# THE EXISTENCE OF SOLUTIONS FOR FREDHOLM/VOLTERRA INTEGRAL EQUATIONS AND FRACTIONAL DIFFERENTIAL EQUATIONS VIA FIXED POINT THEOREMS USING $W$-DISTANCE FUNCTIONS

BY

MR. TEERAWAT WONGYAT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
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| Thesis Title | THE EXISTENCE OF SOLUTIONS FOR FREDHOLM/VOLTERRA INTEGRAL EQUATIONS AND FRACTIONAL DIFFERENTIAL EQUATIONS VIA FIXED POINT THEOREMS USING $W$-DISTANCE FUNCTIONS |
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## ABSTRACT

The aim of this thesis is to investigate new contraction mappings with respect to $w$-distances in complete metric spaces. The main results are distinguish into three parts. In the first part, we introduce the concept of a ceiling distance and use this idea for proving some new fixed point theorems for mappings satisfying new contractive conditions along with $w$-distances in metric spaces. Our theoretical results are extensions of many results in fixed point theory. Some illustrative examples are provided to advocate the usability of our results while Banach contraction principle and some results are not applicable. Also, we give numerical experiments for a fixed point in these examples. In the next part, the received results are used for proving the existence and uniqueness of the solution for nonlinear Fredholm integral equations and Volterra integral equations. In the last part, we apply our theoretical fixed point results to study the nonlinear fractional differential equations of Caputo type.

**Keywords:** $w$-distance; ceiling distance; nonlinear Fredholm integral equation;

nonlinear Volterra integral equation; nonlinear fractional differential equation

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Mr. Teerawat Wongyat

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# CHAPTER 1

# INTRODUCTION

In 1922, a distinguished fixed point theorem was established by the Polish mathematician, Banach [4], known as the **Banach contraction principle** (in short, BCP) which it is a very important result of analysis and is primary sources of metric fixed point theory. Furthermore, it is immensely applied in many branches of mathematics because it requires only the structure of a complete metric space with the contractive condition on the mapping which is easy to test in this setting. It also was used to establish the existence of a solution for an integral equation. Now, we state this principle as follows:

**Theorem A** ([4]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a contraction mapping, i.e.,*

$$d(fx, fy) \leq kd(x,y) \tag{1.0.1}$$

*for all $x,y \in X$, where $k \in [0,1)$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

As the result of its intelligibility and profitableness, it has become a very celebrated and popular tool in solving the existence problems in many branches of mathematical analysis. Many mathematicians extended the Banach contraction principle in the setting of the contractive condition (1.0.1) to another contractive conditions. In 1984, Khan et al. [17] introduced and studied the notion of an altering distance function and it is applied in defining a weak contractive condition and proving the existence of a fixed point. Later, Choudhury et al. [5] proved some fixed point results for generalized weakly contractive mappings by using the idea of an altering distance function.

The another generalization of the condition (1.0.1) appeared in 2014 by Jleli and Samet [15]. They established a new fixed point result for mappings satisfying this contractive condition in the framework of generalized metric spaces. Recently, Hussain et al. [12] investigated several new concepts of generalized contraction mappings and studied sufficient conditions for the existence of a fixed point of these classes in various distance spaces. One of these fixed point results is an extension of a fixed point result due to Jleli and Samet [15] in the framework of metric spaces. Moreover, it is a generalization of many famous fixed point results such as Kannan's result [19], Ciric's result [7], Chatterjea's result [8] and Reich's result [22].

Likewise, the Banach contraction principle was improved by the famous fixed-point's researchers as Boy and Wong [3] and Matkowski's [14]. Afterward, Ri [23] gave some fixed point results for generalized contraction mappings which generalize the Boyd and Wong's fixed point theorem in [3] and the Matkowski's fixed point theorem in [14].

Recently, Sawangsup and Sintunavarat [24] introduced the notion of a weak altering distance function by reducing some condition of an altering distance function and proved the existence of a fixed point by using such function.

On the other hand, in their famed paper, Kada et al. [18] first introduced and studied the notion of a $w$-distance on a metric space. They also used this concept in the improving the Banach contraction principle. In 2008, Du [10] recommended the concept of a $w^0$-distance by adjust conditions in $w$-distance which use it more and he also used the concept of a $w^0$-distance to prove the existence of a fixed point. In the recent, many authors investigated fixed point results for many generalized contraction mappings with respect to $w$-distances on metric spaces (see [1, 20, 21] and references these in).

The aim of this thesis consists of three topics as follows:

First topic, we introduce concept of a ceiling distance with respect to a metric and prove several new fixed point theorems for new contractive conditions along with $w$-distances in metric spaces. Some of these theorems are studied via the idea of ceiling distance. Our theoretical fixed point results are extensions of many results of well-known fixed point's researchers such as Choudhury et al. [5], Hussain et al. [12], Sawangsup and Sintunavarat [24] and Ri [23]. The reader can see the overall of theoretical results in this topic in the Figure 1.1.

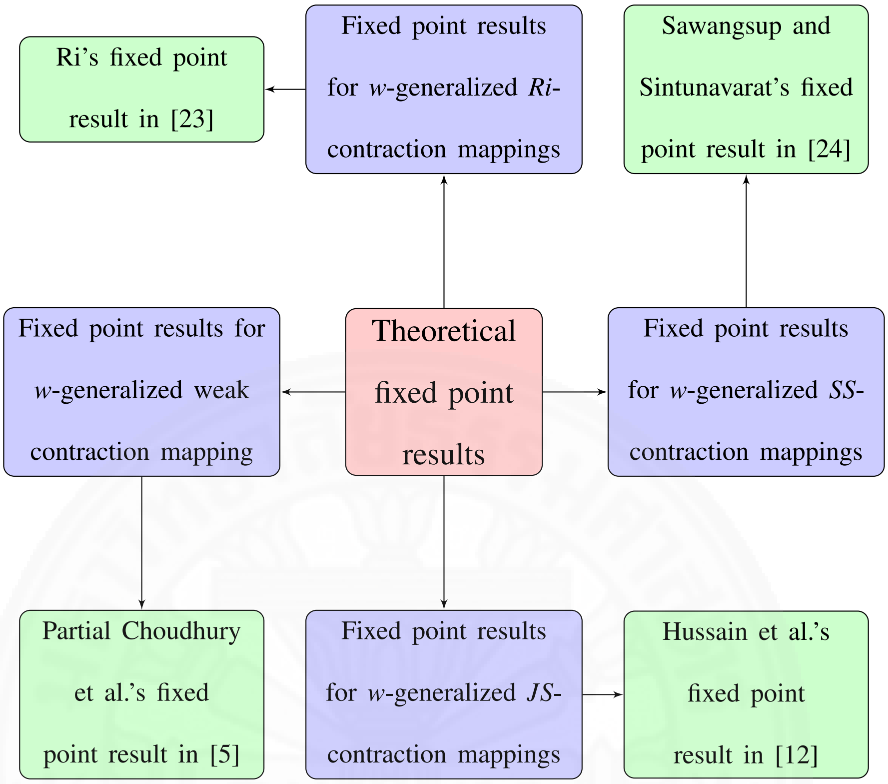


Figure 1.1: The overall of theoretical results in this topic

Some illustrative examples are provided to advocate the usability of our results while Banach contraction principle and some results in the literature are not applicable. Also, we give numerical experiments for a fixed point in these examples.

Second topic, applications to nonlinear Fredholm integral equations and nonlinear Volterra integral equations are given to illustrate the usability of all previous topic.

The last topic, applications to nonlinear fractional differential equations are given to illustrate the usability of all results in the first topic.

# CHAPTER 2

# PRELIMINARIES

In this chapter, we give some important definitions in this research such as fields, vector space, normed space, metric space etc. Throughout this research, we denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{Z}_p$ the sets of positive integers, real numbers, complex numbers, rational numbers, and the set of all congruence classes of the integers for a modulus $p$, respectively.

A notation for the image of a point $x$ under a mapping $f$ would be $f(x)$. Sometime, to simplify formulas in this proposal, customary to take out the bracket and write $fx$.

## 2.1 Fields

**Definition 2.1.1.** The set $\mathbb{F}$ is called a **field** when two binary operations $+$ and $\cdot$ on $\mathbb{F}$, which we call addition and multiplication, satisfy the following conditions for all $a,b,c \in \mathbb{F}$:

1. $a+b=b+a$ and $a\cdot b=b\cdot a$;
2. $(a+b)+c=a+(b+c)$ and $(a\cdot b)\cdot c=a\cdot(b\cdot c)$;
3. there exists an element $0_{\mathbb{F}} \in \mathbb{F}$ such that $a+0_{\mathbb{F}}=a$;
4. there exists an element $1_{\mathbb{F}} \in \mathbb{F}$ such that $a\cdot 1_{\mathbb{F}}=a$;
5. for each $a\in\mathbb{F}$, there exists an element $-a\in\mathbb{F}$ such that $a+(-a)=0_{\mathbb{F}}$;
6. if $a\neq 0_{\mathbb{F}}$, then there exists an element $a^{-1}\in\mathbb{F}$, such that $a\cdot(a^{-1})=1_{\mathbb{F}}$;
7. $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$.

**Example 2.1.2.** $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$ and $(\mathbb{C},+,\cdot)$ are field under the usual addition and multiplication. We note that $(\mathbb{Z},+,\cdot)$ is not a field because 5 has no multiplicative inverse.

**Example 2.1.3.** $(\mathbb{Z}_p,+_p,\cdot_p)$ is a field, where $p$ is a prime number, $+_p$ is addition modulo $p$ and $\cdot_p$ is multiplication modulo $p$.

## 2.2 Vector spaces

**Definition 2.2.1.** A nonempty set $V$ is called a **vector space** (or **linear space**) over a field $\mathbb{F}$ when the vector addition operation $+ : V\times V\to V$ and scalar multiplication operation $\cdot : \mathbb{F}\times V\to V$ satisfy the following properties for all $u,v,w\in V$ and $k,m\in\mathbb{F}$:

1. $(u+v)+w=u+(v+w)$;
2. $u+v=v+u$;
3. there exists an element $0\in V$ such that $u+0=u$;
4. for each $u\in V$, there is an element $(-u)\in V$ such that $u+(-u)=0$;
5. $(km)x=k(mx)$;
6. $k(u+v)=ku+kv$;
7. $(k+m)x=kx+mx$;
8. $1_{\mathbb{F}}u=u$.

The vector space $X$ is called a real vector space when $\mathbb{F}=\mathbb{R}$ and a complex vector space when $\mathbb{F}=\mathbb{C}$.

**Example 2.2.2.** The set $M_{m\times n}(\mathbb{F})$ of all $m\times n$ matrices with entries from a field $\mathbb{F}$ is a vector space under two algebraic operations defined by

$$A+B=(a_{ij}+b_{ij})_{m\times n},$$

$$\alpha \cdot A = (\alpha \cdot a_{ij})_{m\times n}$$

for each $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{F})$ and each scalar $\alpha \in \mathbb{F}$.

For instance, in $M_{2\times 2}(\mathbb{R})$,

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 8 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 5 & 5 \end{pmatrix}, \quad 4\begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 20 & 8 \\ 4 & 12 \end{pmatrix}.$$

**Example 2.2.3.** The set of all real-valued continuous functions defined on $[0,1]$ with function addition and scalar multiplication defined by

$$(f+g)(x) = fx + gx \quad \text{and} \quad (\alpha f)(x) = \alpha fx$$

is a vector spaces over a field $\mathbb{R}$.

**Example 2.2.4.** The set $\mathbb{R}^n$ is a real vector space with the two algebraic operations defined by

$$u + v = (u_1 + v_1, u_2 + v_2, + \cdots, u_n + v_n),$$

$$\alpha u = (\alpha u_1, \alpha u_2 +, + \cdots, \alpha u_n)$$

for each $u = (u_1, u_2, \cdots, u_n), v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and each scalar $\alpha \in \mathbb{R}$.

## 2.3 Normed spaces

**Definition 2.3.1.** Let $V$ be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A **norm** on $V$ is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies the following properties for all $u, v \in V$ and all $\alpha \in \mathbb{K}$:

1. $\|u\| \geq 0$;
2. $\|u\| = 0$ if and only if $u = 0$;
3. $\|\alpha u\| = |\alpha| \|u\|$;
4. $\|u + v\| \leq \|u\| + \|v\|$.

The ordered pair $(V, \|\cdot\|)$ is also called a **normed vector space**.

**Definition 2.3.2.** A sequence $\{x_n\}$ in a normed space $(V, \|\cdot\|)$ is called **converges** to a point $x \in V$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for all $n > N$, denoted by $\lim_{n\to\infty} x_n = x$ or, simply, $x_n \to x$ as $n \to \infty$. In this case, $\{x_n\}$ is called a convergent sequence.

**Remark 2.3.3.** A sequence $\{x_n\}$ in a normed space $(V, \|\cdot\|)$ is a convergent sequence if and only if $\lim_{n\to\infty} \|x_n - x\| = 0$.

**Definition 2.3.4.** A sequence $\{x_n\}$ in a normed space $(V, \|\cdot\|)$ is called a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for all $n, m > N$.

**Remark 2.3.5.** A sequence $\{x_n\}$ in a normed space $(V, \|\cdot\|)$ is a Cauchy sequence if and only if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$.

**Definition 2.3.6.** A normed space $(V, \|\cdot\|)$ is called **complete** if for every Cauchy sequence in $V$ converges.

**Definition 2.3.7.** A complete normed space is called a **Banach space**.

**Example 2.3.8.** The set $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a real (or complex) Banach space with norm defined by

$$\|u\| := \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2},$$

where $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$).

**Example 2.3.9.** The set $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a real (or complex) Banach space with norm defined by

$$\|u\|_p := \left( \sum_{i=1}^{n} |u_i|^p \right)^{\frac{1}{p}}$$

or

$$\|u\|_\infty := \max\{|u_1|, |u_2|, ..., |u_n|\},$$

where $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$) and $1 < p < \infty$. In this case, $\|\cdot\|_p$ is called $p$**-norm** and $\|\cdot\|_\infty$ is called **infinity norm** or **maximum norm**.

**Example 2.3.10.** Let $p \geq 1$ be a fixed real number and $V$ be the set of all sequences of real (or complex) numbers, that is, each element of $V$ is a real (or complex) sequence

$$u = \{u_1, u_2, ...\} \text{ briefly } u = \{u_i\}, \ i = 1, 2, 3, ...$$

such that $\sum_{i=1}^{\infty} |u_i|^p < \infty$. Define a function $\|\cdot\| : V \times V \to \mathbb{R}$ by

$$\|u\|_p := \left( \sum_{i=1}^{\infty} |u_i|^p \right)^{\frac{1}{p}},$$

where $u = \{u_i\} \in V$. Then $(V, \|\cdot\|)$ is a Banach space which is denoted by $\ell^p$.

**Example 2.3.11.** Let $V$ be the set of all bounded sequence of real (or complex) numbers, that is,

$$|u_i| \leq c_u \quad \forall i \in \{1, 2, ...\}$$

for all $u = \{u_i\} \in V$, where $c_u$ is a real numbers which depend on $u$, but not depend on $i$. Define a function $\|\cdot\| : V \times V \to \mathbb{R}$ by

$$\|u\|_\infty := \sup_{i \in \mathbb{N}} |u_i|,$$

where $u = \{u_i\} \in V$ and sup denotes the supremum (least upper bound). Then $(V, \|\cdot\|)$ is a Banach space which is denoted $\ell^\infty$.

## 2.4 Metric spaces

**Definition 2.4.1.** Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{R}$ satisfies the following conditions for all $x, y, z \in X$:

(M1) $d(x,y) \geq 0$ ;

(M2) $d(x,y) = 0$ if and only if $x = y$ ;

(M3) $d(x,y) = d(y,x)$ ;

(M4) $d(x,z) \leq d(x,y) + d(y,z)$ .

Then $d$ is called a **metric** and $(X,d)$ is called a **metric space**.

**Example 2.4.2.** Let $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$d(x,y)=|x-y|$$

for all $x,y\in\mathbb{R}$. Then $d$ is a metric on $\mathbb{R}$. It is known as the **usual** or **standard metric** on $\mathbb{R}$.

**Example 2.4.3.** Let $r>0$ and $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined by

$$d(x,y)=r|x-y|$$

for all $x,y\in\mathbb{R}$. Then $d$ is a metric on $\mathbb{R}$.

**Example 2.4.4.** Let $X$ be a nonempty set and $d:X\times X\to\mathbb{R}$ defined by

$$d(x,y)=\begin{cases}1, & x\neq y\\ 0, & x=y.\end{cases}$$

Then $d$ is a metric on $X$ and it is called the **discrete metric** or the **trivial metric**.

**Example 2.4.5.** Let $d:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ be defined by

$$d(x,y)=|x_1|+|x_2|+|y_1|+|y_2|,$$

where $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^2$. Then $d$ is a metric on $\mathbb{R}^2$.

**Example 2.4.6.** Let $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be defined by

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2},$$

where $x=(x_1,x_2,\ldots,x_n),y=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$. Then $d$ is a metric on $\mathbb{R}^n$ and it is called **Euclidian metric on** $\mathbb{R}^n$.

**Example 2.4.7.** Let $d:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{R}$ be defined by

$$d(x,y)=\sqrt{|x_1-y_1|^2+|x_2-y_2|^2+\cdots+|x_n-y_n|^2},$$

where $x=(x_1,x_2,\ldots,x_n),y=(y_1,y_2,\ldots,y_n)\in\mathbb{C}^n$. Then $d$ is a metric on $\mathbb{C}^n$ and it is called **Euclidian metric on** $\mathbb{C}^n$.

**Example 2.4.8.** Let $[a,b]$ be a closed interval on $\mathbb{R}$ and

$$X = C[a,b] := \{x : [a,b] \to \mathbb{R} : x \text{ is a continuous function}\}.$$

Define a function $d_\infty : X \times X \to \mathbb{R}$ by

$$d_\infty = \sup_{t \in [a,b]} |x(t) - y(t)|$$

for all $x, y \in X$. Then $(X, d_\infty)$ is a metric space. This metric $d_\infty$ is called the **sup metric** or **max metric** or **uniform metric** on $C[a,b]$.

**Definition 2.4.9.** Let $(X,d)$ be a metric space. The **open ball** of radius $r > 0$ and center $x \in X$ is the set $B_r(x) \subset X$ defined by

$$B_r(x) := \{y \in X : d(x,y) < r\}$$

(see some open ball in Figure 2.1).

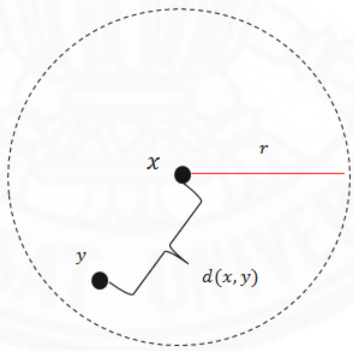


Figure 2.1: Open ball of radius $r$ and center $x$ in the Euclidean space $\mathbb{R}^2$

**Definition 2.4.10.** Let $(X,d)$ be a metric space. The **closed ball** of radius $r > 0$ and center $x \in X$ is the set $B_r[x] \subset X$ defined by

$$B_r[x] := \{y \in X : d(x,y) \leq r\}$$

(see some closed ball in Figure 2.2).

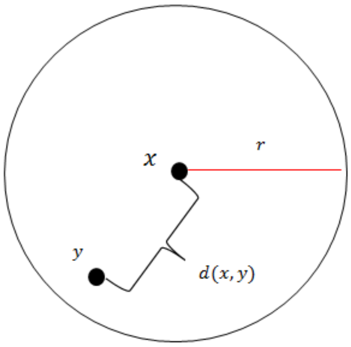


Figure 2.2: Closed ball of radius $r$ and center $x$ in the Euclidean space $\mathbb{R}^2$

**Definition 2.4.11.** Let $(X,d)$ and $(Y,\rho)$ be two metric spaces. A mapping $f : X \to Y$ is called **continuous at $x \in X$** if for every $\varepsilon > 0$, there exist $\delta > 0$ such that

$$f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

Furthermore, a mapping $f$ is called **continuous** if it is continuous at each point in $X$.

The following proposition is the idea of a continuous mapping in terms limits of sequences.

**Proposition 2.4.12.** *Let $(X,d)$, $(Y,\rho)$ be two metric spaces. A mapping $f : X \to Y$ is continuous at a point $a \in X$ if and only if the following condition holds:*

$$x_n \to a \ \ as \ \ n \to \infty \ \ \Rightarrow \ \ f(x_n) \to f(a) \ \ as \ \ n \to \infty.$$

**Definition 2.4.13.** A sequence $\{x_n\}$ in a metric space $(X,d)$ is called **converges** to a point $x \in X$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ for all $n > N$, denoted by $\lim_{n\to\infty} x_n = x$ or, simply, $x_n \to x$. In this case, $\{x_n\}$ is called a convergent sequence.

**Remark 2.4.14.** A sequence $\{x_n\}$ in a metric space $(X,d)$ is a convergent sequence if and only if

$$\lim_{n\to\infty} d(x_n,x) = 0.$$

**Definition 2.4.15.** A sequence $\{x_n\}$ in a metric space $(X,d)$ is called **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n,x_m) < \varepsilon$ for all $n,m > N$.

**Remark 2.4.16.** A sequence $\{x_n\}$ in a metric space $(X,d)$ is Cauchy sequence if and only if

$$\lim_{m,n\to\infty} d(x_m,x_n) = 0.$$

**Definition 2.4.17.** A metric space $(X,d)$ is called **complete** if for every Cauchy sequence in $X$ converges.

**Example 2.4.18.** The usual metric space $\mathbb{R}$ and the Euclidean metric space $\mathbb{C}$ are complete metric spaces.

**Example 2.4.19.** The Euclidean metric space $(\mathbb{R}^n,d)$ is complete.

**Example 2.4.20.** The Euclidean metric space $(\mathbb{C}^n,d)$ is complete.

**Example 2.4.21.** The function space $(C[a,b],d_\infty)$ in Example 2.4.8 is complete.

## 2.5 Weakly continuous functions

**Definition 2.5.1.** Let $(X,d)$ be a metric space. A function $f : X \to \mathbb{R}$ is called **lower semicontinuous at a point *a* ∈ *X*** if and only if the following condition holds:

$$x_n \to a \text{ as } n \to \infty \ \Rightarrow \ f(a) \le \liminf_{n\to\infty} f(x_n).$$

Furthermore, a function $f$ is call **lower semicontinuous** if it is lower semicontinuous at each point in $X$.

**Example 2.5.2.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(t) = \begin{cases} \ln(t^2+2t+1) & \text{if } t \le 1 \\ t+1 & \text{if } t > 1 \end{cases}.$$

Then $f$ is lower semicontinuous (see the graph of a function $f$ in Figure 2.3).

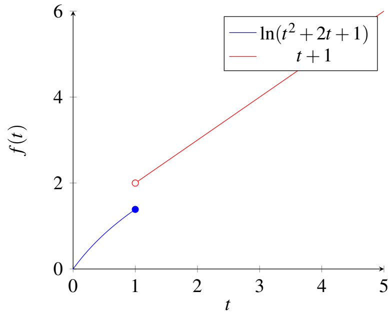


Figure 2.3: The graph of $f$ in Example 2.5.2.

**Example 2.5.3.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(t) = \begin{cases} t^3 & \text{if } t \leq 1 \\ e^t - 1 & \text{if } t > 1 \end{cases}.$$

Then $f$ is lower semicontinuous (see the graph of a function $f$ in Figure 2.4).

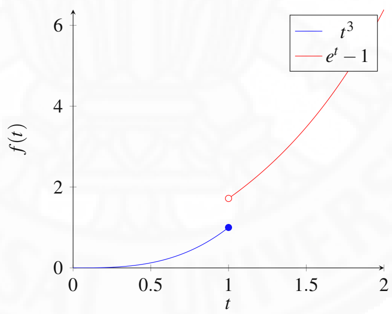


Figure 2.4: The graph of $f$ in Example 2.5.3.

**Example 2.5.4.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{t^2}{2} & \text{if } t \leq 1 \\ t^2 & \text{if } t > 1 \end{cases}.$$

Then $f$ is lower semicontinuous (see the graph of a function $f$ in Figure 2.5).

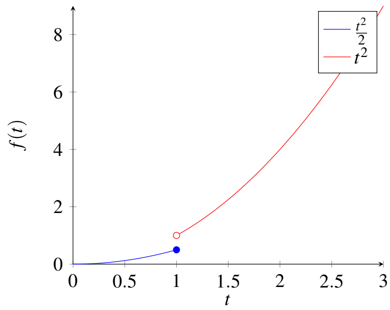


Figure 2.5: The graph of $f$ in Example 2.5.4.

**Definition 2.5.5.** Let $(X,d)$ be a metric space. A function $f : X \to \mathbb{R}$ is called **upper semicontinuous at a point $a \in X$** if and only if the following condition holds:

$$x_n \to a \text{ as } n \to \infty \Rightarrow f(a) \geq \limsup_{n\to\infty} f(x_n).$$

Furthermore, a function $f$ is call **upper semicontinuous** if it is upper semicontinuous at each point in $X$.

**Example 2.5.6.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(t) = \begin{cases} t^3 & \text{if } t < 1 \\ te^t - 1 & \text{if } t \geq 1 \end{cases}.$$

Then $f$ is upper semicontinuous. (see the graph of a function $f$ in Figure 2.6).

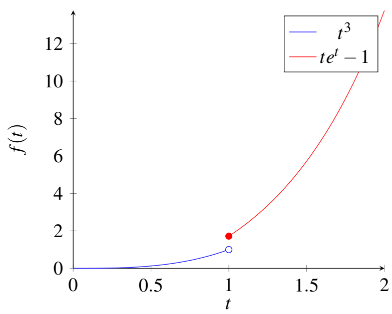


Figure 2.6: The graph of $f$ in Example 2.5.6.

**Example 2.5.7.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(t) = \begin{cases} \ln(1+t) & \text{if } t \leq 1 \\ t & \text{if } t > 1 \end{cases}.$$

Then $f$ is upper semicontinuous. (see the graph of a function $f$ in Figure 2.7).

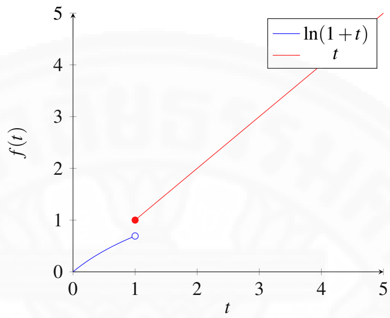


Figure 2.7: The graph of $f$ in Example 2.5.7.

**Example 2.5.8.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a function $f : X \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{2t^3}{3} & \text{if } t < 1 \\ 2t^3 & \text{if } t \geq 1 \end{cases}.$$

Then $f$ is upper semicontinuous and nondecreasing. (see the graph of a functions $f$ in Figure 2.8).

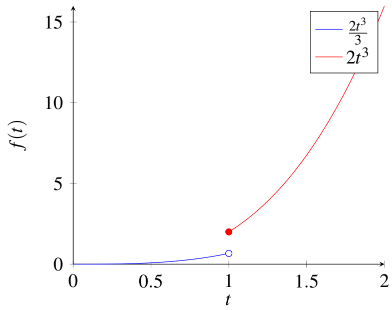


Figure 2.8: The graph of $f$ in Example 2.5.8.

## 2.6 $w$-distances and $w^0$-distances

In 1996, Kada et al.[18] introduced the concept of a $w$-distance on a metric space as follows:

**Definition 2.6.1** ([18])**.** Let $(X,d)$ be a metric space. A function $q : X \times X \to [0,\infty)$ is called a ***w*-distance** on $X$ if it satisfies the following three conditions for all $x,y,z \in X$:

(W1) $q(x,y) \leq q(x,z)+q(z,y)$;

(W2) $q(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous on for all $x \in X$ ;

(W3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$.

**Remark 2.6.2.** In general, for all $x,y \in X$, $q(x,y) \neq q(x,y)$ and not either of the implications $q(x,y) = 0$ if and only if $x = y$ necessarily hold.

**Remark 2.6.3.** Each metric on a nonempty set $X$ is a $w$-distance on $X$.

Here, we give some another examples of a $w$-distance.

**Example 2.6.4.** Let $(X,d)$ be a metric space. The function $q : X \times X \to [0,\infty)$ defined by $q(x,y) = c$ for every $x,y \in X$ is a $w$-distance on $X$, where $c$ is a positive real number. But $q$ is not a metric since $q(x,x) = c \neq 0$ for any $x \in X$.

**Example 2.6.5.** Let $(X,\|\cdot\|)$ be a normed space. The function $q: X \times X \to [0,\infty)$ defined by

$$q(x,y) = \|y\|$$

for all $x,y \in X$, is a $w$-distance on $X$.

**Example 2.6.6.** Let $(X,\|\cdot\|)$ be a normed space. The function $q: X \times X \to [0,\infty)$ defined by

$$q(x,y) = \|x\| + \|y\|$$

for all $x,y \in X$, is a $w$-distance on $X$.

**Example 2.6.7.** Let $a,b \in \mathbb{R}$ with $a < b$ and $X = C[a,b]$ (the set of all continuous functions from $[a,b]$ into $\mathbb{R}$), with the metric $d: X \times X \to \mathbb{R}$ which is defined by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

for all $x,y \in X$. Define the function $q: X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|$$

for all $x,y \in X$. Then $q$ is a $w$-distance on $X$.

The following lemma is a useful tool for proving our main results in this thesis.

**Lemma 2.6.8** ([18])**.** *Let $(X,d)$ be a metric space, $q$ be a $w$-distance on $X$ and $x,y,z \in X$.*

1. *If $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} q(x_n,x) = \lim_{n\to\infty} q(x_n,y) = 0$, then $x = y$. In particular, if $q(z,x) = q(z,y) = 0$, then $x = y$.*
2. *If $q(x_n,y_n) \leq \alpha_n$ and $q(x_n,y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to $0$, then $\{y_n\}$ converges to $y$.*
3. *If $\{x_n\}$ is a sequence in $X$ for which for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $q(x_n,x_m) < \varepsilon$, then $\{x_n\}$ is a Cauchy sequence.*

4. *If* $\{\alpha_n\}$ *is a sequence in* $[0,\infty)$ *such that* $\{\alpha_n\}$ *converging to* $0$ *and* $q(x_n, x_m) \leq \alpha_n$ *for all* $n, m \in \mathbb{N}$ *with* $m > n$*, then* $\{x_n\}$ *is a Cauchy sequence.*

5. *If* $\{\alpha_n\}$ *is a sequence in* $[0,\infty)$ *such that* $\{\alpha_n\}$ *converging to* $0$ *and* $q(y, x_n) \leq \alpha_n$ *for all* $n \in \mathbb{N}$*, then* $\{x_n\}$ *is a Cauchy sequence.*

Recently, Du [10] recommended the concept of $w^0$-distance by adjust conditions in $w$-distance which use it more.

**Definition 2.6.9** ([10])**.** Let $(X,d)$ be a metric space. A function $q : X \times X \to [0,\infty)$ is called a $\boldsymbol{w^0}$**-distance** if it is a $w$-distance on $X$ with $q(x,x) = 0$ for all $x \in X$.

**Remark 2.6.10.** Each metric on nonempty set $X$ is a $w^0$-distance on $X$

Next, we give some other example of a $w^0$-distance.

**Example 2.6.11** ([10])**.** Let $X = \mathbb{R}$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x, y \in X$ and $a, b \geq 1$. Define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{a(y-x), b(x-y)\}$$

for all $x, y \in X$. Then $q$ is nonsymmetric and hence $q$ is not a metric. It is easy to see that $q$ is a $w^0$-distance.

## 2.7 Altering distance functions and weak altering distance functions

In 1984, Khan et al. [17] introduced the concept of an altering distance function as follows:

**Definition 2.7.1** ([17])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is called an **altering distance function** if the following properties hold:

1. $\psi$ is continuous and nondecreasing;

2. $\psi(t) = 0$ if and only if $t = 0$.

**Example 2.7.2.** Define $\psi_1, \psi_2, \psi_3, \psi_4 : [0,\infty) \to [0,\infty)$ by $\psi_1(t) = t$, $\psi_2(t) = t^3$, $\psi_3(t) = te^{3t}$, $\psi_4(t) = \ln(t^2 + 2t + 1)$ for all $t \geq 0$. It is easy to see that $\psi_1, \psi_2, \psi_3$ and $\psi_4$ are altering distance functions because $\psi_1, \psi_2, \psi_3$ and $\psi_4$ are continuous and nondecreasing. Moreover, $\psi_i(t) = 0$ if and only if $t = 0$ for all $i = 1, 2, 3, 4$. (see the graphs of functions $\psi_1, \psi_2, \psi_3$ and $\psi_4$ in Figure 2.9).

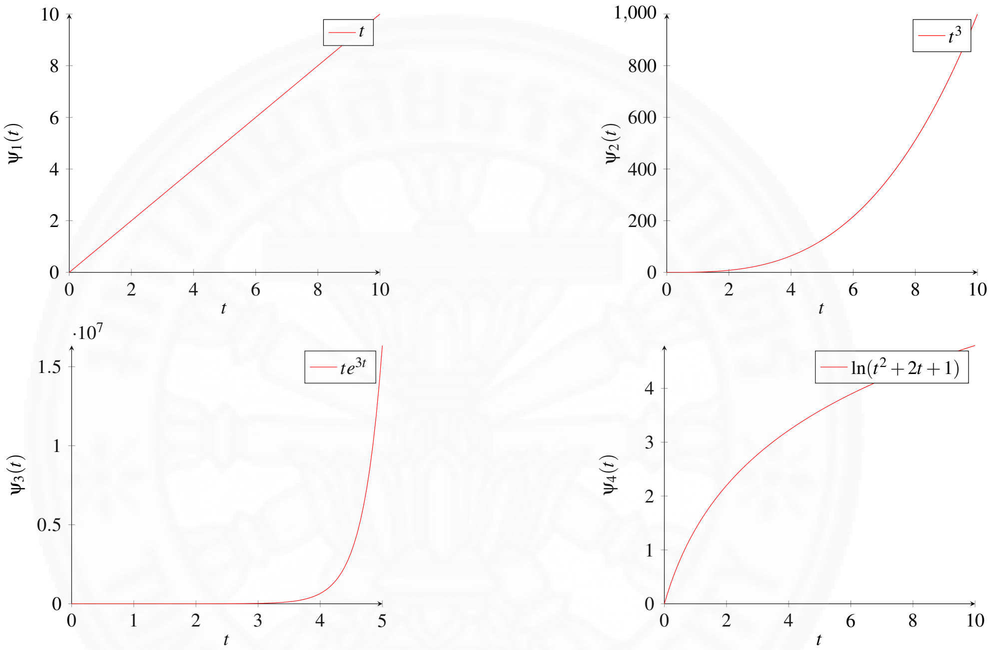


Figure 2.9: Graphs of $\psi_1, \psi_2, \psi_3, \psi_4$ in Example 2.7.2.

Recently, Sawangsup and Sintunavarat [24] introduced the notion of a weak altering distance function by reducing some condition of an altering distance function as follows:

**Definition 2.7.3** ([24])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a **weak altering distance function** if it satisfies the following conditions:

$(a)$ $\psi$ is lower semicontinuous and nondecreasing;

$(b)$ $\psi(t) = 0$ if and only if $t = 0$.

We observe that, if $\psi : [0,\infty) \to [0,\infty)$ is continuous, then $\psi$ is lower semicontinuous. So, the class of weak altering distance functions is wider than the class of altering distance functions.

In general, a weak altering distance function need not necessarily be an altering distance function. Next, we give some examples which guarantee that the class of weak altering distance functions is bigger than the class of altering distance functions.

**Example 2.7.4.** Define $\psi_1, \psi_2, \psi_3 : [0,\infty) \to [0,\infty)$ by

$$\psi_1(t) = \begin{cases} \ln(t^2+2t+1) & \text{if } t \leq 1 \\ t+1 & \text{if } t > 1 \end{cases},$$

$$\psi_2(t) = \begin{cases} t^3 & \text{if } t \leq 1 \\ te^t - 1 & \text{if } t > 1 \end{cases},$$

$$\psi_3(t) = \begin{cases} \frac{2t^3}{3} & \text{if } t \leq 1 \\ 2t^3 & \text{if } t > 1 \end{cases}.$$

It is easy to see that $\psi_1, \psi_2$ and $\psi_3$ are weak altering distance functions because $\psi_1, \psi_2$ and $\psi_3$ are lower semicontinuous and nondecreasing. Moreover, $\psi_i(t) = 0$ if and only if $t = 0$ for all $i = 1,2,3$. (see the graphs of functions $\psi_1, \psi_2$ and $\psi_3$ in Figure 2.10).

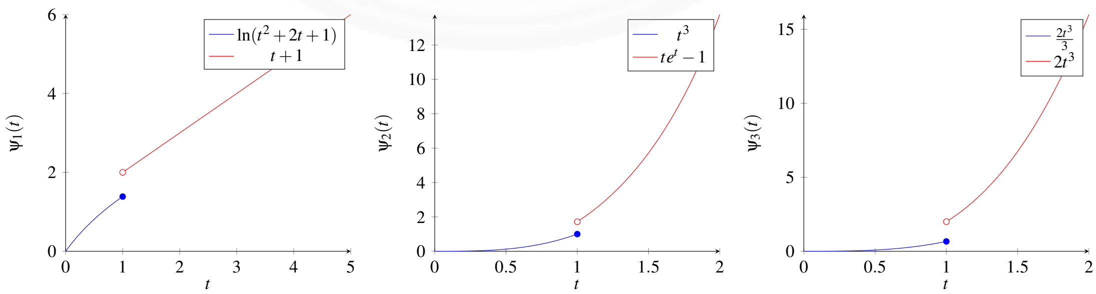


Figure 2.10: Graphs of $\psi_1, \psi_2, \psi_3$ in Example 2.7.4.

## 2.8 Fixed point basics

**Definition 2.8.1.** Let $X$ be a nonempty set. A point $x \in X$ is called **fixed point** of a mapping $f : X \to X$ if and only if $fx = x$.

**Example 2.8.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 2, & \text{if } x > 1. \end{cases}$$

for all $x \in \mathbb{R}$. Then points $0, 1$ and $2$ are fixed points of $f$ (see in Figure 2.11 ).

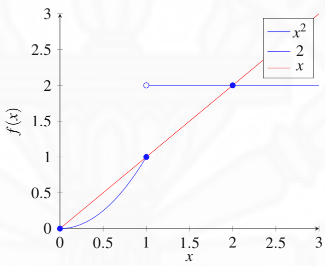


Figure 2.11: The graph of $f$ in Example 2.8.2

**Example 2.8.3.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$fx = x^2 + x + 3$$

for all $x \in \mathbb{R}$. Then $f$ has no a fixed point (see in Figure 2.12 ).

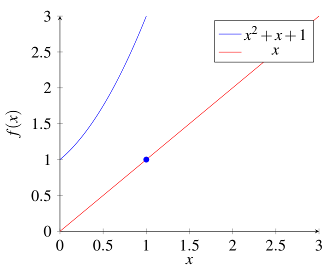


Figure 2.12: The graph of $f$ in Example 2.8.3

**Example 2.8.4.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \leq 1 \\ \frac{1}{2}, & \text{if } x > 1. \end{cases}$$

for all $x \in \mathbb{R}$. Then a point 1 is a unique fixed point of $f$ (see in Figure 2.13 ).

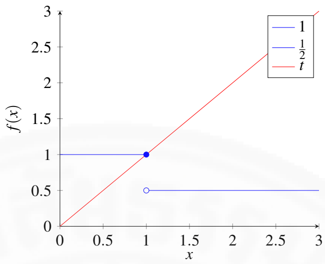


Figure 2.13: The graph of $f$ in Example 2.8.4

## 2.9 Some results on several contraction mappings

In 1922, the following very important result regarding a contractive mapping was proved by Banach [4], Polish mathematician, which is well known as Banach contraction principle.

**Theorem 2.9.1** ([4]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a contraction mapping, i.e.,*

$$d(fx, fy) \leq kd(x,y) \tag{2.9.1}$$

*for all $x,y \in X$, where $k \in [0,1)$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

In 1968, a new type of a contractive mapping was introduced by Kannan [19], which is called the Kannan's contractive type mapping. He also established some fixed point results for such mapping as follows:

**Theorem 2.9.2** ([19]). *Let $(X,d)$ be a complete metric spaces and $f : X \to X$ be a Kannan's contraction mapping, i.e.,*

$$d(fx, fy) \leq k[d(x, fx) + d(y, fy)] \tag{2.9.2}$$

*for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

We observe that Kannan's fixed point theorem is not a generalized of Banach contraction principle. Anywise, Theorem 2.9.2 is significance because Subrahmanyam [27] proved that Theorem 2.9.2 characterizes the metric completeness, that is, a metric space $X$ is complete if and only if every Kannan's contractive mapping on $X$ has a fixed point. Some authors have obtained many of fixed point theorems for Kannan's contractive mappings (see [11], [28] and [25]).

Preferably, the similar contraction condition and fixed point result for Kannan's contractive mappings has been introduced by Chatterjea [8].

**Theorem 2.9.3** ([8]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a Chatterjea's contraction mapping, i.e.,*

$$d(fx, fy) \leq k[d(x, fy) + d(y, fx)] \tag{2.9.3}$$

*for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

Note that the conditions (2.9.1), (2.9.2) and (2.9.3) are independent.

Here are the results that are beneficial for this work.

**Theorem 2.9.4** ([7]). *Let $(X,d)$ be a complete metric space, and let $f : X \to X$ be a mapping. Suppose that there exist nonnegative numbers $q, r, s$ and $t$ such that $q + r + s + 2t < 1$ and*

$$d(fx, fy) \leq qd(x, y) + rd(x, fx) + sd(y, fy) + t[d(x, fy) + d(y, fx)] \tag{2.9.4}$$

*for all $x, y \in X$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

**Theorem 2.9.5** ([22])**.** *Let $(X, d)$ be a complete metric space, and $f : X \to X$ be a mapping with the following property:*

$$d(fx, fy) \leq ad(x, fx) + bd(y, fy) + cd(x, y) \tag{2.9.5}$$

*for all $x \in X$, where $a, b, c$ are nonnegative real numbers such that $a + b + c < 1$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

Now, denote by $\Theta$ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

($\theta_1$) $\theta$ is nondecreasing ;

($\theta_2$) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$ ;

($\theta_3$) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$;

($\theta_4$) $\theta$ is continuous.

**Theorem 2.9.6** ([15])**.** *Let $(X, d)$ be a complete metric space, and $f : X \to X$ be a given mapping. Suppose there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$x, y \in X, \;\; d(fx, fy) \neq 0 \;\; \Rightarrow \;\; \theta(d(fx, fy)) \leq [\theta(d(x, y))]^k. \tag{2.9.6}$$

*Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

Later, some fixed point results for generalized weakly contractive mappings was proved by Choudhury et al. [5] by using a control function via the notion of an altering distance function.

**Theorem 2.9.7** ([5]). *Let $(X,d)$ be a complete metric space and $f$ a self-mapping on $X$. Suppose that $f$ is a generalized weak contractive mapping, i.e., for all $x,y \in X$,*

$$\psi(d(fx,fy)) \leq \psi(m(x,y)) - \phi\Big(\max\Big\{d(x,y), d(y,fy)\Big\}\Big), \tag{2.9.7}$$

*where*

$$m(x,y) := \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,fy), d(y,fx)]\Big\},$$

*$\psi$ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then $f$ has a unique fixed point on $X$.*

Next, denote by $\Psi$ the set of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

($\psi_1$) $\psi$ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$ ;

($\psi_2$) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$ ;

($\psi_3$) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = \ell$;

($\psi_4$) $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a,b > 0$.

**Theorem 2.9.8** ([12]). *Let $(X,d)$ be a complete metric space, and $f : X \to X$ be a continuous JS-contraction, i.e., there are function $\psi \in \Psi$ and nonnegative real numbers $k_1,k_2,k_3,k_4$ with $0 \leq k_1+k_2+k_3+2k_4 < 1$ such that*

$$\psi(d(fx,fy)) \leq [\psi(d(x,y))]^{k_1}[\psi(d(x,fx))]^{k_2}[\psi(d(y,fy))]^{k_3}[\psi(d(x,fy)+d(y,fx))]^{k_4} \tag{2.9.8}$$

*for all $x,y \in X$. Then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

Recently, some fixed point results for generalized contraction mappings was proved by Ri [23]. This result generalized the Boyd and Wong's fixed point theorem in [3] and the Matkowski's fixed point theorem in [14].

**Theorem 2.9.9** ([23]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a mapping. Suppose that there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ and*

$$d(fx, fy) \leq \varphi(d(x,y)), \tag{2.9.9}$$

*for all $x, y \in X$. Then $f$ has a unique fixed point in $X$.*

Most recently, Sawangsup and Sintunavarat [24] introduced the notion of a weak altering distance function and proved the following existence of a fixed point by using such function.

**Theorem 2.9.10** ([24]). *Let $(X,d)$ be a complete metric space. Suppose that $f : X \to X$ is a continuous mappings such that*

$$\psi(d(fx, fy)) \leq \phi(d(x,y)), \tag{2.9.10}$$

*for all $x, y \in X$, where $\psi$ is a weak altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$. Then $T$ has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

# CHAPTER 3

# THEORETICAL FIXED POINT RESULTS

In this chapter, we introduce the new concept of a distance on metric spaces and prove the existence and uniqueness of fixed point results for new contraction mappings along with $w$-distances in complete metric spaces. Some of the results are proved by using the concept of a new purposed distance. First, we give the new definition of a ceiling distance on a metric space.

**Definition 3.0.1.** A $w$-distance $q$ on a metric space $(X,d)$ is said to be a **ceiling distance** of $d$ if and only if

$$q(x,y) \geq d(x,y) \tag{3.0.1}$$

for all $x,y \in X$.

Now we give some examples of a ceiling distance.

**Example 3.0.2.** Each metric on a nonempty set $X$ is a ceiling distance on itself.

**Example 3.0.3.** Let $X = \mathbb{R}$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$ and $a,b \geq 1$. Define a $w$-distance $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{a(y-x), b(x-y)\}$$

for all $x,y \in X$. For each $x,y \in X$, we get

$$\begin{aligned} d(x,y) &= |x-y| \\ &= \begin{cases} x-y, & x \geq y \\ y-x, & x \leq y \end{cases} \\ &\leq \max\{a(y-x), b(x-y)\} \\ &= q(x,y). \end{aligned}$$

Then $q$ is a ceiling distance of $d$.

**Example 3.0.4.** Let $a,b \in \mathbb{R}$ with $a < b$ and $X = C[a,b]$ (the set of all continuous functions from $[a,b]$ into $\mathbb{R}$), with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$ for all $x,y \in X$. Define a $w$-distance $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|$$

for all $x,y \in X$. For each $x,y \in X$ and $t \in [a,b]$, we obtain

$$\begin{aligned} |x(t) - y(t)| &\leq |x(t)| + |y(t)| \\ &\leq \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|. \end{aligned}$$

It yields that $\sup_{t\in[a,b]} |x(t) - y(t)| \leq \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|$ and so $q$ is a ceiling distance of $d$.

## 3.1 Fixed point results for *w*-generalized weak contraction mapping

In this section, we introduce the new concept of a generalized contraction mapping along with $w$-distances in metric spaces. Furthermore, we investigate the sufficient condition for the existence and uniqueness of a fixed point of self mappings on metric spaces satisfying such contractive condition. First, we introduce the definition of the new type of generalized contraction mappings so called a $w$-generalized weak contraction mapping.

**Definition 3.1.1.** Let $q$ be a $w$-distance on a metric space $(X,d)$. A mapping $f : X \to X$ is said to be a ***w*-generalized weak contraction mapping** if for all $x,y \in X$,

$$\psi(q(fx,fy)) \leq \psi(m(x,y)) - \phi(q(x,y)) \tag{3.1.1}$$

where

$$m(x,y) := \max\left\{q(x,y), \frac{1}{2}[q(x,fy) + q(fx,y)]\right\},$$

$\psi : [0,\infty) \to [0,\infty)$ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. If $q = d$, then the mapping $f$ is said to be a generalized weak contraction mapping.

Now, we give the main result in this section.

**Theorem 3.1.2.** *Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a $w^0$-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a continuous $w$-generalized weak contraction mapping. Then $f$ has a unique fixed point in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

*Proof.* Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions in the contractive condition 3.1.1. Starting from a fixed arbitrary point $x_0 \in X$, we put $x_{n+1} = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then $x_{n^*}$ is a fixed point of $f$. Thus we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, i.e., $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $q$ is a ceiling distance of $d$, we obtain $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.1.1), for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \psi(q(x_{n+1}, x_{n+2})) &= \psi(q(fx_n, fx_{n+1})) \\ &\leq \psi(m(x_n, x_{n+1})) - \phi(q(x_n, x_{n+1})) \\ &= \psi\left(\max\left\{q(x_n, x_{n+1}), \frac{1}{2}[q(x_n, fx_{n+1}) + q(fx_n, x_{n+1})]\right\}\right) - \phi(q(x_n x_{n+1})) \\ &\leq \psi\left(\max\left\{q(x_n, x_{n+1}), \frac{1}{2}[q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})]\right\}\right) - \phi(q(x_n, x_{n+1})). \end{aligned} \tag{3.1.2}$$

Suppose that

$$q(x_n, x_{n+1}) \leq q(x_{n+1}, x_{n+2})$$

for some $n \in \mathbb{N} \cup \{0\}$. From (3.1.2), we have

$$\psi(q(x_{n+1}, x_{n+2})) \leq \psi(q(x_{n+1}, x_{n+2})) - \phi(q(x_n, x_{n+1})),$$

that is, $\phi(q(x_n,x_{n+1})) \leq 0$, which implies that $q(x_n,x_{n+1}) = 0$. This implies a contradiction. Therefore, $q(x_{n+1},x_{n+2}) < q(x_n,x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and hence $\{q(x_n,x_{n+1})\}$ is a monotone decreasing and bounded below. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} q(x_n,x_{n+1}) = r. \tag{3.1.3}$$

In view above facts, from (3.1.2) we have for all $n \in \mathbb{N} \cup \{0\}$,

$$\psi(q(x_{n+1},x_{n+2})) \leq \psi(q(x_n,x_{n+1})) - \phi(q(x_n,x_{n+1})).$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of $\phi$ and $\psi$, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} q(x_n,x_{n+1}) = 0. \tag{3.1.4}$$

Similarly, we can prove that

$$\lim_{n\to\infty} q(x_{n+1},x_n) = 0. \tag{3.1.5}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose by contradiction with Lemma 2.6.8, there exists an $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ such that

$$q(x_{m_k},x_{n_k}) \geq \varepsilon \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.1.6}$$

Choosing $n_k$ to be the smallest integer exceeding $m_k$ for which (3.1.6) holds, we obtain that

$$q(x_{m_k},x_{n_k-1}) < \varepsilon. \tag{3.1.7}$$

Now, we get

$$\varepsilon \leq q(x_{m_k},x_{n_k}) \leq q(x_{m_k},x_{n_k-1}) + q(x_{n_k-1},x_{n_k}) \leq \varepsilon + q(x_{n_k-1},x_{n_k}).$$

Taking the limit as $k \to \infty$ in the above inequality and using (3.1.4), we have

$$\lim_{n\to\infty} q(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.1.8}$$

By using (3.1.6) and (W1), we have

$$\varepsilon \le q(x_{m_k}, x_{n_k}) \le q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}),$$

and

$$q(x_{m_k+1}, x_{n_k+1}) \le q(x_{m_k+1}, x_{m_k}) + q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k+1}).$$

Taking the limit as $k \to \infty$ in the above two inequalities and using (3.1.4), (3.1.5) and (3.1.8), we have

$$\lim_{n\to\infty} q(x_{m_k+1}, x_{n_k+1}) = \varepsilon. \tag{3.1.9}$$

Again, by using (W1), we obtain

$$\begin{aligned} q(x_{m_k}, x_{n_k}) &\le q(x_{m_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}) \\ &\le q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}) \end{aligned}$$

and

$$\begin{aligned} q(x_{m_k}, x_{n_k}) &\le q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{n_k}) \\ &\le q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{m_k}) + q(x_{m_k}, x_{n_k}). \end{aligned}$$

Taking $k \to \infty$ in the above two inequalities and using (3.1.4), (3.1.5) and (3.1.8), we have

$$\lim_{n\to\infty} q(x_{m_k}, x_{n_k+1}) = \varepsilon, \;\; \lim_{n\to\infty} q(x_{m_k+1}, x_{n_k}) = \varepsilon. \tag{3.1.10}$$

Substitution $x = x_{m_k}$, $y = x_{n_k}$ in (3.1.1), we have

$$\psi(q(x_{m_k+1}, x_{n_k+1})) \le \psi\Big( \max\Big\{ q(x_{m_k}, x_{n_k}), \frac{1}{2}[q(x_{m_k}, x_{n_k+1}) + q(x_{m_k+1}, x_{n_k})] \Big\}\Big) - \phi(q(x_{m_k}, x_{n_k})).$$

Letting $k \to \infty$ in the above inequality, using (3.1.4), (3.1.5), (3.1.8), (3.1.9), (3.1.10) and using the continuities of $\phi$ and $\psi$, we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),$$

which is a contradiction with the property of $\phi$. Hence by Lemma 2.6.8, we can conclude that $\{x_n\}$ is a Cauchy sequence. Since $(X,d)$ is a complete metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. From the continuity of $f$, we get $x_{n+1} = fx_n \to fp$ as $n \to \infty$, i.e., $p = fp$. Thus, $f$ has a fixed point.

Finally, we prove that the fixed point is unique. Let $p$ and $p^*$ be two fixed point of $f$ and suppose that $p \neq p^*$. Then putting $x = p$ and $y = p^*$ in (3.1.1), we obtain

$$\psi(q(fp,fp^*)) \leq \psi\Big(\max\Big\{q(p,p^*), \frac{1}{2}[q(p,fp^*)+q(fp,p^*)]\Big\}\Big) - \phi(q(p,p^*)),$$

that is, $\psi(q(p,p^*)) \leq \psi(q(p,p^*)) - \phi(q(p,p^*))$, which is a contradiction by property of $\phi$. Therefore, $p = p^*$ and the fixed point is unique. This completes the proof. □

In the next theorem, we omit the continuity hypothesis of $f$.

**Theorem 3.1.3.** *Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a continuous $w^0$-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a w-generalized weak contraction mapping. Then $f$ has a unique fixed point in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

*Proof.* Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions in the contractive condition 3.1.1. Let $x_0$ be an arbitrary point in $X$. Put $x_{n+1} = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then $x_{n^*}$ is a fixed point of $f$. So we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Following the proof of Theorem 3.1.2, we know that $\{x_n\}$ is a Cauchy sequence in $X$. Completeness of $(X,d)$ ensures that there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Assume that $p \neq fp$. Putting $x = x_n$, $y = p$ in (3.1.1), we have

$$\begin{aligned} \psi(q(x_{n+1},fp)) &= \psi(q(fx_n,fp)) \\ &\leq \psi\Big(\max\Big\{q(x_n,p), \frac{1}{2}[q(x_n,fp)+q(x_{n+1},p)]\Big\}\Big) - \phi(q(x_n,p)) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking the limit as $n \to \infty$ in the above inequality and using the continuities of $\phi$, $\psi$ and $q$, we have

$$\psi(q(p, fp)) \leq \psi\Big(\frac{1}{2}q(p, fp)\Big),$$

which is a contradiction. Thus $p = fp$, that is, $p$ is a fixed point of $f$. Following the proof of Theorem 3.1.2, we know that $p$ is a unique fixed point of $f$. This completes the proof. □

Taking $q = d$ in Theorem 3.1.3, we obtain the following result.

**Corollary 3.1.4.** *Let $(X, d)$ be a complete metric space. Suppose that $f : X \to X$ is a generalized weak contraction mapping. Then $f$ has a unique fixed point in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

We can extend the condition of $w^0$-distances in Theorems 3.1.2 and 3.1.3 to $w$-distances if we replace the contractive condition (3.1.1) by some strong condition. Here we give the purposed results.

**Theorem 3.1.5.** *Let $(X, d)$ be a complete metric space and $q : X \times X \to [0, \infty)$ be a $w$-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a continuous mapping such that for all $x, y \in X$,*

$$\psi(q(fx, fy)) \leq \psi(q(x, y)) - \phi(q(x, y)), \tag{3.1.11}$$

*where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then $f$ has a unique fixed point in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

**Theorem 3.1.6.** *Let $(X, d)$ be a complete metric space and $q : X \times X \to [0, \infty)$ be a continuous $w$-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a mapping such that for all $x, y \in X$,*

$$\psi(q(fx, fy)) \leq \psi(q(x, y)) - \phi(q(x, y)), \tag{3.1.12}$$

*where* $\psi : [0,\infty) \to [0,\infty)$ *is an altering distance function and* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$*. Then* $f$ *has a unique fixed point in* $X$*. Moreover, for each* $x_0 \in X$*, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$*, converges to a unique fixed point of* $f$*.*

## 3.2 Fixed point results for *w*-generalized *JS*-contraction mappings

The aim of this section is to introduce the new concept of contractility along with $w$-distances in metric spaces. Furthermore, we investigate the sufficient condition for the existence and uniqueness of a fixed point of self mappings on metric spaces satisfying such contractive condition and give some examples to show the validity of these results.

Throughout this section, we denote by $\Psi$ the set of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

($\psi_1$) $\psi$ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$ ;

($\psi_2$) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim\limits_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim\limits_{n\to\infty} t_n = 0$ ;

($\psi_3$) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim\limits_{t\to 0^+} \frac{\psi(t)-1}{t^r} = \ell$;

($\psi_4$) $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a,b > 0$.

**Example 3.2.1.** Let $\psi_1 : [0,\infty) \to [1,\infty)$ be defined by

$$\psi_1(t) = e^{\sqrt{t}}.$$

It is easy to see that $\psi_1$ satisfies conditions $(\psi_1) - (\psi_4)$. Then $\psi_1 \in \Psi$.

**Example 3.2.2.** Let $\psi_2 : [0,\infty) \to [1,\infty)$ be defined by

$$\psi_2(t) = 1 + \sqrt{t}.$$

It is easy to see that $\psi_2$ satisfies conditions $(\psi_1) - (\psi_4)$. Then $\psi_2 \in \Psi$.

The following is the introducing a new concept which involving with main result of this section.

**Definition 3.2.3.** Let $q$ be a $w$-distance on a metric space $(X,d)$ and $\psi \in \Psi$ . A mapping $f : X \to X$ is said to be a ***w*-generalized *JS*-contraction mapping** whenever there are nonnegative real numbers $k_1, k_2, k_3, k_4$ with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$\psi(q(fx, fy)) \leq [\psi(q(x,y))]^{k_1}[\psi(q(x, fx))]^{k_2}[\psi(q(y, fy))]^{k_3}[\psi(q(x, fy) + q(y, fx))]^{k_4} \tag{3.2.1}$$

for all $x, y \in X$.

**Remark 3.2.4.** The contractive condition in Definition 3.2.3 is an extension of a concept of *JS*-contraction due to Hussain et al [12].

Now, we give the main result in this section.

**Theorem 3.2.5.** *Let $(X,d)$ be a complete metric space, $q : X \times X \to [0,\infty)$ be a $w^0$-distance on X and ceiling distance of d, and $f : X \to X$ be a mapping. Suppose that f is a continuous w-generalized JS-contraction mapping with respect to a function $\psi \in \Psi$ and nonnegative real numbers $k_1, k_2, k_3, k_4$ in (3.2.1). Then f has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converge to a unique fixed point of f.*

*Proof.* Let $x_0 \in X$ be arbitrary. We define the sequence $\{x_n\}$ by

$$x_n = f^n x_0 = f x_{n-1},$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0}$ is a fixed point of $f$ and hence we have nothing to prove. Thus, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, i.e.,

$$d(x_n, x_{n+1}) > 0 \tag{3.2.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $q$ is ceiling distance, we have

$$q(x_n, x_{n+1}) > 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Now we will prove that

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0.$$

By using the condition (3.2.1), we obtain

$$\begin{aligned}
\psi(q(x_n,x_{n+1})) &= \psi(q(fx_{n-1},fx_n)) \\
&\leq [\psi(q(x_{n-1},x_n))]^{k_1}[\psi(q(x_{n-1},fx_{n-1}))]^{k_2}[\psi(q(x_n,fx_n))]^{k_3} \\
&\quad \times[\psi(q(x_{n-1},fx_n)+q(x_n,fx_{n-1}))]^{k_4} \\
&\leq [\psi(q(x_{n-1},x_n))]^{k_1}[\psi(q(x_{n-1},x_n))]^{k_2}[\psi(q(x_n,x_{n+1}))]^{k_3} \\
&\quad \times[\psi(q(x_{n-1},x_n))]^{k_4}[\psi(q(x_n,x_{n+1}))]^{k_4} \\
&= [\psi(q(x_{n-1},x_n))]^{k_1+k_2+k_4}[\psi(q(x_n,x_{n+1}))]^{k_3+k_4}
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we write

$$1 < \psi(q(x_n,x_{n+1})) \leq [\psi(q(x_{n-1},x_n))]^{\frac{k_1+k_2+k_4}{1-k_3-k_4}} \leq [\psi(q(x_0,x_1))]^{\left(\frac{k_1+k_2+k_4}{1-k_3-k_4}\right)^n} \tag{3.2.3}$$

for all $n \in \mathbb{N}$. This implies that

$$\lim_{n\to\infty} \psi(q(x_n,x_{n+1})) = 1.$$

From our assumptions about the function $\psi$, we get $\lim_{n\to\infty} q(x_n,x_{n+1}) = 0$. From condition $(\psi_3)$, there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\psi(q(x_n,x_{n+1}))-1}{[q(x_n,x_{n+1})]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2}$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\psi(q(x_n,x_{n+1}))-1}{[q(x_n,x_{n+1})]^r} - \ell\right| \leq B \quad \text{for all } n \geq n_0.$$

This implies that

$$\frac{\psi(q(x_n,x_{n+1}))-1}{[q(x_n,x_{n+1})]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Then

$$n[q(x_n,x_{n+1})]^r \leq An[\psi(q(x_n,x_{n+1}))-1] \quad \text{for all } \ n \geq n_0,$$

where $A=\frac{1}{B}$.

Suppose now that $\ell=\infty$. Let $B>0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\psi(q(x_n,x_{n+1}))-1}{[q(x_n,x_{n+1})]^r} \geq B \quad \text{for all } \ n \geq n_0.$$

This implies that

$$n[q(x_n,x_{n+1})]^r \leq An[\psi(q(x_n,x_{n+1}))-1] \quad \text{for all } \ n \geq n_0,$$

where $A=\frac{1}{B}$. Thus, in all cases, there exist $A>0$ and $n_0 \in \mathbb{N}$ such that

$$n[q(x_n,x_{n+1})]^r \leq An[\psi(q(x_n,x_{n+1}))-1] \quad \text{for all } \ n \geq n_0.$$

Using (3.2.3), we obtain

$$n[q(x_n,x_{n+1})]^r \leq An([\psi(q(x_0,x_1))]^{\left(\frac{k_1+k_3+k_4}{1-k_2-k_4}\right)^n}-1) \quad \text{for all } \ n \geq n_0.$$

Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} n[q(x_n,x_{n+1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$q(x_n,x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}} \quad \text{for all } \ n \geq n_1.$$

Now, for $m>n>n_1$, we have

$$q(x_n,x_m) \leq \sum_{i=n}^{m-1} q(x_i,x_{i-1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}}.$$

Since $0<r<1$, we get $\sum\limits_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges and hence $q(x_n,x_m) \to 0$ as $m,n \to \infty$. Since $q$ is ceiling distance, we get $d(x_n,x_m) \leq q(x_n,x_m)$ for all $m,n \geq n_1$. This implies that

$d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus, we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of $(X, d)$ ensures that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of $f$, we get $x_{n+1} = fx_n \to fx^*$ as $n \to \infty$, i.e., $x^* = fx^*$. Thus, $f$ has a fixed point.

Finally, suppose that there exists $z \neq x^*$ such that $z = fz$. Clearly, $d(z, x^*) = d(fz, fx^*) \neq 0$ and $d(x^*, z) = d(fx^*, fz) \neq 0$. From condition (3.0.1), we get $q(z, x^*) \neq 0$, $q(x^*, z) \neq 0$ and so we can apply condition (3.2.1) for the pair $(z, x^*)$ and $(x^*, z)$. Now, by (3.2.1) we get

$$\begin{aligned} 1 &< [\psi(q(z,x^*))][\psi(q(x^*,z))] \\ &= [\psi(q(fz,fx^*))][\psi(q(fx^*,fz))] \\ &\leq [\psi(q(z,x^*))]^{k_1+k_4}[\psi(q(x^*,z))]^{k_4}[\psi(q(x^*,z))]^{k_1+k_4}[\psi(q(z,x^*))]^{k_4} \\ &= [\psi(q(z,x^*))]^{k_1+2k_4}[\psi(q(x^*,z))]^{k_1+2k_4} \\ &< [\psi(q(z,x^*))][\psi(q(x^*,z))] \end{aligned}$$

which is a contradiction. Therefore, $f$ has a unique fixed point. This completes the proof. □

By taking $q = d$ in Theorem 3.2.5, we obtain the following result of Hussain [12].

**Theorem 3.2.6** ([12])**.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ be a continuous JS-contraction mapping. Then f has a unique fixed point.*

In the next theorem, we omit the continuity hypothesis of $f$.

**Theorem 3.2.7.** *Let $(X, d)$ be a complete metric space, $q : X \times X \to [0, \infty)$ be a $w^0$-distance on X and ceiling distance on d, and $f : X \to X$ be a mapping. Suppose that f is a w-generalized JS-contraction mapping with respect to a function $\psi \in \Psi$, nonnegative real numbers $k_1, k_2, k_3, k_4$ in (3.2.1) and $\psi$ and q are continuous. Then f has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converge to a unique fixed point of f.*

*Proof.* Let $x_0 \in X$ be arbitrary. We define the sequence $\{x_n\}$ by

$$x_n = f^n x_0 = f x_{n-1},$$

for all $n \in \mathbb{N}$. Following the proof of Theorem 3.2.5, we know that $\{x_n\}$ is a Cauchy sequence in $X$. Completeness of $(X,d)$ ensures that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From condition (3.2.1) we get

$$\begin{aligned}\psi(q(fx_n, fx^*)) &\leq [\psi(q(x_n,x^*))]^{k_1}[\psi(q(x_n,x_{n+1}))]^{k_2}[\psi(q(x^*,fx^*))]^{k_3} \\ &\quad \times[\psi(q(x_n,fx^*)+q(x^*,x_{n+1}))]^{k_4}.\end{aligned}$$

Let $n \to \infty$, since $\psi$ and $q$ are continuous, we obtain

$$\psi(q(x^*,fx^*)) \leq [\psi(q(x^*,fx^*))]^{k_3+k_4}.$$

It yields that, $\psi(q(x^*,fx^*)) = 1$ and so by condition $(\psi_1)$, we get $q(x^*,fx^*) = 0$. From, condition (3.0.1), we get $d(x^*,fx^*) = 0$, i.e., $x^* = fx^*$. Thus, $f$ has a fixed point. The uniqueness of fixed point of $f$ follows from condition (3.2.1). This completes the proof. □

By taking $q = d$ in Theorem 3.2.7, we obtain the following result of Hussain [12].

**Corollary 3.2.8** ([12]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a JS-contraction mapping with respect to a function $\psi \in \Psi$. If $\psi$ is a continuous mapping, then f has a unique fixed point.*

Now we give an example which it is possible to apply by Theorem 3.2.7 but not Banach contraction principle.

**Example 3.2.9.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$ . Define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{y-x, 2(x-y)\}$$

for all $x, y \in X$. Clearly, $q$ is a $w^0$-distance and $q$ is a ceiling distance of $d$. Also, define $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{100}, & x \geq 1 \\ 0, & x < 1. \end{cases}$$

Note that the Banach contractive condition does not hold. Indeed, for $x = 1$ and $y = 0.9999$, we get

$$d(fx, fy) = 0.01 > d(x,y) > kd(x,y)$$

for all $k \in [0,1)$.

Next, we show that Theorem 3.2.7 can be applied in this case. Define a function $\psi : [0,\infty) \to [1,\infty)$ by

$$\psi(t) = e^{\sqrt{t}}$$

for all $t \in [0,\infty)$. It easy to see that $\psi \in \Psi$. Here, we will show that $f$ satisfies the contractive contraction (3.2.1) with $k_1 = \frac{1}{10},\ k_2 = \frac{1}{\sqrt{99}},\ k_3 = \frac{1}{\sqrt{198}}$ *and* $k_4 \in (0, 0.36)$. We will distinguish this claim into the following case.

<u>Case1</u> If $x, y \geq 1$ with $x \geq y$, then

$$\begin{aligned} \psi(q(fx, fy))] &= \Big[\psi\Big(q\Big(\frac{x}{100}, \frac{y}{100}\Big)\Big)\Big] \\ &= e^{\sqrt{\frac{x-y}{50}}} \\ &\leq [e^{\sqrt{2(x-y)}}]^{0.1} \\ &= [e^{\sqrt{2(x-y)}}]^{k_1} \\ &= [\psi(q(x,y))]^{k_1} \\ &\leq [\psi(q(x,y))]^{k_1}[\psi(q(x,fx))]^{k_2}[\psi(q(y,fy))]^{k_3} \\ &\quad \times[\psi(q(x,fy)+q(y,fx))]^{k_4}. \end{aligned}$$

Case2 If $x, y \geq 1$ with $x \leq y$, then

$$
\begin{aligned}
\psi(q(fx,fy))] &= \Big[\psi\Big(q\Big(\frac{x}{100},\frac{y}{100}\Big)\Big)\Big] \\
&= e^{\sqrt{\frac{y-x}{100}}} \\
&\leq [e^{\sqrt{(y-x)}}]^{0.1} \\
&= [e^{\sqrt{(y-x)}}]^{k_1} \\
&= [\psi(q(x,y))]^{k_1} \\
&\leq [\psi(q(x,y))]^{k_1}[\psi(q(x,fx))]^{k_2}[\psi(q(y,fy))]^{k_3} \\
&\quad \times[\psi(q(x,fy)+q(y,fx))]^{k_4}.
\end{aligned}
$$

Case3 If $x, y < 1$, then (3.2.1) holds.

Case4 If $x \geq 1$ and $y < 1$, then

$$
\begin{aligned}
\psi(q(fx,fy))] &= \Big[\psi\Big(q\Big(\frac{x}{100},0\Big)\Big)\Big] \\
&= e^{\sqrt{\frac{x}{50}}} \\
&= [e^{\sqrt{\frac{99x}{50}}}]^{\frac{1}{\sqrt{99}}} \\
&= [e^{\sqrt{\frac{99x}{50}}}]^{k_2} \\
&= [\psi(q(x,fx))]^{k_2} \\
&\leq [\psi(q(x,y))]^{k_1}[\psi(q(x,fx))]^{k_2}[\psi(q(y,fy))]^{k_3} \\
&\quad \times[\psi(q(x,fy)+q(y,fx))]^{k_4}.
\end{aligned}
$$

<u>Case5</u> If $x < 1$ and $y \geq 1$, then

$$\begin{aligned}
\psi(q(fx, fy))] &= \left[\psi\Big(q\Big(0, \frac{y}{100}\Big)\Big)\right] \\
&= e^{\sqrt{\frac{y}{100}}} \\
&\leq [e^{\sqrt{\frac{99y}{50}}}]^{\frac{1}{\sqrt{198}}} \\
&= [e^{\sqrt{\frac{99y}{50}}}]^{k_3} \\
&= [\psi(q(y, fy))]^{k_3} \\
&\leq [\psi(q(x,y))]^{k_1}[\psi(q(x, fx))]^{k_2}[\psi(q(y, fy))]^{k_3} \\
&\quad \times[\psi(q(x, fy) + q(y, fx))]^{k_4}.
\end{aligned}$$

Therefore, all conditions of Theorem 3.2.7 hold and hence $f$ has a unique fixed point. Here, $x = 0$ is a unique fixed point of $f$.

**Example 3.2.10.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by $d(x,y) = |x-y|$ for all $x,y \in X$ . Define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{y-x, 2(x-y)\}$$

for all $x,y \in X$. Clearly, $q$ is a $w^0$-distance and $q$ is a ceiling distance of $d$. Also, define functions $f : X \to X$, and $\psi : [0,\infty) \to [1,\infty)$ by

$$fx = \begin{cases} 0, & x \geq 1 \\ \\ \frac{x}{100}, & x < 1, \end{cases}$$

and,

$$\psi(t) = e^{\sqrt{t}}$$

for all $t \in [0,\infty)$. By using the similarly technique in Example 3.2.9, all conditions of Theorem 3.2.7 hold. So $f$ has a unique fixed point. Here, $x = 0$ is a unique fixed point of $f$.

For specific choices of function $\psi \in \Psi$, we obtain some significant results. First, by taking $\psi(t) = e^{\sqrt{t}}$ in (3.2.1), we state a generalization of *Ćirić*'s result in [7].

**Corollary 3.2.11.** *Let $(X,d)$ be a complete metric space , $\psi \in \Psi$, $q : X \times X \to [0,\infty)$ be a $w^0$-distance on $X$ and ceiling distance on $d$, and $f : X \to X$ be a mapping. Suppose that there exist nonnegative real numbers $k_1,k_2,k_3,k_4$ with $0 \le k_1+k_2+k_3+2k_4 < 1$ such that*

$$\sqrt{q(fx,fy)} \le k_1\sqrt{q(x,y)} + k_2\sqrt{q(x,fx)} + k_3\sqrt{q(y,fy)} + k_4\sqrt{q(x,fy)+q(y,fx)} \tag{3.2.4}$$

*for all $x,y \in X$. If one of the the following conditions hold:*

*(i) $f : X \to X$ is a continuous mapping;*

*(ii) $q$ and $\psi$ are continuous mappings,*

*then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converge to a unique fixed point of $f$.*

**Remark 3.2.12.** Notice that condition (3.2.4) is equivalent to

$$\begin{aligned} q(fx,fy) \;\le\; & k_1^2 q(x,y) + k_2^2 q(x,fx) + k_3^2 q(y,fy) + k_4^2[q(x,fy)+q(y,fx)] \\ & +2k_1k_2\sqrt{q(x,y)q(x,fx)} + 2k_1k_3\sqrt{q(x,y)q(y,fy)} \\ & +2k_1k_4\sqrt{q(x,y)[q(x,fy)+q(y,fx)]} + 2k_2k_3\sqrt{q(x,fx)q(y,fy)} \\ & +2k_2k_4\sqrt{q(x,fx)[q(x,fy)+q(y,fx)]} \\ & +2k_3k_4\sqrt{q(y,fy)[q(x,fy)+q(y,fx)]}. \end{aligned}$$

Next, in view of Remark 3.2.12 , by taking $k_1 = k_4 = 0$ in Corollary 3.2.11, we obtain the following extension of Kannan's result [19].

**Corollary 3.2.13.** *Let $(X,d)$ be a complete metric space , $\psi \in \Psi$, $q : X \times X \to [0,\infty)$ be a $w^0$-distance on $X$ and ceiling distance on $d$, and $f : X \to X$ be a mapping. Suppose that there exist nonnegative real numbers $k_2,k_3$ with $0 \le k_2+k_3 < 1$ such that*

$$q(fx,fy) \le {k_2}^2 q(x,fx) + {k_3}^2 q(y,fy) + 2k_2k_3\sqrt{q(x,fx)+q(y,fy)} \tag{3.2.5}$$

*for all $x,y \in X$. If one of the the following conditions hold:*

*(i)* $f : X \to X$ *is a continuous mapping;*

*(ii)* $q$ *and* $\psi$ *are continuous mappings,*

*then* $f$ *has a unique fixed point. Moreover, for each* $x_0 \in X$*, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$*, converge to a unique fixed point of* $f$*.*

On the other hand, by taking $k_1 = k_2 = k_3 = 0$ in Corollary 3.2.11, we obtain the extension of Chatterjea's result in [8].

**Corollary 3.2.14.** *Let* $(X,d)$ *be a complete metric space ,* $\psi \in \Psi$*,* $q : X \times X \to [0,\infty)$ *be a* $w^0$*-distance on* $X$ *and ceiling distance on* $d$*, and* $f : X \to X$ *be a mapping. Suppose that there exist* $k_4 \in [0, \frac{1}{2})$ *such that*

$$q(fx, fy) \leq {k_4}^2[q(x, fy) + q(y, fx)]$$

*for all* $x, y \in X$*. If one of the the following conditions hold:*

*(i)* $f : X \to X$ *is a continuous mapping;*

*(ii)* $q$ *and* $\psi$ *are continuous mappings,*

*then* $f$ *has a unique fixed point. Moreover, for each* $x_0 \in X$*, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$*, converge to a unique fixed point of* $f$*.*

From Corollary 3.2.11, by taking $k_4 = 0$, we obtain the extension of Reich's result in [22].

**Corollary 3.2.15.** *Let* $(X,d)$ *be a complete metric space ,* $\psi \in \Psi$*,* $q : X \times X \to [0,\infty)$ *be a* $w^0$*-distance on* $X$ *and ceiling distance on* $d$*, and* $f : X \to X$ *be a mapping. Suppose that there exist nonnegative real numbers* $k_1, k_2, k_3$ *with* $0 \leq k_1 + k_2 + k_3 < 1$ *such that*

$$\begin{aligned} q(fx, fy) \leq\ & {k_1}^2 q(x,y) + {k_2}^2 q(x, fx) + {k_3}^2 q(y, fy) \\ & + 2k_1k_2\sqrt{q(x,y)q(x,fx)} + 2k_1k_3\sqrt{q(x,y)q(y,fy)} \\ & + 2k_2k_3\sqrt{q(x,fx)q(y,fy)} \end{aligned}$$

*for all* $x, y \in X$*. If one of the the following conditions hold:*

*(i)* $f : X \to X$ *is a continuous mapping;*

*(ii)* $q$ *and* $\psi$ *are continuous mappings,*

*then* $f$ *has a unique fixed point. Moreover, for each* $x_0 \in X$*, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$*, converge to a unique fixed point of* $f$.

Finally, by taking $\psi(t) = e^{\sqrt[n]{t}}$ in (3.2.1), we have the following corollary.

**Corollary 3.2.16.** *Let* $(X,d)$ *be a complete metric space ,* $\psi \in \Psi$*,* $q : X \times X \to [0,\infty)$ *be a* $w^0$*-distance on* $X$ *and ceiling distance on* $d$*, and* $f : X \to X$ *be a mapping. Suppose that there exist nonnegative real numbers* $k_1, k_2, k_3, k_4$ *with* $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ *such that*

$$\sqrt[n]{q(fx,fy)} \leq k_1 \sqrt[n]{q(x,y)} + k_2 \sqrt[n]{q(x,fx)} + k_3 \sqrt[n]{q(y,fy)} + k_4 \sqrt[n]{q(x,fy)+q(y,fx)}$$

*for all* $x,y \in X$*. If one of the the following conditions hold:*

*(i)* $f : X \to X$ *is a continuous mapping;*

*(ii)* $q$ *and* $\psi$ *are continuous mappings,*

*then* $f$ *has a unique fixed point. Moreover, for each* $x_0 \in X$*, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$*, converge to a unique fixed point of* $f$.

We can extend condition of $w^0$-distances in Theorem 3.2.5 and 3.2.7 to $w$-distances if we replace the contractive condition (3.2.1) by some strong condition.

**Theorem 3.2.17.** *Let* $(X,d)$ *be a complete metric space ,* $\psi \in \Psi$*,* $q : X \times X \to [0,\infty)$ *be a* $w$*-distance on* $X$ *and ceiling distance on* $d$*, and* $f : X \to X$ *be a mapping. Suppose that there exist* $k_1 \in [0,1)$ *such that*

$$\psi(q(fx,fy)) \leq [\psi(q(x,y))]^{k_1} \tag{3.2.6}$$

*for all* $x,y \in X$*. If one of the the following conditions hold:*

*(i)* $f : X \to X$ *is a continuous mapping;*

*(ii)* $q$ *and* $\psi$ *are continuous mappings,*

*then* $f$ *has a unique fixed point. Moreover, for each* $x_0 \in X$*, the Picard iteration* $\{x_n\}$*, which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$*, converge to a unique fixed point of* $f$*.*

## 3.3 Fixed point results for *w*-generalized *SS*-contraction mappings

In this section, we introduce the new concept of contractility along with $w$-distances in metric spaces and establish a new fixed point theorem for generalized contraction mappings with respect to $w$-distances in complete metric spaces by using the concept of a weak altering distance function. Two illustrative examples are provided to advocate the usability of our results while Banach contraction principle is not applicable. We also give numerical experiments for a fixed point in these examples. First, we introduce the definition of the new type of generalized contraction mappings so called a $w$-generalized $SS$-contraction mappings.

**Definition 3.3.1.** Let $q$ be a $w$-distance on a metric space $(X,d)$. A mapping $f : X \to X$ is said to be a ***w*-generalized *SS*-contraction mapping** if

$$\psi(q(fx, fy)) \leq \phi(q(x,y)), \tag{3.3.1}$$

for all $x, y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a weak altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$.

Now, we give the main result in this section.

**Theorem 3.3.2.** *Let* $(X,d)$ *be a complete metric space and* $q : X \times X \to [0,\infty)$ *be a* $w$*-distance on* $X$ *and a ceiling distance of* $d$*. Suppose that* $f : X \to X$ *is a continuous* $w$*-generalized* $SS$*-contraction mapping. Then* $f$ *has a unique fixed point on* $X$*. Moreover,* *for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

*Proof.* Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions satisfying the condition (3.3.1). Let $x_0$ be an arbitrary point in $X$. Put $x_n = fx_{n-1} = f^n x_0$ for all $n \in \mathbb{N}$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then $x_{n^*}$ is a fixed point of $f$. Thus we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, i.e., $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $q$ is a ceiling distance of $d$, we obtain $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.3.1), we have

$$\begin{aligned} \psi(q(x_n, x_{n+1})) &= \psi(q(fx_{n-1}, fx_n)) \\ &\leq \phi(q(x_{n-1}, x_n)) \\ &< \psi(q(x_{n-1}, x_n)) \end{aligned} \tag{3.3.2}$$

for all $n \in \mathbb{N}$. Since $\psi$ is a nondecreasing function, we have

$$q(x_n, x_{n+1}) < q(x_{n-1}, x_n) \tag{3.3.3}$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{q(x_n, x_{n+1})\}$ is decreasing and bounded below. Therefore, there exists $s \geq 0$ such that

$$q(x_n, x_{n+1}) \to s \quad \text{as} \quad n \to \infty.$$

From (3.3.2), letting $n \to \infty$ and using the property of $\psi$ and $\phi$ we get

$$\psi(s) \leq \liminf_{n\to\infty} \psi(q(x_n, x_{n+1})) \leq \limsup_{n\to\infty} \psi(q(x_n, x_{n+1})) \leq \limsup_{n\to\infty} \phi(q(x_{n-1}, x_n)) \leq \phi(s).$$

Since $\psi(t) > \phi(t)$ for all $t > 0$, we have $s = 0$ and so $\{q(x_n, x_{n+1})\}$ converges to 0. Similarly, it can be show that $\{q(x_{n+1}, x_n)\}$ converges to 0. Now, we will show that $\{x_n\}$ is a Cauchy sequence. Assume this contrary, there is an $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ such that

$$q(x_{m_k}, x_{n_k}) \geq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.3.4}$$

Choosing $n_k$ to be the smallest integer exceeding $m_k$ for which (3.3.4) holds, we obtain that

$$q(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{3.3.5}$$

Using (3.3.4) and (3.3.5), we get

$$\varepsilon \leq q(x_{m_k}, x_{n_k}) \leq q(x_{m_k}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k}) < \varepsilon + q(x_{n_k-1}, x_{n_k}).$$

Hence, $q(x_{m_k}, x_{n_k}) \to \varepsilon$ as $k \to \infty$. Furthermore, we have

$$q(x_{m_k}, x_{n_k}) \leq q(x_{m_k}, x_{m_k-1}) + q(x_{m_k-1}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k}) \tag{3.3.6}$$

and

$$q(x_{m_k-1}, x_{n_k-1}) \leq q(x_{m_k-1}, x_{m_k}) + q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k-1}). \tag{3.3.7}$$

Letting $k \to \infty$ in (3.3.6) and (3.3.7) and using the fact that $\lim_{n\to\infty} q(x_n, x_{n+1}) = 0$, $\lim_{n\to\infty} q(x_{n+1}, x_n) = 0$ and $\lim_{k\to\infty} q(x_{m_k}, x_{n_k}) = \varepsilon$, we have

$$\lim_{k\to\infty} q(x_{m_k-1}, x_{n_k-1}) = \varepsilon.$$

From (3.3.1), we obtain

$$\psi(q(x_{m_k}, x_{n_k})) \leq \phi(q(x_{m_k-1}, x_{n_k-1})). \tag{3.3.8}$$

From (3.3.8), letting $k \to \infty$ and using the property of $\psi$ and $\phi$ we get

$$\psi(\varepsilon) \leq \liminf_{k\to\infty} \psi(q(x_{m_k}, x_{n_k})) \leq \limsup_{k\to\infty} \psi(q(x_{m_k}, x_{n_k})) \leq \limsup_{k\to\infty} \phi(q(x_{m_k-1}, x_{n_k-1})) \leq \phi(\varepsilon).$$

It yields that $\varepsilon = 0$, which is a contradiction. By using Lemma 2.6.8, we can conclude that $\{x_n\}$ is a Cauchy sequence. Since $(X,d)$ is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of $f$, we get $x_{n+1} = fx_n \to fx^*$ as $n \to \infty$, i.e., $x^* = fx^*$. Thus, $f$ has a fixed point. Finally, we claim that $x^*$ is a unique fixed point of $f$. Suppose that $y^* \in X$ is a fixed point of $f$. By (3.3.1), we obtain

$$\psi(q(x^*, y^*)) = \psi(q(fx^*, fy^*)) \leq \phi(q(x^*, y^*))$$

for all $n \in \mathbb{N}$. From the fact that $\psi(t) > \phi(t)$ for all $t > 0$, we get $q(x^*, y^*) = 0$. Similarly, we have $q(x^*, x^*) = 0$. From Lemma 2.6.8, we get $x^* = y^*$. This completes the proof.

□

In the next theorem, we omit the continuity hypothesis of $f$.

**Theorem 3.3.3.** *Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a w-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a w-generalized SS-contraction mapping such that $\phi(0) = 0$. Then $f$ has a unique fixed point on $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converge to a unique fixed point of $f$.*

*Proof.* Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions satisfying the condition (3.3.1). Let $x_0 \in X$ be arbitrary. We define the sequence $\{x_n\}$ by

$$x_n = f^n x_0 = f x_{n-1},$$

for all $n \in \mathbb{N}$. Following the proof of Theorem 3.3.2, we know that $\{x_n\}$ is a Cauchy sequence in $X$. Completeness of $(X,d)$ ensures that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. For each $k \in \mathbb{N}$, there exists $N_k$ such that $m_k > n_k > N_k$ and $q(x_{n_k}, x_{m_k}) < \frac{1}{k}$. Since $q(x,\cdot)$ is lower semicontinuous, we get

$$q(x_{n_k}, x^*) \leq \liminf_{k\to\infty} q(x_{n_k}, x_{m_k}) \leq \frac{1}{k}.$$

It implies that

$$\lim_{k\to\infty} q(x_{n_k}, x^*) = 0. \tag{3.3.9}$$

Setting $x = x_{n_k+1}$ and $y = x^*$ in (3.3.1), we get

$$\psi(q(x_{n_k+1}, fx^*)) \leq \phi(q(x_{n_k}, x^*)). \tag{3.3.10}$$

From (3.3.10), letting $k \to \infty$ and using the property of $\psi$ and $\phi$, we get

$$\begin{aligned} \limsup_{k\to\infty} \psi(q(x_{n_k+1}, fx^*))) &\leq \limsup_{k\to\infty} \phi(q(x_{n_k}, x^*))) \\ &\leq \phi(0) \\ &= 0. \end{aligned}$$

Therefore, $\lim_{k\to\infty} q(x_{n_k+1}, fx^*) = 0$. By the triangle inequality, we get

$$q(x_{n_k}, fx^*) \leq q(x_{n_k}, x_{n_k+1}) + q(x_{n_k+1}, fx^*)$$

and so

$$\lim_{k\to\infty} q(x_{n_k}, fx^*) = 0 \tag{3.3.11}$$

Using Lemma 2.6.8, (3.3.9) and (3.3.11), we conclude that $x^* = fx^*$. Following the proof of Theorem 3.3.2, we know that $x^*$ is a unique fixed point of $f$. This completes the proof. □

Now, we give two examples where it is possible to apply contractive condition (3.3.1) but not Banach contraction principle.

**Example 3.3.4.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by

$$d(x,y) = |x-y|$$

for all $x,y \in X$ . Define a mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$

Then $f$ is not continuous and so $f$ does not satisfy Banach contractive condition. Next, we define two functions $\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \begin{cases} \frac{t^2}{2} & \text{if } 0 \leq t \leq 1 \\ t^2 & \text{if } t > 1 \end{cases}$$

and

$$\phi(t) = \frac{t^2}{4}.$$

From the Figure 3.1, we observe that $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$ and $\phi(0) = 0$.

Also, we define a $w$-distance $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{x,y\}$$

for all $x,y \in X$. It is easy to see that $q$ is a ceiling distance of $d$. Now, we will show that $f$ satisfies the contractive condition (3.3.1) . We will distinguish this claim into the following cases.

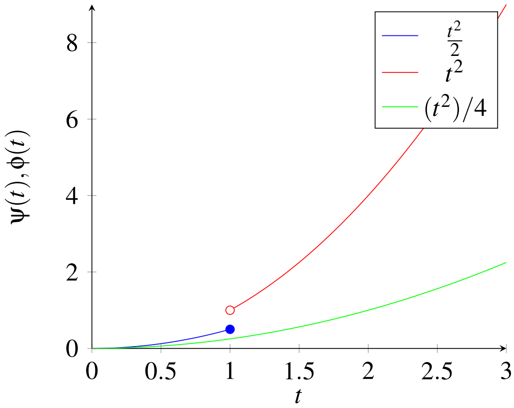


Figure 3.1: Graphs of $\psi$ and $\phi$ in example 3.3.4

Case 1. If $x, y \in (2, \infty)$, then

$$\begin{aligned}\psi(q(fx, fy)) &= \psi\left(\max\left\{\frac{x}{2}, \frac{y}{2}\right\}\right)\\ &= \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\}\\ &= \phi(\max\{x,y\})\\ &= \phi(q(x,y)).\end{aligned}$$

Case 2. If $x, y \in [1, 2]$, then

$$\begin{aligned}\psi(q(fx, fy)) &= \psi\left(\max\left\{\frac{x}{2}, \frac{y}{2}\right\}\right)\\ &= \max\left\{\frac{x^2}{8}, \frac{y^2}{8}\right\}\\ &\leq \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\}\\ &= \phi(\max\{x,y\})\\ &= \phi(q(x,y)).\end{aligned}$$

Case 3. If $x, y \in [0, 1)$, then

$$\begin{aligned}\psi(q(fx, fy)) &= \psi(0)\\ &= 0\\ &\leq \phi(q(x,y)).\end{aligned}$$

Case 4. If $x \in [0,1)$ and $y \in [1,2]$, then

$$\begin{aligned}
\psi(q(fx,fy)) &= \psi\Big(\max\Big\{0,\frac{y}{2}\Big\}\Big) \\
&= \frac{y^2}{8} \\
&\leq \frac{y^2}{4} \\
&= \phi(y) \\
&= \phi(q(x,y)).
\end{aligned}$$

Case5. If $x \in [1,2]$ and $y \in [0,1)$, then the proof is similar to Case 4.

Case 6. If $x \in [0,1)$ and $y \in (2,\infty)$, then

$$\begin{aligned}
\psi(q(fx,fy)) &= \psi\Big(\max\Big\{0,\frac{y}{2}\Big\}\Big) \\
&= \frac{y^2}{4} \\
&= \phi(y) \\
&= \phi(q(x,y)).
\end{aligned}$$

Case 7. If $x \in (2,\infty)$ and $y \in [0,1)$, then the proof is similar to Case 6.

Case 8. If $x \in [1,2]$ and $y \in (2,\infty)$, then

$$\begin{aligned}
\psi(q(fx,fy)) &= \psi\Big(\max\Big\{\frac{x}{2},\frac{y}{2}\Big\}\Big) \\
&= \frac{y^2}{4} \\
&= \phi(y) \\
&= \phi(q(x,y)).
\end{aligned}$$

Case 9. If $x \in (2,\infty)$ and $y \in [1,2]$, then the proof is similar to Case 8.

Therefore, all conditions of Theorem 3.3.3 hold and hence $f$ has a unique fixed point. Here, $x = 0$ is a unique fixed point of $f$.

Some numerical experiments for the unique fixed point of $f$ is given in Figure 3.2. Furthermore, the convergence behavior of these iterations is shown in Figure 3.3.

| | $x_0 = 50$ | $x_0 = 100$ | $x_0 = 150$ | $x_0 = 200$ |
|---|---|---|---|---|
| $x_1$ | 25.000000 | 50.000000 | 75.000000 | 100.000000 |
| $x_2$ | 12.500000 | 25.000000 | 37.500000 | 50.000000 |
| $x_3$ | 6.250000 | 12.500000 | 18.750000 | 25.000000 |
| $x_4$ | 3.125000 | 6.250000 | 9.375000 | 12.500000 |
| $x_5$ | 1.562500 | 3.125000 | 4.687500 | 6.250000 |
| $x_6$ | 0.781250 | 1.562500 | 2.343750 | 3.125000 |
| $x_7$ | 0.000000 | 0.781250 | 1.171875 | 1.562500 |
| $x_8$ | 0.000000 | 0.000000 | 0.585938 | 0.781250 |
| $x_9$ | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| $x_{10}$ | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Figure 3.2: Iterates of Picard iterations

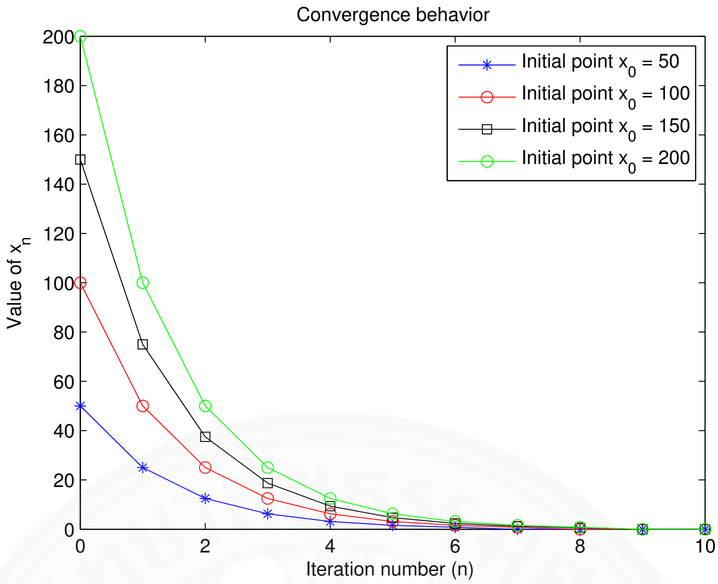


Figure 3.3: The convergence behavior in Example 3.3.4

**Example 3.3.5.** Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by

$$d(x,y) = |x-y|$$

for all $x,y \in X$ . Define a mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{\sin x}{2} & \text{if } 0 \leq x \leq 1 \\ \ln x & \text{if } 1 < x \leq 2 \\ \frac{1}{x} & \text{if } x > 2. \end{cases}$$

Then $f$ is not continuous and so $f$ is not satisfied Banach contractive condition. Next, we define two functions $\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \begin{cases} \frac{t^2}{2} & \text{if } 0 \leq t \leq 1 \\ t^2 & \text{if } t > 1 \end{cases}$$

and

$$\phi(t) = \frac{t^2}{4}.$$

From the Figure 3.4, we observe that $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$ and $\phi(0) = 0$.

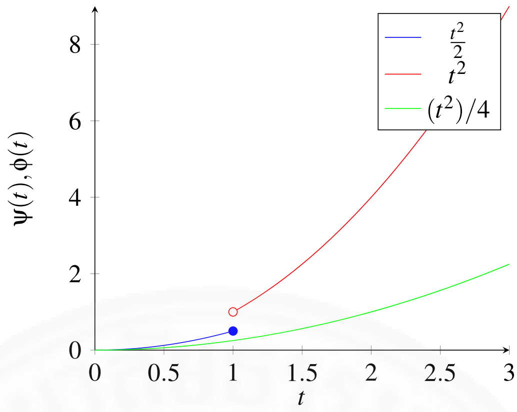


Figure 3.4: Graphs of $\psi$ and $\phi$ in Example 3.3.5

Also, we define a $w$-distance $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{x,y\}$$

for all $x,y \in X$. It easy to see that $q$ is a ceiling distance of $d$. Now we will show that $f$ satisfies the contractive condition (3.3.1) . We will distinguish this claim into the following cases.

Case 1. If $x,y \in [0,1]$, then

$$\begin{aligned}
\psi(q(fx,fy)) &= \psi\Big(\max\Big\{\frac{\sin x}{2},\frac{\sin y}{2}\Big\}\Big) \\
&= \max\Big\{\frac{\sin^2 x}{8},\frac{\sin^2 y}{8}\Big\} \\
&\leq \max\Big\{\frac{x^2}{4},\frac{y^2}{4}\Big\} \\
&= \phi(\max\{x,y\}) \\
&= \phi(q(x,y)).
\end{aligned}$$

Case 2. If $x, y \in (1,2]$, then

$$\begin{aligned}\psi(q(fx,fy)) &= \psi(\max\{\ln x, \ln y\}) \\ &= \max\left\{\frac{(\ln x)^2}{2}, \frac{(\ln y)^2}{2}\right\} \\ &\leq \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\} \\ &= \phi(\max\{x,y\}) \\ &= \phi(q(x,y)).\end{aligned}$$

Case 3. If $x, y \in [2,\infty)$, then

$$\begin{aligned}\psi(q(fx,fy)) &= \psi\left(\max\left\{\frac{1}{y}, \frac{1}{x}\right\}\right) \\ &= \max\left\{\frac{1}{2y^2}, \frac{1}{2x^2}\right\} \\ &\leq \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\} \\ &= \phi(\max\{x,y\}) \\ &= \phi(q(x,y)).\end{aligned}$$

Case 4. If $x \in [0,1]$ and $y \in (1,2]$, then

$$\begin{aligned}\psi(q(fx,fy)) &= \psi\left(\max\left\{\frac{\sin x}{2}, \ln y\right\}\right) \\ &\leq \max\left\{\frac{\sin^2 x}{8}, \frac{(\ln y)^2}{2}\right\} \\ &\leq \max\left\{\frac{x^2}{4}, \frac{y^2}{4}\right\} \\ &\leq \frac{y^2}{4} \\ &= \phi(y) \\ &= \phi(q(x,y)).\end{aligned}$$

Case 5. If $x \in (1,2]$ and $y \in [0,1]$, then the proof is similar to Case 4.

<u>Case 6.</u> If $x \in [0,1]$ and $y \in (2,\infty)$, then

$$\begin{aligned}
\psi(q(fx,fy)) &= \psi\Big(\max\Big\{\frac{\sin x}{2}, \frac{1}{y},\Big\}\Big) \\
&\leq \max\Big\{\frac{\sin^2 x}{8}, \frac{1}{2y^2}\Big\} \\
&\leq \max\Big\{\frac{x^2}{4}, \frac{y^2}{4}\Big\} \\
&\leq \frac{y^2}{4} \\
&= \phi(y) \\
&= \phi(q(x,y)).
\end{aligned}$$

<u>Case 7.</u> If $x \in (2,\infty)$ and $y \in [0,1]$, then the proof is similar to Case 6.

<u>Case 8.</u> If $x \in (1,2]$ and $y \in (2,\infty)$, then

$$\begin{aligned}
\psi(q(fx,fy)) &= \psi\Big(\max\Big\{\ln x, \frac{1}{y}\Big\}\Big) \\
&\leq \max\Big\{\frac{(\ln x)^2}{2}, \frac{1}{2y^2}\Big\} \\
&\leq \max\Big\{\frac{x^2}{4}, \frac{y^2}{4}\Big\} \\
&\leq \frac{y^2}{4} \\
&= \phi(y) \\
&= \phi(q(x,y)).
\end{aligned}$$

<u>Case 9.</u> If $x \in (2,\infty)$ and $y \in (1,2]$, then the proof is similar to Case 8.

Therefore, all conditions of Theorem 3.3.3 hold and hence $f$ has a unique fixed point. Here, $x = 0$ is a unique fixed point of $f$.

Some numerical experiments for the unique fixed point of $f$ is given in Figure 3.5. Furthermore, the convergence behavior of these iterations is shown in Figure 3.6.

| | $x_0 = 0.5$ | $x_0 = 1.5$ | $x_0 = 3$ |
|---|---|---|---|
| $x_1$ | 0.239713 | 0.405465 | 0.333333 |
| $x_2$ | 0.118712 | 0.197223 | 0.163597 |
| $x_3$ | 0.059217 | 0.097974 | 0.081434 |
| $x_4$ | 0.029591 | 0.048908 | 0.040672 |
| $x_5$ | 0.014793 | 0.024444 | 0.020330 |
| $x_6$ | 0.007396 | 0.012221 | 0.010165 |
| $x_7$ | 0.003698 | 0.006110 | 0.005082 |
| $x_8$ | 0.001849 | 0.003055 | 0.002541 |
| $x_9$ | 0.000925 | 0.001528 | 0.001271 |
| $x_{10}$ | 0.000462 | 0.000764 | 0.000635 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Figure 3.5: Iterates of Picard iterations

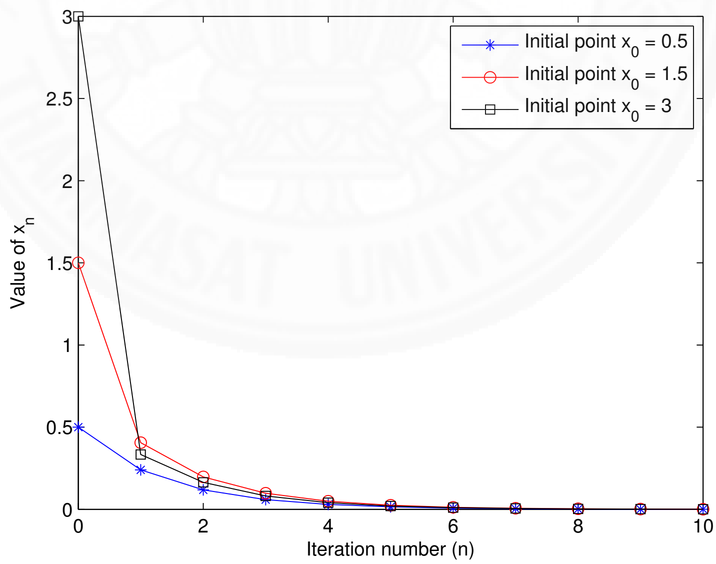


Figure 3.6: The convergence behavior in Example 3.3.5

Taking $q = d$ in Theorems 3.3.2 and 3.3.3, we obtain the following results.

**Corollary 3.3.6.** *Let* $(X,d)$ *be a complete metric space. Suppose that* $f : X \to X$ *is a* *continuous mapping satisfying the following condition:*

$$\psi(d(fx,fy)) \leq \phi(d(x,y)) \tag{3.3.12}$$

*for all* $x,y \in X$, *where* $\psi : [0,\infty) \to [0,\infty)$ *is a weak altering distance function and* $\phi : [0,\infty) \to [0,\infty)$ *is a right upper semicontinuous function such that* $\psi(t) > \phi(t)$ *for all* $t > 0$. *Then* $f$ *has a unique fixed point on* $X$. *Moreover, for each* $x_0 \in X$, *the Picard iteration* $\{x_n\}$, *which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$, *converges to a unique fixed point of* $f$.

**Corollary 3.3.7.** *Let* $(X,d)$ *be a complete metric space. Suppose that* $f : X \to X$ *is a mapping satisfying the following condition:*

$$\psi(d(fx,fy)) \leq \phi(d(x,y)) \tag{3.3.13}$$

*for all* $x,y \in X$, *where* $\psi : [0,\infty) \to [0,\infty)$ *is a weak altering distance function and* $\phi : [0,\infty) \to [0,\infty)$ *is a right upper semicontinuous function such that* $\psi(t) > \phi(t)$ *for all* $t > 0$ *and* $\phi(0) = 0$. *Then* $f$ *has a unique fixed point on* $X$. *Moreover, for each* $x_0 \in X$, *the Picard iteration* $\{x_n\}$, *which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$, *converges to a unique fixed point of* $f$.

## 3.4 Fixed point results for *w*-generalized *Ri*-contraction mappings

In this section, we introduce the new concept of a generalized contraction mapping along with $w$-distances in metric spaces and prove new fixed point theorems for generalized contraction mappings with respect to $w$-distances in complete metric spaces. First, we introduce the definition of the new type of generalized contraction mappings so called a $w$-generalized Ri-contraction mappings.

**Definition 3.4.1.** Let $q$ be a $w$-distance on a metric space $(X,d)$. A mapping $f : X \to X$

is said to be a ***w*-generalized *Ri*-contraction mapping** if

$$q(fx, fy) \leq \varphi(q(x,y)) \tag{3.4.1}$$

for all $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a mapping such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$.

Now, we will create two lemmas for proving the main result in this section.

**Lemma 3.4.2.** *Let $q$ be a w-distance on a metric space $(X,d)$. Suppose that $f : X \to X$ is a w-generalized Ri-contraction mapping. Then $\lim_{n\to\infty} q(f^n x, f^{n+1}x) = 0$ and $\lim_{n\to\infty} q(f^{n+1}x, f^n x) = 0$ for each $x \in X$.*

*Proof.* Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is mapping satisfying the condition (3.4.1). Let $x \in X$ be arbitrary. We define the sequence $\{x_n\} \subset X$ by

$$x_n = f^n x$$

for all $n \in \mathbb{N}$. Set $a_n := q(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$, then $\varphi(a_{n_0}) = 0$ and so

$$\begin{aligned} 0 &\leq a_{n_0+1} \\ &= q(fx_{n_0}, fx_{n_0+1}) \\ &\leq \varphi(a_{n_0}) \\ &= 0. \end{aligned}$$

This implies that $a_{n_0+1} = 0$. By similar process, we obtain $a_n = 0$ for all $n > n_{n_0+1}$ and hence

$$\lim_{n\to\infty} q(f^n x, f^{n+1}x) = 0$$

for all $x \in X$. Now we may suppose that $a_n > 0$ for each $n \in \mathbb{N}$. From condition (3.4.1) and $\varphi(t) < t$ for all $t > 0$, we obtain

$$\begin{aligned} 0 &< a_{n+2} \\ &\leq \varphi(a_{n+1}) \\ &< a_{n+1} \\ &\leq \varphi(a_n) \\ &< a_n \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\{a_n\}$ and $\{\varphi(a_n)\}$ are strictly decreasing and bounded below. It yields that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} \varphi(a_n)$ exist. We assume that $0 < a = \lim_{n\to\infty} a_n$ and $a_n = a + \varepsilon_n$, where $\varepsilon_n > 0$. Note that if $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, then for each sequence $\{t_n\}$ with $t_n \downarrow a^+$ as $n \to \infty$; $\limsup_{t_n \to a^+} \varphi(t_n) < a$. Therefore,

$$\begin{aligned} 0 &< a \\ &= \lim_{n\to\infty} a_{n+1} \\ &\leq \lim_{n\to\infty} \varphi(a_n) \\ &\leq \lim_{n\to\infty} \sup_{s\in(a,a_{n+1})} \varphi(s) \\ &= \lim_{\varepsilon_{n+1}\to 0} \sup_{s\in(a,a+\varepsilon_{n+1})} \varphi(s) \\ &\leq \lim_{\varepsilon\to 0} \sup_{s\in(a,a+\varepsilon)} \varphi(s) \\ &< a. \end{aligned}$$

This is a contradiction. Thus $\lim_{n\to\infty} a_n = 0$, that is, $\lim_{n\to\infty} q(f^n x, f^{n+1}x) = 0$ for each $x \in X$. Similarly, we can conclude that $\lim_{n\to\infty} q(f^{n+1}x, f^n x) = 0$ for each $x \in X$. □

**Lemma 3.4.3.** *Let $q$ be a w-distance on a metric space $(X,d)$. Suppose that $f : X \to X$ is a w-generalized Ri-contraction mapping. Then for each $x \in X$, $\{f^n x\}_{n=0}^{n=\infty}$ is a Cauchy sequence.*

*Proof.* Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is mapping satisfying the condition (3.4.1). We want to proof that $\lim_{m,n\to\infty} q(x_n, x_m) = 0$. Suppose this by contradiction. Assume that $\{x_n = f^n x\}_{n=1}^{n=\infty}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and integers $m_k, n_k \in \mathbb{N}$

such that $m_k > n_k > k$ and $q(x_{n_k}, x_{m_k}) \geq \varepsilon$ for $k = 0, 1, 2, \ldots$. Also, we can choose $m_k$ in order to be assume that $q(x_{n_k}, x_{m_k - 1}) < \varepsilon$. Hence for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\varepsilon &\leq q(x_{n_k}, x_{m_k})\\ &\leq q(x_{n_k}, x_{m_k-1}) + q(x_{m_k-1}, x_{m_k})\\ &\leq \varepsilon + q(x_{m_k-1}, x_{m_k}).\end{aligned}$$

By Lemma 3.4.2, we obtain

$$\lim_{k\to\infty} q(x_{n_k}, x_{m_k}) = \varepsilon.$$

We observe that

$$\begin{aligned}q(x_{n_k}, x_{m_k}) &\leq q(x_{n_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{m_k+1}) + q(x_{m_k+1}, x_{m_k})\\ &\leq q(x_{n_k}, x_{n_k+1}) + \varphi(q(x_{n_k}, x_{m_k})) + q(x_{m_k+1}, x_{m_k}).\end{aligned}$$

Letting $k \to \infty$ and using $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} q(x_{n_k}, x_{m_k})\\ &\leq \lim_{k\to\infty} \varphi(q(x_{n_k}, x_{m_k}))\\ &\leq \lim_{\bar{\varepsilon}\to+0} \sup_{s\in(\varepsilon,\varepsilon+\bar{\varepsilon})} \varphi(s)\\ &< \varepsilon.\end{aligned}$$

This is a contradiction. Hence, $\lim\limits_{m,n\to\infty} q(x_n, x_m) = 0$. From Lemma 2.6.8, we get $\{f^n x\}_{n=1}^{n=\infty}$ is a Cauchy sequence in $X$. □

Next, we give the main result in this section.

**Theorem 3.4.4.** *Let $(X, d)$ be a complete metric space and $q : X \times X \to [0, \infty)$ be a w-distance on $X$. Suppose that $f : X \to X$ is a continuous w-generalized Ri-contraction mapping. Then $f$ has a unique fixed point in $X$. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

*Proof.* Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is mapping satisfying the condition (3.4.1). Let $x \in X$ be an arbitrary point in $X$. From Lemma 3.4.3, we obtain $\{f^n x\}_{n=0}^{n=\infty}$ is a Cauchy sequence. Since $(X,d)$ is a complete metric space, get $\lim\limits_{n\to\infty} f^n x = p$ for some $p \in X$. From the continuity of $f$, we get

$$\begin{aligned} p &= \lim_{n\to\infty} f^{n+1}x \\ &= \lim_{n\to\infty} f(f^n x) \\ &= f\left(\lim_{n\to\infty} f^n x\right) \\ &= f(p) \end{aligned}$$

Thus, $p$ is a fixed point of $f$. Furthermore, from the condition (3.4.1), we obtain

$$q(p,p) = q(fp,fp) \leq \varphi(q(p,p)).$$

It implies that $q(p,p) = 0$. Next, we will show the uniqueness of the fixed point of $T$. Suppose that $u \in X$ is an another fixed point of $f$. From the condition (3.4.1), we obtain

$$q(p,u) = q(fp,fu) \leq \varphi(q(p,u)).$$

This implies that $q(p,u) = 0$. By Lemma 2.6.8, we get $p = u$. Therefore, $f$ has a unique fixed point $p$. This completes the proof. □

Here, we give the well-known lemma about the relation between some conditions of the control function without the proof. (see more details from [3]).

**Lemma 3.4.5.** *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be a function.*

*(♠$_1$) If* $\varphi$ *is right continuous such that* $\varphi(t) < t$ *for all* $t > 0$*, then* $\varphi(0) = 0$.

*(♠$_2$) If* $\varphi$ *is increasing and right continuous, then* $\varphi$ *is upper semi-continuous.*

*(♠$_3$) If* $\varphi$ *is upper semi-continuous from the right such that* $\varphi(t) < t$ *for all* $t > 0$*, then* $\limsup\limits_{s\to t^+} \varphi(s) < t$ *for all* $t > 0$.

By using Theorem 3.4.4 and Lemma 3.4.5, we get the following results.

**Corollary 3.4.6.** *Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a w-distance on $X$. Suppose that $f : X \to X$ is a continuous mapping and $\varphi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function from the right such that $\varphi(0) = 0$, $\varphi(t) < t$ for all $t > 0$ and*

$$q(fx, fy) \leq \varphi(q(x,y)) \tag{3.4.2}$$

*for all $x, y \in X$. Then $f$ has a unique fixed point in $X$. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

**Corollary 3.4.7.** *Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a w-distance on $X$. Suppose that $f : X \to X$ is a continuous mapping and $\varphi : [0,\infty) \to [0,\infty)$ is increasing and right continuous such that $\varphi(t) < t$ for all $t > 0$ and*

$$q(fx, fy) \leq \varphi(q(x,y)) \tag{3.4.3}$$

*for all $x, y \in X$. Then $f$ has a unique fixed point $p$ in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

Taking $q = d$ in Theorem 3.4.4 and Corollaries 3.4.6, 3.4.7, we obtain the following results.

**Corollary 3.4.8** ([23]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a mapping. Suppose that there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$ and*

$$d(fx, fy) \leq \varphi(d(x,y)), \tag{3.4.4}$$

*for all $x, y \in X$. Then $f$ has a unique fixed point $p$ in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.*

**Corollary 3.4.9.** *Let* $(X,d)$ *be a complete metric space. Suppose that* $f : X \to X$ *is a mapping and* $\varphi : [0,\infty) \to [0,\infty)$ *is upper semi-continuous from the right such that* $\varphi(0) = 0$, $\varphi(t) < t$ *for all* $t > 0$ *and*

$$d(fx, fy) \leq \varphi(d(x,y)) \tag{3.4.5}$$

*for all* $x,y \in X$. *Then* $f$ *has a unique fixed point in* $X$. *Moreover, for each* $x \in X$, *the Picard iteration* $\{x_n\}$, *which is defined by* $x_n = f^n x$ *for all* $n \in \mathbb{N}$, *converges to a unique fixed point of* $f$.

**Corollary 3.4.10** ([14])**.** *Let* $(X,d)$ *be a complete metric space. Suppose that* $f : X \to X$ *is a mapping and* $\varphi : [0,\infty) \to [0,\infty)$ *is increasing and right continuous such that* $\varphi(t) < t$ *for all* $t > 0$ *and*

$$d(fx, fy) \leq \varphi(d(x,y)) \tag{3.4.6}$$

*for all* $x,y \in X$. *Then* $f$ *has a unique fixed point* $p$ *in* $X$. *Moreover, for each* $x_0 \in X$, *the Picard iteration* $\{x_n\}$, *which is defined by* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$, *converges to a unique fixed point of* $f$.

# CHAPTER 4

# AN APPLICATION TO NONLINEAR FREDHOLM INTEGRAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS

The aim of this chapter is to present the applications of our theoretical results in the previous chapter for guaranteeing the existence and uniqueness of a solution for various problems which regarded by the following equations:

- nonlinear Fredholm integral equations;
- nonlinear Volterra integral equations.

Throughout this section, let us denote $C[a,b]$, where $a,b \in \mathbb{R}$ with $a<b$, by the set of all continuous functions from $[a,b]$ into $\mathbb{R}$.

## 4.1 Solutions of nonlinear Fredholm and Volterra integral equations arising from $w$-generalized weak contraction mappings

In this section, we prove the existence and uniqueness results of a solution for the nonlinear Fredholm integral equations and nonlinear Volterra integral equations by using Theorem 3.1.5 in the previous chapter.

**Theorem 4.1.1.** *Consider the nonlinear Fredholm integral equation*

$$x(t) = \varphi(t) + \int_a^b K(t,s,x(s))ds \tag{4.1.1}$$

*where $a,b \in \mathbb{R}$ with $a<b$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given mappings. Suppose that the following condition holds:*

*(i) $\varphi$ and $K$ are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

*is a continuous mapping;*

*(iii) there are two functions* $\psi, \phi : [0,\infty) \to [0,\infty)$ *with* $\psi$ *is an altering distance function and* $\phi$ *is a continuous function such that* $\psi(t) < t$ *for all* $t > 0$ *and* $\phi(t) = 0$ *if and only if* $t = 0$*, and for each* $x, y \in C[a,b]$*, we have*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{\Big[\psi(|x(s)| + |y(s)|)\Big] - \left[\phi\left(\sup_{l\in[a,b]}|x(l)| + \sup_{l\in[a,b]}|y(l)|\right)\right] - 2|\varphi(t)|}{b-a}$$

*for all* $t, s \in [a,b]$.

*Then the nonlinear Fredholm integral equation (4.1.1) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$*, the Picard iteration* $\{x_n\}$*, which is defined by*

$$(x_n)(t) = \varphi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$*, converges to a unique solution of the nonlinear Fredholm integral equation (4.1.1).*

*Proof.* Let $X = C[a,b]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

for all $x, y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$ and a ceiling distance of $d$. Here, we will show that $f$ satisfies the contractive condition (3.1.11). Assume that $x, y \in X$ and

$t \in [a,b]$. Then we get

$$\begin{aligned}
|(fx)(t)|+|(fy)(t)| &= \left|\varphi(t)+\int_a^b K(t,s,x(s))ds\right| + \left|\varphi(t)+\int_a^b K(t,s,y(s))ds\right| \\
&\leq |\varphi(t)| + \left|\int_a^b K(t,s,x(s))ds\right| + |\varphi(t)| + \left|\int_a^b K(t,s,y(s))ds\right| \\
&\leq 2|\varphi(t)| + \int_a^b |K(t,s,x(s))|ds + \int_a^b |K(t,s,y(s))|ds \\
&= 2|\varphi(t)| + \int_a^b (|K(t,s,x(s))| + |K(t,s,y(s))|)ds \\
&\leq 2|\varphi(t)| + \int_a^b \left(\frac{[\psi(|x(s)|+|y(s)|)] - \phi(q(x,y)) - 2|\varphi(t)|}{b-a}\right)ds \\
&\leq 2|\varphi(t)| + \frac{1}{b-a}\int_a^b \Big(\psi(q(x,y)) - \phi(q(x,y)) - 2|\varphi(t)|\Big)ds \\
&= \psi(q(x,y)) - \phi(q(x,y)).
\end{aligned}$$

This implies that

$$\sup_{t\in[a,b]} |(fx)(t)| + \sup_{t\in[a,b]} |(fy)(t)| \leq \psi(q(x,y)) - \phi(q(x,y))$$

and so

$$q(fx,fy) \leq \psi(q(x,y)) - \phi(q(x,y))$$

for all $x,y \in X$. Hence we have

$$\psi(q(fx,fy)) \leq q(fx,fy)) \leq \psi(q(x,y)) - \phi(q(x,y))$$

for all $x,y \in X$. It follows that $f$ satisfies the condition (3.1.11). Therefore, all conditions of Theorem 3.1.5 are satisfied and thus $f$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (4.1.1). This completes the proof. □

By using the identical method in the proof of the above theorem, we get the following result.

**Theorem 4.1.2.** *Consider the nonlinear Volterra integral equation*

$$x(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \tag{4.1.2}$$

*where $a,b \in \mathbb{R}$ with $a < b$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given mapping. Suppose that the following conditions hold:*

*(i)* $\varphi$ *and* $K$ *are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } \ x \in C[a,b] \ \text{ and } \ t \in [a,b]$$

*is a continuous mapping;*

*(ii) there are two functions* $\psi, \phi : [0,\infty) \to [0,\infty)$ *with* $\psi$ *is an altering distance function and* $\phi$ *is a continuous function such that* $\psi(t) \leq t$ *for all* $t \geq 0$, $\phi(t) = 0$ *if and only if* $t = 0$, *and for each* $x,y \in C[a,b]$, *we have*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\psi(|x(s)| + |y(s)|)] - \left[\phi\left(\sup_{l\in[a,b]} |x(l)| + \sup_{l\in[a,b]} |y(l)|\right)\right] - 2|\varphi(t)|}{b-a}$$

*for all* $t,s \in [a,b]$.

*Then the nonlinear Volterra integral equation (4.1.2) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$(x_n)(t) = \varphi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear Volterra integral equation (4.1.2).*

## 4.2 Solutions of nonlinear Fredholm and Volterra integral equations arising from $w$-generalized $JS$-contraction mappings

In this section, we prove the existence and uniqueness result of a solution for the nonlinear Fredholm integral equations and nonlinear Volterra integral equations by using Theorem 3.2.17.

**Theorem 4.2.1.** *Consider the nonlinear Fredholm integral equation*

$$x(t) = \phi(t) + \int_a^b K(t,s,x(s))ds \tag{4.2.1}$$

*where $a,b \in \mathbb{R}$ with $a < b$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following conditions hold:*

*(i)* $\phi$ *and* $K$ *are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } \ x \in C[a,b] \ \text{ and } \ t \in [a,b]$$

*is a continuous mapping;*

*(iii) there is* $\psi \in \Psi$ *such that* $\psi(t) < t$ *for all* $t \geq 1$ *and for each* $x,y \in C[a,b]$*, we have*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\psi(|x(t)| + |y(t)|)]^{k_1} - 2|\phi(t)|}{b-a}$$

*for all* $t,s \in [a,b]$*, where* $k_1 \in [0,1)$.

*Then the nonlinear Fredholm integral equation (4.2.1) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$*, the Picard iteration* $\{x_n\}$*, which is defined by*

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$*, converges to a unique solution of the nonlinear Fredholm integral equation (4.2.1).*

*Proof.* Let $X = C[a,b]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|$$

for all $x,y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t \in [a,b]} |x(t)| + \sup_{t \in [a,b]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$ and ceiling distance of $d$. Here, we will show that $f$ is satisfies the condition (3.2.6). Assume that $x, y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}
|(fx)(t)| + |(fy)(t)| &= \left|\phi(t) + \int_a^b K(t,s,x(s))ds\right| + \left|\phi(t) + \int_a^b K(t,s,y(s))ds\right| \\
&\leq |\phi(t)| + \left|\int_a^b K(t,s,x(s))ds\right| + |\phi(t)| + \left|\int_a^b K(t,s,y(s))ds\right| \\
&\leq 2|\phi(t)| + \int_a^b |K(t,s,x(s))|ds + \int_a^b |K(t,s,y(s))|ds \\
&= 2|\phi(t)| + \int_a^b (|K(t,s,x(s))| + |K(t,s,y(s))|)ds \\
&\leq 2|\phi(t)| + \int_a^b \left(\frac{[\psi(|x(t)| + |y(t)|)]^{k_1} - 2|\phi(t)|}{b-a}\right) ds \\
&\leq 2|\phi(t)| + \frac{1}{b-a}\left[\int_a^b [\psi(q(x,y))]^{k_1} ds - \int_a^b 2|\phi(t)|ds\right] \\
&= [\psi(q(x,y))]^{k_1}.
\end{aligned}$$

This implies that $\sup_{t\in[a,b]} |(fx)(t)| + \sup_{t\in[a,b]} |(fy)(t)| \leq [\psi(q(x,y))]^{k_1}$ and so

$$q(fx, fy) \leq [\psi(q(x,y))]^{k_1}$$

for all $x, y \in X$. Hence we have

$$\psi(q(fx, fy)) \leq \psi([\psi(q(x,y))]^{k_1}) \leq [\psi(q(x,y))]^{k_1}$$

for all $x, y \in X$. It follows that $f$ satisfies the condition (3.2.6). Therefore, all condition of Theorem 3.2.17 are satisfied and thus $f$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (4.2.1). This completes the proof. □

By using the identical method in the proof of the above theorem, we get the following result.

**Theorem 4.2.2.** *Consider the nonlinear Volterra integral equation*

$$x(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \tag{4.2.2}$$

*where $a, b \in \mathbb{R}$ with $a < b$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following conditions hold:*

*(i)* $\phi$ *and* $K$ *are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

*is a continuous mapping;*

*(iii) there is* $\psi \in \Psi$ *such that* $\psi(t) < t$ *for all* $t \geq 1$ *and for each* $x,y \in C[a,b]$*, we have*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\psi(|x(t)|+|y(t)|)]^{k_1} - 2|\phi(t)|}{b-a}$$

*for all* $t,s \in [a,b]$*, where* $k_1 \in [0,1)$.

*Then the nonlinear Volterra integral equation (4.2.2) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$*, the Picard iteration* $\{x_n\}$*, which is defined by*

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$*, converges to a unique solution of the nonlinear Volterra integral equation (4.2.2).*

## 4.3 Solutions of nonlinear Fredholm and Volterra integral equations arising from *w*-generalized *SS*-contraction mappings

In this section, we prove the existence and uniqueness result of a solution for the nonlinear Fredholm integral equations and nonlinear Volterra integral equations by using Theorem 3.3.3.

**Theorem 4.3.1.** *Consider the nonlinear Fredholm integral equation*

$$x(t) = \varphi(t) + \int_a^b K(t,s,x(s))ds \tag{4.3.1}$$

*where* $a,b \in \mathbb{R}$ *with* $a < b$, $\varphi : [a,b] \to \mathbb{R}$ *and* $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ *are two given functions. Suppose that the following condition holds:*

*(i)* $\varphi$ *and* $K$ *are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

*is a continuous mapping;*

*(iii) there are two functions* $\psi, \phi : [0,\infty) \to [0,\infty)$ *with* $\psi$ *is a weak altering distance function and* $\phi$ *is a right upper semicontinuous function such that* $\psi(t) > \phi(t)$ *for all* $t > 0$ *and* $\psi(t) < t$ *for all* $t \geq 0$, $\phi(0) = 0$ *and for each* $x, y \in C[a,b]$, *we have*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\phi(\sup_{s\in[a,b]} |x(t)| + \sup_{s\in[a,b]} |y(t)|)] - 2|\varphi(t)|}{b-a}$$

*for all* $t, s \in [a,b]$.

*Then the nonlinear Fredholm integral equation (4.3.1) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$(x_n)(t) = \varphi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear Fredholm integral equation (4.3.1).*

*Proof.* Let $X = C[a,b]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

for all $x, y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$ and a ceiling distance of $d$ . Here, we will show that $f$ satisfies the contractive condition (3.3.1). Assume that $x, y \in X$ and

$t \in [a,b]$. Then we get

$$
\begin{aligned}
|(fx)(t)| + |(fy)(t)| &= \left|\varphi(t) + \int_a^b K(t,s,x(s))ds\right| + \left|\varphi(t) + \int_a^b K(t,s,y(s))ds\right| \\
&\leq |\varphi(t)| + \left|\int_a^b K(t,s,x(s))ds\right| + |\varphi(t)| + \left|\int_a^b K(t,s,y(s))ds\right| \\
&\leq 2|\varphi(t)| + \int_a^b |K(t,s,x(s))|ds + \int_a^b |K(t,s,y(s))|ds \\
&= 2|\varphi(t)| + \int_a^b (|K(t,s,x(s))| + |K(t,s,y(s))|)ds \\
&\leq 2|\varphi(t)| + \int_a^b \Big(\frac{[\phi(\sup_{s\in[a,b]}|x(t)| + \sup_{s\in[a,b]}|y(t)|)] - 2|\varphi(t)|}{b-a}\Big)ds \\
&\leq 2|\varphi(t)| + \frac{1}{b-a}\Big[\int_a^b [\phi(q(x,y))]ds - \int_a^b 2|\varphi(t)|ds\Big] \\
&= \phi(q(x,y)).
\end{aligned}
$$

This implies that $\sup_{t\in[a,b]}|(fx)(t)| + \sup_{t\in[a,b]}|(fy)(t)| \leq \phi(q(x,y))$ and so

$$q(fx,fy) \leq [\phi(q(x,y))]$$

for all $x,y \in X$. Hence we have

$$\psi(q(fx,fy)) \leq \psi(\phi(q(x,y))) \leq \phi(q(x,y))$$

for all $x,y \in X$. It follows that $f$ satisfies the condition (3.3.1). Therefore, all conditions of Theorem 3.3.3 are satisfied and thus $f$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (4.3.1). This completes the proof. □

By using the identical method in the proof of the above theorem, we get the following result.

**Theorem 4.3.2.** *Consider the nonlinear Volterra integral equation*

$$x(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \tag{4.3.2}$$

*where $a,b \in \mathbb{R}$ with $a < b$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following condition holds:*

*(i)* $\varphi$ *and* $K$ *are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

*is a continuous mapping;*

*(iii) there are two functions* $\psi, \phi : [0,\infty) \to [0,\infty)$ *with* $\psi$ *is a weak altering distance function and* $\phi$ *is a right upper semicontinuous function such that* $\psi(t) > \phi(t)$ *for all* $t > 0$ *and* $\psi(t) < t$ *for all* $t \geq 0$, $\phi(0) = 0$ *and for each* $x, y \in C[a,b]$, *we have*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\phi(|x(t)| + |y(t)|)] - 2|\varphi(t)|}{b-a}$$

*for all* $t, s \in [a,b]$.

*Then the nonlinear Volterra integral equation (4.3.2) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$(x_n)(t) = \varphi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear Volterra integral equation (4.3.2).*

## 4.4 Solutions of nonlinear Fredholm and Volterra integral equations arising from *w*-generalized *Ri*-contraction mappings

In this section, we prove the existence and uniqueness results of a solution for the nonlinear Fredholm integral equations and nonlinear Volterra integral equations by using Theorem 3.4.4.

**Theorem 4.4.1.** *Consider the nonlinear Fredholm integral equation*

$$x(t) = \phi(t) + \int_a^b K(t,s,x(s))ds \tag{4.4.1}$$

*where* $a,b \in \mathbb{R}$ *with* $a < b$, $\phi : [a,b] \to \mathbb{R}$ *and* $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ *are given mappings. Suppose that the following condition holds:*

*(i)* $\phi$ *and* $K$ *are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

*is a continuous mapping;*

*(iii) there is a function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(0) = 0$, $\varphi(t) < t$ *and* $\limsup_{s \to t^+} \varphi(s) < t$ *for all* $t > 0$ *and*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{\left[\varphi\left(\sup_{s\in[a,b]} |x(s)| + \sup_{s\in[a,b]} |y(s)|\right)\right] - 2\phi(t)}{b-a}$$

*for all* $x,y \in C[a,b]$ *and for all* $t,s \in [a,b]$.

*Then the nonlinear Fredholm integral equation (4.4.1) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear Fredholm integral equation (4.4.1).*

*Proof.* Let $X = C[a,b]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

for all $x,y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$. Here, we will show that $f$ satisfies the contractive condition (3.4.1). Assume that $x, y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}
|(fx)(t)| + |(fy)(t)| &= \left|\phi(t) + \int_a^b K(t,s,x(s))ds\right| + \left|\phi(t) + \int_a^b K(t,s,y(s))ds\right| \\
&\leq |\phi(t)| + \left|\int_a^b K(t,s,x(s))ds\right| + |\phi(t)| + \left|\int_a^b K(t,s,y(s))ds\right| \\
&\leq 2|\phi(t)| + \int_a^b |K(t,s,x(s))|ds + \int_a^b |K(t,s,y(s))|ds \\
&= 2|\phi(t)| + \int_a^b (|K(t,s,x(s))| + |K(t,s,y(s))|)ds \\
&\leq 2|\phi(t)| + \int_a^b \left( \frac{\left[\varphi\Big(\sup\limits_{s\in[a,b]} |x(s)| + \sup\limits_{s\in[a,b]} |y(s)|\Big)\right] - 2|\phi(t)|}{b-a} \right) ds \\
&= 2|\phi(t)| + \frac{1}{b-a}\left[\int_a^b [\varphi(q(x,y))] - 2|\phi(t)|ds\right] \\
&= \varphi(q(x,y)).
\end{aligned}$$

This implies that $\sup\limits_{t\in[a,b]} |(fx)(t)| + \sup\limits_{t\in[a,b]} |(fy)(t)| \leq \varphi(q(x,y))$ and so

$$q(fx, fy) \leq [\varphi(q(x,y))]$$

for all $x, y \in X$. It follows that $f$ satisfies the condition (3.4.1). Therefore, all conditions of Theorem 3.4.4 are satisfied and thus $f$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (4.4.1). This completes the proof. □

Using the identical method in the proof of the above theorem, we get the following result.

**Theorem 4.4.2.** *Consider the nonlinear Volterra integral equation*

$$x(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \tag{4.4.2}$$

*where $a, b \in \mathbb{R}$ with $a < b$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$. Suppose that the following conditions hold:*

*(i) $\phi$ and $K$ are continuous mappings;*

*(ii) the mapping* $f : C[a,b] \to C[a,b]$ *defined by*

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

*is a continuous mapping;*

*(iii) there is a function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(0) = 0$, $\varphi(t) < t$ *and* $\limsup_{s \to t^+} \varphi(s) < t$ *for all* $t > 0$ *and*

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{\left[\varphi\left(\sup_{s\in[a,b]} |x(s)| + \sup_{s\in[a,b]} |y(s)|\right)\right] - 2\phi(t)}{b-a}$$

*for all* $x, y \in C[a,b]$ *and for all* $t, s \in [a,b]$.

*Then the nonlinear Volterra integral equation (4.4.2) has a unique solution. Moreover, for each* $x_0 \in C[a,b]$*, the Picard iteration* $\{x_n\}$*, which is defined by*

$$(x_n)(t) = \phi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

*for all* $n \in \mathbb{N}$*, converges to a unique solution of the nonlinear Volterra integral equation (4.4.2).*

# CHAPTER 5

# AN APPLICATION TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

The aim of this chapter is to present the application of our theoretical results in Chapter 3 for guaranteeing the existence and uniqueness of a solution for nonlinear fractional differential equations of Caputo type.

Throughout this section, let us denote $C[a,b]$, where $a,b \in \mathbb{R}$ with $a < b$, by the set of all continuous functions from $[a,b]$ into $\mathbb{R}$.

First, let us recall some basic definitions of fractional calculus (see [16, 26]). For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of $g$ order $\beta > 0$ is denote by ${}^{C}D^{\beta}(g(t))$ and it is defined as

$$ {}^{C}D^{\beta}(g(t)) := \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)ds, $$

where $n = [\beta]+1$ such that $[\beta]$ denotes the integer part of the positive real number $\beta$ and $\Gamma$ is a gamma function.

Consider the nonlinear fractional differential equation of Caputo type:

$$ {}^{C}D^{\beta}(x(t)) = f(t,x(t)) \tag{5.0.1} $$

via the integral boundary conditions

$$ x(0) = 0, \quad x(1) = \int_0^{\eta} x(s)ds, $$

where $1 < \beta \leq 2$, $0 < \eta < 1$, $x \in C[0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function (see [2]). It is well-known that if $f$ is continuous, then (5.0.1) is immediately inverted as the very familiar integral equation

$$ \begin{aligned} x(t) \;=\; & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ & -\frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ & +\frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \Big( \int_0^s (s-m)^{\beta-1} f(m,x(m))dm \Big) ds. \end{aligned} \tag{5.0.2} $$

## 5.1 Solutions of nonlinear fractional differential equations arising from *w*-generalized weak contraction mappings

The aim of this section is to present an application of Theorem 3.1.5 for proving the existence and uniqueness of a solution for the nonlinear fractional differential equation of Caputo type.

**Theorem 5.1.1.** *Consider the nonlinear fractional differential equation (5.0.1). Suppose that the following conditions hold:*

*(i) the mapping $T : C[0,1] \to C[0,1]$ defined by*

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

*for all $x \in C[a,b]$ and $t \in [a,b]$,*

*is a continuous mapping;*

*(ii) there are two functions $\psi,\phi : [0,\infty) \to [0,\infty)$ with $\psi$ is an altering distance function and $\phi$ is a continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$, and for each $x,y \in C[0,1]$, we have*

$$|f(s,x(s))|+|f(s,y(s))| \le \frac{\Gamma(\beta+1)}{5}[\psi(|x(s)|+|y(s)|)] - \left[\phi\left(\sup_{t\in[0,1]}|x(t)| + \sup_{t\in[0,1]}|y(t)|\right)\right]$$

*for all $s \in [0,1]$.*

*Then the nonlinear fractional differential equation of Caputo type (5.0.1) has a unique* *solution. Moreover, for each $x_0 \in C[0,1]$, the Picard iteration $\{x_n\}$, which is defined by*

$$
\begin{aligned}
(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, x_{n-1}(s))ds \\
&\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s, x_{n-1}(s))ds \\
&\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m, x_{n-1}(m))dm\Big)ds
\end{aligned}
$$

*for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1).*

*Proof.* Let $X = C[0,1]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|$$

for all $x, y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$ and a ceiling distance of $d$ . Here, we will show that $T$ satisfies the contractive condition (3.1.11). Assume that $x, y \in X$ and $t \in [0,1]$. Then we get

$$
\begin{aligned}
&|(Tx)(t)|+|(Ty)(t)| \\
&= \Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,y(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,y(m))dm\Big)ds\Big| \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}(|f(s,x(s))|+|f(s,y(s))|)ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}(|f(s,x(s))|+|f(s,y(s))|)ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big|\int_0^s (s-m)^{\beta-1}(f(m,x(m))+f(m,y(m)))dm\Big|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\frac{\Gamma(\beta+1)}{5}([\psi(|x(s)|+|y(s)|)]-\phi(q(x,y)))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\frac{\Gamma(\beta+1)}{5}([\psi(|x(s)|+|y(s)|)]-\phi(q(x,y)))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big|\int_0^s (s-m)^{\beta-1}\frac{\Gamma(\beta+1)}{5}([\psi(|x(s)|+|y(s)|)] \\
&\qquad -\phi(q(x,y)))dm\Big|ds \\
&\le \frac{\Gamma(\beta+1)}{5}([\psi(q(x,y))]-\phi(q(x,y))) \\
&\quad \times \sup_{t\in(0,1)}\Big(\frac{1}{\Gamma(\beta)}\int_0^1 |t-s|^{\beta-1}ds \\
&\qquad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}ds+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s |s-m|^{\beta-1}dmds\Big) \\
&\le \psi(q(x,y))-\phi(q(x,y)).
\end{aligned}
$$

This implies that

$$\sup_{t\in[a,b]}|(Tx)(t)|+\sup_{t\in[a,b]}|(Ty)(t)|\le\psi(q(x,y))-\phi(q(x,y))$$

and so

$$q(Tx,Ty)\le\psi(q(x,y))-\phi(q(x,y))$$

for all $x, y \in X$. Hence we have

$$\psi(q(Tx,Ty)) \leq q(Tx,Ty) \leq \psi(q(x,y)) - \phi(q(x,y))$$

for all $x, y \in X$. It follows that $T$ satisfies the condition (3.1.11). Therefore, all conditions of Theorem 3.1.5 are satisfied and thus $T$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1). This completes the proof. □

## 5.2 Solutions of nonlinear fractional differential equations arising from *w*-generalized *JS*-contraction mappings

The aim of this section is to prove the existence and uniqueness result of solutions for the nonlinear fractional differential equations of Caputo type by using Theorem 3.2.17.

**Theorem 5.2.1.** *Consider the nonlinear fractional differential equation (5.0.1). Suppose that the following conditions hold:*

*(i) the mapping $T : C[0,1] \to C[0,1]$ defined by*

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

*for all $x \in C[a,b]$ and $t \in [a,b]$,*

*is a continuous mapping;*

*(ii) there is $\psi \in \Psi$ such that $\psi(t) < t$ for all $t \geq 1$ and for each $x, y \in C[0,1]$, we have*

$$|f(s,x(s))| + |f(s,y(s))| \leq \frac{\Gamma(\beta+1)}{5}\left[\psi\Big(|x(s)| + |y(s)|\Big)\right]^{k_1}$$

*for all $s \in [0,1]$, where $k_1 \in [0,1)$.*

*Then the nonlinear fractional differential equation of Caputo type (5.0.1) has a unique solution. Moreover, for each* $x_0 \in C[0,1]$*, the Picard iteration* $\{x_n\}$*, which is defined by*

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

*for all* $n \in \mathbb{N}$*, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1).*

*Proof.* Let $X = C[0,1]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|$$

for all $x,y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |y(t)|$$

for all $x,y \in X$. Clearly, $q$ is a $w$-distance on $X$. Here, we will show that $T$ satisfies the contractive condition (3.2.6). Assume that $x,y \in X$ and $t \in [0,1]$. Then we get

$$
\begin{aligned}
&|(Tx)(t)|+|(Ty)(t)| \\
&= \Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,y(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,y(m))dm\Big)ds\Big| \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}(|f(s,x(s))|+|f(s,y(s))|)ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}(|f(s,x(s))|+|f(s,y(s))|)ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big|\int_0^s (s-m)^{\beta-1}(f(m,x(m))+f(m,y(m)))dm\Big|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left[\psi\left(|x(s)|+|y(s)|\right)\right]^{k_1} ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left[\psi\left(|x(s)|+|y(s)|\right)\right]^{k_1} ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Bigg|\int_0^s (s-m)^{\beta-1}\frac{\Gamma(\beta+1)}{5} \\
&\qquad \times\left[\psi\left(|x(s)|+|y(s)|\right)\right]^{k_1} dm\Bigg|ds \\
&\le \frac{\Gamma(\beta+1)}{5}[\psi(q(x,y))]^{k_1} \\
&\quad \times \sup_{t\in(0,1)}\Big(\frac{1}{\Gamma(\beta)}\int_0^1 |t-s|^{\beta-1}ds \\
&\qquad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}ds+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s |s-m|^{\beta-1}dmds\Big) \\
&\le [\psi(q(x,y))]^{k_1}.
\end{aligned}
$$

This implies that $\sup_{t\in[a,b]}|(Tx)(t)|+\sup_{t\in[a,b]}|(Ty)(t)|\le[\psi(q(x,y))]^{k_1}$ and so

$$\psi(q(Tx,Ty))\le\psi([\psi(q(x,y))]^{k_1})\le[\psi(q(x,y))]^{k_1}$$

for all $x, y \in X$. It follows that $T$ satisfies the condition (3.2.6). Therefore, all conditions of Theorem 3.2.17 are satisfied and thus $T$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1). This completes the proof. □

## 5.3 Solutions of nonlinear fractional differential equations arising from *w*-generalized *SS*-contraction mappings

The aim of this section is to present an application of Theorem 3.3.3 for proving the existence and uniqueness of a solution for the nonlinear fractional differential equation of Caputo type.

**Theorem 5.3.1.** *Consider the nonlinear fractional differential equation (5.0.1). Suppose that the following conditions hold:*

*(i) the mapping $T : C[0,1] \to C[0,1]$ defined by*

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

*for all $x \in C[a,b]$ and $t \in [a,b]$,*

*is a continuous mapping;*

*(ii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi(t) < t$ for all $t \geq 0$, $\phi(0) = 0$ and for each $x, y \in C[0,1]$, we have*

$$|f(s,x(s))| + |f(s,y(s))| \leq \frac{\Gamma(\beta+1)}{5}\left[\phi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]$$

*for all* $s \in [0,1]$.

*Then the nonlinear fractional differential equation of Caputo type (5.0.1) has a unique solution. Moreover, for each* $x_0 \in C[0,1]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

*for all* $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1).*

*Proof.* Let $X = C[0,1]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|$$

for all $x, y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$. Here, we will show that $T$ satisfies the contractive condition (3.3.1). Assume that $x, y \in X$ and $t \in [0,1]$. Then we get

$$
\begin{aligned}
&|(Tx)(t)|+|(Ty)(t)| \\
&= \Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,y(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,y(m))dm\Big)ds\Big| \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}(|f(s,x(s))|+|f(s,y(s))|)ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}(|f(s,x(s))|+|f(s,y(s))|)ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big|\int_0^s (s-m)^{\beta-1}(f(m,x(m))+f(m,y(m)))dm\Big|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left[\phi\left(\sup_{s\in[0,1]}|x(s)|+\sup_{s\in[0,1]}|y(s)|\right)\right]ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left[\phi\left(\sup_{s\in[0,1]}|x(s)|+\sup_{s\in[0,1]}|y(s)|\right)\right]ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Bigg|\int_0^s (s-m)^{\beta-1}\frac{\Gamma(\beta+1)}{5} \\
&\qquad \times\left[\phi\left(\sup_{s\in[0,1]}|x(s)|+\sup_{s\in[0,1]}|y(s)|\right)\right]dm\Bigg|ds \\
&\le \frac{\Gamma(\beta+1)}{5}[\phi(q(x,y))] \\
&\quad \times\sup_{t\in(0,1)}\Big(\frac{1}{\Gamma(\beta)}\int_0^1 |t-s|^{\beta-1}ds \\
&\qquad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}ds+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s |s-m|^{\beta-1}dmds\Big) \\
&\le \phi(q(x,y)).
\end{aligned}
$$

This implies that $\sup_{t\in[a,b]}|(Tx)(t)|+\sup_{t\in[a,b]}|(Ty)(t)|\le\phi(q(x,y))$ and so

$$\psi(q(Tx,Ty))\le\psi(\phi(q(x,y)))\le\phi(q(x,y))$$

for all $x, y \in X$. It follows that $T$ satisfies the condition (3.3.1). Therefore, all conditions of Theorem 3.3.3 are satisfied and thus $T$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1). This completes the proof. □

## 5.4 Solutions of nonlinear fractional differential equations arising from *w*-generalized *Ri*-contraction mappings

The aim of this section is to prove the existence and uniqueness result of solutions for the nonlinear fractional differential equations of Caputo type by using Theorem 3.4.4.

**Theorem 5.4.1.** *Consider the nonlinear fractional differential equation (5.0.1). Suppose that the following conditions hold:*

*(i) the mapping $T : C[0,1] \to C[0,1]$ defined by*

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

*for all $x \in C[a,b]$ and $t \in [a,b]$,*

*is a continuous mapping;*

*(ii) there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, and for each $x, y \in C[0,1]$, we have*

$$|f(s,x(s))| + |f(s,y(s))| \le \frac{\Gamma(\beta+1)}{5}\left[\varphi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]$$

*for all $s \in [0,1]$.*

*Then the nonlinear fractional differential equation of Caputo type (5.0.1) has a unique solution. Moreover, for each* $x_0 \in C[0,1]$*, the Picard iteration* $\{x_n\}$*, which is defined by*

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

*for all* $n \in \mathbb{N}$*, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1).*

*Proof.* Let $X = C[0,1]$. Clearly, $X$ with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|$$

for all $x, y \in X$, is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |y(t)|$$

for all $x, y \in X$. Clearly, $q$ is a $w$-distance on $X$. Here, we will show that $T$ satisfies the contractive condition (3.4.1). Assume that $x, y \in X$ and $t \in [0,1]$. Then we get

$$
\begin{aligned}
&|(Tx)(t)| + |(Ty)(t)| \\
&= \Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\Big| \\
&\quad +\Big|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,y(s))ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,y(m))dm\Big)ds\Big| \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1} (|f(s,x(s))| + |f(s,y(s))|)ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} (|f(s,x(s))| + |f(s,y(s))|)ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big|\int_0^s (s-m)^{\beta-1} (f(m,x(m)) + f(m,y(m)))dm\Big|ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1} \frac{\Gamma(\beta+1)}{5}\left[\varphi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]ds \\
&\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} \frac{\Gamma(\beta+1)}{5}\left[\varphi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]ds \\
&\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Bigg|\int_0^s (s-m)^{\beta-1} \frac{\Gamma(\beta+1)}{5} \\
&\quad\quad \times\left[\varphi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]dm\Bigg|ds \\
&\le \frac{\Gamma(\beta+1)}{5}[\varphi(q(x,y))] \\
&\quad \times \sup_{t\in(0,1)}\Big(\frac{1}{\Gamma(\beta)}\int_0^1 |t-s|^{\beta-1}ds \\
&\quad\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s |s-m|^{\beta-1}dmds\Big) \\
&\le \varphi(q(x,y)).
\end{aligned}
$$

This implies that $\sup_{t\in[a,b]}|(Tx)(t)| + \sup_{t\in[a,b]}|(Ty)(t)| \le \varphi(q(x,y))$ and so

$$q(Tx,Ty) \le \varphi(q(x,y))$$

for all $x, y \in X$. It follows that $T$ satisfies the condition (3.4.1). Therefore, all conditions of Theorem 3.4.4 are satisfied and thus $T$ has a unique fixed point. This implies that there exists a unique solution of the nonlinear fractional differential equation of Caputo type (5.0.1). This completes the proof. □

# CHAPTER 6

# CONCLUSIONS AND OPEN PROBLEMS

In this chapter, we will summarize all of the results of this thesis and give some open problems for further verification.

## 6.1 Conclusions

In Chapter 3, we introduce the concept of a ceiling distance and establish the existence and uniqueness of fixed point results for some new contraction mappings in complete metric spaces as follows:

$(A1)$ Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a $w^0$-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a continuous $w$-generalized weak contraction mapping. Then $f$ has a unique fixed point in $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.

$(A2)$ Let $(X,d)$ be a complete metric space , $\psi \in \Psi$, $q : X \times X \to [0,\infty)$ be a $w^0$-distance on $X$ and ceiling distance of $d$, and $f : X \to X$ be a mapping. Suppose that $f$ is a $w$-generalized $JS$-contraction mapping. If $f$ is continuous, then $f$ has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converge to a unique fixed point of $f$.

$(A3)$ Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a $w$-distance on $X$ and a ceiling distance of $d$. Suppose that $f : X \to X$ is a continuous $w$-generalized $SS$-contraction mapping. Then $f$ has a unique fixed point on $X$. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.

$(A4)$ Let $(X,d)$ be a complete metric space and $q : X \times X \to [0,\infty)$ be a $w$-distance on $X$.

Suppose that $f : X \to X$ is a continuous $w$-generalized $Ri$-contraction mapping. Then $f$ has a unique fixed point in $X$. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of $f$.

In Chapter 4, we apply fixed point results for proving the existence and uniqueness results of a solution for the nonlinear Fredholm integral equations and nonlinear Volterra integral equations as follows:

$(B1)$ Consider the nonlinear Fredholm integral equation

$$x(t) = \varphi(t) + \int_a^b K(t,s,x(s))ds \tag{6.1.1}$$

where $a,b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given mappings. Suppose that the following condition holds:

(i) $\varphi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is an altering distance function and $\phi$ is a continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$, and for each $x,y \in C[a,b]$, we have

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{\Big[\psi(|x(s)| + |y(s)|)\Big] - \left[\phi\left(\sup_{l\in[a,b]}|x(l)| + \sup_{l\in[a,b]}|y(l)|\right)\right] - 2|\varphi(t)|}{b-a}$$

for all $t,s \in [a,b]$.

Then the nonlinear integral equation (6.1.1) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \varphi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear integral equation (6.1.1).

$(B2)$ Consider the nonlinear Volterra integral equation

$$x(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \tag{6.1.2}$$

where $a,b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given mapping. Suppose that the following conditions hold:

(i) $\varphi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is an altering distance function and $\phi$ is a continuous function such that $\psi(t) \le t$ for all $t \ge 0$, $\phi(t) = 0$ if and only if $t = 0$, and for each $x,y \in C[a,b]$, we have

$$|K(t,s,x(s))| + |K(t,s,y(s))| \le \frac{[\psi(|x(s)| + |y(s)|)] - \left[\phi\left(\sup_{l\in[a,b]} |x(l)| + \sup_{l\in[a,b]} |y(l)|\right)\right] - 2|\varphi(t)|}{b-a}$$

for all $t,s \in [a,b]$.

Then the nonlinear integral equation (6.1.2) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \varphi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear integral equation (6.1.2).

$(B3)$ Consider the nonlinear Fredholm integral equation

$$x(t) = \phi(t) + \int_a^b K(t,s,x(s))ds \tag{6.1.3}$$

where $a,b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following conditions hold:

(i) $\phi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there is $\psi \in \Psi$ such that $\psi(t) < t$ for all $t \geq 1$ and for each $x, y \in C[a,b]$, we have

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\psi(|x(t)| + |y(t)|)]^{k_1} - 2|\phi(t)|}{b-a}$$

for all $t, s \in [a,b]$, where $k_1 \in [0,1)$.

Then the nonlinear Fredholm integral equation (6.1.3) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear Fredholm integral equation (6.1.3).

$(B4)$ Consider the nonlinear Volterra integral equation

$$x(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \tag{6.1.4}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following conditions hold:

(i) $\phi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there is $\psi \in \Psi$ such that $\psi(t) < t$ for all $t \geq 1$ and for each $x, y \in C[a,b]$, we have

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\psi(|x(t)| + |y(t)|)]^{k_1} - 2|\phi(t)|}{b-a}$$

for all $t, s \in [a,b]$, where $k_1 \in [0,1)$.

Then the nonlinear Volterra integral equation (6.1.4) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear Volterra integral equation (6.1.4).

$(B5)$ Consider the nonlinear Fredholm integral equation

$$x(t) = \varphi(t) + \int_a^b K(t,s,x(s))ds \tag{6.1.5}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following condition holds:

(i) $\varphi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi(t) < t$ for all $t \geq 0$, $\phi(0) = 0$ and for each $x, y \in C[a,b]$, we have

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\phi(|x(t)| + |y(t)|)] - 2|\varphi(t)|}{b-a}$$

for all $t, s \in [a,b]$.

Then the nonlinear Fredholm integral equation (6.1.5) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \varphi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear Fredholm integral equation (6.1.5).

$(B6)$ Consider the nonlinear Volterra integral equation

$$x(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \tag{6.1.6}$$

where $a,b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are two given functions. Suppose that the following condition holds:

(i) $\varphi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \varphi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi(t) < t$ for all $t \geq 0$, $\phi(0) = 0$ and for each $x,y \in C[a,b]$, we have

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{[\phi(|x(t)| + |y(t)|)] - 2|\varphi(t)|}{b-a}$$

for all $t,s \in [a,b]$.

Then the nonlinear Volterra integral equation (6.1.6) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \varphi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear Volterra integral equation (6.1.6).

$(B7)$ Consider the nonlinear Fredholm integral equation

$$x(t) = \phi(t) + \int_a^b K(t,s,x(s))ds \tag{6.1.7}$$

where $a,b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given mappings. Suppose that the following condition holds:

(i) $\phi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$ and

$$|K(t,s,x(s))| + |K(t,s,y(s))| \le \frac{\left[\varphi\left(\sup_{s\in[a,b]} |x(s)| + \sup_{s\in[a,b]} |y(s)|\right)\right] - 2\phi(t)}{b-a}$$

for all $x,y \in C[a,b]$ and for all $t,s \in [a,b]$.

Then the nonlinear integral equation (6.1.7) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \phi(t) + \int_a^b K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear integral equation (6.1.7).

$(B8)$ Consider the nonlinear Volterra integral equation

$$x(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \tag{6.1.8}$$

where $a,b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$. Suppose that the following conditions hold:

(i) $\phi$ and $K$ are continuous mappings;

(ii) the mapping $f : C[a,b] \to C[a,b]$ defined by

$$(fx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(iii) there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ and

$$|K(t,s,x(s))| + |K(t,s,y(s))| \leq \frac{\left[\varphi\left(\sup_{s\in[a,b]} |x(s)| + \sup_{s\in[a,b]} |y(s)|\right)\right] - 2\phi(t)}{b-a}$$

for all $x,y \in C[a,b]$ and for all $t,s \in [a,b]$.

Then the nonlinear integral equation (6.1.8) has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by

$$(x_n)(t) = \phi(t) + \int_a^t K(t,s,x_{n-1}(s))ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear integral equation (6.1.8).

In Chapter 5, we apply fixed point results for proving the existence and uniqueness results of a solution for the following nonlinear fractional differential equations of Caputo type

$$^{C}D^{\beta}(x(t)) = f(t,x(t)) \tag{6.1.9}$$

via the integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^{\eta} x(s)ds,$$

where $1 < \beta \leq 2$, $0 < \eta < 1$, $x \in C[0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

$(C1)$ Consider the nonlinear fractional differential equation of Caputo type (6.1.9). Suppose that the following conditions hold:

(i) the mapping $T : C[0,1] \to C[0,1]$ defined by

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

for all $x \in C[a,b]$ and $t \in [a,b]$,

is a continuous mapping;

(ii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is an altering distance function and $\phi$ is a continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$, and for each $x, y \in C[0,1]$, we have

$$|f(s,x(s))| + |f(s,y(s))| \leq \frac{\Gamma(\beta+1)}{5}[\psi(|x(s)|+|y(s)|)] - \left[\phi\left(\sup_{t\in[0,1]}|x(t)| + \sup_{t\in[0,1]}|y(t)|\right)\right]$$

for all $s \in [0,1]$.

Then the nonlinear fractional differential equation of Caputo type (6.1.9) has a unique solution. Moreover, for each $x_0 \in C[0,1]$, the Picard iteration $\{x_n\}$, which is defined by

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (6.1.9).

$(C2)$ Consider the nonlinear fractional differential equation of Caputo type (6.1.9). Suppose that the following conditions hold:

(i) the mapping $T : C[0,1] \to C[0,1]$ defined by

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

for all $x \in C[a,b]$ and $t \in [a,b]$,

is a continuous mapping;

(ii) there is $\psi \in \Psi$ such that $\psi(t) < t$ for all $t \geq 1$ and for each $x, y \in C[0,1]$, we have

$$|f(s,x(s))| + |f(s,y(s))| \leq \frac{\Gamma(\beta+1)}{5}\left[\psi\bigg(|x(s)| + |y(s)|\bigg)\right]^{k_1}$$

for all $s \in [0,1]$, where $k_1 \in [0,1)$.

Then the nonlinear fractional differential equation of Caputo type (6.1.9) has a unique solution. Moreover, for each $x_0 \in C[0,1]$, the Picard iteration $\{x_n\}$, which is defined by

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (6.1.9).

$(C3)$ Consider the nonlinear fractional differential equation of Caputo type (6.1.9). Suppose that the following conditions hold:

(i) the mapping $T : C[0,1] \to C[0,1]$ defined by

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

for all $x \in C[a,b]$ and $t \in [a,b]$,

is a continuous mapping;

(ii) there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$ and $\psi(t) < t$ for all $t \geq 0$, $\phi(0) = 0$ and for each $x, y \in C[0,1]$, we have

$$|f(s,x(s))| + |f(s,y(s))| \leq \frac{\Gamma(\beta+1)}{5}\left[\phi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]$$

for all $s \in [0,1]$.

Then the nonlinear fractional differential equation of Caputo type (6.1.9) has a unique solution. Moreover, for each $x_0 \in C[0,1]$, the Picard iteration $\{x_n\}$, which is defined by

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (6.1.9).

$(C4)$ Consider the nonlinear fractional differential equation of Caputo type (6.1.9). Suppose that the following conditions hold:

(i) the mapping $T : C[0,1] \to C[0,1]$ defined by

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm\Big)ds\end{aligned}$$

for all $x \in C[a,b]$ and $t \in [a,b]$,

is a continuous mapping;

(ii) there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s\to t^+}\varphi(s) < t$ for all $t > 0$, and for each $x, y \in C[0,1]$, we have

$$|f(s,x(s))| + |f(s,y(s))| \leq \frac{\Gamma(\beta+1)}{5}\left[\varphi\left(\sup_{s\in[0,1]}|x(s)| + \sup_{s\in[0,1]}|y(s)|\right)\right]$$

for all $s \in [0,1]$.

Then the nonlinear fractional differential equation of Caputo type (6.1.9) has a unique solution. Moreover, for each $x_0 \in C[0,1]$, the Picard iteration $\{x_n\}$, which is defined by

$$\begin{aligned}(x_n)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad -\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,x_{n-1}(s))ds \\ &\quad +\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \Big(\int_0^s (s-m)^{\beta-1} f(m,x_{n-1}(m))dm\Big)ds\end{aligned}$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (6.1.9).

## 6.2 Open problems

In this section, we have the following problems in the further investigation.

- Can we extend the condition of $w^0$-distances in some results of this thesis to $w$-distances and prove the existence and uniqueness of a fixed point?

- Can the main idea in this thesis be used to create new research for other nonlinear mappings?
- Can the main idea in this thesis be used to create new research in generalized metric space?
- Can the main idea in this thesis be used to study via other distances such as $\tau$-distances, $\Omega$-distances etc?

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