \beta_1 \triangle \cdots \triangle \beta_k \triangle \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots(w^{\beta_1})^{\cdots})^{\beta_k})^{\alpha_1})^{\cdots})^{\alpha_m}, & y = \big((\cdots((w,n-1)^{\beta_1})^{\cdots})^{\beta_k}, n\big), w \in \bar{\hat{Q}}^{n-1}. \end{cases}$$

and $s_{\bar{\hat{Q}}}^{n-1} t_{\bar{\hat{Q}}}^{n}\big((\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$= \begin{cases} (\cdots(((s_Q^{n-1} t_Q^n(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n-1, n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s_Q^{n-1} s_Q^n(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m \ni n-1; \\ (\cdots(((t_Q^{n-1} t_Q^n(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n-1 \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m \ni n; \\ (\cdots(((t_Q^{n-1} s_Q^n(y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & y \in Q^{n+1}, n-1, n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots((s_Q^{n-1}(z))^{\beta_1})^{\cdots})^{\beta_k})^{\alpha_1})^{\cdots})^{\alpha_m}, & y = ((\cdots(z^{\beta_1})^{\cdots})^{\beta_k}), n), z \in Q^n, \\ & n-1 \notin \beta_1 \triangle \cdots \triangle \beta_k \triangle \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots((t_Q^{n-1}(z))^{\beta_1})^{\cdots})^{\beta_k})^{\alpha_1})^{\cdots})^{\alpha_m}, & y = ((\cdots(z^{\beta_1})^{\cdots})^{\beta_k}), n), z \in Q^n, \\ & n-1 \in \beta_1 \triangle \cdots \triangle \beta_k \triangle \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots(w^{\beta_1})^{\cdots})^{\beta_k})^{\alpha_1})^{\cdots})^{\alpha_m}, & y = \big((\cdots((w, n-1)^{\beta_1})^{\cdots})^{\beta_k}, n\big), w \in \bar{\hat{Q}}^{n-1}. \end{cases}$$

By the globularity condition of Q, we get $s_{\bar{\hat{Q}}}^{n-1} s_{\bar{\hat{Q}}}^{n} = s_{\bar{\hat{Q}}}^{n-1} t_{\bar{\hat{Q}}}^{n}$.

Applying a similar method, we obtain $t_{\bar{\hat{Q}}}^{n-1} s_{\bar{\hat{Q}}}^{n} = t_{\bar{\hat{Q}}}^{n-1} t_{\bar{\hat{Q}}}^{n}$.

Thus, $\bar{\hat{Q}}^0 \underset{t_{\bar{\hat{Q}}}^0}{\overset{s_{\bar{\hat{Q}}}^0}{\leftleftarrows}} \bar{\hat{Q}}^1 \underset{t_{\bar{\hat{Q}}}^1}{\overset{s_{\bar{\hat{Q}}}^1}{\leftleftarrows}} \cdots \underset{t_{\bar{\hat{Q}}}^{n-1}}{\overset{s_{\bar{\hat{Q}}}^{n-1}}{\leftleftarrows}} \bar{\hat{Q}}^n \underset{t_{\bar{\hat{Q}}}^n}{\overset{s_{\bar{\hat{Q}}}^n}{\leftleftarrows}} \cdots$ is an ω-globular set.

Next, we establish the following recursive family.

For every $m \in \mathbb{N}$, we define

$$\begin{aligned} \hat{\mathscr{E}} &:= \{(\cdots((w^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid w \in \mathscr{E},\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0\}, \\ \hat{\mathscr{E}}^0 &:= \{(\cdots(((\ldots,(1,(0,x))\cdots)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \bar{\hat{Q}}^0,\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0\}, \\ \hat{\mathscr{E}}^1 &:= \{(\cdots(((\ldots,(2,(1,y))\cdots)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid y \in \bar{\hat{Q}}^1,\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0\}, \\ &\vdots \qquad \vdots \\ \hat{\mathscr{E}}^n &:= \{(\cdots(((\ldots,(n+1,(n,z))\cdots)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid z \in \bar{\hat{Q}}^n,\ \alpha_1, \alpha_2, \ldots, \alpha_m \subseteq \mathbb{N}_0\}, \\ &\vdots \qquad \vdots \end{aligned}$$

Next, define $\bar{\hat{\mathscr{E}}} := \hat{\mathscr{E}} \cup \bigcup_{n=0}^{\infty} \hat{\mathscr{E}}^n$.

To obtain a globular cone, we define the following sources and targets.

For each $k,m,n \in \mathbb{N}_0$, $s^n_{\bar{\bar{\mathscr{E}}}} : \bar{\bar{\mathscr{E}}} \to \bar{\bar{Q}}^n$ is defined by $\big(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) \mapsto$

$$\begin{cases}
\big(\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
\big(\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^n_{\bar{\bar{Q}}} \cdots s^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \bar{\bar{Q}}^k,\ k > n,\ n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((t^n_{\bar{\bar{Q}}} \cdots t^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \bar{\bar{Q}}^k,\ k > n,\ n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),n-1)^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}}, & z \in \bar{\bar{Q}}^k,\ l \geq n \\
\qquad x = (\ldots,((\cdots(n-1,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}})^{\cdots}). &
\end{cases}$$

Similarly, $t^n_{\bar{\bar{\mathscr{E}}}} : \bar{\bar{\mathscr{E}}} \to \bar{\bar{Q}}^n$ is defined by $\big(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) \mapsto$

$$\begin{cases}
\big(\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
\big(\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots((t^n_{\bar{\bar{Q}}} \cdots t^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \bar{\bar{Q}}^k,\ k > n,\ n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^n_{\bar{\bar{Q}}} \cdots s^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \bar{\bar{Q}}^k,\ k > n,\ n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),n-1)^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}}, & z \in \bar{\bar{Q}}^k,\ l \geq n \\
\qquad x = (\ldots,((\cdots(n-1,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}})^{\cdots}). &
\end{cases}$$

We will verify that

$$\begin{tikzcd}
 & & & & & & \bar{\bar{\mathscr{E}}} \arrow[llllll, shift right, "s^0_{\bar{\bar{\mathscr{E}}}}"'] \arrow[llllll, shift left, "t^0_{\bar{\bar{\mathscr{E}}}}"] \arrow[lllld, shift right, "s^1_{\bar{\bar{\mathscr{E}}}}"'] \arrow[lllld, shift left, "t^1_{\bar{\bar{\mathscr{E}}}}"] \arrow[d, shift right, "s^n_{\bar{\bar{\mathscr{E}}}}"'] \arrow[d, shift left, "t^n_{\bar{\bar{\mathscr{E}}}}"] & \cdots \\
\bar{\bar{Q}}^0 & & \bar{\bar{Q}}^1 \arrow[ll, shift right, "s^0_{\bar{\bar{Q}}}"'] \arrow[ll, shift left, "t^0_{\bar{\bar{Q}}}"] & & \cdots \arrow[ll, shift right, "s^1_{\bar{\bar{Q}}}"'] \arrow[ll, shift left, "t^1_{\bar{\bar{Q}}}"] & & \bar{\bar{Q}}^n \arrow[ll, shift right, "s^{n-1}_{\bar{\bar{Q}}}"'] \arrow[ll, shift left, "t^{n-1}_{\bar{\bar{Q}}}"] & & \cdots \arrow[ll, shift right, "s^n_{\bar{\bar{Q}}}"'] \arrow[ll, shift left, "t^n_{\bar{\bar{Q}}}"]
\end{tikzcd}$$

is a globular cone.

For each $k,m,n \in \mathbb{N}_0$, $y \in Q^k$, and $z \in \bar{\hat{Q}}^k$, $s^n_{\bar{\hat{Q}}} s^{n+1}_{\bar{\hat{\mathscr{E}}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$= \begin{cases} (\cdots(((s^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\ (\cdots(((s^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\ (\cdots(((t^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\ (\cdots(((t^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\ (\cdots((s^n_{\bar{\hat{Q}}} s^{n+1}_{\bar{\hat{Q}}} \cdots s^{k-1}_{\bar{\hat{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\hat{Q}}^k, \\ & k > n,\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\ (\cdots((t^n_{\bar{\hat{Q}}} s^{n+1}_{\bar{\hat{Q}}} \cdots s^{k-1}_{\bar{\hat{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\hat{Q}}^k, \\ & k > n,\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\ (\cdots((s^n_{\bar{\hat{Q}}} t^{n+1}_{\bar{\hat{Q}}} \cdots t^{k-1}_{\bar{\hat{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\hat{Q}}^k, \\ & k > n,\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\ (\cdots((t^n_{\bar{\hat{Q}}} t^{n+1}_{\bar{\hat{Q}}} \cdots t^{k-1}_{\bar{\hat{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\hat{Q}}^k, \\ & k > n,\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\ (\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),n-2)^{\gamma_1})^{\cdots})^{\gamma_{m_{n-2}}}, & z \in \bar{\hat{Q}}^k,\ l \geq n \\ \quad x = (\ldots,((\cdots(n-2,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{n-2}}})^{\cdots}). \end{cases}$$

$$= \begin{cases} (\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^n_{\bar{\hat{Q}}} \cdots s^{k-1}_{\bar{\hat{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\ & y \in \bar{\hat{Q}}^k,\ k > n,\ n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^n_{\bar{\hat{Q}}} \cdots t^{k-1}_{\bar{\hat{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\ & y \in \bar{\hat{Q}}^k,\ k > n,\ n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),n-1)^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}}, & z \in \bar{\hat{Q}}^k,\ l \geq n \\ \quad x = (\ldots,((\cdots(n-1,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}})^{\cdots}). \end{cases}$$

$$= s^n_{\bar{\hat{\mathscr{E}}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).$$

It follows that $s^n_{\bar{\hat{Q}}} s^{n+1}_{\bar{\hat{\mathscr{E}}}} = s^n_{\bar{\hat{\mathscr{E}}}}$.

Similarly, we see that $s^n_{\bar{\bar{Q}}} t^{n+1}_{\bar{\bar{\mathscr{E}}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) =$

$$
= \begin{cases}
\big(\cdots(((s^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\
\big(\cdots(((s^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\
\big(\cdots(((t^n_Q t^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\
\big(\cdots(((t^n_Q s^{n+1}_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\
(\cdots((s^n_{\bar{\bar{Q}}} t^{n+1}_{\bar{\bar{Q}}} \cdots t^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\bar{Q}}^k, \\
 & k > n,\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\
(\cdots((t^n_{\bar{\bar{Q}}} t^{n+1}_{\bar{\bar{Q}}} \cdots t^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\bar{Q}}^k, \\
 & k > n,\ n+1 \notin \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\
(\cdots((s^n_{\bar{\bar{Q}}} s^{n+1}_{\bar{\bar{Q}}} \cdots s^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\bar{Q}}^k, \\
 & k > n,\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \not\ni n; \\
(\cdots((t^n_{\bar{\bar{Q}}} s^{n+1}_{\bar{\bar{Q}}} \cdots s^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}),\ y \in \bar{\bar{Q}}^k, \\
 & k > n,\ n+1 \in \alpha_1 \triangle \cdots \triangle \alpha_m \ni n; \\
(\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),n-2)^{\gamma_1})^{\cdots})^{\gamma_{m_{n-2}}}, & z \in \bar{\bar{Q}}^k,\ l \geq n \\
\qquad x = (\ldots,((\cdots(n-2,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{n-2}}})^{\cdots}). &
\end{cases}
$$

$$
= \begin{cases}
\big(\cdots(((s^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
\big(\cdots(((t^n_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big), & x \in \mathscr{E},\ n \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^n_{\bar{\bar{Q}}} \cdots s^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \bar{\bar{Q}}^k,\ k > n,\ n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((t^n_{\bar{\bar{Q}}} \cdots t^{k-1}_{\bar{\bar{Q}}}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \bar{\bar{Q}}^k,\ k > n,\ n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),n-1)^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}}, & z \in \bar{\bar{Q}}^k,\ l \geq n \\
\qquad x = (\ldots,((\cdots(n-1,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{n-1}}})^{\cdots}). &
\end{cases}
$$

$= s^n_{\bar{\bar{\mathscr{E}}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).$

This implies that $s^n_{\bar{\bar{Q}}} t^{n+1}_{\bar{\bar{\mathscr{E}}}} = s^n_{\bar{\bar{\mathscr{E}}}}$.

Using a similar argument, we get $t^n_{\bar{\bar{Q}}} s^{n+1}_{\bar{\bar{\mathscr{E}}}} = t^n_{\bar{\bar{\mathscr{E}}}} = t^n_{\bar{\bar{Q}}} t^{n+1}_{\bar{\bar{\mathscr{E}}}}$.

Then we have a globular cone as promised.

Now define $\iota^n_{\bar{\hat{Q}}} : \bar{\hat{Q}}^n \to \bar{\hat{Q}}^{n+1}$ by $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\cdots(((x,n)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$.

It is easy to see that $s^n_{\bar{\hat{Q}}} \circ \iota^n_{\bar{\hat{Q}}} = \mathrm{Id}_{\bar{\hat{Q}}^n} = t^n_{\bar{\hat{Q}}} \circ \iota^n_{\bar{\hat{Q}}}$ for every $n \in \mathbb{N}_0$.

Then, for all $n \in \mathbb{N}_0$, we define $\iota^n_{\bar{\hat{\mathscr{E}}}} : \bar{\hat{Q}}^n \to \bar{\hat{\mathscr{E}}}$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\ldots,(n+1,(\cdots(((n,x)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}.$$

Due to the reflexivity, we first construct the following components:

1. $\bar{\hat{Q}}^n \times_G \bar{\hat{Q}}^n := \{(x,y) \in \bar{\hat{Q}}^n \times \bar{\hat{Q}}^n \mid s^n_{\bar{\hat{Q}}}(x) = s^n_{\bar{\hat{Q}}}(y) \text{ and } t^n_{\bar{\hat{Q}}}(x) = t^n_{\bar{\hat{Q}}}(y)\}$,
2. $\pi^n_{\bar{\hat{\mathscr{E}}}} : \bar{\hat{\mathscr{E}}} \to \bar{\hat{Q}}^n \times_G \bar{\hat{Q}}^n$ is defined by $\pi^n_{\bar{\hat{\mathscr{E}}}}(x) := \big(s^n_{\bar{\hat{\mathscr{E}}}}(x), t^n_{\bar{\hat{\mathscr{E}}}}(x)\big)$,
3. $\Delta^n : \bar{\hat{Q}}^n \to \bar{\hat{Q}}^n \times_G \bar{\hat{Q}}^n$ is defined by $\Delta^n(x) := (x,x)$.

We need to check that $(\iota^n_{\bar{\hat{\mathscr{E}}}})_{n\in\mathbb{N}_0}$ are identity maps in the globular cone.

Consider $\pi^{n+k}_{\bar{\hat{\mathscr{E}}}} \circ \iota^n_{\bar{\hat{\mathscr{E}}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$\begin{aligned}
&= \pi^{n+k}_{\bar{\hat{\mathscr{E}}}}\big((\ldots,(n+1,(\cdots(((n,x)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}\big)\\
&= \Big(s^{n+k}_{\bar{\hat{\mathscr{E}}}}\big((\ldots,(n+1,(\cdots(((n,x)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}\big),\\
&\qquad t^{n+k}_{\bar{\hat{\mathscr{E}}}}\big((\ldots,(n+1,(\cdots(((n,x)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}\big)\Big)\\
&= \big(((\cdots(\cdots(((x,n)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m},\ldots),n+k-1)^{\varnothing},\\
&\qquad ((\cdots(\cdots(((x,n)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m},\ldots),n+k-1)^{\varnothing}\big)\\
&= \Delta^{n+k}\big(((\cdots(\cdots(((x,n)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m},\ldots),n+k-1)^{\varnothing}\big)\\
&= \Delta^{n+k} \circ \iota^{n+k-1}_{\bar{\hat{Q}}} \circ \cdots \circ \iota^n_{\bar{\hat{Q}}}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).
\end{aligned}$$

This means that $\pi^{n+k}_{\bar{\hat{\mathscr{E}}}} \circ \iota^n_{\bar{\hat{\mathscr{E}}}} = \Delta^{n+k} \circ \iota^{n+k-1}_{\bar{\hat{Q}}} \circ \cdots \circ \iota^n_{\bar{\hat{Q}}}$.

For each $\alpha \subseteq \mathbb{N}_0$, define $\bar{*}_\alpha : \bar{\hat{\mathscr{E}}} \to \bar{\hat{\mathscr{E}}}$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}.$$

Hence, $(\bar{\hat{\mathscr{E}}}, (\bar{*}_\alpha)_{\alpha\subseteq\mathbb{N}_0}, (\iota^n_{\bar{\hat{\mathscr{E}}}})_{n\in\mathbb{N}_0})$ is a reflexive self-dual globular cone.

Next, define $i : \mathscr{E} \to \bar{\hat{\mathscr{E}}}$ by $x \mapsto x^{\varnothing}$.

Suppose that $f : \mathscr{E} \to (\mathscr{R}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0}, (\iota^n_{\mathscr{R}})_{n\in\mathbb{N}_0}), (\iota^n_{Q_{\mathscr{R}}})_{n\in\mathbb{N}_0})$ is a morphism from the original globular cone into another reflexive self-dual globular cone.

The only choice of morphism of reflexive self-dual globular cones holding its universal factorization property is given by the following.

Let $k,m,n \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_k \subseteq \mathbb{N}_0$, $w \in \mathscr{E}$, $x \in Q^0$, and $y \in Q^1$.

Define $\psi : \bar{\bar{\mathscr{E}}} \to \mathscr{R}$ recursively by

- $(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\cdots(((f(x))^{*\alpha_1})^{*\alpha_2})^{\cdots})^{*\alpha_m}$,
- $(\cdots(((\cdots(x^{\beta_1})^{\cdots})^{\beta_n},0)^{\alpha_1})^{\cdots})^{\alpha_m} \mapsto (\cdots((\iota^0_{Q_{\mathscr{R}}}((\cdots((f(x))^{*\beta_1})^{\cdots})^{*\beta_n})^{*\alpha_1})^{\cdots})^{*\alpha_m}$,
- $(\cdots((y^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\cdots(((f(y))^{*\alpha_1})^{*\alpha_2})^{\cdots})^{*\alpha_m}$,
- $(\cdots(((\cdots(y^{\beta_1})^{\cdots})^{\beta_n},1)^{\alpha_1})^{\cdots})^{\alpha_m} \mapsto (\cdots((\iota^1_{Q_{\mathscr{R}}}((\cdots((f(y))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}$,
- $(\cdots(((\cdots(((\cdots(x^{\gamma_1})^{\cdots})^{\gamma_k},0)^{\beta_1})^{\cdots})^{\beta_n},1)^{\alpha_1})^{\cdots})^{\alpha_m} \mapsto$
  $(\cdots((\iota^1_{Q_{\mathscr{R}}}((\cdots((\iota^0_{Q_{\mathscr{R}}}((\cdots((f(x))^{*\gamma_1})^{\cdots})^{*\gamma_k}))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}$,

$$\vdots$$

- $(\cdots((w^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\cdots(((f(w))^{*\alpha_1})^{*\alpha_2})^{\cdots})^{*\alpha_m}$ and in particular, $w^{\varnothing} \mapsto f(w)$,
- $(\cdots((\ldots,(1,(0,(\cdots(x^{\beta_1})^{\cdots})^{\beta_n}))\ldots)^{\alpha_1})^{\cdots})^{\alpha_m} \mapsto$
  $(\cdots((\iota^0_{\mathscr{R}}((\cdots((f(x))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}$,
- $(\cdots((\ldots,(2,(1,(\cdots(y^{\beta_1})^{\cdots})^{\beta_n}))\ldots)^{\alpha_1})^{\cdots})^{\alpha_m} \mapsto$
  $(\cdots((\iota^1_{\mathscr{R}}((\cdots((f(y))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}$,
- $(\cdots((\ldots,(2,(1,(\cdots(((\cdots(x^{\gamma_1})^{\cdots})^{\gamma_k},0)^{\beta_1})^{\cdots})^{\beta_n}))\ldots)^{\alpha_1})^{\cdots})^{\alpha_m} \mapsto$
  $(\cdots((\iota^1_{\mathscr{R}}((\cdots((\iota^0_{Q_{\mathscr{R}}}((\cdots((f(x))^{*\gamma_1})^{\cdots})^{*\gamma_k}))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}$,

$$\vdots$$

We will verify that ψ is a morphism of reflexive self-dual globular cones.

First of all, it is easy to see that, for each $w \in \mathscr{E}$,

$$\begin{aligned}\psi\big((\cdots((w^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big) &= (\cdots(((f(w))^{*\alpha_1})^{*\alpha_2})^{\cdots})^{*\alpha_m}\\ &= (\cdots(((\psi(w))^{*\alpha_1})^{*\alpha_2})^{\cdots})^{*\alpha_m}.\end{aligned}$$

Then, for all $(\cdots((\ldots,(1,(0,(\cdots(x^{\beta_1})^{\cdots})^{\beta_n}))\cdots)^{\alpha_1})^{\cdots})^{\alpha_m} \in \hat{\mathscr{E}}^0$, we have

$$\begin{aligned}&\psi\big((\cdots((\ldots,(1,(0,(\cdots(x^{\beta_1})^{\cdots})^{\beta_n}))\cdots)^{\alpha_1})^{\cdots})^{\alpha_m}\big)\\ =\ &(\cdots((\iota^0_{\mathscr{R}}((\cdots((f(x))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}\\ =\ &(\cdots((\iota^0_{\mathscr{R}}(\psi((\cdots(x^{\beta_1})^{\cdots})^{\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}.\end{aligned}$$

Next, for any $(\cdots((\ldots,(2,(1,(\cdots(y^{\beta_1})^{\cdots})^{\beta_n}))\cdots)^{\alpha_1})^{\cdots})^{\alpha_m} \in \hat{\mathscr{E}}^1$, we get

$$\begin{aligned}&\psi\big((\cdots((\ldots,(2,(1,(\cdots(y^{\beta_1})^{\cdots})^{\beta_n}))\cdots)^{\alpha_1})^{\cdots})^{\alpha_m}\big)\\ &=\begin{cases}(\cdots((\iota^1_{\mathscr{R}}((\cdots((f(y))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}, & y \in Q^1;\\ (\cdots((\iota^1_{\mathscr{R}}((\cdots((\iota^0_{Q\mathscr{R}}((\cdots((f(x))^{*\gamma_1})^{\cdots})^{*\gamma_k}))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}, & \\ \qquad y = ((\cdots(x^{\gamma_1})^{\cdots})^{\gamma_k},0),\ x \in Q^0.\end{cases}\\ &= (\cdots((\iota^1_{\mathscr{R}}(\psi((\cdots(y^{\beta_1})^{\cdots})^{\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}.\end{aligned}$$

Suppose that $\psi\big((\cdots((\ldots,(h+1,(h,z))\cdots)^{\alpha_1})^{\cdots})^{\alpha_m}\big) =$ $(\cdots((\iota^h_{\mathscr{R}}(\psi(z))^{*\alpha_1})^{\cdots})^{*\alpha_m}$ holds for every $h = 0,1,\ldots,k-1$ and $z \in \hat{\mathscr{E}}^h$.

For $x \in \bar{\bar{Q}}^k$, by the hypothesis, we also obtain

$$\begin{aligned}&\psi\big((\cdots((\ldots,(h+1,(h,(\cdots(z^{\beta_1})^{\cdots})^{\beta_n}))\cdots)^{\alpha_1})^{\cdots})^{\alpha_m}\big)\\ =\ &\begin{cases}(\cdots((\iota^k_{\mathscr{R}}((\cdots((f(z))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}, & z \in Q^k;\\ (\cdots((\iota^k_{\mathscr{R}}((\cdots((\psi(z))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}, & z \in (\bar{\bar{Q}}^{k-1})^k.\end{cases}\\ =\ &(\cdots((\iota^k_{\mathscr{R}}((\cdots((\psi(z))^{*\beta_1})^{\cdots})^{*\beta_n}))^{*\alpha_1})^{\cdots})^{*\alpha_m}.\end{aligned}$$

This means that ψ is a unique morphism of reflexive self-dual globular cones such that $f = \psi \circ i$.

Therefore, $((\bar{\bar{\mathscr{E}}}, (\bar{*}_\alpha)_{\alpha\subseteq\mathbb{N}_0}, (\iota^n_{\bar{\bar{\mathscr{E}}}})_{n\in\mathbb{N}_0}), i)$ is a free reflexive self-dual globular cone over a globular cone. $\square$

4.3.3 Free Reflexive Globular-Cone ω-Magmas

Now we turn our attention to the construction of a free reflexive globular-cone ω-magma over a globular cone.

Definition 4.3.3.1. A globular cone $\mathscr{E}$ is called a **reflexive globular-cone ω-magma** if it is a globular-cone ω-magma equipped with a family of identities $\iota^n_{\mathscr{E}} : Q^n \to \mathscr{E}$, for all $n \in \mathbb{N}_0$, such that $(\mathscr{E}, (\iota^n_{\mathscr{E}})_{n \in \mathbb{N}_0})$ is a reflexive globular cone.

Proposition 4.3.3.2. *A free reflexive globular-cone* ω*-magma over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{E}}"] \arrow[dll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

First of all, we introduce a construction of a new ω-globular set.

Set $\langle \bar{Q}^0 \rangle := Q^0$, $\left(\langle \bar{Q}^0 \rangle\right)^1 := \{(x,0) \mid x \in \langle \bar{Q}^0 \rangle\}$, and $\langle \bar{Q}^1[1] \rangle := Q^1 \cup \left(\langle \bar{Q}^0 \rangle\right)^1$.

Define $s^0[1] : \langle \bar{Q}^1[1] \rangle \to \langle \bar{Q}^0 \rangle$ by

$$s^0[1](y) := \begin{cases} s^0_Q(y), & y \in Q^1; \\ x, & y = (x,0),\ x \in \langle \bar{Q}^0 \rangle. \end{cases}$$

and also $t^0[1] : \langle \bar{Q}^1[1] \rangle \to \langle \bar{Q}^0 \rangle$ by

$$t^0[1](y) := \begin{cases} t^0_Q(y), & y \in Q^1; \\ x, & y = (x,0),\ x \in \langle \bar{Q}^0 \rangle. \end{cases}$$

Let $\langle \bar{Q}^1[2] \rangle := \{(x,0,y) \mid x,y \in \langle \bar{Q}^1[1] \rangle, s^0[1](x) = t^0[1](y)\}$.

Define $s^0[2] : \langle \bar{Q}^1[2] \rangle \to \langle \bar{Q}^0 \rangle$ by $s^0[2]\big((x,0,y)\big) := s^0[1](y)$

and $t^0[2] : \langle \bar{Q}^1[2] \rangle \to \langle Q^0 \rangle$ by $t^0[2]\big((x,0,y)\big) := t^0[1](x)$.

Suppose that we have $\langle \bar{Q}^1[l] \rangle$, $s^0[l]$, and $t^0[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle \bar{Q}^1[k] \rangle := \{(x,0,y) \mid x \in \langle \bar{Q}^1[i] \rangle, y \in \langle \bar{Q}^1[j] \rangle, i+j=k, s^0[i](x) = t^0[j](y)\}$.

If $(x,0,y) \in \langle \bar{Q}^1[k] \rangle$, we define $s^0[k], t^0[k] : \langle \bar{Q}^1[k] \rangle \to \langle \bar{Q}^0 \rangle$ by $s^0[k]\big((x,0,y)\big) := s^0[j](y)$ and $t^0[k]\big((x,0,y)\big) := t^0[i](x)$.

Set $\langle \bar{Q}^1 \rangle := \bigcup\limits_{k=1}^{\infty} \langle \bar{Q}^1[k] \rangle$, $s^0_{\langle \bar{Q} \rangle} := \bigcup\limits_{k=1}^{\infty} s^0[k]$, and $t^0_{\langle \bar{Q} \rangle} := \bigcup\limits_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle \bar{Q}^m \rangle$, $s^{m-1}_{\langle \bar{Q} \rangle}$, and $t^{m-1}_{\langle \bar{Q} \rangle}$ for every $m = 1,2,\ldots,n-1$.

Let $\langle \bar{Q}^n[1] \rangle := Q^n \cup \big(\langle \bar{Q}^{n-1} \rangle\big)^n$ and $s^{n-1}[1], t^{n-1}[1] : \langle \bar{Q}^n[1] \rangle \to \langle \bar{Q}^{n-1}[1] \rangle$ be defined by

$$s^{n-1}[1](y) := \begin{cases} s^{n-1}_Q(y), & y \in Q^n; \\ x, & y = (x,n),\ x \in \langle \bar{Q}^{n-1} \rangle. \end{cases}$$

$$t^{n-1}[1](y) := \begin{cases} t^{n-1}_Q(y), & y \in Q^n; \\ x, & y = (x,n),\ x \in \langle \bar{Q}^{n-1} \rangle. \end{cases}$$

Let $\langle \bar{Q}^n[2] \rangle := \bigcup\limits_{p=0}^{n-1} \{(x,p,y) \mid x,y \in \langle \bar{Q}^n[1] \rangle, s^p_Q \cdots s^{n-1}[1](x) = t^p_Q \cdots t^{n-1}[1](y)\}$.

If $(x,p,y) \in \langle \bar{Q}^n[2] \rangle$, then we define $s^{n-1}[2], t^{n-1}[2] : \langle \bar{Q}^n[2] \rangle \to \langle \bar{Q}^{n-1} \rangle$ by

- $s^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[1](x), p, s^{n-1}[1](y)), & n-1 > p; \\ s^{n-1}[1](y), & n-1 = p. \end{cases}$

- $t^{n-1}[2]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[1](x), p, t^{n-1}[1](y)), & n-1 > p; \\ t^{n-1}[1](x), & n-1 = p. \end{cases}$

Suppose that we have $\langle \bar{Q}^n[l] \rangle$, $s^{n-1}[l]$, and $t^{n-1}[l]$ for each $l = 1,2,\ldots,k-1$

Let $\langle \bar{Q}^n[k] \rangle := \bigcup\limits_{p=0}^{n-1} \{(x,p,y) \mid x \in \langle \bar{Q}^n[i] \rangle,\ y \in \langle \bar{Q}^n[j] \rangle,\ i+j=k,$

$$s^p s^{p+1} \cdots s^{n-1}[i](x) = t^p t^{p+1} \cdots t^{n-1}[j](y)\}.$$

If $(x,p,y) \in \langle \bar{Q}^n[k] \rangle$, then we define $s^{n-1}[k], t^{n-1}[k] : \langle \bar{Q}^n[k] \rangle \to \langle \bar{Q}^{n-1} \rangle$ by

- $s^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (s^{n-1}[i](x), p, s^{n-1}[j](y)), & n-1 > p; \\ s^{n-1}[j](y), & n-1 = p. \end{cases}$

- $t^{n-1}[k]\big((x,p,y)\big) := \begin{cases} (t^{n-1}[i](x), p, t^{n-1}[j](y)), & n-1 > p; \\ t^{n-1}[i](x), & n-1 = p. \end{cases}$

Set $\langle \bar{Q}^n \rangle := \dot{\bigcup\limits_{k=1}^{\infty}} \langle \bar{Q}^n[k] \rangle$, $s^{n-1}_{\langle \bar{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} s^{n-1}[k]$, and $t^{n-1}_{\langle \bar{Q} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} t^{n-1}[k]$.

By Proposition 3.1.2.2, we get that $\langle \bar{Q}^0 \rangle \underset{t^0_{\langle \bar{Q} \rangle}}{\overset{s^0_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \langle \bar{Q}^1 \rangle \underset{t^1_{\langle \bar{Q} \rangle}}{\overset{s^1_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle \bar{Q} \rangle}}{\overset{s^{n-1}_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \langle \bar{Q}^n \rangle \underset{t^n_{\langle \bar{Q} \rangle}}{\overset{s^n_{\langle \bar{Q} \rangle}}{\leftleftarrows}} \cdots$ is an ω-globular set.

Next, we establish the following recursive family:

$$\begin{array}{rcl} \bar{\mathscr{E}}^0 & := & \{(\ldots,(1,(0,x))\cdots) \mid x \in \langle \bar{Q}^0 \rangle\}, \\ \bar{\mathscr{E}}^1 & := & \{(\ldots,(2,(1,y))\cdots) \mid y \in \langle \bar{Q}^1 \rangle\}, \\ \vdots & \vdots & \\ \bar{\mathscr{E}}^n & = & \{(\ldots,(n+1,(n,z))\cdots) \mid z \in \langle \bar{Q}^n \rangle\}, \\ \vdots & \vdots & \end{array}$$

Let $\langle \bar{\mathscr{E}}[1] \rangle := \mathscr{E} \cup \dot{\bigcup\limits_{n=0}^{\infty}} \bar{\mathscr{E}}^n$.

For convenience, let us assume that $q \in \mathbb{N}_0$.

Define $s^q[1], t^q[1] : \langle \bar{\mathscr{E}}[1] \rangle \to \langle \bar{Q}^q \rangle$ by

$$s^q[1](x) := \begin{cases} s^q_{\mathscr{E}}(x), & x \in \mathscr{E}; \\ s^q_{\langle \bar{Q} \rangle} \cdots s^{k-1}_{\langle \bar{Q} \rangle}(y), & x = (\ldots,(k,y)\cdots),\ y \in \langle \bar{Q}^k \rangle,\ q < k; \\ z, & x = (\ldots,(q,z)\cdots),\ z \in \langle \bar{Q}^q \rangle; \\ ((\cdots(w,k),\ldots),q-1), & x = (\ldots,(k,w)\cdots),\ w \in \langle \bar{Q}^k \rangle,\ 0 \leq k < q. \end{cases}$$

$$t^q[1](x) := \begin{cases} t^q_{\mathscr{E}}(x), & x \in \mathscr{E}; \\ t^n_{\langle \bar{Q} \rangle} \cdots t^{k-1}_{\langle \bar{Q} \rangle}(y), & x = (\ldots,(k,y)\cdots),\ y \in \langle \bar{Q}^k \rangle,\ q < k; \\ z, & x = (\ldots,(q,z)\cdots),\ z \in \langle \bar{Q}^q \rangle; \\ ((\cdots(w,k),\ldots),q-1), & x = (\ldots,(k,w)\cdots),\ w \in \langle \bar{Q}^k \rangle,\ 0 \leq k < q. \end{cases}$$

Now set $\langle \mathscr{E}[2] \rangle := \dot{\bigcup}_{p=0}^{\infty} \big\{ (x,p,y) \mid x,y \in \langle \mathscr{E}[1] \rangle,\ s^p[1](x) = t^p[1](y) \big\}$.

If $(x,p,y) \in \langle \bar{\mathscr{E}}[2] \rangle$, then we define $s^q[2], t^q[2] : \langle \bar{\mathscr{E}}[2] \rangle \to \langle \bar{Q}^q \rangle$ by

- $s^q[2]\big((x,p,y)\big) := \begin{cases} (s^q[1](x), p, s^q[1](y)), & q > p; \\ s^q[1](y), & q \le p. \end{cases}$

- $t^q[2]\big((x,p,y)\big) := \begin{cases} (t^q[1](x), p, t^q[1](y)), & q > p; \\ t^q[1](x), & q \le p. \end{cases}$

Suppose that we have $\langle \bar{\mathscr{E}}[k] \rangle$, $s^q[k]$, and $t^q[k]$ for all $k = 1, 2, \ldots, n-1$.

Let $\langle \bar{\mathscr{E}}[n] \rangle := \dot{\bigcup}_{p=0}^{\infty} \big\{ (x,p,y) \mid x \in \langle \bar{\mathscr{E}}[i] \rangle,\ y \in \langle \bar{\mathscr{E}}[j] \rangle,\ i+j=n,\ s^p[i](x) = t^p[j](y) \big\}$.

If $(x,p,y) \in \langle \bar{\mathscr{E}}[n] \rangle$, then we define $s^q[n], t^q[n] : \langle \bar{\mathscr{E}}[n] \rangle \to \langle \bar{Q}^q \rangle$ by

- $s^q[n]\big((x,p,y)\big) := \begin{cases} (s^q[i](x), p, s^q[j](y)), & q > p; \\ s^q[j](y), & q \le p. \end{cases}$

- $t^q[n]\big((x,p,y)\big) := \begin{cases} (t^q[i](x), p, t^q[j](y)), & q > p; \\ t^q[i](x), & q \le p. \end{cases}$

Let $\langle \bar{\mathscr{E}} \rangle := \dot{\bigcup}_{n=1}^{\infty} \langle \bar{\mathscr{E}}[n] \rangle$, $s^q_{\langle \bar{\mathscr{E}} \rangle} := \dot{\bigcup}_{n=1}^{\infty} s^q[n]$, and $t^q_{\langle \bar{\mathscr{E}} \rangle} := \dot{\bigcup}_{n=1}^{\infty} t^q[n]$.

We will show show that $s^n_{\langle \bar{Q} \rangle} s^{n+1}_{\langle \bar{\mathscr{E}} \rangle} = s^n_{\langle \bar{\mathscr{E}} \rangle} = s^n_{\langle \bar{Q} \rangle} t^{n+1}_{\langle \bar{\mathscr{E}} \rangle}$ and $t^n_{\langle \bar{Q} \rangle} s^{n+1}_{\langle \bar{\mathscr{E}} \rangle} = t^n_{\langle \bar{\mathscr{E}} \rangle} = t^n_{\langle \bar{Q} \rangle} t^{n+1}_{\langle \bar{\mathscr{E}} \rangle}$ for all $n \in \mathbb{N}_0$.

Suppose that $n \in \mathbb{N}_0$ and $(x,p,y) \in \langle \bar{\mathscr{E}} \rangle$.

Then $(x,p,y) \in \langle \bar{\mathscr{E}}[k] \rangle$ for some $k \in \mathbb{N}$.

For the case $k=1$, we have the following equalities:

$$
\begin{aligned}
& s^n_{\langle\bar{Q}\rangle} s^{n+1}[1](x) \\
= & \begin{cases}
s^n_Q s^{n+1}_{\mathscr{E}}(x), & x\in\mathscr{E}; \\
s^n_{\langle\bar{Q}\rangle} s^{n+1}_{\langle\bar{Q}\rangle}\cdots s^{k-1}_{\langle\bar{Q}\rangle}(y), & x=(\ldots,(k,y)\cdots),\ y\in\left\langle\bar{Q}^k\right\rangle,\ n+1<k; \\
s^n_{\langle\bar{Q}\rangle}(z), & x=(\ldots,(n+1,z)\cdots),\ z\in\left\langle\bar{Q}^{n+1}\right\rangle; \\
v, & x=(\ldots,(n,v)\cdots),\ v\in\left\langle\bar{Q}^n\right\rangle; \\
((\cdots(w,l),\ldots),n-1), & x=(\ldots,(k,w)\cdots),\ w\in\left\langle\bar{Q}^k\right\rangle,\ 0\le k<n.
\end{cases} \\
= & \begin{cases}
s^n_{\mathscr{E}}(x), & x\in\mathscr{E}; \\
s^n_{\langle\bar{Q}\rangle}\cdots s^{k-1}_{\langle\bar{Q}\rangle}(y), & x=(\ldots,(k,y)\cdots),\ y\in\left\langle\bar{Q}^k\right\rangle,\ n<k; \\
v, & x=(\ldots,(n,v)\cdots),\ v\in\left\langle\bar{Q}^n\right\rangle; \\
((\cdots(w,l),\ldots),n-1), & x=(\ldots,(k,w)\cdots),\ w\in\left\langle\bar{Q}^k\right\rangle,\ 0\le k<n.
\end{cases} \\
= & \ s^n[1](x)
\end{aligned}
$$

and

$$
\begin{aligned}
& s^n_{\langle\bar{Q}\rangle} t^{n+1}[1](x) \\
= & \begin{cases}
s^n_Q t^{n+1}_{\mathscr{E}}(x), & x\in\mathscr{E}; \\
t^n_{\langle\bar{Q}\rangle} t^{n+1}_{\langle\bar{Q}\rangle}\cdots t^{k-1}_{\langle\bar{Q}\rangle}(y), & x=(\ldots,(k,y)\cdots),\ y\in\left\langle\bar{Q}^k\right\rangle,\ n+1<k; \\
t^n_{\langle\bar{Q}\rangle}(z), & x=(\ldots,(n+1,z)\cdots),\ z\in\left\langle\bar{Q}^{n+1}\right\rangle; \\
v, & x=(\ldots,(n,v)\cdots),\ v\in\left\langle\bar{Q}^n\right\rangle; \\
((\cdots(w,l),\ldots),n-1), & x=(\ldots,(k,w)\cdots),\ w\in\left\langle\bar{Q}^k\right\rangle,\ 0\le k<n.
\end{cases} \\
= & \begin{cases}
s^n_{\mathscr{E}}(x), & x\in\mathscr{E}; \\
s^n_{\langle\bar{Q}\rangle}\cdots s^{k-1}_{\langle\bar{Q}\rangle}(y), & x=(\ldots,(k,y)\cdots),\ y\in\left\langle\bar{Q}^k\right\rangle,\ n<k; \\
v, & x=(\ldots,(n,v)\cdots),\ v\in\left\langle\bar{Q}^n\right\rangle; \\
((\cdots(w,l),\ldots),n-1), & x=(\ldots,(k,w)\cdots),\ w\in\left\langle\bar{Q}^k\right\rangle,\ 0\le k<n.
\end{cases} \\
= & \ s^n[1](x).
\end{aligned}
$$

Thus, $s^n_{\langle\bar{Q}\rangle} s^{n+1}[1] = s^n[1] s^n_{\langle\bar{Q}\rangle} t^{n+1}[1]$.

Suppose that $s^n_{\langle\bar{Q}\rangle}s^{n+1}[m] = s^n[m] = s^n_{\langle\bar{Q}\rangle}t^{n+1}[m]$ for $m = 1,2,\dots,k-1$.

Note that $n < p$ and $s^p[i](x) = t^p[j](y)$ imply

$$s^n[i](x) = s^n_{\langle\bar{Q}\rangle}\cdots s^{p-1}_{\langle\bar{Q}\rangle}s^p[i](x) = s^n_{\langle\bar{Q}\rangle}\cdots s^{p-1}_{\langle\bar{Q}\rangle}t^p[j](y) = s^n[j](y).$$

It follows from assumption that

$$\begin{aligned}
s^n_{\langle\bar{Q}\rangle}s^{n+1}[k]\big((x,p,y)\big) &= \begin{cases} (s^n_{Q'}s^{n+1}[i](x), p, s^n_{Q'}s^{n+1}[j](y)), & n > p; \\ s^n_{Q'}s^{n+1}[j](y), & n \le p. \end{cases} \\
&= \begin{cases} (s^n_{\langle\bar{Q}\rangle}s^{n+1}[i](x), p, s^n_{\langle\bar{Q}\rangle}s^{n+1}[j](y)), & n > p; \\ s^n_{\langle\bar{Q}\rangle}s^{n+1}[j](y), & n \le p. \end{cases} \\
&= \begin{cases} (s^n[i](x), p, s^n[j](y)), & n > p; \\ s^n[j](y), & n \le p. \end{cases} \\
&= s^n[k]\big((x,p,y)\big)
\end{aligned}$$

and also

$$\begin{aligned}
s^n_{\langle\bar{Q}\rangle}t^{n+1}[k]\big((x,p,y)\big) &= \begin{cases} (s^n_{Q'}t^{n+1}[i](x), p, s^n_{Q'}t^{n+1}[j](y)), & n > p; \\ s^n_{Q'}t^{n+1}[j](y), & n = p; \\ s^n_{Q'}t^{n+1}[i](x), & n < p. \end{cases} \\
&= \begin{cases} (s^n_{\langle\bar{Q}\rangle}t^{n+1}[i](x), p, s^n_{\langle\bar{Q}\rangle}t^{n+1}[j](y)), & n > p; \\ s^n_{\langle\bar{Q}\rangle}t^{n+1}[j](y), & n = p; \\ s^n_{\langle\bar{Q}\rangle}t^{n+1}[i](x), & n < p. \end{cases} \\
&= \begin{cases} (s^n[i](x), p, s^n[j](y)), & n > p; \\ s^n[j](y), & n \le p. \end{cases} \\
&= s^n[k]\big((x,p,y)\big)
\end{aligned}$$

This yields $s^n_{\langle\bar{Q}\rangle}s^{n+1}[k] = s^n[k] = s^n_{\langle\bar{Q}\rangle}t^{n+1}[k]$ and so $s^n_{\langle\bar{Q}\rangle}s^{n+1}_{\langle\bar{\mathscr{E}}\rangle} = s^n_{\langle\bar{\mathscr{E}}\rangle} = s^n_{\langle\bar{Q}\rangle}t^{n+1}_{\langle\bar{\mathscr{E}}\rangle}$.

Applying a similar argument, we obtain $t^n_{\langle\bar{Q}\rangle}s^{n+1}_{\langle\bar{\mathscr{E}}\rangle} = t^n_{\langle\bar{\mathscr{E}}\rangle} = t^n_{\langle\bar{Q}\rangle}t^{n+1}_{\langle\bar{\mathscr{E}}\rangle}$.

As a consequence,

$$\begin{tikzcd}[column sep=large, row sep=huge]
 & & & \langle \bar{\mathscr{E}} \rangle \arrow[dlll, shift right, "s^0_{\langle \bar{\mathscr{E}} \rangle}"'] \arrow[dlll, shift left, "t^0_{\langle \bar{\mathscr{E}} \rangle}"] \arrow[dll, shift right, "s^1_{\langle \bar{\mathscr{E}} \rangle}"'] \arrow[dll, shift left, "t^1_{\langle \bar{\mathscr{E}} \rangle}"] \arrow[d, shift right, "s^n_{\langle \bar{\mathscr{E}} \rangle}"'] \arrow[d, shift left, "t^n_{\langle \bar{\mathscr{E}} \rangle}"] & \cdots \\
\langle \bar{Q}^0 \rangle & \langle \bar{Q}^1 \rangle \arrow[l, shift right, "s^0_{\langle \bar{Q} \rangle}"'] \arrow[l, shift left, "t^0_{\langle \bar{Q} \rangle}"] & \cdots \arrow[l, shift right, "s^1_{\langle \bar{Q} \rangle}"'] \arrow[l, shift left, "t^1_{\langle \bar{Q} \rangle}"] & \langle \bar{Q}^n \rangle \arrow[l, shift right, "s^{n-1}_{\langle \bar{Q} \rangle}"'] \arrow[l, shift left, "t^{n-1}_{\langle \bar{Q} \rangle}"] & \cdots \arrow[l, shift right, "s^n_{\langle \bar{Q} \rangle}"'] \arrow[l, shift left, "t^n_{\langle \bar{Q} \rangle}"]
\end{tikzcd}$$

is a globular cone.

For all $p \in \mathbb{N}_0$, set

$$\langle \bar{\mathscr{E}} \rangle \times_p \langle \bar{\mathscr{E}} \rangle := \{(x,y) \in \langle \bar{\mathscr{E}} \rangle \times \langle \bar{\mathscr{E}} \rangle \mid s^p_{\langle \bar{\mathscr{E}} \rangle}(x) = t^p_{\langle \bar{\mathscr{E}} \rangle}(y)\},$$

Define a family of partial operations $\circ_p : \langle \bar{\mathscr{E}} \rangle \times_p \langle \bar{\mathscr{E}} \rangle \to \langle \bar{\mathscr{E}} \rangle$ by $x \circ_p y \mapsto (x,p,y)$.

Now we define $\iota^n_{\langle \bar{Q} \rangle} : \langle \bar{Q}^n \rangle \to \langle \bar{Q}^{n+1} \rangle$ by $x \mapsto (x,n)$ for every $n \in \mathbb{N}_0$.

Then define $\iota^n_{\langle \bar{\mathscr{E}} \rangle} : \langle \bar{Q}^n \rangle \to \langle \bar{\mathscr{E}} \rangle$ by $x \mapsto (\ldots,(n+1,(n,x))\cdots)$ for all $n \in \mathbb{N}_0$.

Next, we establish the following components:

1. $\langle \bar{Q}^n \rangle \times_G \langle \bar{Q}^n \rangle := \{(x,y) \in \langle \bar{Q}^n \rangle \times \langle \bar{Q}^n \rangle \mid s^n_{\langle \bar{Q} \rangle}(x) = s^n_{\langle \bar{Q} \rangle}(y) \text{ and } t^n_{\langle \bar{Q} \rangle}(x) = t^n_{\langle \bar{Q} \rangle}(y)\}$,
2. $\pi^n_{\langle \bar{\mathscr{E}} \rangle} : \langle \bar{\mathscr{E}} \rangle \to \langle \bar{Q}^n \rangle \times_G \langle \bar{Q}^n \rangle$ is defined by $\pi^n_{\langle \bar{\mathscr{E}} \rangle}(x) := \big(s^n_{\langle \bar{\mathscr{E}} \rangle}(x), t^n_{\langle \bar{\mathscr{E}} \rangle}(x)\big)$,
3. $\Delta^n : \langle \bar{Q}^n \rangle \to \langle \bar{Q}^n \rangle \times_G \langle \bar{Q}^n \rangle$ is defined by $\Delta^n(x) := (x,x)$.

These components induce the following equations:

$$\begin{aligned}
\pi^{n+k}_{\langle \bar{\mathscr{E}} \rangle} \circ \iota^n_{\langle \bar{\mathscr{E}} \rangle}(x) &= \pi^{n+k}_{\langle \bar{\mathscr{E}} \rangle}\big((\ldots,(n+1,(n,x))\cdots)\big) \\
&= \Big(s^{n+k}_{\langle \bar{\mathscr{E}} \rangle}\big((\ldots,(n+1,(n,x))\cdots)\big), t^{n+k}_{\langle \bar{\mathscr{E}} \rangle}\big((\ldots,(n+1,(n,x))\cdots)\big)\Big) \\
&= \Big((\cdots((x,n),\ldots),n+k-1),(\cdots((x,n),\ldots),n+k-1)\Big) \\
&= \Delta^{n+k}\Big(((\cdots((x,n),n+1),\ldots),n+k-1)\Big) \\
&= \Delta^{n+k} \circ \iota^{n+k-1}_{\langle \bar{Q} \rangle} \circ \cdots \circ \iota^n_{\langle \bar{Q} \rangle}(x).
\end{aligned}$$

This means that $\pi^{n+k}_{\langle \bar{\mathscr{E}} \rangle} \circ \iota^n_{\langle \bar{\mathscr{E}} \rangle} = \Delta^{n+k} \circ \iota^{n+k-1}_{\langle \bar{Q} \rangle} \circ \cdots \circ \iota^n_{\langle \bar{Q} \rangle}$.

Hence, $(\langle\bar{\mathscr{E}}\rangle, (\circ_p)_{p\in\mathbb{N}_0}, (\iota^n_{\langle\bar{\mathscr{E}}\rangle})_{p\in\mathbb{N}_0})$ is a reflexive globular-cone ω-magma.

Next, we define $i : \mathscr{E} \to \langle\bar{\mathscr{E}}\rangle$ by $x \mapsto x$.

Assume that $f : \mathscr{E} \to (\mathscr{R}, (\hat{\circ}_p)_{p\in\mathbb{N}_0}, (\iota^n_{\mathscr{R}})_{n\in\mathbb{N}_0}), (\iota^n_{Q_{\mathscr{R}}})_{n\in\mathbb{N}_0})$ is a morphism from the original globular cone into another reflexive globular-cone ω-magma.

The only choice of morphism of reflexive globular-cone ω-magmas holding its universal factorization property is given by the following.

Define $\psi : \langle\bar{\mathscr{E}}\rangle \to \mathscr{R}$ recursively by, for each $u, v, w \in \mathscr{E}$, $x, z \in Q^0$, and $y \in Q^1$,

$$\begin{array}{rcl}
x & \mapsto & f(x), \\
(x,0) & \mapsto & \iota^0_{Q_{\mathscr{R}}}(f(x)), \\
y & \mapsto & f(y), \\
(y,1) & \mapsto & \iota^1_{Q_{\mathscr{R}}}(f(y)), \\
((x,0),1) & \mapsto & \iota^1_{Q_{\mathscr{R}}}\iota^0_{Q_{\mathscr{R}}}(f(x)), \\
\vdots & & \vdots
\end{array}$$

$$\begin{array}{rcl}
w & \mapsto & f(w), \\
(\ldots,(1,(0,x))\cdots) & \mapsto & \iota^0_{\mathscr{R}}(f(x)), \\
(\ldots,(2,(1,y))\cdots) & \mapsto & \iota^1_{\mathscr{R}}(f(y)), \\
(\ldots,(1,(x,0))\cdots) & \mapsto & \iota^1_{\mathscr{R}}\iota^0_{Q_{\mathscr{R}}}(f(x)), \\
\vdots & & \vdots
\end{array}$$

$$\begin{array}{rcl}
(w,p,v) & \mapsto & f(w)\hat{\circ}_p f(v), \\
(w,p,(\ldots,(1,(0,x))\cdots)) & \mapsto & f(w)\hat{\circ}_p \iota^0_{\mathscr{R}}(f(x)), \\
(w,p,(\ldots,(2,(1,y))\cdots)) & \mapsto & f(w)\hat{\circ}_p \iota^1_{\mathscr{R}}(f(y)), \\
\vdots & & \vdots
\end{array}$$

$$\begin{array}{rcl}
((\ldots,(1,(0,x))\cdots),p,w) & \mapsto & \iota^0_{\mathscr{R}}(f(x))\hat{\circ}_p f(w), \\
((\ldots,(1,(0,x))\cdots),p,(\ldots,(1,(0,z))\cdots)) & \mapsto & \iota^0_{\mathscr{R}}(f(x))\hat{\circ}_p \iota^0_{\mathscr{R}}(f(z)), \\
\vdots & & \vdots
\end{array}$$

$$\begin{array}{rcl} ((u,p,v),q,w) & \mapsto & (f(u)\hat{\circ}_p f(v))\hat{\circ}_q f(w), \\ (u,p,(v,q,w)) & \mapsto & f(u)\hat{\circ}_p(f(v)\hat{\circ}_q f(w)), \\ \vdots & & \vdots \end{array}$$

First, it is easy to see that, for each $x \in \langle \bar{Q}^0 \rangle$,

$$\psi\big((\ldots,(1,(0,x))\cdots)\big) = \iota^0_{\mathscr{R}}(f(x)) = \iota^0_{\mathscr{R}}(\psi(x)).$$

Next, for every $x \in \bar{Q}^1$,

$$\begin{aligned} \psi\big((\ldots,(2,(1,x))\cdots)\big) &= \begin{cases} \iota^1_{\mathscr{R}}(f(x)), & x \in Q^1; \\ \iota^1_{\mathscr{R}}\iota^0_{Q_{\mathscr{R}}}(f(y)), & x = (y,0),\ y \in Q^0. \end{cases} \\ &= \begin{cases} \iota^1_{\mathscr{R}}(\psi(x)), & x \in Q^1; \\ \iota^1_{\mathscr{R}}(\psi((y,0))), & x = (y,0),\ y \in Q^0. \end{cases} \\ &= \iota^1_{\mathscr{R}}(\psi(x)). \end{aligned}$$

Suppose that $\psi\big((\ldots,(k+1,(k,x))\cdots)\big) = \iota^k_{\mathscr{R}}(\psi(x))$ for all $k = 0,1,\ldots,n-1$ and $x \in \langle \bar{Q}^k \rangle$.

For $z \in \langle \bar{Q}^n \rangle$, we have $\psi\big((\ldots,(n+1,(n,z))\cdots)\big)$

$$\begin{aligned} &= \begin{cases} \iota^n_{\mathscr{R}}(f(z)), & z \in Q^n; \\ \iota^n_{\mathscr{R}}\iota^{n-1}_{Q_{\mathscr{R}}}(f(z_{n-1})), & z = (z_{n-1},n-1),\ z_{n-1} \in Q^{n-1}; \\ \vdots & \vdots \\ \iota^n_{\mathscr{R}}\cdots\iota^1_{Q_{\mathscr{R}}}(f(z_1)), & z = ((\cdots(z_1,1),\ldots),n-1),\ z_1 \in Q^1; \\ \iota^n_{\mathscr{R}}\cdots\iota^0_{Q_{\mathscr{R}}}(f(z_0)), & z = ((\cdots(z_0,0),\ldots),n-1),\ z_0 \in Q^0. \end{cases} \\ &= \iota^n_{\mathscr{R}}(\psi(z)). \end{aligned}$$

Next, for every $w \in \mathscr{E}$, we have $\psi\big((w,p,v)\big) = f(w)\hat{\circ}_p f(v) = \psi(w)\hat{\circ}_p\psi(v)$ and

$$\psi\big((w,p,(\ldots,(1,(0,x))\cdots))\big) = f(w)\hat{\circ}_p\iota^0_{\mathscr{R}}(f(x)) = \psi(w)\hat{\circ}_p\psi\big((\ldots,(1,(0,x))\cdots)\big).$$

Assume that $\psi\big((w,p,(\ldots,(k+1,(k,x))\cdots))\big) = \psi(w)\hat{\circ}_p\psi\big((\ldots,(k+1,(k,x))\cdots)\big)$ for all $k = 1,2,\ldots,n-1$ and $x \in \langle \bar{Q}^k \rangle$.

For $z \in \langle \bar{Q}^n \rangle$, we have $\psi\big(w, p, (\ldots, (n+1, (n,z)) \cdots)\big)$

$$= \begin{cases} f(w) \hat{\circ}_p \iota^n_{\mathscr{R}}(f(z)), & z \in Q^n; \\ f(w) \hat{\circ}_p \iota^n_{\mathscr{R}} \iota^{n-1}_{Q_{\mathscr{R}}}(f(z_{n-1})), & z = (z_{n-1}, n-1),\ z_{n-1} \in Q^{n-1}; \\ \vdots & \vdots \\ f(w) \hat{\circ}_p \iota^n_{\mathscr{R}} \cdots \iota^1_{Q_{\mathscr{R}}}(f(z_1)), & z = ((\cdots(z_1, 1), \ldots), n-1),\ z_1 \in Q^1; \\ f(w) \hat{\circ}_p \iota^n_{\mathscr{R}} \cdots \iota^0_{Q_{\mathscr{R}}}(f(z_0)), & z = ((\cdots(z_0, 0), \ldots), n-1),\ z_0 \in Q^0. \end{cases}$$
$$= \psi(w) \hat{\circ}_p \iota^n_{\mathscr{R}}(\psi(z)).$$

Assume that $\psi\big(((\ldots, (l+1, (l,y)) \cdots), p, (\ldots, (k+1, (k,x)) \cdots))\big) = \psi\big((\ldots, (l+1, (l,y)) \cdots)\big) \hat{\circ}_p \psi\big((\ldots, (k+1, (k,x)) \cdots)\big)$ for all $l = 1, 2, \ldots, m-1$, $k = 1, 2, \ldots, n$, $y \in \langle \bar{Q}^l \rangle$ and $x \in \langle \bar{Q}^k \rangle$.

For each $n \in \mathbb{N}$, $x \in \langle \bar{Q}^n \rangle$, and $z \in \langle \bar{Q}^m \rangle$, we have

$$\psi\big((\ldots, (m+1, (m,z)) \cdots), p, (\ldots, (n+1, (n,x)) \cdots)\big)$$
$$= \begin{cases} \iota^m_{\mathscr{R}}(f(z)) \hat{\circ}_p \psi(x), & z \in Q^m; \\ \iota^m_{\mathscr{R}} \iota^{m-1}_{Q_{\mathscr{R}}}(f(z_{m-1})) \hat{\circ}_p \psi(x), & z = (z_{m-1}, m-1),\ z_{m-1} \in Q^{m-1}; \\ \vdots & \vdots \\ \iota^m_{\mathscr{R}} \cdots \iota^1_{Q_{\mathscr{R}}}(f(z_1)) \hat{\circ}_p \psi(x), & z = ((\cdots(z_1, 1), \ldots), m-1),\ z_1 \in Q^1; \\ \iota^m_{\mathscr{R}} \cdots \iota^0_{Q_{\mathscr{R}}}(f(z_0)) \hat{\circ}_p \psi(x), & z = ((\cdots(z_0, 0), \ldots), m-1),\ z_0 \in Q^0. \end{cases}$$
$$= \iota^m_{\mathscr{R}}(\psi(z)) \hat{\circ}_p \psi(x).$$

Next, we will verify that $\psi\big((x,p,y)\big) = \psi(x) \hat{\circ}_p \psi(y)$ for every $(x,p,y) \in \langle \bar{\mathscr{E}} \rangle$.

Notice that if $(x,p,y) \in \langle \bar{\mathscr{E}} \rangle$, then $x \in \langle \bar{\mathscr{E}}[i] \rangle$ and $y \in \langle \bar{\mathscr{E}}[j] \rangle$, where $i + j = k$ for some $k \in \mathbb{N} \setminus \{1\}$.

Suppose that this equality holds for $i = 1$ and $j = 1, \ldots, t-1$ for some $t \in \mathbb{N}$.

We have $\psi\big((x,p,y)\big) = \psi((x)) \hat{\circ}_p \psi(y)$.

Assume that the equation holds for $i = 1, \ldots, s-1$ and $j \in \mathbb{N}$ for some $s \in \mathbb{N}$.

We have $\psi\big((x,p,y)\big) = \psi(x) \hat{\circ}_p \psi(y)$.

This means that $\psi\big((x,p,y)\big) = \psi(x) \hat{\circ}_p \psi(y)$ for each $(x,p,y) \in \langle \bar{\mathscr{E}} \rangle$.

As a result, ψ is a unique morphism of reflexive globular-cone ω-magmas satisfying $f = \psi \circ i$.

Therefore, $\big((\langle \bar{\mathscr{E}} \rangle, (\circ_p)_{p \in \mathbb{N}_0}, (\iota^n_{\langle \bar{\mathscr{E}} \rangle})_{p \in \mathbb{N}_0}), i\big)$ is a free reflexive globular-cone ω-magma over a globular cone. $\square$

4.3.4 Free Reflexive Self-Dual Globular-Cone ω-Magmas

Definition 4.3.4.1. A globular cone $\mathscr{E}$ is called a **reflexive self-dual globular-cone ω-magma** if it is a self-dual globular-cone ω-magma equipped with a family of identities $\iota^n_{\mathscr{E}} : Q^n \to \mathscr{E}$, for all $n \in \mathbb{N}_0$, such that $(\mathscr{E}, (\iota^n_{\mathscr{E}})_{n \in \mathbb{N}_0})$ is a reflexive globular cone.

Combining Proposition 4.3.2.2 with Proposition 4.3.3.2, we can easily have a free reflexive self-dual globular-cone ω-magma over a globular cone and then we will utilize this result to obtain a free strict involutive globular-cone ω-category over a globular cone.

Proposition 4.3.4.2. *A free reflexive self-dual globular ω-magma over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}
 & & & \mathscr{E} \arrow[dlll, shift left, "s^0_{\mathscr{E}}"'] \arrow[dlll, shift right, "t^0_{\mathscr{E}}"] \arrow[dll, shift left, "s^1_{\mathscr{E}}"'] \arrow[dll, shift right, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & Q^1 \arrow[l, shift right, "s^0_Q"'] \arrow[l, shift left, "t^0_Q"] & \cdots \arrow[l, shift right, "s^1_Q"'] \arrow[l, shift left, "t^1_Q"] & Q^n \arrow[l, shift right, "s^{n-1}_Q"'] \arrow[l, shift left, "t^{n-1}_Q"] & \cdots \arrow[l, shift right, "s^n_Q"'] \arrow[l, shift left, "t^n_Q"]
\end{tikzcd}$$

For each $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we first set

$$\hat{Q}^n := \{(\cdots((y^{\beta_1})^{\beta_2})^{\cdots})^{\beta_m} \mid y \in Q^n,\ \beta_j \subseteq \mathbb{N}_0,\ j = 1, 2, \ldots, m\}.$$

Then we establish a new ω-globular set as follows: $\left\langle \bar{\hat{Q}}^0 \right\rangle := \hat{Q}^0$,

$$\left(\widehat{\left\langle \bar{\hat{Q}}^0 \right\rangle}\right)^1 := \{(\cdots(x,0)^{\beta_1})^{\cdots})^{\beta_m} \mid x \in \left\langle \bar{\hat{Q}}^0 \right\rangle,\ \beta_j \subseteq \mathbb{N}_0,\ j = 1,2,\ldots,m\}.$$

Set $\left\langle \bar{\hat{Q}}^1[1] \right\rangle := \hat{Q}^1 \cup \left(\widehat{\left\langle \bar{\hat{Q}}^0 \right\rangle}\right)^1$ and define $s^0[1], t^0[1] : \left\langle \bar{\hat{Q}}^1[1] \right\rangle \to \left\langle \bar{\hat{Q}}^0 \right\rangle$ by

- $s^0[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((s_Q^0(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^1,\ 0 \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((t_Q^0(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^1,\ 0 \in \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots(x^{\beta_1})^{\cdots})^{\beta_m}, & y = (x,0),\ x \in \left\langle \bar{\hat{Q}}^0 \right\rangle. \end{cases}$

- $t^0[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((t_Q^0(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^1,\ 0 \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((s_Q^0(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^1,\ 0 \in \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots(x^{\beta_1})^{\cdots})^{\beta_m}, & y = (x,0),\ x \in \left\langle \bar{\hat{Q}}^0 \right\rangle. \end{cases}$

Now let $\left\langle \bar{\hat{Q}}^1[2] \right\rangle := \{(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x,y \in \left\langle \bar{\hat{Q}}^1[1] \right\rangle,\ \alpha_j \subseteq \mathbb{N}_0,$

$$j = 1,2,\ldots,m,\ s^0[1](x) = t^0[1](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \in \left\langle \bar{\hat{Q}}^1[2] \right\rangle$, we define $s^0[2], t^0[2] : \left\langle \bar{\hat{Q}}^1[2] \right\rangle \to \left\langle \bar{\hat{Q}}^0 \right\rangle$ by

- $s^0[2]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((s^0[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^0[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

- $t^0[2]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((t^0[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^0[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

Suppose that we have $\left\langle \bar{\hat{Q}}^1[l] \right\rangle$, $s^0[l]$, and $t^0[l]$ for every $l = 1,2,\ldots,k-1$.

Let $\left\langle \bar{\hat{Q}}^1[k] \right\rangle := \{(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x \in \left\langle \bar{\hat{Q}}^1[i] \right\rangle,\ y \in \left\langle \bar{\hat{Q}}^1[j] \right\rangle,\ i+j = k,$

$$\alpha_h \subseteq \mathbb{N}_0,\ h = 1,2,\ldots,m,\ s^0[i](x) = t^0[j](y)\}.$$

If $(\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m} \in \left\langle \bar{\hat{Q}}^1[k] \right\rangle$, we define $s^0[k], t^0[k] : \left\langle \bar{\hat{Q}}^1[k] \right\rangle \to \left\langle \bar{\hat{Q}}^0 \right\rangle$ by

- $s^0[k]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((s^0[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^0[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

- $t^0[k]\big((\cdots((x,0,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots((t^0[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^0[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & 0 \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$

Set $\left\langle \bar{\hat{Q}}^1 \right\rangle := \mathop{\dot{\bigcup}}\limits_{k=1}^{\infty} \left\langle \bar{\hat{Q}}^1[k] \right\rangle$, $s^0_{\left\langle \bar{\hat{Q}} \right\rangle} := \mathop{\dot{\bigcup}}\limits_{k=1}^{\infty} s^0[k]$, and $t^0_{\left\langle \bar{\hat{Q}} \right\rangle} := \mathop{\dot{\bigcup}}\limits_{k=1}^{\infty} t^0[k]$.

Assume that we have $\langle \hat{Q}^r \rangle$, $s^{r-1}_{\left\langle \bar{\hat{Q}} \right\rangle}$, and $t^{r-1}_{\left\langle \bar{\hat{Q}} \right\rangle}$ for every $r = 1, 2, \ldots, n$.

Let $\left\langle \bar{\hat{Q}}^{n+1}[1] \right\rangle := \hat{Q}^{n+1} \,\dot{\cup}\, \left(\widehat{\left\langle \bar{\hat{Q}}^n \right\rangle}\right)^{n+1}$.

Define $s^n[1], t^n[1] : \left\langle \bar{\hat{Q}}^{n+1}[1] \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by

$$s^n[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((s^n_Q(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^{n+1},\ n \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((t^n_Q(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^{n+1},\ n \in \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots(x^{\beta_1})^{\cdots})^{\beta_m}, & y = (x,n),\ x \in \left\langle \bar{\hat{Q}}^n \right\rangle. \end{cases}$$

$$t^n[1]\big((\cdots(y^{\beta_1})^{\cdots})^{\beta_m}\big) := \begin{cases} (\cdots((t^n_Q(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^{n+1},\ n \notin \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots((s^n_Q(y))^{\beta_1})^{\cdots})^{\beta_m}, & y \in Q^{n+1},\ n \in \beta_1 \triangle \cdots \triangle \beta_m; \\ (\cdots(x^{\beta_1})^{\cdots})^{\beta_m}, & y = (x,n),\ x \in \left\langle \bar{\hat{Q}}^n \right\rangle. \end{cases}$$

Now let $\left\langle \bar{\hat{Q}}^{n+1}[2] \right\rangle := \mathop{\dot{\bigcup}}\limits_{p=0}^{n} \{(\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x, y \in \langle \hat{Q}^{n+1}[1] \rangle,$

$\alpha_j \subseteq \mathbb{N}_0,\ j = 1, 2, \ldots, m,\ s^p[1] \cdots s^n[1](x) = t^p[1] \cdots t^n[1](y)\}$.

Define $s^n[2], t^n[2] : \left\langle \bar{\hat{Q}}^{n+1}[2] \right\rangle \to \left\langle \bar{\hat{Q}}^n \right\rangle$ by

$$s^n[2]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((s^n[1](x), p, s^n[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^n[1](x), p, t^n[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^n[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p = n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^n[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p = n \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$$

$$t^n[2]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((t^n[1](x), p, t^n[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^n[1](x), p, s^n[1](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p < n \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((t^n[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p = n \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\ (\cdots((s^n[1](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p = n \in \alpha_1 \triangle \cdots \triangle \alpha_m. \end{cases}$$

Suppose that we have $\left\langle \bar{\hat{Q}}^{n+1}[l] \right\rangle$, $s^n[l]$, and $t^n[l]$ for every $l = 1,2,\ldots,k-1$.

Let $\left\langle \bar{\hat{Q}}^{n+1}[k] \right\rangle := \dot{\bigcup\limits_{p=0}^{n}} \{(\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x \in \langle \hat{Q}^{n+1}[i] \rangle,\ y \in \langle \hat{Q}^{n+1}[j] \rangle,$

$i+j=k,\ \alpha_h \subseteq \mathbb{N}_0,\ h=1,2,\ldots,m,\ s^p[i]\cdots s^n[i](x) = t^p[j]\cdots t^n[j](y)\}.$

Define $s^n[k], t^n[k] : \left\langle \bar{\hat{Q}}^{n+1}[k] \right\rangle \to \left\langle \bar{\hat{Q}}^{n} \right\rangle$ by

$$s^n[k]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((s^n[i](x),p,s^n[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n\notin \alpha_1\triangle\cdots\triangle\alpha_m; \\ (\cdots((t^n[i](x),p,t^n[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n\in \alpha_1\triangle\cdots\triangle\alpha_m; \\ (\cdots((s^n[i](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n\notin \alpha_1\triangle\cdots\triangle\alpha_m; \\ (\cdots((t^n[i](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n\in \alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

$$t^n[k]\big((\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m}\big)$$
$$:= \begin{cases} (\cdots((t^n[i](x),p,t^n[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n\notin \alpha_1\triangle\cdots\triangle\alpha_m; \\ (\cdots((s^n[i](x),p,s^n[j](y))^{\alpha_1})^{\cdots})^{\alpha_m}, & p<n\in \alpha_1\triangle\cdots\triangle\alpha_m; \\ (\cdots((t^n[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n\notin \alpha_1\triangle\cdots\triangle\alpha_m; \\ (\cdots((s^n[i](x))^{\alpha_1})^{\cdots})^{\alpha_m}, & p=n\in \alpha_1\triangle\cdots\triangle\alpha_m. \end{cases}$$

Set $\left\langle \bar{\hat{Q}}^{n+1} \right\rangle := \dot{\bigcup\limits_{k=1}^{\infty}} \left\langle \bar{\hat{Q}}^{n+1}[k] \right\rangle$, $s^n_{\langle \bar{\hat{Q}} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} s^n[k]$, and $t^n_{\langle \bar{\hat{Q}} \rangle} := \dot{\bigcup\limits_{k=1}^{\infty}} t^n[k]$.

It follows from Proposition 3.1.2.2 that we get an ω-globular set

$$\left\langle \bar{\hat{Q}}^0 \right\rangle \underset{t^0_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^0_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^1 \right\rangle \underset{t^1_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^1_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots \underset{t^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^{n-1}_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \left\langle \bar{\hat{Q}}^n \right\rangle \underset{t^n_{\langle \bar{\hat{Q}} \rangle}}{\overset{s^n_{\langle \bar{\hat{Q}} \rangle}}{\leftleftarrows}} \cdots.$$

For every $m \in \mathbb{N}$, we define

$$\begin{aligned}
\bar{\hat{\mathscr{E}}} &:= \{(\cdots((w^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid w \in \mathscr{E},\ \alpha_1,\alpha_2,\ldots,\alpha_m \subseteq \mathbb{N}_0\}, \\
\bar{\hat{\mathscr{E}}}^0 &:= \{(\cdots(((\ldots,(1,(0,x))\cdots)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid x \in \left\langle \bar{\hat{Q}}^0 \right\rangle,\ \alpha_1,\alpha_2,\ldots,\alpha_m \subseteq \mathbb{N}_0\}, \\
\bar{\hat{\mathscr{E}}}^1 &:= \{(\cdots(((\ldots,(2,(1,y))\cdots)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid y \in \left\langle \bar{\hat{Q}}^1 \right\rangle,\ \alpha_1,\alpha_2,\ldots,\alpha_m \subseteq \mathbb{N}_0\}, \\
&\vdots \quad \vdots \\
\bar{\hat{\mathscr{E}}}^n &:= \{(\cdots(((\ldots,(n+1,(n,z))\cdots)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mid z \in \left\langle \bar{\hat{Q}}^n \right\rangle,\ \alpha_1,\alpha_2,\ldots,\alpha_m \subseteq \mathbb{N}_0\}, \\
&\vdots \quad \vdots
\end{aligned}$$

Now we establish the following inductive family.

Let $\left\langle \bar{\hat{\mathscr{E}}}[1] \right\rangle := \hat{\mathscr{E}} \cup \dot{\bigcup}_{n=0}^{\infty} \bar{\hat{\mathscr{E}}}^n$.

For any $q \in \mathbb{N}_0$, define $s^q[1] : \left\langle \bar{\hat{\mathscr{E}}}[1] \right\rangle \to \left\langle \bar{\hat{Q}}^q \right\rangle$ by $\left(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\right) \mapsto$

$$\begin{cases}
\left(\cdots(((s^q_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\right), & x \in \mathscr{E},\ q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
\left(\cdots(((t^q_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\right), & x \in \mathscr{E},\ q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^q_{\left\langle \bar{\hat{Q}} \right\rangle} \cdots s^{k-1}_{\left\langle \bar{\hat{Q}} \right\rangle}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \left\langle \bar{\hat{Q}}^k \right\rangle,\ k > q,\ q \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((t^q_{\left\langle \bar{\hat{Q}} \right\rangle} \cdots t^{k-1}_{\left\langle \bar{\hat{Q}} \right\rangle}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \left\langle \bar{\hat{Q}}^k \right\rangle,\ k > q,\ q \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),q-1)^{\gamma_1})^{\cdots})^{\gamma_{m_{q-1}}}, & z \in \left\langle \bar{\hat{Q}}^k \right\rangle,\ k \geq q \\
\qquad x = (\ldots,((\cdots(q-1,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{q-1}}})^{\cdots}). &
\end{cases}$$

Similarly, define $t^q[1] : \left\langle \bar{\hat{\mathscr{E}}}[1] \right\rangle \to \left\langle \bar{\hat{Q}}^q \right\rangle$ by $\left(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\right) \mapsto$

$$\begin{cases}
\left(\cdots(((t^q_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\right), & x \in \mathscr{E},\ q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
\left(\cdots(((s^q_{\mathscr{E}}(x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\right), & x \in \mathscr{E},\ q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\
(\cdots((t^q_{\left\langle \bar{\hat{Q}} \right\rangle} \cdots t^{k-1}_{\left\langle \bar{\hat{Q}} \right\rangle}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \left\langle \bar{\hat{Q}}^k \right\rangle,\ k > q,\ q \notin \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots((s^q_{\left\langle \bar{\hat{Q}} \right\rangle} \cdots s^{k-1}_{\left\langle \bar{\hat{Q}} \right\rangle}(y))^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (\ldots,((\cdots((k,y)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots}), \\
 & y \in \left\langle \bar{\hat{Q}}^k \right\rangle,\ k > q,\ q \in \alpha_1 \triangle \cdots \triangle \alpha_m; \\
(\cdots(((\cdots((\cdots((z,k)^{\beta_1})^{\cdots})^{\beta_{m_k}}),\ldots),q-1)^{\gamma_1})^{\cdots})^{\gamma_{m_{q-1}}}, & z \in \left\langle \bar{\hat{Q}}^k \right\rangle,\ k \geq q \\
\qquad x = (\ldots,((\cdots(q-1,(\ldots,((\cdots((k,z)^{\beta_1})^{\cdots})^{\beta_{m_k}})^{\cdots})^{\gamma_1})^{\cdots})^{\gamma_{m_{q-1}}})^{\cdots}). &
\end{cases}$$

Suppose that we have $\left\langle \bar{\hat{\mathscr{E}}}[k] \right\rangle$, $s^q[k]$, and $t^q[k]$ for all $k = 1,2,\ldots,n-1$.

Let $\left\langle \bar{\hat{\mathscr{E}}}[n] \right\rangle := \dot{\bigcup}_{p=0}^{\infty} \big\{ (\cdots((x,p,y)^{\alpha_1})^{\cdots})^{\alpha_m} \mid x \in \left\langle \bar{\hat{\mathscr{E}}}[i] \right\rangle, y \in \left\langle \bar{\hat{\mathscr{E}}}[j] \right\rangle$

$s^p[i](x) = t^p[j](y),\ \alpha_k \subseteq \mathbb{N}_0,\ k = 1,2,\ldots,m \big\}$.

If $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \left\langle \bar{\hat{\mathscr{E}}}[n] \right\rangle$, then we define

- $s^q[n] : \left\langle \bar{\hat{\mathscr{E}}}[n] \right\rangle \to \left\langle \bar{\hat{Q}}^q \right\rangle$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((s^q[i](x),p,s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\notin \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m; \\ (\cdots(((t^q[i](x),p,t^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\in \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m; \\ (\cdots(((s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\notin \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m; \\ (\cdots(((t^q[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\in \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m, \end{cases}$$

- $t^q[n] : \left\langle \bar{\bar{\mathscr{E}}}[n] \right\rangle \to \left\langle \bar{\bar{Q}}^q \right\rangle$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((t^q[i](x),p,t^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\notin \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m; \\ (\cdots(((s^q[i](x),p,s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p<q\in \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m; \\ (\cdots(((t^q[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\notin \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m; \\ (\cdots(((s^q[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p\geq q\in \alpha_1\triangle\alpha_2\triangle\cdots\triangle\alpha_m. \end{cases}$$

Let $\left\langle \bar{\bar{\mathscr{E}}} \right\rangle := \dot{\bigcup}_{n=1}^{\infty} \left\langle \bar{\bar{\mathscr{E}}}[n] \right\rangle$, $s^q_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle} := \dot{\bigcup}_{n=1}^{\infty} s^q[n]$, and $t^q_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle} := \dot{\bigcup}_{n=1}^{\infty} t^q[n]$.

By Proposition 4.3.2.2 and Proposition 4.3.3.2, we obtain a new globular cone

$$\begin{tikzcd}
 & & & \left\langle \bar{\bar{\mathscr{E}}} \right\rangle \arrow[dlll, shift right, "s^0_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}"'] \arrow[dlll, shift left, "t^0_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}"] \arrow[dll, shift right, "s^1_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}"'] \arrow[dll, shift left, "t^1_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}"] \arrow[d, shift right, "s^n_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}"'] \arrow[d, shift left, "t^n_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}"] & \cdots \\
\left\langle \bar{\bar{Q}}^0 \right\rangle & \left\langle \bar{\bar{Q}}^1 \right\rangle \arrow[l, shift right, "s^0_{\left\langle \bar{\bar{Q}} \right\rangle}"'] \arrow[l, shift left, "t^0_{\left\langle \bar{\bar{Q}} \right\rangle}"] & \cdots \arrow[l, shift right, "s^1_{\left\langle \bar{\bar{Q}} \right\rangle}"'] \arrow[l, shift left, "t^1_{\left\langle \bar{\bar{Q}} \right\rangle}"] & \left\langle \bar{\bar{Q}}^n \right\rangle \arrow[l, shift right, "s^{n-1}_{\left\langle \bar{\bar{Q}} \right\rangle}"'] \arrow[l, shift left, "t^{n-1}_{\left\langle \bar{\bar{Q}} \right\rangle}"] & \cdots \arrow[l, shift right, "s^n_{\left\langle \bar{\bar{Q}} \right\rangle}"'] \arrow[l, shift left, "t^n_{\left\langle \bar{\bar{Q}} \right\rangle}"]
\end{tikzcd}$$

Then, for each $\alpha \subseteq \mathbb{N}_0$, we define $\hat{*}_\alpha : \left\langle \bar{\bar{\mathscr{E}}} \right\rangle \to \left\langle \bar{\bar{\mathscr{E}}} \right\rangle$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}.$$

We also define $\hat{\circ}_p : \left\langle \bar{\bar{\mathscr{E}}} \right\rangle \times_p \left\langle \bar{\bar{\mathscr{E}}} \right\rangle \to \left\langle \bar{\bar{\mathscr{E}}} \right\rangle$ by $x \;\hat{\circ}_p\; y \mapsto (x,p,y)^{\varnothing}$, where

$$\left\langle \bar{\bar{\mathscr{E}}} \right\rangle \times_p \left\langle \bar{\bar{\mathscr{E}}} \right\rangle := \left\{ (x,y) \in \left\langle \bar{\bar{\mathscr{E}}} \right\rangle \times \left\langle \bar{\bar{\mathscr{E}}} \right\rangle \mid s^p_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}(x) = t^p_{\left\langle \bar{\bar{\mathscr{E}}} \right\rangle}(y) \right\}.$$

Next we define $\iota^n_{\left\langle \bar{\bar{Q}} \right\rangle} : \left\langle \bar{\bar{Q}}^n \right\rangle \to \left\langle \bar{\bar{Q}}^{n+1} \right\rangle$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto \left((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, n\right)^{\varnothing}.$$

It is easy to see that $s^n_{\langle\bar{\hat{Q}}\rangle} \circ \iota^n_{\langle\bar{\hat{Q}}\rangle} = \mathrm{Id}_{\langle\bar{\hat{Q}}^n\rangle} = t^n_{\langle\bar{\hat{Q}}\rangle} \circ \iota^n_{\langle\bar{\hat{Q}}\rangle}$ for every $n \in \mathbb{N}_0$.

Then, for all $n \in \mathbb{N}_0$, we define $\iota^n_{\bar{\hat{\mathscr{E}}}} : \bar{\hat{Q}}^n \to \bar{\hat{\mathscr{E}}}$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\ldots,(n+1,(n,(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}))^{\varnothing})^{\cdots})^{\varnothing}.$$

We now form the following components:

1. $\left\langle\bar{\hat{Q}}^n\right\rangle \times_G \left\langle\bar{\hat{Q}}^n\right\rangle := \left\{(x,y) \in \left\langle\bar{\hat{Q}}^n\right\rangle \times \left\langle\bar{\hat{Q}}^n\right\rangle \mid (s/t)^n_{\langle\bar{\hat{Q}}\rangle}(x) = (s/t)^n_{\langle\bar{\hat{Q}}\rangle}(y)\right\}$,

2. $\pi^n_{\langle\bar{\hat{\mathscr{E}}}\rangle} : \left\langle\bar{\hat{\mathscr{E}}}\right\rangle \to \left\langle\bar{\hat{Q}}^n\right\rangle \times_G \left\langle\bar{\hat{Q}}^n\right\rangle$ is defined by $\pi^n_{\langle\bar{\hat{\mathscr{E}}}\rangle}(x) := \left(s^n_{\langle\bar{\hat{\mathscr{E}}}\rangle}(x), t^n_{\langle\bar{\hat{\mathscr{E}}}\rangle}(x)\right)$,

3. $\Delta^n : \left\langle\bar{\hat{Q}}^n\right\rangle \to \left\langle\bar{\hat{Q}}^n\right\rangle \times_G \left\langle\bar{\hat{Q}}^n\right\rangle$ is defined by $\Delta^n(x) := (x,x)$.

Consider $\pi^{n+k}_{\langle\bar{\hat{\mathscr{E}}}\rangle} \circ \iota^n_{\langle\bar{\hat{\mathscr{E}}}\rangle}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big)$

$$\begin{aligned}
&= \pi^{n+k}_{\langle\bar{\hat{\mathscr{E}}}\rangle}\big((\ldots,(n+1,(n,(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}\big)\\
&= \Big(s^{n+k}_{\langle\bar{\hat{\mathscr{E}}}\rangle}\big((\ldots,(n+1,(n,(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}\big),\\
&\qquad t^{n+k}_{\langle\bar{\hat{\mathscr{E}}}\rangle}\big((\ldots,(n+1,(n,(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\varnothing})^{\cdots})^{\varnothing}\big)\Big)\\
&= \big(((\cdots((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m},n)^{\varnothing},\ldots),n+k-1)^{\varnothing},\\
&\qquad ((\cdots((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m},n)^{\varnothing},\ldots),n+k-1)^{\varnothing}\big)\\
&= \Delta^{n+k}\big(((\cdots((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m},n)^{\varnothing},\ldots),n+k-1)^{\varnothing}\big)\\
&= \Delta^{n+k} \circ \iota^{n+k-1}_{\langle\bar{\hat{Q}}\rangle} \circ \cdots \circ \iota^n_{\langle\bar{\hat{Q}}\rangle}\big((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}\big).
\end{aligned}$$

This means that $\pi^{n+k}_{\langle\bar{\hat{\mathscr{E}}}\rangle} \circ \iota^n_{\langle\bar{\hat{\mathscr{E}}}\rangle} = \Delta^{n+k} \circ \iota^{n+k-1}_{\langle\bar{\hat{Q}}\rangle} \circ \cdots \circ \iota^n_{\langle\bar{\hat{Q}}\rangle}$.

Hence, $\left(\left\langle\bar{\hat{\mathscr{E}}}\right\rangle, (\iota^n_{\langle\bar{\hat{\mathscr{E}}}\rangle})_{n\in\mathbb{N}_0}, (\hat{\circ}_p)_{p\in\mathbb{N}_0}, (\hat{*}_\alpha)_{\alpha\subseteq\mathbb{N}_0}\right)$ becomes a reflexive self-dual globular-cone ω-magma.

Next, we define $i : \mathscr{E} \to \left\langle\bar{\hat{\mathscr{E}}}\right\rangle$ by $x \mapsto x^{\varnothing}$.

Assume that $f : \mathscr{E} \to \left(\mathscr{R}, (\iota^n_{Q_{\mathscr{R}}})_{n\in\mathbb{N}_0}, (\iota^n_{\mathscr{R}})_{n\in\mathbb{N}_0}, (\circ_p)_{p\in\mathbb{N}_0}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0}\right)$ is a morphism of globular cones into another reflexive self-dual globular-cone ω-magma.

As morphisms of reflexive self-dual globular cones and reflexive globular-cone ω-magmas have already discussed in detail before, we will explain the unique construction of a morphism of reflexive self-dual globular-cone ω-magmas $\psi : \left\langle \bar{\hat{\mathscr{E}}} \right\rangle \to \mathscr{R}$ in short as follows.

For any $x \in Q^n$, $w \in \mathscr{E}$, and $(y, p, z) \in \mathscr{E} \times_p \mathscr{E}$, we associate the simple elements in $\left\langle \bar{\hat{\mathscr{E}}} \right\rangle$ to the simple elements in $\mathscr{R}$:

$$\begin{array}{rcl} w & \mapsto & f(w), \\ (x,n) & \mapsto & \iota^n_{Q_{\mathscr{R}}}(f(x)), \\ (\ldots,(n+1,(n,x))\cdots) & \mapsto & \iota^n_{\mathscr{R}}(f(x)), \\ (\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m} & \mapsto & (\cdots((f(x))^{*\alpha_1})^{\cdots})^{*\alpha_m}, \\ (y,p,z) & \mapsto & f(y) \circ_p f(z). \end{array}$$

Combining Proposition 4.3.2.2 with Proposition 4.3.3.2, we get a unique morphism of reflexive self-dual globular-cone ω-magmas satisfying $f = \psi \circ i$.

As a result, $\left(\left(\left\langle \bar{\hat{\mathscr{E}}} \right\rangle, (\iota^n_{\left\langle \bar{\hat{\mathscr{E}}} \right\rangle})_{n\in\mathbb{N}_0}, (\hat{\circ}_p)_{p\in\mathbb{N}_0}, (\hat{*}\alpha)_{\alpha\subseteq\mathbb{N}_0} \right), i \right)$ is a free reflexive self-dual globular-cone ω-magma over a globular cone as desired. $\square$

4.4 Free Involutive Penon Cone-Contractions

At this point we are about to complete the construction of free components referring some arguments done before. This section discusses on a free strict involutive globular-cone ω-category and a free involutive Penon cone-contraction over the original globular cone as before.

4.4.1 Free Strict Involutive Globular-Cone ω-Categories

Let's turn our attention now to another significant part of our work of the existence of a *free strict involutive globular-cone* ω*-category* over a globular cone. We start here with its definition.

Definition 4.4.1.1. A **strict involutive globular-cone ω-category** is a reflexive self-dual globular-cone ω-magma $\mathscr{C}$ satisfying the following axioms:

1. (associativity) if $(z,y),(y,x) \in \mathscr{C} \times_p \mathscr{C}$, then $(z \circ_p y) \circ_p x = z \circ_p (y \circ_p x)$,

2. (unitality) if $0 \le p < m$ and $x \in \mathscr{C}$, then

$$\iota_{\mathscr{C}}^{m-1} \cdots \iota_Q^{p+1} \iota_Q^p t_Q^p t_Q^{p+1} \cdots t_{\mathscr{C}}^{m-1}(x) \circ_p x = x = x \circ_p \iota_{\mathscr{C}}^{m-1} \cdots \iota_Q^{p+1} \iota_Q^p s_Q^p \cdots s_{\mathscr{C}}^{m-1}(x),$$

3. (binary exchange) if $(y',y),(x',x) \in \mathscr{C} \times_p \mathscr{C}$ and $(y',x'),(y,x) \in \mathscr{C} \times_q \mathscr{C}$ with $0 \le q < p$, then $(y' \circ_p y) \circ_q (x' \circ_p x) = (y' \circ_q x') \circ_p (y \circ_q x)$,

4. (functoriality of identities) if $0 \le q < p$ and $(x',x) \in Q^p \times_q Q^p$, then

$$\iota_{\mathscr{C}}^p(x') \circ_q \iota_{\mathscr{C}}^p(x) = \iota_{\mathscr{C}}^p(x' \circ_q^p x),$$

5. (involutivity) if $x \in \mathscr{C}$, then $(x^{*\alpha})^{*\alpha} = x$,

6. (commutativity of involutions) if $x \in \mathscr{C}$, then $(x^{*\alpha})^{*\beta} = (x^{*\beta})^{*\alpha}$,

7. (covariance/contravariance of involutions) if $(x,y) \in \mathscr{C} \times_p \mathscr{C}$, then

$$(x \circ_p y)^{*\alpha} = \begin{cases} x^{*\alpha} \circ_p y^{*\alpha}, & p \notin \alpha; \\ y^{*\alpha} \circ_p x^{*\alpha}, & p \in \alpha. \end{cases}$$

8. (functoriality of involutions) if $w \in Q^n$, then $\iota_{\mathscr{C}}^n(w^{\overset{n}{*\alpha}}) = (\iota_{\mathscr{C}}^n(w))^{*\alpha}$.

Applying a similar argument as Proposition 3.2.1.3, we get the result.

Proposition 4.4.1.2. *A free strict involutive globular-cone* ω*-category over a globular cone exists.*

Proof. Let $(\mathscr{M}, i)$ be the free reflexive self-dual globular-cone ω-magma over a globular cone $\mathscr{E}$ constructed before.

Consider the smallest congruence R in $\mathscr{M}$ generated by

$$
\begin{aligned}
\mathrm{Ax} := {} & \{\big((x \circ_p y) \circ_p z, x \circ_p (y \circ_p z)\big) \mid (x,y),(y,z) \in \mathscr{M} \times_p \mathscr{M}, p = 0,1,\dots\} \\
& \cup \{(\iota^{n-1}_{\mathscr{M}} \cdots \iota^p_Q t^p_Q \cdots t^{n-1}_{\mathscr{M}}(x) \circ_p x, x) \mid x \in \mathscr{M},\ p = 0,1,\dots,n-1, n \in \mathbb{N}\} \\
& \cup \{(x \circ_p \iota^{n-1}_{\mathscr{M}} \cdots \iota^p_Q s^p_Q \cdots s^{n-1}\mathscr{M}(x), x) \mid x \in \mathscr{M},\ p = 0,1,\dots,n-1,\ n \in \mathbb{N}\} \\
& \cup \{\big(\iota^n_{\mathscr{M}}(x \circ^n_p y), \iota^n_{\mathscr{M}}(x) \circ_p \iota^n_{\mathscr{M}}(y)\big) \mid (x,y) \in \mathscr{M} \times \mathscr{M}, p = 0,1,\dots,n-1,\ n \in \mathbb{N}\} \\
& \cup \{\big((y' \circ_p y) \circ_q (x' \circ_p x), (y' \circ_q x') \circ_p (y \circ_q x)\big) \mid (y',y),(x',x) \in \mathscr{M} \times_p \mathscr{M}, \\
& \qquad (y',x'),(y,x) \in \mathscr{M} \times_q \mathscr{M}\} \cup \{\big((x^{*\alpha})^{*\beta}, (x^{*\beta})^{*\alpha}\big) \mid \alpha,\beta \subseteq \mathbb{N}_0\} \\
& \cup \{\big((x \circ_p y)^{*\alpha}, x^{*\alpha} \circ_p y^{*\alpha}\big) \mid (x,y) \in \mathscr{M} \times_p \mathscr{M},\ \mathbb{N}_0 \supseteq \alpha \not\ni p = 0,1,\dots\} \\
& \cup \{\big((x \circ_p y)^{*\alpha}, y^{*\alpha} \circ_p x^{*\alpha}\big) \mid (x,y) \in \mathscr{M} \times_p \mathscr{M},\ \mathbb{N}_0 \supseteq \alpha \ni p = 0,1,\dots\} \\
& \cup \{(\iota^n_{\mathscr{M}}(x) \circ_p \iota^n_{\mathscr{M}}(y), \iota^n_{\mathscr{M}}(x \circ^n_p y))\} \cup \{\iota^n_{\mathscr{M}}(w^{*^n_\alpha}), (\iota^n_{\mathscr{M}}(w))^{*\alpha} \mid w \in Q^n\}.
\end{aligned}
$$

Since $\mathrm{Ax} \subseteq R$ and all algebraic axioms already hold in $\mathscr{M}/R$, $\mathscr{M}/R$ becomes a strict involutive globular-cone ω-category.

Let $f : \mathscr{E} \to \hat{\mathscr{C}}$ be a morphism of globular cones into another strict involutive globular-cone ω-category.

As $(\mathscr{M}, i)$ is a free reflexive self-dual globular-cone ω-magma over a globular cone $\mathscr{E}$, there exists a unique morphism of reflexive self-dual globular-cone ω-magmas $\phi : \mathscr{M} \to \hat{\mathscr{C}}$ such that $f = \phi \circ i$.

Consider $R_\phi := \{(x,y) \in \mathscr{M} \times \mathscr{M} \mid \phi(x) = \phi(y)\}$.

It follows that R_ϕ is a congruence in $\mathscr{M}$.

Since $\hat{\mathscr{C}}$ is a strict involutive globular-cone ω-category, $\mathrm{Ax} \subseteq R_\phi$ and so $\mathscr{M}/R_\phi$ becomes a strict involutive globular-cone ω-category.

If $\tilde{\phi} : \mathscr{M}/R_\phi \to \hat{\mathscr{C}}$ is defined by $[x]_\phi \mapsto \phi(x)$, it is a unique map such that $\tilde{\phi} \circ \pi_\phi = \phi$, where $\pi_\phi : \mathscr{M} \to \mathscr{M}/R_\phi$ is defined by $x \mapsto [x]_\phi$.

As R is the smallest congruence containing Ax, $R \subseteq R_\phi$ and so $\theta : \mathscr{M}/R \to \mathscr{M}/R_\phi$, defined $[x] \mapsto [x]_\phi$, is a unique map such that $\pi_\phi = \theta \circ \pi$.

Combining all the previous maps, we get that $\hat{\phi} := \tilde{\phi} \circ \theta : \mathscr{M}/R \to \hat{\mathscr{C}}$ is a unique morphism of strict involutive globular-cone ω-categories satisfying

$$f = \phi \circ i = \tilde{\phi} \circ \pi_\phi \circ i = \tilde{\phi} \circ \theta \circ \pi \circ i = \hat{\phi} \circ (\pi \circ i).$$

Therefore, $(\mathscr{M}/R, \pi \circ i)$ is a free strict involutive globular-cone ω-category over a globular cone $\mathscr{E}$. $\square$

4.4.2 Free Involutive Penon Cone-Contractions

This subsection is devoted to establishing a category $\hat{\mathscr{Q}}^*$ whose objects are of the form $\left(\mathscr{M}^* \xrightarrow{f^*} \mathscr{C}^*, [\cdot,\cdot]^*\right)$, where $\mathscr{M}^*$ is a self-dual globular-cone ω-magma, $\mathscr{C}^*$ is a strict involutive globular-cone ω-category, and f^* is a morphism of self-dual globular-cone ω-magmas, together with a modified Penon contraction $[\cdot,\cdot]^*$. We simply call these objects *involutive Penon cone-contractions*. First, we give a modification of such Penon contraction as follows.

Definition 4.4.2.1. Let

$$\begin{tikzcd}
&& \cdots & \mathscr{M} \arrow[dlll, shift right, "s^0_{\mathscr{M}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{M}}"] \arrow[dll, shift right, "s^1_{\mathscr{M}}"'] \arrow[dll, shift left, "t^1_{\mathscr{M}}"] \arrow[d, shift right, "s^n_{\mathscr{M}}"'] \arrow[d, shift left, "t^n_{\mathscr{M}}"] & \cdots \\
M^0 & M^1 \arrow[l, shift right, "s^0_M"'] \arrow[l, shift left, "t^0_M"] & \cdots \arrow[l, shift right, "s^1_M"'] \arrow[l, shift left, "t^1_M"] & M^n \arrow[l, shift right, "s^{n-1}_M"'] \arrow[l, shift left, "t^{n-1}_M"] & \cdots \arrow[l, shift right, "s^n_M"'] \arrow[l, shift left, "t^n_M"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
&& \cdots & \mathscr{C} \arrow[dlll, shift right, "s^0_{\mathscr{C}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{C}}"] \arrow[dll, shift right, "s^1_{\mathscr{C}}"'] \arrow[dll, shift left, "t^1_{\mathscr{C}}"] \arrow[d, shift right, "s^n_{\mathscr{C}}"'] \arrow[d, shift left, "t^n_{\mathscr{C}}"] & \cdots \\
C^0 & C^1 \arrow[l, shift right, "s^0_C"'] \arrow[l, shift left, "t^0_C"] & \cdots \arrow[l, shift right, "s^1_C"'] \arrow[l, shift left, "t^1_C"] & C^n \arrow[l, shift right, "s^{n-1}_C"'] \arrow[l, shift left, "t^{n-1}_C"] & \cdots \arrow[l, shift right, "s^n_C"'] \arrow[l, shift left, "t^n_C"]
\end{tikzcd}$$

be a reflexive self-dual globular-cone ω-magma and a strict involutive globular-cone ω-category, respectively. Suppose that $\pi : \mathscr{M} \to \mathscr{C}$ is a morphism of reflexive self-dual globular-cone ω-magmas. For every $n \in \mathbb{N}$, consider

$$D([\cdot,\cdot]_n^{\mathscr{M}}) := \{(x,y) \in M^n \times M^n \mid s_M^{n-1}(x) = s_M^{n-1}(y), t_M^{n-1}(x) = t_M^{n-1}(y), \pi_M^n(x) = \pi_M^n(y)\}.$$

An **involutive Penon cone-contraction** is a family of maps $[\cdot,\cdot]_n^{\mathscr{M}} : D([\cdot,\cdot]_n^{\mathscr{M}}) \to M^{n+1}$ such that the following requirements are satisfied:

1. $s^n_{\mathscr{M}}([x,y]_n^{\mathscr{M}}) = x$, $t^n_{\mathscr{M}}([x,y]_n^{\mathscr{M}}) = y$,
2. $[x,y]_n^{\mathscr{M}} \in \iota^{n+1}_{\mathscr{M}}(M^{n+1})$,
3. $x = y \in M^n$ implies $[x,y]_n^{\mathscr{M}} = \iota^n_{\mathscr{M}}(x) = \iota^n_{\mathscr{M}}(y)$.

Remark 4.4.2.2. Observe that the conditions 1 and 3 are exactly the same as in the definition of involutive Penon contractions. It is easy to see that the condition 2 together with being a morphism of π imply $\pi^{n+1}_{\mathscr{M}}([x,y]_n^{\mathscr{M}}) = \iota^n_{\mathscr{C}}(\pi^n_C(x)) = \iota^n_{\mathscr{C}}(\pi^n_C(y))$. Moreover, if we define $[x,y]_n^M := s^{n+1}_{\mathscr{M}}([x,y]_n^{\mathscr{M}}) = t^{n+1}_{\mathscr{M}}([x,y]_n^{\mathscr{M}})$, it reduces to the original involutive Penon contraction.

Theorem 4.4.2.3. *A free involutive Penon cone-contraction over a globular cone exists.*

Proof. Let a globular cone over an ω-globular set be given

$$\begin{tikzcd}[column sep=large, row sep=huge]
 & & & & & & \mathscr{E} \arrow[dllllll, shift right, "s^0_{\mathscr{E}}"'] \arrow[dllllll, shift left, "t^0_{\mathscr{E}}"] \arrow[dllll, shift right, "s^1_{\mathscr{E}}"'] \arrow[dllll, shift left, "t^1_{\mathscr{E}}"] \arrow[d, shift right, "s^n_{\mathscr{E}}"'] \arrow[d, shift left, "t^n_{\mathscr{E}}"] & \cdots \\
Q^0 & & Q^1 \arrow[ll, shift right, "s^0_Q"'] \arrow[ll, shift left, "t^0_Q"] & & \cdots \arrow[ll, shift right, "s^1_Q"'] \arrow[ll, shift left, "t^1_Q"] & & Q^n \arrow[ll, shift right, "s^{n-1}_Q"'] \arrow[ll, shift left, "t^{n-1}_Q"] & & \cdots \arrow[ll, shift right, "s^n_Q"'] \arrow[ll, shift left, "t^n_Q"]
\end{tikzcd}$$

Referring to Theorem 3.2.2.1, we already have a free involutive Penon contraction $\left(\left(M \overset{\pi}{\to} C, [\cdot,\cdot]\right), g\right)$ over an ω-globular set Q.

Now let's establish $\left(\mathscr{M} \overset{\Pi}{\to} \mathscr{C}, [\cdot,\cdot]^{\mathscr{M}}\right)$ analogously to Proposition 4.3.4.2.

Let $\mathscr{M}[1] := \left\langle \bar{\bar{\mathscr{E}}}[1] \right\rangle$ and define $s^q_{\mathscr{M}}[1], t^q_{\mathscr{M}}[1] : \mathscr{M}[1] \to M^q$ by

$$s^q_{\mathscr{M}}[1]\big((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots(y^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (y,z)_n \in \mathrm{Ax}^n; \\ s^q[1]\big((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}\big), & \text{otherwise.} \end{cases}$$

$$t^q_{\mathscr{M}}[1]\big((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}\big) := \begin{cases} (\cdots(z^{\alpha_1})^{\cdots})^{\alpha_m}, & x = (y,z)_n \in \mathrm{Ax}^n; \\ t^q[1]\big((\cdots(x^{\alpha_1})^{\cdots})^{\alpha_m}\big), & \text{otherwise.} \end{cases}$$

Suppose that we have $\mathscr{M}[k]$, $s^q_{\mathscr{M}}[k]$, and $t^q_{\mathscr{M}}[k]$ for every $k = 1,2,\ldots,n-1$.

Set $\mathscr{M}[n] := \left\langle \bar{\bar{\mathscr{E}}}[n] \right\rangle$.

If $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \in \mathscr{M}[n]$, where $x \in \mathscr{M}[i]$ and $y \in \mathscr{M}[j]$, we define

- $s^q_{\mathscr{M}}[n] : \mathscr{M}[n] \to M^q$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((s^q_{\mathscr{M}}[i](x), p, s^q_{\mathscr{M}}[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q_{\mathscr{M}}[i](x), p, t^q_{\mathscr{M}}[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q_{\mathscr{M}}[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q_{\mathscr{M}}[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m, \end{cases}$$

- $t^q_{\mathscr{M}}[n] : \mathscr{M}[n] \to M^q$ by $(\cdots(((x,p,y)^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}$

$$\mapsto \begin{cases} (\cdots(((t^q_{\mathscr{M}}[i](x), p, t^q_{\mathscr{M}}[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q_{\mathscr{M}}[i](x), p, s^q_{\mathscr{M}}[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p < q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((t^q_{\mathscr{M}}[i](x))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \notin \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m; \\ (\cdots(((s^q_{\mathscr{M}}[j](y))^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, & p \geq q \in \alpha_1 \triangle \alpha_2 \triangle \cdots \triangle \alpha_m. \end{cases}$$

Let $\mathscr{M} := \dot{\bigcup\limits_{n=0}^{\infty}} \mathscr{M}[n]$, $s^q_{\mathscr{M}} := \dot{\bigcup\limits_{n=0}^{\infty}} s^q_{\mathscr{M}}[n]$, and $t^q_{\mathscr{M}} := \dot{\bigcup\limits_{n=0}^{\infty}} t^q_{\mathscr{M}}[n]$.

By Proposition 4.3.4.2, we obtain a globular cone

$$\begin{tikzcd}
 & & \cdots & \mathscr{M} \arrow[dlll, shift right, "s^0_{\mathscr{M}}"'] \arrow[dlll, shift left, "t^0_{\mathscr{M}}"] \arrow[dll, shift right, "s^1_{\mathscr{M}}"'] \arrow[dll, shift left, "t^1_{\mathscr{M}}"] \arrow[d, shift right, "s^n_{\mathscr{M}}"'] \arrow[d, shift left, "t^n_{\mathscr{M}}"] & \cdots \\
M^0 & M^1 \arrow[l, shift right, "s^0_M"'] \arrow[l, shift left, "t^0_M"] & \cdots \arrow[l, shift right, "s^1_M"'] \arrow[l, shift left, "t^1_M"] & M^n \arrow[l, shift right, "s^{n-1}_M"'] \arrow[l, shift left, "t^{n-1}_M"] & \cdots \arrow[l, shift right, "s^n_M"'] \arrow[l, shift left, "t^n_M"]
\end{tikzcd}$$

For each $\alpha \subseteq \mathbb{N}_0$, we define $*_\alpha : \mathscr{M} \to \mathscr{M}$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto ((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m})^{\alpha}.$$

We also define $\circ_p : \mathscr{M} \times_p \mathscr{M} \to \mathscr{M}$ by $x \circ_p y \mapsto (x,p,y)^{\varnothing}$, where

$$\mathscr{M} \times_p \mathscr{M} := \left\{(x,y) \in \mathscr{M} \times \mathscr{M} \;\middle|\; s^p_{\mathscr{M}}(x) = t^p_{\mathscr{M}}(y)\right\}.$$

Next we define $\iota^n_M : M^n \to M^{n+1}$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto \left((\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}, n\right)^{\varnothing}.$$

Then, for all $n \in \mathbb{N}_0$, we define $\iota^n_{\mathscr{M}} : M^n \to \mathscr{M}$ by

$$(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m} \mapsto (\ldots,(n+1,(n,(\cdots((x^{\alpha_1})^{\alpha_2})^{\cdots})^{\alpha_m}))^{\varnothing})^{\cdots})^{\varnothing}.$$

It follows from Proposition 4.3.4.2 again that we get a free reflexive self-dual globular-cone ω-magma $\left(\left(\mathscr{M}, (\iota^n_{\mathscr{M}})_{n\in\mathbb{N}_0}, (\circ_p)_{p\in\mathbb{N}_0}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0}\right), i\right)$ over $\mathscr{E}$.

As the list of axioms Ax^n has already been defined in Theorem 4.4.1.2, for each $n \in \mathbb{N}$, we define $[\mathrm{Ax}^1] := \mathrm{Ax}$ and $[\mathrm{Ax}^{n+1}] := \mathrm{Ax}^{n+1} \cup \{(s^n_{\mathscr{M}}((x,y)^{\mathscr{M}}_n), x) \mid (x,y) \in \mathrm{Ax}^n\}$ $\cup \{(t^n_{\mathscr{M}}((x,y)^{\mathscr{M}}_n), y) \mid (x,y) \in \mathrm{Ax}^n\} \cup \{((x,x)_n, (x,n)) \mid x \in M^n\}$.

In addition, we set $[R^{n+1}]$ to be the smallest congruence generated by $[\mathrm{Ax}^{n+1}]$, for any $n \in \mathbb{N}_0$, and also $R := \bigcup_{n=0}^{\infty} [R^{n+1}]$.

Defining $\mathscr{C} := \mathscr{M}/R$, we get a strict involutive globular-cone ω-category and so $\Pi : \mathscr{M} \to \mathscr{C}$ plays the role of the quotient map.

Furthermore, we define $[\cdot,\cdot]^{\mathscr{M}}_n : [\mathrm{Ax}^n] \to \mathscr{M}$ by $(x,y) \mapsto (\ldots,(n,(x,y)_n)\cdots)$.

It is easy to see that $[\cdot,\cdot]^{\mathscr{M}}_n$ is an involutive Penon cone-contraction.

Next, we define $g : \mathscr{E} \to \left(\mathscr{M} \overset{\Pi}{\to} \mathscr{C}, [\cdot,\cdot]^{\mathscr{M}}\right)$ by $x \mapsto x^{\varnothing}$.

The rest of the proof follows from Proposition 3.2.2.1 and Proposition 4.3.4.2.

As a result, $\left(\left(\mathscr{M} \overset{\Pi}{\to} \mathscr{C}, [\cdot,\cdot]^{\mathscr{M}}\right), g\right)$ becomes a free involutive Penon cone-contraction over a globular cone $\mathscr{E}$. $\square$

4.5 Involutive Weak Globular-Cone ω-Categories

Analogous to Theorem 4.4.2.3, we can easily get a couple of free-forgetful functors $\mathbf{GCone} \underset{U^*}{\overset{F^*}{\rightleftarrows}} \hat{\mathscr{Q}}^*$, where **GCone** is the category of globular cones and $\hat{\mathscr{Q}}^*$ is the category of involutive Penon cone-contractions. This provides us an important adjunction.

4.5.1 Adjunction between Free-Forgetful Functors

Theorem 4.5.1.1. *The free functor $F^* : \mathbf{GCone} \to \hat{\mathscr{Q}}^*$ is left adjoint to the forgetful functor $U^* : \hat{\mathscr{Q}}^* \to \mathbf{GCone}$.*

Proof. Let $\mathscr{E} \in \mathrm{Ob}_{\mathbf{GCone}}$ and $\left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right) \in \mathrm{Ob}_{\hat{\mathscr{Q}}^*}$.

First of all, let us separate the functor F^* into the following components: $F^*_{\mathfrak{M}} : \mathscr{E} \mapsto \left\langle \bar{\hat{\mathscr{E}}} \right\rangle$ and $F^*_{\mathfrak{C}} : \mathscr{E} \mapsto [\mathscr{E}]$.

Consider $U^* : \mathscr{M} \mapsto U(\mathscr{M})$ forgetting the reflexivity, self-duality, and compositions and $U^* : \mathscr{C} \mapsto U(\mathscr{C})$ remaining the original globular cone.

So, $(\mathscr{M} \to F^*_{\mathfrak{M}} U^*(\mathscr{M}))$ is a free reflexive self-dual globular-cone ω-magma over the underlying globular cone of a reflexive self-dual globular-cone ω-magma $\mathscr{M}$.

Suppose that $\theta : \mathscr{M} \to \mathscr{N}$ is a morphism of reflexive self-dual globular-cone ω-magmas.

We get that $F^*_{\mathfrak{M}} U^*(\theta) : F^*_{\mathfrak{M}} U^*(\mathscr{M}) \to F^*_{\mathfrak{M}} U^*(\mathscr{N})$, defined similarly as in Proposition 4.3.4.2, becomes a morphism of reflexive self-dual globular-cone ω-magmas.

Now consider a free strict involutive globular-cone ω-category $(\mathscr{C} \to F^*_{\mathfrak{C}} U^*(\mathscr{C}))$ over the underlying globular cone of a strict involutive globular-cone ω-category $\mathscr{C}$.

Assume that $\nu : \mathscr{C} \to \mathscr{D}$ is a morphism of strict involutive globular-cone ω-categories.

We obtain a map $F^*_{\mathfrak{C}} U^*(\nu) : F^*_{\mathfrak{C}} U^*(\mathscr{C}) \to F^*_{\mathfrak{C}} U^*(\mathscr{D})$ defined by $[x] \mapsto [\nu(x)]$.

It follows from Proposition 4.4.1.2 that $F^*_{\mathfrak{C}}U^*(\nu)$ is a morphism of strict involutive globular-cone ω-categories.

Define $\varepsilon : \mathrm{Ob}_{\hat{\mathscr{Q}}^*} \to \mathrm{Hom}_{\hat{\mathscr{Q}}^*}$ by $\left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right) \mapsto \varepsilon_{\left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right)}$, where $\varepsilon_{\left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right)} : F^*U^*\left(\left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right)\right) \to \left(\mathscr{M} \xrightarrow{f} \mathscr{C}, [\cdot,\cdot]\right)$ is divided as follows.

For each $\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}, (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_k}\right) \in \langle \mathscr{M}[i]\rangle \times_p \langle \mathscr{M}[j]\rangle$, we define $\varepsilon^{\mathfrak{M}}_{\mathscr{M}} : F^*_{\mathfrak{M}}U^*(\mathscr{M}) \to \mathscr{M}$ by $\left(\cdots\left(\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m} \,\bar{\circ}_p (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_k}\right)^{\bar{*}\gamma_1}\right)^{\cdots}\right)^{\bar{*}\gamma_t}$ $\mapsto \left(\cdots\left(\left((\cdots(x^{*\alpha_1})^{\cdots})^{*\alpha_m} \circ_p (\cdots(y^{*\beta_1})^{\cdots})^{*\beta_k}\right)^{*\gamma_1}\right)^{\cdots}\right)^{*\gamma_t}$.

For every $\left(\left[(\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m}\right], \left[(\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_k}\right]\right) \in [\mathscr{C}[i]] \times_p [\mathscr{C}[j]]$, define $\varepsilon^{\mathfrak{C}}_{\mathscr{C}} : F^*_{\mathfrak{C}}U^*(\mathscr{C}) \to \mathscr{C}$ by $\left[\left(\cdots\left(\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m} \,\bar{\circ}_p (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_k}\right)^{\bar{*}\gamma_1}\right)^{\cdots}\right)^{\bar{*}\gamma_t}\right]$ $\mapsto \left(\cdots\left(\left((\cdots(x^{*\alpha_1})^{\cdots})^{*\alpha_m} \circ_p (\cdots(y^{*\beta_1})^{\cdots})^{*\beta_k}\right)^{*\gamma_1}\right)^{\cdots}\right)^{*\gamma_t}$.

First, consider the diagram

$$\begin{tikzcd}
\left(F^*_{\mathfrak{M}}U^*(\mathscr{M}), (\bar{\circ}_p)_{p\in\mathbb{N}_0}, (\bar{*}\alpha)_{\alpha\subseteq\mathbb{N}_0}\right) \arrow[r, "\varepsilon^{\mathfrak{M}}_{\mathscr{M}}"] \arrow[d, "F^*_{\mathfrak{M}}U^*(\theta)"'] & \left(\mathscr{M}, (\circ_p)_{p\in\mathbb{N}_0}, (*\alpha)_{\alpha\subseteq\mathbb{N}_0}\right) \arrow[d, "\theta"] \\
\left(F^*_{\mathfrak{M}}U^*(\mathscr{N}), (\tilde{\circ}_p)_{p\in\mathbb{N}_0}, (\tilde{*}\alpha)_{\alpha\subseteq\mathbb{N}_0}\right) \arrow[r, "\varepsilon^{\mathfrak{M}}_{\mathscr{N}}"'] & \left(\mathscr{N}, (\hat{\circ}_p)_{p\in\mathbb{N}_0}, (\hat{*}\alpha)_{\alpha\subseteq\mathbb{N}_0}\right)
\end{tikzcd}$$

To prove commutativity, we need to show that $\varepsilon^{\mathfrak{M}}_{\mathscr{N}} \circ F^*_{\mathfrak{M}}U^*(\theta) = \theta \circ \varepsilon^{\mathfrak{M}}_{\mathscr{M}}$.

We see that

$$\begin{aligned}
& \varepsilon^{\mathfrak{M}}_{\mathscr{N}} \circ F^*_{\mathfrak{M}}U^*(\theta)\left(\left(\cdots\left(\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m} \,\bar{\circ}_p (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_k}\right)^{\bar{*}\gamma_1}\right)^{\cdots}\right)^{\bar{*}\gamma_t}\right) \\
= \;& \varepsilon^{\mathfrak{M}}_{\mathscr{N}}\left(\left(\cdots\left(\left((\cdots((\theta(x))^{\tilde{*}\alpha_1})^{\cdots})^{\tilde{*}\alpha_m} \,\tilde{\circ}_p (\cdots((\theta(y))^{\tilde{*}\beta_1})^{\cdots})^{\tilde{*}\beta_k}\right)^{\tilde{*}\gamma_1}\right)^{\cdots}\right)^{\tilde{*}\gamma_t}\right) \\
= \;& \left(\cdots\left(\left((\cdots((\theta(x))^{\tilde{*}\alpha_1})^{\cdots})^{\hat{*}\alpha_m} \,\hat{\circ}_p (\cdots((\theta(y))^{\hat{*}\beta_1})^{\cdots})^{\hat{*}\beta_k}\right)^{\hat{*}\gamma_1}\right)^{\cdots}\right)^{\hat{*}\gamma_t} \\
= \;& \theta\left(\left(\cdots\left(\left((\cdots(x^{*\alpha_1})^{\cdots})^{*\alpha_m} \circ_p (\cdots(y^{*\beta_1})^{\cdots})^{*\beta_k}\right)^{*\gamma_1}\right)^{\cdots}\right)^{*\gamma_t}\right) \\
= \;& \theta \circ \varepsilon^{\mathfrak{M}}_{\mathscr{M}}\left(\left(\cdots\left(\left((\cdots(x^{\bar{*}\alpha_1})^{\cdots})^{\bar{*}\alpha_m} \,\bar{\circ}_p (\cdots(y^{\bar{*}\beta_1})^{\cdots})^{\bar{*}\beta_k}\right)^{\bar{*}\gamma_1}\right)^{\cdots}\right)^{\bar{*}\gamma_t}\right).
\end{aligned}$$

Then we will show that this diagram commutes:

$$\begin{tikzcd}
\left(F^*_{\mathfrak{C}}U^*(\mathscr{C}), (\bar{\circ}_p)_{p\in\mathbb{N}_0}, (\bar{*}\alpha)_{\alpha\subseteq\mathbb{N}_0}\right) \arrow[r, "\varepsilon^{\mathfrak{C}}_{\mathscr{C}}"] \arrow[d, "F^*_{\mathfrak{C}}U^*(\nu)"'] & \left(\mathscr{C}, (\circ_p)_{p\in\mathbb{N}_0}, (*\alpha)_{\alpha\subseteq\mathbb{N}_0}\right) \arrow[d, "\nu"] \\
\left(F^*_{\mathfrak{C}}U^*(\mathscr{D}), (\tilde{\circ}_p)_{p\in\mathbb{N}_0}, (\tilde{*}\alpha)_{\alpha\subseteq\mathbb{N}_0}\right) \arrow[r, "\varepsilon^{\mathfrak{C}}_{\mathscr{D}}"'] & \left(\mathscr{D}, (\hat{\circ}_p)_{p\in\mathbb{N}_0}, (\hat{*}\alpha)_{\alpha\subseteq\mathbb{N}_0}\right)
\end{tikzcd}$$

We also see that

$$\begin{aligned}
& \varepsilon^{\mathfrak{C}}_{\mathscr{D}} \circ F^*_{\mathfrak{C}} U^*(\nu) \left(\left[\left(\cdots \left(\left(\left(\cdots \left(x^{\bar{*}\alpha_1} \right)^{\cdots} \right)^{\bar{*}\alpha_m} \bar{\circ}_p \left(\cdots \left(y^{\bar{*}\beta_1} \right)^{\cdots} \right)^{\bar{*}\beta_k} \right)^{\bar{*}\gamma_1} \right)^{\cdots} \right)^{\bar{*}\gamma_t} \right] \right) \\
= \; & \varepsilon^{\mathfrak{C}}_{\mathscr{D}} \left(\left[\left(\cdots \left(\left(\left(\cdots \left((\nu(x))^{\bar{*}\alpha_1} \right)^{\cdots} \right)^{\bar{*}\alpha_m} \bar{\circ}_p \left(\cdots \left((\nu(y))^{\bar{*}\beta_1} \right)^{\cdots} \right)^{\bar{*}\beta_k} \right)^{\bar{*}\gamma_1} \right)^{\cdots} \right)^{\bar{*}\gamma_t} \right]' \right) \\
= \; & \left(\cdots \left(\left(\left(\cdots \left((\nu(x))^{\tilde{*}\alpha_1} \right)^{\cdots} \right)^{\hat{*}\alpha_m} \hat{\circ}_p \left(\cdots \left((\nu(y))^{\hat{*}\beta_1} \right)^{\cdots} \right)^{\hat{*}\beta_k} \right)^{\hat{*}\gamma_1} \right)^{\cdots} \right)^{\hat{*}\gamma_t} \\
= \; & \nu \left(\left(\cdots \left(\left(\left(\cdots \left(x^{*\alpha_1} \right)^{\cdots} \right)^{*\alpha_m} \circ_p \left(\cdots \left(y^{*\beta_1} \right)^{\cdots} \right)^{*\beta_k} \right)^{*\gamma_1} \right)^{\cdots} \right)^{*\gamma_t} \right) \\
= \; & \nu \circ \varepsilon^{\mathfrak{C}}_{\mathscr{C}} \left(\left[\left(\cdots \left(\left(\left(\cdots \left(x^{\bar{*}\alpha_1} \right)^{\cdots} \right)^{\bar{*}\alpha_m} \bar{\circ}_p \left(\cdots \left(y^{\bar{*}\beta_1} \right)^{\cdots} \right)^{\bar{*}\beta_k} \right)^{\bar{*}\gamma_1} \right)^{\cdots} \right)^{\bar{*}\gamma_t} \right] \right).
\end{aligned}$$

Thus, ε is a natural transformation.

Define $\eta : \mathrm{Ob}_{\mathbf{GCone}} \to \mathrm{Hom}_{\mathbf{GCone}}$ by $\mathscr{E} \mapsto \eta_{\mathscr{E}}$, where $\eta_{\mathscr{E}} : \mathscr{E} \to U^*F^*(\mathscr{E})$ is defined by $x \mapsto x^{*\varnothing}$ for every $x \in \mathscr{E}$.

Now consider the diagram

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "\eta_{\mathscr{E}}"] \arrow[d, "\lambda"'] & U^*F^*(\mathscr{E}) \arrow[d, "U^*F^*(\lambda)"] \\
\mathscr{G} \arrow[r, "\eta_{\mathscr{G}}"'] & U^*F^*(\mathscr{G})
\end{tikzcd}$$

To prove commutativity, we need to show that $\eta_{\mathscr{G}} \circ \lambda = U^*F^*(\lambda) \circ \eta_{\mathscr{G}}$.

For any $x \in \mathscr{E}$, we see that

$$U^*F^*(\lambda) \circ \eta_{\mathscr{E}}(x) = U^*F^*(\lambda)\left(x^{*\varnothing}\right) = (\lambda(x))^{\hat{*}\varnothing} = \eta_{\mathscr{G}}(\lambda(x)) = \eta_{\mathscr{G}} \circ \lambda(x).$$

Hence, η is a natural transformation.

Finally, consider the following diagrams

$$\begin{tikzcd}
F^* \arrow[r, "F^*\eta"] \arrow[dr, "1_{F^*}"'] & F^*U^*F^* \arrow[d, "\varepsilon F^*"] \\
& F^*
\end{tikzcd}
\qquad
\begin{tikzcd}
U^* \arrow[r, "\eta U^*"] \arrow[dr, "1_{U^*}"'] & U^*F^*U^* \arrow[d, "U^*\varepsilon"] \\
& U^*
\end{tikzcd}$$

To obtain an adjunction, we have to show that both triangles are commutative; that is, $\varepsilon F^* \circ F^*\eta = 1_{F^*}$ and $U^*\varepsilon \circ \eta U^* = 1_{U^*}$.

But verifying that the diagrams commute is equivalent to demonstrating that the following triangles commute:

$$\begin{tikzcd} F^*_{\mathfrak{M}}(\mathscr{E}) \arrow[r, "F^*_{\mathfrak{M}}\eta^{\mathfrak{M}}_{\mathscr{E}}"] \arrow[dr, "1_{F^*_{\mathfrak{M}}(\mathscr{E})}"'] & F^*_{\mathfrak{M}}U^*F^*_{\mathfrak{M}}(\mathscr{E}) \arrow[d, "\varepsilon^{\mathfrak{M}}_{\mathscr{E}}F^*_{\mathfrak{M}}"] \\ & F^*_{\mathfrak{M}}(\mathscr{E}) \end{tikzcd} \qquad \begin{tikzcd} F^*_{\mathfrak{C}}(\mathscr{E}) \arrow[r, "F^*_{\mathfrak{C}}\eta^{\mathfrak{C}}_{\mathscr{E}}"] \arrow[dr, "1_{F^*_{\mathfrak{C}}(\mathscr{E})}"'] & F^*_{\mathfrak{C}}U^*F^*_{\mathfrak{C}}(\mathscr{E}) \arrow[d, "\varepsilon^{\mathfrak{C}}_{\mathscr{E}}F^*_{\mathfrak{C}}"] \\ & F^*_{\mathfrak{C}}(\mathscr{E}) \end{tikzcd}$$

$$\begin{tikzcd} U^*(\mathscr{M}) \arrow[r, "\eta^{\mathfrak{M}}_{\mathscr{M}}U^*"] \arrow[dr, "1_{U^*(\mathscr{M})}"'] & U^*F^*_{\mathfrak{M}}U^*(\mathscr{M}) \arrow[d, "U^*\varepsilon^{\mathfrak{M}}_{\mathscr{M}}"] \\ & U^*(\mathscr{M}) \end{tikzcd} \qquad \begin{tikzcd} U^*(\mathscr{C}) \arrow[r, "\eta^{\mathfrak{C}}_{\mathscr{C}}U^*"] \arrow[dr, "1_{U^*(\mathscr{C})}"'] & U^*F^*_{\mathfrak{C}}U^*(\mathscr{C}) \arrow[d, "U^*\varepsilon^{\mathfrak{C}}_{\mathscr{C}}"] \\ & U^*(\mathscr{C}) \end{tikzcd}$$

First, using the same notation as before, we get $F^*_{\mathfrak{M}}(\mathscr{E}) = \left\langle \bar{\hat{\mathscr{E}}} \right\rangle$.

Thus, $U^*F^*_{\mathfrak{M}}(\mathscr{E}) = \left\langle \bar{\hat{\mathscr{E}}} \right\rangle$ as a globular cone and so $F^*_{\mathscr{M}}U^*F^*_{\mathscr{M}}(\mathscr{E}) = \left\langle \overline{\widehat{\left\langle \bar{\hat{\mathscr{M}}} \right\rangle}} \right\rangle$.

Define $F^*_{\mathfrak{M}}\eta^{\mathfrak{M}}_{\mathscr{E}} : F^*_{\mathfrak{M}}(\mathscr{E}) \to F^*_{\mathfrak{M}}U^*F^*_{\mathfrak{M}}(\mathscr{E})$ by $x \mapsto (x)^{\hat{*}\varnothing}$ for every $x \in \left\langle \bar{\hat{\mathscr{E}}} \right\rangle$.

For each $(x,y) \in \left\langle \bar{\hat{\mathscr{E}}}[i] \right\rangle \times_p \left\langle \bar{\hat{\mathscr{E}}}[j] \right\rangle$ and $z \in \left\langle \bar{\hat{\mathscr{E}}}[1] \right\rangle$, we define $\varepsilon^{\mathfrak{M}}_{\mathscr{E}}F^*_{\mathfrak{M}} : F^*_{\mathfrak{M}}U^*F^*_{\mathfrak{M}}(\mathscr{E}) \to F^*_{\mathfrak{M}}(\mathscr{E})$ by

$$\begin{aligned} (\cdots((x\hat{\circ}_p y)^{\hat{*}\gamma_1})^{\cdots})^{\hat{*}\gamma_t}) &\mapsto (\cdots((x \circ_p y)^{*\gamma_1})^{\cdots})^{*\gamma_t}, \\ ((\cdots((z)^{\hat{*}\gamma_1})^{\cdots})^{\hat{*}\gamma_t})^{\hat{*}\varnothing} &\mapsto (\cdots(z^{*\gamma_1})^{\cdots})^{*\gamma_t}. \end{aligned}$$

It follows that $\varepsilon^{\mathfrak{M}}_{\mathscr{E}}F^*_{\mathfrak{M}} \circ F^*_{\mathfrak{M}}\eta^{\mathfrak{M}}_{\mathscr{E}}(x) = \varepsilon^{\mathfrak{M}}_{\mathscr{E}}F^*_{\mathfrak{M}}\left((x)^{\hat{*}\varnothing}\right) = x$.

This implies that $\varepsilon^{\mathfrak{M}}_{\mathscr{E}}F^*_{\mathfrak{M}} \circ F^*_{\mathfrak{M}}\eta^{\mathfrak{M}}_{\mathscr{E}} = 1_{F^*_{\mathfrak{M}}(\mathscr{E})}$.

Second, since $F^*_{\mathfrak{C}}(\mathscr{E}) = [\mathscr{E}]$, $U^*F^*_{\mathfrak{C}}(\mathscr{E}) = [\mathscr{E}]$ as a globular cone and so $F^*_{\mathfrak{C}}U^*F^*_{\mathfrak{C}}(\mathscr{E}) = [[\mathscr{E}]]$.

Define $F^*_{\mathfrak{C}}\eta^{\mathfrak{C}}_{\mathscr{E}} : F^*_{\mathfrak{C}}(\mathscr{E}) \to F^*_{\mathfrak{C}}U^*F^*_{\mathfrak{C}}(\mathscr{E})$ by $[y] \mapsto [[y]]'$.

Define $\varepsilon^{\mathfrak{C}}_{\mathscr{E}}F^*_{\mathfrak{C}} : F^*_{\mathfrak{C}}U^*F^*_{\mathfrak{C}}(\mathscr{E}) \to F^*_{\mathfrak{C}}(\mathscr{E})$ by

$$\begin{aligned} \left[(\cdots(([x]\tilde{\circ}_p[y])^{\tilde{*}\gamma_1})^{\cdots})^{\tilde{*}\gamma_t}\right]' &\mapsto (\cdots(([x]\circ_p[y])^{*\gamma_1})^{\cdots})^{*\gamma_t}, \\ [[z]]' &\mapsto [z]. \end{aligned}$$

for any $(x,y) \in [\mathscr{C}[i]] \times_p [\mathscr{C}[j]]$ and $z \in [\mathscr{C}[1]]$.

It is obvious that $\varepsilon^{\mathfrak{C}}_{\mathscr{E}} F^*_{\mathfrak{C}} \circ F^*_{\mathfrak{C}} \eta^{\mathfrak{C}}_{\mathscr{E}} = 1_{F^*_{\mathfrak{C}}(\mathscr{E})}$.

Third, consider $U^*(\mathscr{M}) = \mathscr{M}$ as a globular cone.

So $F^*_{\mathfrak{M}} U^*(\mathscr{M}) = \left\langle \bar{\bar{\mathscr{M}}} \right\rangle$ constructed from elements of $\mathscr{M}$.

Thus, $U^* F^*_{\mathfrak{M}} U^*(\mathscr{M}) = \left\langle \bar{\bar{\mathscr{E}}} \right\rangle$ as a globular cone.

Define $\eta^{\mathfrak{M}}_{\mathscr{M}} U^* : U^*(\mathscr{M}) \to U^* F^*_{\mathfrak{M}} U^*(\mathscr{M})$ by $x \mapsto (x)^{\bar{*}\varnothing}$ for every $x \in \mathscr{M}$.

Define $U^* \varepsilon^{\mathfrak{M}}_{\mathscr{M}} : U^* F^*_{\mathfrak{M}} U^*(\mathscr{M}) \to U^*(\mathscr{M})$ by

$$\begin{aligned} (\cdots((x \bar{\circ}_p y)^{\bar{*}\gamma_1})^{\cdots})^{\bar{*}\gamma_t} &\mapsto (\cdots((x \circ_p y)^{*\gamma_1})^{\cdots})^{*\gamma_t}, \\ ((\cdots((z)^{\bar{*}\gamma_1})^{\cdots})^{\bar{*}\gamma_t})^{\bar{*}\varnothing} &\mapsto (\cdots(z^{*\gamma_1})^{\cdots})^{*\gamma_t}. \end{aligned}$$

for any $(x,y) \in \left\langle \bar{\bar{\mathscr{M}}}[i] \right\rangle \times_p \left\langle \bar{\bar{\mathscr{M}}}[j] \right\rangle$ and $z \in \left\langle \bar{\bar{\mathscr{M}}}[1] \right\rangle$.

We see that $U^* \varepsilon^{\mathfrak{M}}_{\mathscr{M}} \circ \eta^{\mathfrak{M}}_{\mathscr{M}} U^*(x) = U^* \varepsilon^{\mathfrak{M}}_{\mathscr{M}} \left(x^{\bar{*}\varnothing} \right) = x$.

This means that $U^* \varepsilon^{\mathfrak{M}}_{\mathscr{M}} \circ \eta^{\mathscr{M}}_{\mathscr{M}} U^* = 1_{U^*(\mathscr{M})}$.

Fourth, consider $U^*(\mathscr{C}) = \mathscr{C}$ as a globular cone.

Then $F^*_{\mathfrak{C}} U^*(\mathscr{C}) = [\mathscr{C}]$ established from elements of $\mathscr{C}$ and so $U^* F^*_{\mathfrak{C}} U^*(\mathscr{C}) = [\mathscr{C}]$ as a globular cone.

Define $\eta^{\mathfrak{C}}_{\mathscr{C}} U^* : U^*(\mathscr{E}) \to U^* F^*_{\mathfrak{C}} U^*(\mathscr{E})$ by $x \mapsto [x]$.

Define $U^* \varepsilon^{\mathfrak{C}}_{\mathscr{C}} : U^* F^*_{\mathfrak{C}} U^*(\mathscr{C}) \to U^*(\mathscr{C})$ by

$$\begin{aligned} \left[(\cdots(([x] \tilde{\circ}_p [y])^{\tilde{*}\gamma_1})^{\cdots})^{\tilde{*}\gamma_t} \right]' &\mapsto (\cdots(([x] \circ_p [y])^{*\gamma_1})^{\cdots})^{*\gamma_t}, \\ [[z]]' &\mapsto [z]. \end{aligned}$$

for any $(x,y) \in [\mathscr{C}[i]] \times_p [\mathscr{C}[j]]$ and $z \in [\mathscr{C}[1]]$.

It is easy to see that $U^* \varepsilon^{\mathfrak{C}}_{\mathscr{C}} \circ \eta^{\mathfrak{C}}_{\mathscr{C}} U^* = 1_{U^*(\mathscr{C})}$.

This means that $\varepsilon F^* \circ F^* \eta = 1_{F^*}$ and $U^* \varepsilon \circ \eta U^* = 1_{U^*}$.

Therefore, F^* is left adjoint to U^*. $\square$

4.5.2 Involutive Weak Globular-Cone ω-Categories

Now we can provide our main definition as follows.

Definition 4.5.2.1. An **involutive weak globular-cone ω-category** is an algebra for the monad $(U^*F^*, U^*\varepsilon^*F^*, \eta^*)$.

Finally, we list here some examples of Penon involutive weak globular-cone ω-categories but discuss some of them in detail.

Example 4.5.2.2. Every strict involutive globular-cone ω-category is a very particular trivial case of weak involutive globular-cone ω-category.

Proof. Let $\big(\mathscr{C}, (\iota^n_{\mathscr{C}})_{n\in\mathbb{N}_0}, (\circ_p)_{p\in\mathbb{N}_0}, (*_\alpha)_{\alpha\subseteq\mathbb{N}_0}\big)$ be a strict involutive globular-cone ω-category over an ω-globular set Q.

By definition, $\mathscr{C}$ is a globular cone over Q.

Let **GCone** $\underset{U^*}{\overset{F^*}{\rightleftarrows}} \hat{\mathscr{Q}}^*$ be the pair of free-forgetful functors and $T^* = U^*F^*$.

Then $T^*(\mathscr{C})$ is the family of all possible concatenated elements of $\mathscr{C}$ attached by new operations of compositions, involutions, and identities.

Now we define the evaluation map $\theta^* : T^*(\mathscr{C}) \to \mathscr{C}$ on simple elements by

$$\begin{array}{rcl}
(x) & \mapsto & x, \\
(x,p,y) & \mapsto & x \circ_p y, \\
x^{\alpha} & \mapsto & x^{*_\alpha}, \\
(x,n) & \mapsto & \iota^n_Q(x), \\
(\ldots,(n,x)\cdots) & \mapsto & \iota^n_{\mathscr{C}}(x), \\
(x,y)_n & \mapsto & \iota^n_{\mathscr{C}}(x) = \iota^n_{\mathscr{C}}(y).
\end{array}$$

We see that $(\mathscr{C}, \theta^*)$ is an algebra for the monad $(U^*F^*, \eta^*, U^*\varepsilon^*F^*)$. □

Example 4.5.2.3. Globular ω-quivers are an example of strict involutive globular-cone ω-category. Globular propagators of globular ω-quivers give an example of weak involutive globular-cone ω-categories (see [BJ]).

Example 4.5.2.4. Let $\mathscr{M}^0$ be a family of involutive monoids $A, B, C, \dots$ and $\mathscr{M}^1$ the family of the bimodules ${}_AM_B$, with $A, B \in \mathscr{M}^0$. Composition $\circ^1_0$ of bimodules is given by the Rieffel tensor product ${}_AM_B \otimes_B {}_BN_C$ and involution $*^1_0$ of bimodules is provided by the Rieffel dual ${}_B\overline{M}_A$ where $\overline{M} := \{\overline{x} \mid x \in M\}$ is just a (specific) disjoint copy of M and the bimodule actions are $b \cdot \overline{x} \cdot a := \overline{a^*xb^*}$, for all $a \in A$, $b \in B$ and $x \in M$. Similarly starting from a class $\mathscr{M}^0$ of strict involutive 1-categories, the family $\mathscr{M}^1$ of "bimodules" between them is an involutive weak 1-category. Introducing a suitable notion of "bimodule" between strict involutive globular-cone n-categories, we obtain an involutive weak globular-cone n-category. If $\mathscr{M}^0$ is a family of strict globular-cone ω-categories, the family $\mathscr{M}^1$ of "bimodules" between them is an involutive weak globular-cone ω-category.

CHAPTER 5

OUTLOOK

There are plenty of further directions that emerge from this work. In the near future, we plan to study/explore;

- further details of the several examples briefly introduced here.
- the involutive versions of weak globular ω-categories in the Batanin's approach.
- the involutive versions of weak globular ω-categories in the Leinster's approach.
- the (involutive) versions of weak cubical ω-categories.
- the possibility to define globular/cubical weak ω-C*-categories.
- tentative notions of noncommutative higher topos theory adapted to C*-algebraic environments.
- possible applications (vertically categorified) to noncommutative geometry.
- possible applications to relational quantum theory.

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[1] Note the change of terminology: "ω" here refers to cubical groupoids (with "connections") and "∞" to globular groupoids. Today the two symbols are equivalent and the prefix "cubical" / "globular" is added.

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