PEXIDERIZED VERSIONS OF FUNCTIONAL EQUATIONS ARISING FROM THE DETERMINANT AND PERMANENT OF CERTAIN MATRICES

BY

MR. WUTTICHAI SURIYACHAROEN

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
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ENTITLED

PEXIDERIZED VERSIONS OF FUNCTIONAL EQUATIONS ARISING FROM THE DETERMINANT AND PERMANENT OF CERTAIN MATRICES

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ABSTRACT

General solution functions $f, g, h, \ell, n : \mathbb{R}^3 \to \mathbb{R}$ of the following pexiderized functional equations

$$f(ux+vy, uy+vx, wz) = g(x,y,z) + h(u,v,w) + \ell(x,y,z)n(u,v,w),$$

$$f(ux-vy, uy-vx, wz) = g(x,y,z) + h(u,v,w) + \ell(x,y,z)n(u,v,w),$$

$$f(ux+vy, uy-vx, wz) = f(x,y,z) + f(v,u,w) + g(x,y,z)g(u,v,w)$$

are determined without any regularity assumptions on the unknown functions. These equations arise from identities satified by the determinant and permanent of certain symmetric matrices.

Keywords: Determinant, permanent, multiplicative functions, logarithmic functions, functional equations.

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Wuttichai Suriyacharoen

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CHAPTER 1

INTRODUCTION

We first introduce the notion of functional equations and some definitions related to functional equations. We then give some basic examples of functional equations including some basic concepts for solving each example.

1.1 Functional Equations

The study of functional equations originated more than 260 years ago. Functional equations arise in many fields of applied science and engineering, such as Mechanics, Geometry, Statistics, Hydraulics, and Economics [1], [3]. From 1747 to 1750, J. d'Alembert started published three papers on functional equations. These three papers were regarded as first papers on functional equations. Functional equations were studied by many celebreted mathematicians including d'Alembert (1747), Euler (1768), Poisson (1804), Cauchy (1821), Abel (1823), Darboux (1875). The field of functional equations includes differential equations, difference equations and iterations, and integral equations. Functional equations are equations in which the unknowns variables are functions [3].

Example 1.1. The following equations are typical examples of functional equations.

1. Cauchy's Equation

$$f(x+y) = f(x) + f(y) \qquad (x, y \in \mathbb{R}).$$

This equation involves only one unknown function.

2. Pexider's Equation

$$f(x+y) = g(x) + h(y) \qquad (x, y \in \mathbb{R}).$$

This is an equation of three unknown functions.

3. Homogeneous Equation

$$f(zx, zy) = z^n f(x, y) \qquad (x, y \in \mathbb{R}_{\geq 0},\ n \in \mathbb{R}_+).$$

Notice that the unknown function f depends on several variables.

4. Transformation Equation

$$f(f(x,y), z) = f(x, g(y,z)) \qquad (x, y, z \in \mathbb{R}).$$

This is an example of an equation with several functions of several variables.

$\square$

Solving a functional equation means to find all functions satisfying the functional equation. Unlike the field of differential equations, where a clear methodology to solve them exists, in functional equations such a methodology does not exist. The main operation is the substitution of known or unknown functions into known or unknown functions, for see example [2].

Example 1.2. Let a, b be fixed non-zero real numbers. Find all function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x - \frac{b}{a}\right) + 2x \ \leq\ \frac{a}{b}x^2 + \frac{2b}{a} \ \leq\ f\left(x + \frac{b}{a}\right) - 2x \qquad (x \in \mathbb{R}). \tag{1.1}$$

Solution. Putting $y = x - \dfrac{b}{a}$ in (1.1). The left inequality becomes

$$f(y) \leq \frac{a}{b}y^2 + \frac{b}{a}.$$

Similarly, writting $y = x + \dfrac{b}{a}$ in (1.1). The right inequality becomes

$$f(y) \geq \frac{a}{b}y^2 + \frac{b}{a}.$$

Hence,

$$f(y) = \frac{a}{b}y^2 + \frac{b}{a} \qquad (y \in \mathbb{R}).$$

It is easily verified that this solution function satisfies (1.1). $\square$

Example 1.3. Find all function $f : \mathbb{R} \to \mathbb{R}$ such that

$$2f(x+y) + 6y^3 = f(x+2y) + x^3 \qquad (x, y \in \mathbb{R}). \tag{1.2}$$

Solution. Putting $y = 0$ in (1.2), we get

$$2f(x) = f(x) + x^3.$$

Thus,

$$f(x) = x^3 \qquad (x \in \mathbb{R}).$$

In the other hand, for $f(x) = x^3$, the left hand of (1.2) equals

$$2x^3 + 6x^2y + 6xy^2 + 8y^3,$$

while the right hand side equals

$$2x^3 + 6x^2y + 12xy^2 + 8y^3.$$

This implies that the functional equation (1.2) has no solution. $\square$

The previous problem points out that, when we solve any functional equations, we must check that the obtained functions indeed satisfy the given functional equation.

Example 1.4. Determine all function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + 2f(x-y) + f(x) + 2f(y) = 4x + y \qquad (x, y \in \mathbb{R}). \tag{1.3}$$

Solution. Putting $y = 0$ in (1.3), we have

$$4f(x) + 2c = 4x \qquad (x \in \mathbb{R}),$$

where $f(0) = c$ is a constant. Thus,

$$f(x) = x - \frac{c}{2} \qquad (x \in \mathbb{R}).$$

Next, using this solution back into (1.3) we see that $c = 0$. Hence,

$$f(x) = x \qquad (x \in \mathbb{R}).$$

It is easily checked that this result satisfies the functional equation (1.3). $\square$

For more detailed description, we refer the reader to the book by Castillo, Iglesias, and Ruiz-Cobo [1] and the book by Sahoo and Kannappan [3].

In next section, we review the literature of functional equations. Our motivations and proposed problems are outlined in the last section.

1.2 Literature Reviews

It is easily checked that the 2×2 determinant function $d : \mathbb{R}^2 \to \mathbb{R}$, which is defined by

$$d(x,y) = \begin{vmatrix} x & y \\ y & x \end{vmatrix} = x^2 - y^2,$$

satisfies the two functional equations

$$f(ux + vy, uy + vx) = f(x,y)f(u,v), \tag{1.4}$$

$$f(ux - vy, uy - vx) = f(x,y)f(u,v), \tag{1.5}$$

while the 3×3 determinant function $D : \mathbb{R}^3 \to \mathbb{R}$, which is defined by

$$D(x,y,z) = \begin{vmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{vmatrix} = (x^2 - y^2)z,$$

satisfies the functional equation

$$f(ux+vy, uy+vx, wz) = f(x,y,z)f(u,v,w). \tag{1.6}$$

Henceforth, functional equations like (1.4) or (1.5) or (1.6), which are satisfied by such determinants, will be collectively refered to as ***determinantal equations***.

In 2002, Chung and Sahoo [6] solved (1.4) and (1.6) as well as their pexiderized forms

$$f(ux+vy, uy+vx) = g(x,y)h(u,v), \tag{1.7}$$

$$f(ux+vy, uy+vx, wz) = g(x,y,z)h(u,v,w). \tag{1.8}$$

In another direction, the 2×2 permanent function $p : \mathbb{R}^2 \to \mathbb{R}$, which is defined by

$$p(x,y) := \operatorname{per}\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 + y^2,$$

satisfies the functional equation

$$f(ux+vy, uy-vx) = f(x,y)f(u,v), \tag{1.9}$$

while the 3×3 permanent function $P : \mathbb{R}^3 \to \mathbb{R}$, which is defined by

$$P(x,y,z) := \operatorname{per}\begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix} = (x^2+y^2)z,$$

satisfies the functional equation

$$f(ux+vy, uy-vx, wz) = f(x,y,z)f(u,v,w). \tag{1.10}$$

In 2016, Choi, Kim and Lee [4] solved (1.10) and the functional equation

$$f(ux-vy, uy-vx, wz) = f(x,y,z)f(u,v,w), \tag{1.11}$$

which are the 3-dimensional extensions of the functional equations (1.9) and (1.5), respectively. We shall refer collectively to functional equations like (1.9) or (1.10) as ***permanental equations***.

Closely related to (1.5) and (1.9) are the following functional equations

$$f(ux-vy,uy-vx) = f(x,y)+f(u,v)+f(x,y)f(u,v) \tag{1.12}$$

$$f(ux+vy,uy-vx) = f(x,y)+f(u,v)+f(x,y)f(u,v), \tag{1.13}$$

which were solved in 2007-2008 by Houston and Sahoo [7, 8]; they also solved the pexiderized form of (1.12):

$$f(ux-vy,uy-vx) = g(x,y)+h(u,v)+\ell(x,y)m(u,v). \tag{1.14}$$

In 2014, Chung and Chang [5] solved the following two functional equations

$$f(x_1,y_1)f(x_2,y_2) = f(x_1x_2+y_1y_2, x_1y_2-x_2y_1), \tag{1.15}$$

and

$$\begin{aligned} &f(x_1,y_1,u_1,v_1)f(x_2,y_2,u_2,v_2) \\ &\quad = f(x_1x_2+y_1y_2+u_1u_2+v_1v_2, x_1y_2-y_1x_2+u_1v_2-v_1u_2, \\ &\qquad x_1u_2-y_1v_2-u_1x_2+v_1y_2, x_1v_2+y_1u_2-u_1y_2-v_1x_2) \end{aligned} \tag{1.16}$$

Obviously, a permanent function $p:\mathbb{R}^2\to\mathbb{R}$ also satisfies the functional equation (1.15). These two functional equations arise from a well-known theorem in number theory: *A positive integer of the from m^2n, where each divisor of n is not a squares of integer, can be represented as a sum of two squares of integer if and only if every prime factor of n is not of the form $4k+3$.*

1.3 Motivations and Proposed Problems

Now we consider the following three functional equations:

$$f(ux+vy,uy+vx,wz) = f(x,y,z)+f(u,v,w)+f(x,y,z)f(u,v,w) \tag{1.17}$$

$$f(ux-vy,uy-vx,wz) = f(x,y,z)+f(u,v,w)+f(x,y,z)f(u,v,w) \tag{1.18}$$

$$f(ux+vy,uy-vx,wz) = f(x,y,z)+f(u,v,w)+f(x,y,z)f(u,v,w). \tag{1.19}$$

It is easily verified that a function

$$f(x,y,z) = \begin{vmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{vmatrix} - 1 = (x^2-y^2)z-1 \tag{1.20}$$

satisfies the functional equations (1.17) and (1.18), while a function

$$f(x,y,z) = \mathrm{per}\begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix} - 1 = (x^2+y^2)z-1 \tag{1.21}$$

satisfies the functional equation (1.19). Note that the functional equations (1.18) and (1.19) are the 3-dimensional extensions of the functional equations (1.12), and (1.13), respectively.

In this thesis, we embrace all the afore-mentioned results by solving the following two pexiderized functional equations:

$$f(ux+vy,uy+vx,wz) = g(x,y,z)+h(u,v,w)+\ell(x,y,z)n(u,v,w) \tag{1.22}$$

$$f(ux-vy,uy-vx,wz) = g(x,y,z)+h(u,v,w)+\ell(x,y,z)n(u,v,w) \tag{1.23}$$

$$f(ux+vy,uy-vx,wz) = f(x,y,z)+f(v,u,w)+g(x,y,z)g(u,v,w), \tag{1.24}$$

without any regularity assumptions on the unknown functions $f,g,h,\ell,n : \mathbb{R}^3 \to \mathbb{R}$. These three equations are the 3-dimensional generalizations of the functional equations (1.17), (1.18) and (1.19), respectively.

Our strategy is to first solve the corresponding equations of only one unknown function, which are the functional equations (1.17)-(1.19).

This thesis is organized as follows: in Chapter 2, the some definitions of some functions which involve the main results and the preliminaries results are given. In chapter 3, we solve the determinantal functional equations. In Chapter 4, general solutions of the permanental functional equations are determined. The conclusions and recommendations are given in Chapter 5.

CHAPTER 2

PRELIMINARY RESULTS

Let D be a non-empty set of real numbers having the properties that D contains at least one non-zero element, and whenever $x, y\ (\neq 0) \in D$, then both xy and x/y are also in the set D. The solutions of our results involve the use of the ***logarithmic*** and ***multiplicative functions*** which are the functions $L, M : D \to \mathbb{R}$ satisfying, respectively,

$$L(xy) = L(x) + L(y) \quad (x, y \in D),$$
$$M(xy) = M(x)M(y) \quad (x, y \in D).$$

In passing, note that the only logarithmic function defined over the entire set of real numbers, i.e., when $D = \mathbb{R}$, is the zero function.

Lemma 2.1. *A) The general solution $f : D^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1x_2, y_1y_2, wz) = f(x_1,\ y_1,\ z)\ f(x_2,\ y_2,\ w) \tag{2.1}$$

is given by

$$f(x, y, z) = M_1(x)M_2(y)M_3(z), \tag{2.2}$$

where $M_1, M_2, M_3 : D \to \mathbb{R}$ are multiplicative functions.

B) The general solution $f : D^3 \to \mathbb{R}$ of the functional equation

$$f(x_1y_2, y_1x_2, wz) = f(x_1, y_1, z)f(x_2, y_2, w) \tag{2.3}$$

is given by

$$f(x, y, z) = M_1(xy)M_3(z), \tag{2.4}$$

where $M_1, M_3 : D \to \mathbb{R}$ are multiplicative functions.

C) The general solution $f : D^3 \to \mathbb{R}$ of the functional equation

$$f(x_1x_2,\ y_1y_2,\ wz) = f(x_1,\ y_1,\ z) + f(x_2,\ y_2,\ w) \tag{2.5}$$

is given by

$$f(x,y,z) = L_1(x) + L_2(y) + L_3(z), \tag{2.6}$$

where $L_1, L_2, L_3 : D \to \mathbb{R}$ *are logarithmic functions. If* $D = \mathbb{R}$*, then* $f \equiv 0$ *is the only solution of the functional equation* (2.5).

Proof. It is easily checked that the functions (2.2), (2.4), and (2.6) satisfy the functional equations (2.1), (2.3) and (2.5), respectively.

A) If f is a constant function solution of (2.1), then either $f \equiv 0$ or $f \equiv 1$, both of which are of the form (2.2). Assume now that f is a non-constant solution of (2.1). Clearly, (2.1) implies that $f(0,0,0) = 0$.

We claim that there exists $a \in D \setminus \{0\}$ such that $f(a,a,a) \neq 0$. For if not, then $f(a,a,a) = 0$ for all $a \in D \setminus \{0\}$. Putting $x_1 = y_1 = z = a$ in (2.1), we have

$$f(ax_2, ay_2, aw) = f(a,a,a)f(x_2,y_2,w) = 0,$$

showing that $f \equiv 0$, which is a contradiction, and the claimed is verified. Take such an a as in the claim and using (2.1) repeatedly, we get

$$\begin{aligned}
f(x,y,z) &= f(x,y,z)f(a,a,a)f(a,a,a)^2 f(a,a,a)^{-3} \\
&= f(xa,ya,za)f(a,a,a)f(a,a,a)f(a,a,a)^{-3} \\
&= f((xa)a,(ya)a,(za)a)f(a,a,a)f(a,a,a)^{-3} \\
&= f((xaa)a,(yaa)a,(zaa)a)f(a,a,a)^{-3} \\
&= f(xa(aa),a(yaa),a(zaa))f(a,a,a)^{-3} \\
&= f(xa,a,a)f(aa,yaa,zaa)f(a,a,a)^{-3} \\
&= f(xa,a,a)f(aa,(ya)a,a(za))f(a,a,a)^{-3} \\
&= f(xa,a,a)f(a,ya,a)f(a,a,za)f(a,a,a)^{-3} \\
&= M_1(x)M_2(y)M_3(z), \qquad (2.7)
\end{aligned}$$

where

$$M_1(x) := f(xa,a,a)f(a,a,a)^{-1},$$
$$M_2(y) := f(a,ya,a)f(a,a,a)^{-1},$$
$$M_3(z) := f(a,a,za)f(a,a,a)^{-1}.$$

There remains to show that the functions M_1, M_2, M_3 are multiplicative. We show this only for M_1 as the others are similar. Using (2.1) repeatedly, we get

$$\begin{aligned}
M_1(xy) &= f((xy)a,a,a)f(a,a,a)^{-1} \\
&= f((xy)a,a,a)f(a,a,a)f(a,a,a)^{-2} \\
&= f((xya)a,aa,aa)f(a,a,a)^{-2} \\
&= f((xa)(ya),aa,aa)f(a,a,a)^{-2} \\
&= f(xa,a,a)f(ya,a,a)f(a,a,a)^{-2} \\
&= f(xa,a,a)f(a,a,a)^{-1}f(ya,a,a)f(a,a,a)^{-1} \\
&= M_1(x)M_1(y).
\end{aligned}$$

B) The proof is the same as that of part A) except the step leading to (2.7), which we now elaborate. Let $a \in D \setminus \{0\}$ be such that $f(a,a,a) \neq 0$. Using (2.3) repeatedly, we get

$$\begin{aligned}
f(x,y,z) &= f(x,y,z)f(a,a,a)f(a,a,a)^{-1} \\
&= f(xa,ya,za)f(a,a,a)f(a,a,a)^{-2} \\
&= f((xa)a,(ya)a,(za)a)f(a,a,a)f(a,a,a)^{-3} \\
&= f((xaa)a,a(yaa),(zaa)a)f(a,a,a)^{-3} \\
&= f((xa)aa,a(yaa),a(zaa))f(a,a,a)^{-3} \\
&= f(xa,a,a)f(yaa,aa,zaa)f(a,a,a)^{-3} \\
&= f(xa,a,a)f((ya)a,aa,a(za))f(a,a,a)^{-2} \\
&= f(xa,a,a)f(ya,a,a)f(a,a,za)f(a,a,a)^{-3} \\
&= M_1(x)M_1(y)M_3(z),
\end{aligned}$$

where

$$M_1(x) := f(xa,a,a)f(a,a,a)^{-1},$$
$$M_3(z) := f(a,a,za)f(a,a,a)^{-1}.$$

C) If f is a constant function solution of (2.5), then $f \equiv 0$, which is included in (2.6). Assume now that f is a non-constant solution of (2.5). Let $a \in D$ be fixed. Suppose that $f : D^3 \to \mathbb{R}$ satisfies the functional equation (2.5). Then

$$\begin{aligned}
f(x,y,z) &= f(x,y,z) + f(a,a,a) + 2f(a,a,a) - 3f(a,a,a)\\
&= f(xa,ya,za) + f(a,a,a) + f(a,a,a) - 3f(a,a,a)\\
&= f((xa)a,(ya)a,(za)a) + f(a,a,a) - 3f(a,a,a)\\
&= f((xaa)a,(yaa)a,(zaa)a) - 3f(a,a,a)\\
&= f(xa(aa),a(yaa),a(zaa)) - 3f(a,a,a)\\
&= f(xa,a,a) + f(aa,yaa,zaa) - 3f(a,a,a)\\
&= f(xa,a,a) + f(aa,(ya)a,a(za)) - 3f(a,a,a)\\
&= f(xa,a,a) + f(a,ya,a) + f(a,a,za) - 3f(a,a,a)\\
&= L_1(x) + L_2(y) + L_3(z),
\end{aligned}$$

where

$$L_1(x) := f(xa,a,a) - f(a,a,a),$$
$$L_2(y) := f(a,ya,a) - f(a,a,a),$$
$$L_3(z) := f(a,a,za) - f(a,a,a).$$

There remains to show that the functions L_1, L_2, L_3 are logarithmic. We show this only for L_1 as the others are similar. If $D = \mathbb{R}$, the result follows from the above observation that $L \equiv 0$, henceforth assume that $D \neq \mathbb{R}$. Using (2.5) repeatedly, we get

$$
\begin{aligned}
L_1(xy) &= f((xy)a,a,a) - f(a,a,a) \\
&= f((xy)a,a,a) + f(a,a,a) - 2f(a,a,a) \\
&= f((xya)a,aa,aa) - 2f(a,a,a) \\
&= f((xa)(ya),aa,aa) - 2f(a,a,a) \\
&= f(xa,a,a) + f(ya,a,a) - 2f(a,a,a) \\
&= f(xa,a,a) - f(a,a,a) + f(ya,a,a) - f(a,a,a) \\
&= L_1(x) + L_1(y).
\end{aligned}
$$

□

Part A) and C) of Lemma 2.1 are generalized versions of Lemma 2 and Lemma 3 in [7] on page 62-63, respectively, while part B) is a new functional equation.

Lemma 2.2. *A.) The general solutions $f,\ g : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1x_2, y_1y_2, wz) = f(x_1, y_1, z) + f(x_2, y_2, w) + g(x_1, y_1, z)g(x_2, y_2, w) \tag{2.8}$$

are given by

$$\begin{cases} f(x,y,z) = \delta^2 \left[M_1(x)M_2(y)M_3(z) - 1\right] \\ g(x,y,z) = \delta \left[M_1(x)M_2(y)M_3(z) - 1\right], \end{cases} \tag{2.9}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions, and δ is an arbitrary constant.

B.) The general solutions $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation

$$f(x_1x_2, y_1y_2, wz) = f(x_1, y_1, z) + f(x_2, y_2, w) + g(x_1, y_1, z)h(x_2, y_2, w) \tag{2.10}$$

are given by

$$f \equiv g \equiv 0,\ h \text{ is arbitrary}, \tag{2.11}$$

or

$$f \equiv h \equiv 0,\ g \text{ is arbitrary}, \tag{2.12}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x)M_2(y)M_3(z)-1\right] \\ g(x,y,z) = \dfrac{1}{k_2}\left[M_1(x)M_2(y)M_3(z)-1\right] \\ h(x,y,z) = \dfrac{1}{k_1}\left[M_1(x)M_2(y)M_3(z)-1\right], \end{cases} \tag{2.13}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are nonzero constants.

Proof. A.) That the functions (2.9) satisfies (2.8) is easily checked. If $g \equiv -\delta$ is a constant function solution of (2.8), then the equation (2.8) becomes

$$f(x_1x_2, y_1y_2, wz) = f(x_1, y_1, z) + f(x_2, y_2, w) + \delta^2 \qquad (x_1, x_2, y_1, y_2, w, z \in \mathbb{R}). \tag{2.14}$$

Defining $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x,y,z) = f(x,y,z) + \delta^2 \qquad (x,y,z \in \mathbb{R}), \tag{2.15}$$

enables us to rewrite (2.14) as

$$F(x_1x_2, y_1y_2, wz) = F(x_1, y_1, z) + F(x_2, y_2, w) \qquad (x_1, x_2, y_1, y_2, w, z \in \mathbb{R}). \tag{2.16}$$

Using Lemma 2.1 C), we get $F(x,y,z) \equiv 0$, and so (2.15) shows that $f(x,y,z) \equiv -\delta^2$, which is included in (2.9).

Assume now that g is a non-constant solution of (2.8). Putting $x_2 = y_2 = w = 0$ in (2.8) we have

$$f(x_1, y_1, z) = -g(0,0,0)g(x_1, y_1, z) \qquad (x_1, y_1, z \in \mathbb{R}). \tag{2.17}$$

We assert now that $g(0,0,0) := -1/\alpha \neq 0$. For otherwise $f(x_1, y_1, z) = 0$ for all $x_1, y_1, z \in \mathbb{R}$. This imples, by (2.8), that g is a constant function, which is a contradiction. Thus, (2.17) can be rewritten as

$$g(x,y,z) = \alpha f(x,y,z) \qquad (x,y,z \in \mathbb{R}). \tag{2.18}$$

Substituting (2.18) back into (2.8), we have

$$f(x_1x_2,y_1y_2,wz)=f(x_1,y_1,z)+f(x_2,y_2,w)+\alpha^2 f(x_1,y_1,z)f(x_2,y_2,w) \qquad (x_1,x_2,y_1,y_2,w,z\in\mathbb{R}). \tag{2.19}$$

Using the substitution

$$\tilde{F}(x,y,z)=\alpha^2 f(x,y,z)+1 \qquad (x,y,z\in\mathbb{R}), \tag{2.20}$$

the functional equation (2.19) becomes

$$\tilde{F}(x_1x_2,y_1y_2,wz)=\tilde{F}(x_1,y_1,z)\tilde{F}(x_2,y_2,w) \qquad (x_1,x_2,y_1,y_2,w,z\in\mathbb{R}). \tag{2.21}$$

Using Lemma 2.1 A) and (2.20), we get

$$\alpha^2 f(x,y,z)+1=\tilde{F}(x,y,z)=M_1(x)M_2(y)M_3(z) \qquad (x,y,z\in\mathbb{R}), \tag{2.22}$$

where $M_1,M_2,M_3:\mathbb{R}\to\mathbb{R}$ are multiplicative functions. Finally, using (2.18) and (2.22) we obatain

$$\begin{cases} f(x,y,z)=\dfrac{1}{\alpha^2}\left[M_1(x)M_2(y)M_3(z)-1\right] \\ g(x,y,z)=\dfrac{1}{\alpha}\left[M_1(x)M_2(y)M_3(z)-1\right], \end{cases} \tag{2.23}$$

where $M_1,M_2,M_3:\mathbb{R}\to\mathbb{R}$ are multiplicative functions. If g is a non-constant solution of (2.8), then $M_i\neq 0$ for all $i\in\{1,2,3\}$.

B) It is easily checked that the functions (2.11)-(2.13) satisfy the functional equations (2.10). If $g(x,y,z)h(u,v,w)=0$, then the functional equation (2.10) becomes

$$f(x_1x_2,y_1y_2,wz)=f(x_1,y_1,z)+f(x_2,y_2,w) \qquad (x_1,x_2,y_1,y_2,w,z\in\mathbb{R}). \tag{2.24}$$

Using Lemma 2.1 C), we obtain $f(x,y,z)\equiv 0$. Assume now that $g(x,y,z)h(u,v,w)\neq 0$, there are two possible cases.

Case I : If $g\equiv h$, then (2.10) yields

$$f(x_1x_2,y_1y_2,wz)=f(x_1,y_1,z)+f(x_2,y_2,w)+g(x_1,y_1,z)g(x_2,y_2,w) \qquad (x_1,x_2,y_1,y_2,w,z\in\mathbb{R}). \tag{2.25}$$

The desired result thus follows at once from Lemma 2.2 A):

$$\begin{cases} f(x,y,z) = \delta^2 \left[M_1(x)M_2(y)M_3(z) - 1\right] \\ g(x,y,z) = \delta \left[M_1(x)M_2(y)M_3(z) - 1\right] \\ h(x,y,z) = \delta \left[M_1(x)M_2(y)M_3(z) - 1\right], \end{cases} \tag{2.26}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions, and δ is an arbitrary constant.

Case II : If $g \not\equiv h$, then putting $x_2 = y_2 = w = 0$ into (2.10) we have

$$f(x_1, y_1, z) = -h(0,0,0)g(x_1, y_1, z) \qquad (x_1, y_1, z \in \mathbb{R}). \tag{2.27}$$

We claim that $h(0,0,0) \neq 0$; for if not, then $f(x_1, y_1, z) \equiv 0$, and so (2.10) gives $g(x,y,z)h(u,v,w) = 0$, which is a contradiction. Thus, (2.27) becomes

$$g(x,y,z) = k_1 f(x,y,z), \qquad k_1 := -1/h(0,0,0) \neq 0. \tag{2.28}$$

Similarly, putting $x_1 = y_1 = z = 0$ in (2.10) we get

$$h(x,y,z) = k_2 f(x,y,z), \qquad k_2 := -1/g(0,0,0) \neq 0. \tag{2.29}$$

Substituting (2.28) and (2.29) back into (2.10), we have

$$f(x_1x_2, y_1y_2, wz) = f(x_1, y_1, z) + f(x_2, y_2, w) + k_1k_2 f(x_1, y_1, z) f(x_2, y_2, w) \qquad (x_1, x_2, y_1, y_2, w, z \in \mathbb{R}). \tag{2.30}$$

Using the substitution

$$F(x,y,z) = k_1k_2 f(x,y,z) + 1 \qquad (x,y,z \in \mathbb{R}), \tag{2.31}$$

we can write (2.30) as

$$F(x_1x_2, y_1y_2, wz) = F(x_1, y_1, z)F(x_2, y_2, w) \qquad (x_1, x_2, y_1, y_2, w, z \in \mathbb{R}). \tag{2.32}$$

Invoking upon Lemma 2.1 A), and using (2.28), (2.29) and (2.31) we obtain

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2} \left[M_1(x)M_2(y)M_3(z) - 1\right] \\ g(x,y,z) = \dfrac{1}{k_2} \left[M_1(x)M_2(y)M_3(z) - 1\right] \\ h(x,y,z) = \dfrac{1}{k_1} \left[M_1(x)M_2(y)M_3(z) - 1\right], \end{cases} \tag{2.33}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions, which are the asserted solutions. $\square$

Part A) of Lemma 2.2 is a new functional equation, while part B) is generalized version of Lemma 3 in [7] on page 64.

Corollary 2.1. *A) The general solutions $f, g : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1y_2, x_2y_1, wz) = f(x_1, y_1, z) + f(y_2, x_2, w) + g(x_1, y_1, z)g(x_2, y_2, w) \tag{2.34}$$

are given by $f \equiv g \equiv 0$, or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k^2}\left[M_1(xy)M_3(z) - 1\right] \\ g(x,y,z) = \dfrac{1}{k}\left[M_1(xy)M_3(z) - 1\right], \end{cases} \tag{2.35}$$

where $M_1, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k is a non-zero constant.

B) The general solutions $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation

$$f(x_1y_2, x_2y_1, wz) = f(x_1, y_1, z) + f(y_2, x_2, w) + g(x_1, y_1, z)h(x_2, y_2, w) \tag{2.36}$$

are given by

$$f \equiv g \equiv 0, \ h \ \text{ is arbitrary}, \tag{2.37}$$

or

$$f \equiv h \equiv 0, \ g \ \text{ is arbitrary}, \tag{2.38}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ g(x,y,z) = \dfrac{1}{k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ h(x,y,z) = \dfrac{1}{k_1}\left[M_1(y)M_2(x)M_3(z) - 1\right], \end{cases} \tag{2.39}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are nonzero constants.

Proof. A) It is easily verified that the functions (2.35) satisfies the functional equation (2.34). Interchanging x_2 with y_2 in (2.34), we get

$$f(x_1x_2, y_1y_2, wz) = f(x_1, y_1, z) + f(x_2, y_2, w) + g(x_1, y_1, z)g(y_2, x_2, w) \qquad (2.40)$$
$$(x_1, x_2, y_1, y_2, w, z \in \mathbb{R}).$$

Using the substitution

$$h(x, y, z) = g(y, x, z) \qquad (x, y, z \in \mathbb{R}), \qquad (2.41)$$

the functional equation (2.40) becomes

$$f(x_1x_2, y_1y_2, wz) = f(x_1, y_1, z) + f(x_2, y_2, w) + g(x_1, y_1, z)h(x_2, y_2, w) \qquad (2.42)$$
$$(x_1, x_2, y_1, y_2, w, z \in \mathbb{R}).$$

The solution function follows by appealing to Lemma 2.2 B), and so (2.41) shows that

$$f \equiv g \equiv 0, \ h \ \text{is arbitrary}, \qquad (2.43)$$

or

$$f \equiv h \equiv 0, \ g \ \text{is arbitrary}, \qquad (2.44)$$

or

$$\begin{cases} f(x, y, z) = \dfrac{1}{k_1k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ g(x, y, z) = \dfrac{1}{k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ h(x, y, z) = \dfrac{1}{k_1}\left[M_1(y)M_2(x)M_3(z) - 1\right], \end{cases} \qquad (2.45)$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are nonzero constants. Finally, using (2.45) back into (2.34) we see that $M_1 = M_2$ and $k_1 = k_2 = k$, and so (2.45) becomes

$$\begin{cases} f(x, y, z) = \dfrac{1}{k^2}\left[M_1(xy)M_3(z) - 1\right] \\ g(x, y, z) = \dfrac{1}{k}\left[M_1(xy)M_3(z) - 1\right] \\ h(x, y, z) = \dfrac{1}{k}\left[M_1(yx)M_3(z) - 1\right], \end{cases} \qquad (2.46)$$

which are the desired solutions.

B) That the function (2.37)-(2.39) satisfies the functional equation (2.36). Interchanging x_2 with y_2 in (2.36), we get

$$f(x_1x_2,y_1y_2,wz) = f(x_1,y_1,z)+f(x_2,y_2,w)+g(x_1,y_1,z)h(y_2,x_2,w) \qquad (x_1,x_2,y_1,y_2,w,z \in \mathbb{R}). \tag{2.47}$$

Defining $\tilde{h}:\mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{h}(x,y,z) = h(y,x,z) \qquad (x,y,z \in \mathbb{R}), \tag{2.48}$$

the functional equation (2.47) becomes

$$f(x_1x_2,y_1y_2,wz) = f(x_1,y_1,z)+f(x_2,y_2,w)+g(x_1,y_1,z)\tilde{h}(x_2,y_2,w) \qquad (x_1,x_2,y_1,y_2,w,z \in \mathbb{R}). \tag{2.49}$$

The solution function follows by appealing to Lemma 2.2 B), and so (2.48) shows that

$$f \equiv g \equiv 0, \ h \ \text{ is } \ \text{arbitrary}, \tag{2.50}$$

or

$$f \equiv h \equiv 0, \ g \ \text{ is } \ \text{arbitrary}, \tag{2.51}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x)M_2(y)M_3(z)-1\right] \\ g(x,y,z) = \dfrac{1}{k_2}\left[M_1(x)M_2(y)M_3(z)-1\right] \\ h(x,y,z) = \dfrac{1}{k_1}\left[M_1(y)M_2(x)M_3(z)-1\right], \end{cases} \tag{2.52}$$

where $M_1,M_2,M_3:\mathbb{R}\to\mathbb{R}$ are multiplicative functions and k_1,k_2 are nonzero constants.

□

CHAPTER 3

DETERMINANTAL FUNCTIONAL EQUATIONS

In this chapter, the general solutions of the functional equations (1.17) and (1.18) as well as their pexiderized functional equations (1.22) and (1.23) are determined.

In 2002, Chung and Sahoo [6] solved the functional equation (1.6) and its pexiderized form, which are the functional equation (1.8). Next, Choi, Kim and Lee [4] solved the functional equation (1.11), in 2016, which are contained in the following theorem.

Theorem 3.1. *A)* (Chung and Sahoo [6]) *The general solution $f:\mathbb{R}^3\to\mathbb{R}$ of the functional equation*

$$f(ux+vy,uy+vx,wz)=f(x,y,z)f(u,v,w) \tag{3.1}$$

is given by

$$f(x,y,z)=M_1(x+y)M_2(x-y)M_3(z), \tag{3.2}$$

where $M_1,M_2,M_3:\mathbb{R}\to\mathbb{R}$ are multiplicative functions.

B) (Chung and Sahoo [6]) *The general solutions $f,g,h:\mathbb{R}^3\to\mathbb{R}$ of the functional equation*

$$f(ux+vy,uy+vx,wz)=g(x,y,z)h(u,v,w) \tag{3.3}$$

are given by

$$f\equiv g\equiv 0,\ h \text{ is arbitrary}, \tag{3.4}$$

or

$$f\equiv h\equiv 0,\ g \text{ is arbitrary}, \tag{3.5}$$

or

$$\begin{cases} f(x,y,z) = k_1k_2\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ g(x,y,z) = k_2\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ h(x,y,z) = k_1\left[M_1(x+y)M_2(x-y)M_3(z)-1\right], \end{cases} \tag{3.6}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ *are multiplicative functions and* k_1, k_2 *are nonzero constants.*

C) (Choi, Kim and Lee [4]) *The general solution* $f : \mathbb{R}^3 \to \mathbb{R}$ *of the functional equation*

$$f(ux-vy, uy-vx, wz) = f(x,y,z)f(u,v,w) \tag{3.7}$$

is given by

$$f(x,y,z) = M_1(x^2-y^2)M_2(z), \tag{3.8}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ *are multiplicative functions.*

We are now ready to prove our first main results. The general solutions of functional equations extending those similar to (1.12) and (1.13) are contained in the following corollary.

Corollary 3.1. *A) The general solution* $f : \mathbb{R}^3 \to \mathbb{R}$ *of the functional equation*

$$f(ux-vy, uy-vx, wz) = g(x,y,z)h(u,v,w) \tag{3.9}$$

are given by

$$f \equiv g \equiv 0,\ h \text{ is arbitrary}, \tag{3.10}$$

or

$$f \equiv h \equiv 0,\ g \text{ is arbitrary}, \tag{3.11}$$

or

$$\begin{cases} f(x,y,z) = k_1k_2\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ g(x,y,z) = k_2\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ h(x,y,z) = k_1\left[M_1(x-y)M_2(x+y)M_3(z)-1\right], \end{cases} \tag{3.12}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are nonzero constants.

B) The general solution $f : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation

$$f(ux+vy, uy+vx, wz) = f(x,y,z) + f(u,v,w) + f(x,y,z)f(u,v,w) \tag{3.13}$$

is given by

$$f(x,y,z) = M_1(x+y)M_2(x-y)M_3(z) - 1, \tag{3.14}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

C) The general solution $f : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation

$$f(ux-vy, uy-vx, wz) = f(x,y,z) + f(u,v,w) + f(x,y,z)f(u,v,w) \tag{3.15}$$

is given by

$$f(x,y,z) = M_1(x^2-y^2)M_2(z) - 1, \tag{3.16}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

Proof. A) Replacing v by $-v$ in the functional equation (3.9), we get

$$f(ux+vy, uy+vx, wz) = g(x,y,z)h(u,-v,w) \qquad (x,y,u,v,w,z \in \mathbb{R}). \tag{3.17}$$

Defining $\tilde{h} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{h}(x,y,z) = h(x,-y,z) \qquad (x,y,z \in \mathbb{R}), \tag{3.18}$$

enables us to rewrite (3.17) as

$$f(ux+vy, uy+vx, wz) = g(x,y,z)\tilde{h}(u,v,w) \qquad (x,y,u,v,w,z \in \mathbb{R}). \tag{3.19}$$

Using Theorem 3.1 B), and so (3.18) shows that

$$f \equiv g \equiv 0,\ h \text{ arbitrary}, \tag{3.20}$$

or

$$f \equiv h \equiv 0,\ g \text{ arbitrary}, \tag{3.21}$$

or

$$\begin{cases} f(x,y,z) = k_1k_2\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ g(x,y,z) = k_2\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ h(x,y,z) = k_1\left[M_1(x-y)M_2(x+y)M_3(z)-1\right], \end{cases} \tag{3.22}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are nonzero constants, which are the asserted solutions. The solution function and the justification of the substitution are verified by direct checking with the functional equation (3.9).

B) It is easily checked that the function (3.14) satisfies the functional equation (3.13). If f is a constant solution of (3.13), then either $f \equiv 0$ or $f \equiv -1$, both of which are of the form (3.14). Assume now that f is a non-constant function solution of (3.13). Defining $g : \mathbb{R}^3 \to \mathbb{R}$ by

$$g(x,y,z) = f\left(\frac{x+y}{2}, \frac{x-y}{2}, z\right) + 1 \qquad (x,y,z \in \mathbb{R}), \tag{3.23}$$

allows us to replace (3.13) by

$$g((x+y)(u+v),(x-y)(u-v),wz) = g(x+y,x-y,z)g(u+v,u-v,w) \qquad (x,y,u,v,w,z \in \mathbb{R}). \tag{3.24}$$

Substituting $x_1 = x+y,\ y_1 = x-y,\ x_2 = u+v$ and $y_2 = u-v$ into (3.24), we get

$$g(x_1x_2, y_1y_2, wz) = g(x_1,y_1,z)g(x_2,y_2,w) \qquad (x_1,x_2,y_1,y_2,w,z \in \mathbb{R}). \tag{3.25}$$

The shape of the function solution follows immediately from Lemma 2.1 A):

$$g(x,y,z) = M_1(x)M_2(y)M_3(z), \tag{3.26}$$

and so (3.23) yields

$$f(x,y,z) = M_1(x+y)M_2(x-y)M_3(z) - 1, \tag{3.27}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions, which is the asserted solutions.

C) That the function (3.16) satisfies the functional equation (3.15). Making the variable change

$$g(x,y,z) = f(x,y,z) + 1 \qquad (x,y,z \in \mathbb{R}), \tag{3.28}$$

the functional equation (3.15) becomes

$$g(ux - vy, uy - vx, wz) = g(x,y,z)g(u,v,w) \qquad (x,y,u,v,w,z \in \mathbb{R}), \tag{3.29}$$

which is of the form (3.7), and so Theorem 3.1 C) and (3.28) yield the results follows:

$$f(x,y,z) = M_1(x^2 - y^2)M_2(z) - 1, \tag{3.30}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions. □

Theorem 3.2. *A) The general solutions $f, g, h, \ell, n : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(ux + vy, uy + vx, wz) = g(x,y,z) + h(u,v,w) + \ell(x,y,z)n(u,v,w) \tag{3.31}$$

are given by

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 \\ h(x,y,z) = \beta_1 + \alpha_1\alpha_2 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \quad \text{is arbitrary,} \end{cases} \tag{3.32}$$

or

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 + \alpha_1\alpha_2 - \alpha_1\ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \quad \text{is arbitrary} \\ n(x,y,z) = \alpha_1, \end{cases} \tag{3.33}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] + \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_1}{k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] + \beta_2 \\ h(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] + \beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] + \alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] + \alpha_1, \end{cases} \tag{3.34}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ *are multiplicative functions and* $\alpha_1, \alpha_2, \beta_1, \beta_2, k_1 (\neq 0), k_2 (\neq 0)$ *are constants.*

B) The general solutions $f, g, h, \ell, n : \mathbb{R}^3 \to \mathbb{R}$ *of the functional equation*

$$f(ux-vy, uy-vx, wz) = g(x,y,z) + h(u,v,w) + \ell(x,y,z)n(u,v,w) \tag{3.35}$$

are given by

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 \\ h(x,y,z) = \beta_1 + \alpha_1\alpha_2 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \quad \text{is arbitrary}, \end{cases} \tag{3.36}$$

or

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 + \alpha_1\alpha_2 - \alpha_1\ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \quad \text{is arbitrary} \\ n(x,y,z) = \alpha_1, \end{cases} \tag{3.37}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right]+\beta_1+\beta_2+\alpha_1\alpha_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_1}{k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right]+\beta_2 \\ h(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x-y)M_2(x+y)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}\left[M_1(x-y)M_2(x+y)M_3(z)-1\right]+\beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right]+\alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}\left[M_1(x-y)M_2(x+y)M_3(z)-1\right]+\alpha_1, \end{cases} \tag{3.38}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ *are multiplicative functions and* $\alpha_1, \alpha_2, \beta_1, \beta_2, k_1(\neq 0), k_2(\neq 0)$ *are constants.*

Proof. It is easily checked that the equations (3.32) - (3.34) and (3.36)-(3.38) satisfy the functional equations (3.31) and (3.35), respectively.

A) The substitution

$$\begin{cases} F(x,y,z) = f\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ G(x,y,z) = g\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ H(x,y,z) = h\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ L(x,y,z) = \ell\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ N(x,y,z) = n\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \qquad (x,y,z \in \mathbb{R}), \end{cases} \tag{3.39}$$

enables us to replace (3.31) by

$$\begin{aligned} F((x+y)(u+v),(x-y)(u-v),wz) &= G(x+y,x-y,z)+H(u+v,u-v,w) \\ &\quad + L(x+y,x-y,z)N(u+v,u-v,w) \qquad (3.40) \\ &\qquad (x,y,u,v,w,z \in \mathbb{R}). \end{aligned}$$

Substituting $x_1 = x+y$, $y_1 = x-y$, $x_2 = u+v$ and $y_2 = u-v$ in (3.40), we get

$$F(x_1x_2, y_1y_2, wz) = G(x_1, y_1, z) + H(x_2, y_2, w) + L(x_1, y_1, z)N(x_2, y_2, w) \qquad (3.41)$$
$$(x_1, x_2, y_1, y_2, w, z \in \mathbb{R}).$$

Putting $x_2 = y_2 = w = 1$ in (3.41) we get

$$F(x_1, y_1, z) = G(x_1, y_1, z) + \alpha_1 L(x_1, y_1, z) + \beta_1 \qquad (x_1, y_1, z \in \mathbb{R}), \qquad (3.42)$$

where $\alpha_1 = N(1,1,1), \beta_1 = H(1,1,1)$. Similarly, setting $x_1 = y_1 = z = 1$ in (3.41) we get

$$F(x_2, y_2, w) = H(x_2, y_2, w) + \alpha_2 N(x_2, y_2, w) + \beta_2 \qquad (x_2, y_2, w \in \mathbb{R}), \qquad (3.43)$$

where $\alpha_2 = L(1,1,1), \beta_2 = G(1,1,1)$. Putting (3.42) and (3.43) back into (3.41), we get

$$\begin{aligned} F(x_1x_2, y_1y_2, wz) = {} & F(x_1, y_1, z) - \alpha_1 L(x_1, y_1, z) + F(x_2, y_2, w) \\ & - \alpha_2 N(x_2, y_2, w) + L(x_1, y_1, z)N(x_2, y_2, w) - \beta_1 - \beta_2 \qquad (3.44) \\ & (x_1, x_2, y_1, y_2, w, z \in \mathbb{R}). \end{aligned}$$

Defining $\tilde{F}, \tilde{L}, \tilde{N} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{cases} \tilde{F}(x,y,z) = F(x,y,z) - \beta_1 - \beta_2 - \alpha_1\alpha_2 \\ \tilde{L}(x,y,z) = L(x,y,z) - \alpha_2 \\ \tilde{N}(x,y,z) = N(x,y,z) - \alpha_1 \qquad (x,y,z \in \mathbb{R}), \end{cases} \qquad (3.45)$$

allows us to relplace (3.44) by

$$\tilde{F}(x_1x_2, y_1y_2, wz) = \tilde{F}(x_1, y_1, z) + \tilde{F}(x_2, y_2, w) + \tilde{L}(x_1, y_1, z)\tilde{N}(x_2, y_2, w) \qquad (3.46)$$
$$(x_1, x_2, y_1, y_2, w, z \in \mathbb{R}).$$

Invoking upon Lemma 2.2 B), we obtain

$$\tilde{F} \equiv \tilde{L} \equiv 0, \ \ \tilde{N} \ \text{ is arbitrary}, \qquad (3.47)$$

or

$$\tilde{F} \equiv \tilde{N} \equiv 0, \;\; \tilde{L} \;\text{ is arbitrary}, \tag{3.48}$$

or

$$\begin{cases} \tilde{F}(x,y,z) = \dfrac{1}{k_1 k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ \tilde{L}(x,y,z) = \dfrac{1}{k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ \tilde{N}(x,y,z) = \dfrac{1}{k_1}\left[M_1(x)M_2(y)M_3(z) - 1\right], \end{cases} \tag{3.49}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are nonzero constants.

Finally, using (3.39), (3.42), (3.43), (3.45), (3.47), (3.48) and (3.49), we obtain

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 \\ h(x,y,z) = \beta_1 + \alpha_1\alpha_2 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \;\; \text{is arbitrary}, \end{cases} \tag{3.50}$$

or

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 + \alpha_1\alpha_2 - \alpha_1 \ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \;\; \text{is arbitrary} \\ n(x,y,z) = \alpha_1, \end{cases} \tag{3.51}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right]+\beta_1+\beta_2+\alpha_1\alpha_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_1}{k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right]+\beta_2 \\ h(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right]+\beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right]+\alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}\left[M_1(x+y)M_2(x-y)M_3(z)-1\right]+\alpha_1, \end{cases} \tag{3.52}$$

which are the desired solutions.

B) Replacing v by $-v$ into (3.35) we obtain

$$f(ux+vy,uy+vx,wz) = g(x,y,z)+h(u,-v,w)+\ell(x,y,z)n(u,-v,w) \tag{3.53}$$
$$(x,y,u,v,w,z \in \mathbb{R}).$$

Interchanging x with y in (3.53), we get

$$f(uy+vx,ux+vy,wz) = g(y,x,z)+h(u,-v,w)+\ell(y,x,z)n(u,-v,w) \tag{3.54}$$
$$(x,y,u,v,w,z \in \mathbb{R}).$$

The substitution

$$\begin{cases} F(x,y,z) = f(y,x,z) \\ G(x,y,z) = g(y,x,z) \\ H(x,y,z) = h(x,-y,z) \\ L(x,y,z) = \ell(y,x,z) \\ N(x,y,z) = n(x,-y,z) \qquad (x,y,z \in \mathbb{R}) \end{cases} \tag{3.55}$$

allows us to replace (3.54) by

$$F(ux+vy,uy+vx,wz) = G(x,y,z)+H(u,v,w)+L(x,y,z)N(u,v,w) \tag{3.56}$$
$$(x,y,u,v,w,z \in \mathbb{R}),$$

which is a functional equation of the form (3.31), and the results follows from part A and (3.55):

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 \\ h(x,y,z) = \beta_1 + \alpha_1\alpha_2 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \quad \text{is arbitrary,} \end{cases} \tag{3.57}$$

or

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 + \alpha_1\alpha_2 - \alpha_1 \ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \quad \text{is arbitrary} \\ n(x,y,z) = \alpha_1, \end{cases} \tag{3.58}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right] + \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_1}{k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right] + \beta_2 \\ h(x,y,z) = \dfrac{1}{k_1k_2}\left[M_1(x-y)M_2(x+y)M_3(z)-1\right] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}\left[M_1(x-y)M_2(x+y)M_3(z)-1\right] + \beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}\left[M_1(x+y)M_2(y-x)M_3(z)-1\right] + \alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}\left[M_1(x-y)M_2(x+y)M_3(z)-1\right] + \alpha_1, \end{cases} \tag{3.59}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary constant and k_1, k_2 are nonzero constants. $\square$

CHAPTER 4

PERMANENTAL FUNCTIONAL EQUATIONS

In this chapter, the general solution of the functional equation (1.19) and its partial pexiderized functional equation (1.24) are determined.

In 2016, Choi, Kim and Lee [4] solved the functional equation (1.10), which is contained in the following theorem.

Theorem 4.1. (Choi, Kim and Lee [4]) *The general solution $f : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(ux+vy, uy-vx, wz) = f(x,y,z)f(u,v,w) \tag{4.1}$$

is given by

$$f(x,y,z) = M_1\left(\sqrt{x^2+y^2}\right)M_2(z) \tag{4.2}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

We turn now to our last batch of results. We begin with

Lemma 4.1. *The only solution function $f : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(ux+vy,\ uy-vx,\ wz) = f(x,\ y,\ z) + f(v,\ u,\ w) \tag{4.3}$$

is the trivial zero function $f \equiv 0$.

Proof. Clearly, the zero function is a solution of the functional equation (4.3). On the other hand, if $f \equiv c$ is a constant solution of the functional equation (4.3), substituting into the functional equation (4.3) shows that $c = 0$. If f is a non-constant solution, then putting $v = u = w = 0$ into the functional equation (4.3) yields $f \equiv 0$. $\square$

Applying Theorem 4.1, we obtain the extension of the functional equation (4.1), which is contained in the following corollary.

Corollary 4.1. *The general solution $f : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(ux+vy, uy-vx, wz) = f(x,y,z) + f(u,v,w) + f(x,y,z)f(u,v,w) \tag{4.4}$$

is given by

$$f(x,y,z) = M_1\left(\sqrt{x^2+y^2}\right)M_2(z) - 1, \tag{4.5}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

Proof. If f is a constant function solution of (4.4), then either $f \equiv 0$ or $f \equiv -1$, both of which are of the form (4.5). Assume now that f is a non-constant function solution of (4.4). The substitution

$$g(x,y,z) = f(x,y,z) + 1 \qquad (x,y,z \in \mathbb{R}) \tag{4.6}$$

allows us to replace (4.4) by

$$g(ux+vy, uy-vx, wz) = g(x,y,z)g(u,v,w). \tag{4.7}$$

Invoking upon Theorem 4.1, the desired solution function follows at once:

$$f(x,y,z) + 1 = g(x,y,z) = M_1\left(\sqrt{x^2+y^2}\right)M_2(z), \tag{4.8}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions. The solution function and the justification of the substitution are verified by direct checking with the functional equation (4.4). $\square$

Theorem 4.2. *The general solutions $f, g : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(ux+vy,\ uy-vx,\ wz) = f(x,y,z) + f(v,u,w) + g(x,y,z)g(u,v,w) \tag{4.9}$$

are given by

$$\begin{cases} f(x,y,z) = \delta^2[M_1(\sqrt{x^2+y^2})M_2(z) - 1] \\ g(x,y,z) = \delta[M_1(\sqrt{x^2+y^2})M_2(z) - 1], \end{cases} \tag{4.10}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and δ is an arbitrary constant.

Proof. It is easily checked that the functions (4.10) satisfy the functional equation (4.9). On the other hand, if $g \equiv -c$ is a constant function satisfying the functional equation (4.9), then the equation (4.9) becomes

$$f(ux+vy,uy-vx,wz) = f(x,y,z)+f(v,u,w)+c^2 \qquad (x,y,u,v,w,z \in \mathbb{R}). \tag{4.11}$$

Defining $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x,y,z) = f(x,y,z)+c^2 \qquad (x,y,z \in \mathbb{R}), \tag{4.12}$$

the functional equation (4.11) becomes

$$F(ux+vy,uy-vx,wz) = F(x,y,z)+F(v,u,w) \qquad (x,y,u,v,w,z \in \mathbb{R}). \tag{4.13}$$

By Lemma 4.1, we get $F(x,y,z) = 0$, i.e., $f(x,y,z) = -c^2$, which is included in (4.10).

Assume now that g is a non-constant function satisfying (4.9). Putting $u = v = w = 0$ in (4.9) we have

$$f(x,y,z) = -g(0,0,0)g(x,y,z) \qquad (x,y,z \in \mathbb{R}). \tag{4.14}$$

In the equation (4.14), $g(0,0,0) \neq 0$; for if not, then $f(x,y,z) = 0$, implying, by (4.9), that $g \equiv 0$, which is a contradiction. Rewrite the equation (4.14) as

$$g(x,y,z) = kf(x,y,z), \qquad k := -1/g(0,0,0) \neq 0. \tag{4.15}$$

Putting $x = y = z = 0$ in (4.9), we get

$$f(v,u,w) = -g(0,0,0)g(u,v,w) \qquad (u,v,w \in \mathbb{R}), \tag{4.16}$$

and use the same arguments, we obtain

$$g(y,x,z) = kf(x,y,z) \qquad (x,y,z \in \mathbb{R}), \tag{4.17}$$

which by virtue of (4.15) yields

$$g(y,x,z) = g(x,y,z) \qquad (x,y,z \in \mathbb{R}), \tag{4.18}$$

and so

$$f(x,y,z) = f(y,x,z) \qquad (x,y,z \in \mathbb{R}). \tag{4.19}$$

Substituting (4.15) back into (4.9), and using (4.19), we obtain

$$f(ux+vy, uy-vx, wz) = f(x,y,z) + f(u,v,w) + k^2 f(x,y,z) f(u,v,w) \qquad (x,y,u,v,w,z \in \mathbb{R}). \tag{4.20}$$

Making the variable change

$$\tilde{F}(x,y,z) = k^2 f(x,y,z) + 1 \qquad (x,y,z \in \mathbb{R}), \tag{4.21}$$

the functional equation (4.20) becomes

$$\tilde{F}(ux+vy, uy-vx, wz) = \tilde{F}(x,y,z)\tilde{F}(u,v,w) \qquad (x,y,u,v,w,z \in \mathbb{R}), \tag{4.22}$$

and the desired results follows from Theorem 4.1 and (4.21):

$$k^2 f(x,y,z) + 1 = \tilde{F}(x,y,z) = M_1\left(\sqrt{x^2+y^2}\right) M_2(z) \qquad (x,y,z \in \mathbb{R}), \tag{4.23}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions. Finally, using (4.15) and (4.23) we obtain

$$\begin{cases} f(x,y,z) = \dfrac{1}{k^2}[M_1(\sqrt{x^2+y^2})M_2(z) - 1] \\ g(x,y,z) = \dfrac{1}{k}[M_1(\sqrt{x^2+y^2})M_2(z) - 1], \end{cases} \tag{4.24}$$

which are the desired solutions. $\square$

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

General solution function $f : \mathbb{R}^3 \to \mathbb{R}$ of the following three functional equations

$$f(ux+vy,uy+vx,wz) = f(x,y,z)+f(u,v,w)+f(x,y,z)f(u,v,w) \tag{1.17}$$

$$f(ux-vy,uy-vx,wz) = f(x,y,z)+f(u,v,w)+f(x,y,z)f(u,v,w) \tag{1.18}$$

$$f(ux+vy,uy-vx,wz) = f(x,y,z)+f(u,v,w)+f(x,y,z)f(u,v,w) \tag{1.19}$$

are as follows:

- for the functional equation (1.17):

$$f(x,y,z) = M_1(x+y)M_2(x-y)M_3(z)-1,$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

- for the functional equation (1.18):

$$f(x,y,z) = M_1(x^2-y^2)M_2(z)-1,$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

- for the functional equation (1.19):

$$f(x,y,z) = M_1\left(\sqrt{x^2+y^2}\right)M_2(z)-1,$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

General solution functions $f, g, h, \ell, n : \mathbb{R}^3 \to \mathbb{R}$ of the following two pexiderized functional equations

$$f(ux+vy,uy+vx,wz) = g(x,y,z)+h(u,v,w)+\ell(x,y,z)n(u,v,w) \tag{1.22}$$

$$f(ux-vy,uy-vx,wz) = g(x,y,z)+h(u,v,w)+\ell(x,y,z)n(u,v,w) \tag{1.23}$$

are as follows:

- for the functional equation (1.22):

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 \\ h(x,y,z) = \beta_1 + \alpha_1\alpha_2 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \quad is \ \ arbitrary, \end{cases}$$

or

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 + \alpha_1\alpha_2 - \alpha_1 \ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \quad is \ \ arbitrary \\ n(x,y,z) = \alpha_1, \end{cases}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}[M_1(x+y)M_2(x-y)M_3(z)-1] + \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}[M_1(x+y)M_2(x-y)M_3(z)-1] \\ \qquad\qquad -\dfrac{\alpha_1}{k_2}[M_1(x+y)M_2(x-y)M_3(z)-1] + \beta_2 \\ h(x,y,z) = \dfrac{1}{k_1k_2}[M_1(x+y)M_2(x-y)M_3(z)-1] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}[M_1(x+y)M_2(x-y)M_3(z)-1] + \beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}[M_1(x+y)M_2(x-y)M_3(z)-1] + \alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}[M_1(x+y)M_2(x-y)M_3(z)-1] + \alpha_1, \end{cases}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary constant and k_1, k_2 are nonzero constants.

- for the functional equation (1.23):

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 \\ h(x,y,z) = \beta_1 + \alpha_1\alpha_2 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \quad is \ \ arbitrary, \end{cases}$$

or

$$\begin{cases} f(x,y,z) = \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \beta_2 + \alpha_1\alpha_2 - \alpha_1 \ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \quad is \ \ arbitrary \\ n(x,y,z) = \alpha_1, \end{cases}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}[M_1(x+y)M_2(y-x)M_3(z)-1] + \beta_1 + \beta_2 + \alpha_1\alpha_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}[M_1(x+y)M_2(y-x)M_3(z)-1] \\ \qquad\qquad -\dfrac{\alpha_1}{k_2}[M_1(x+y)M_2(y-x)M_3(z)-1] + \beta_2 \\ h(x,y,z) = \dfrac{1}{k_1k_2}[M_1(x-y)M_2(x+y)M_3(z)-1] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}[M_1(x-y)M_2(x+y)M_3(z)-1] + \beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}[M_1(x+y)M_2(y-x)M_3(z)-1] + \alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}[M_1(x-y)M_2(x+y)M_3(z)-1] + \alpha_1, \end{cases}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary constant and k_1, k_2 are nonzero constants.

General solution functions $f, g : \mathbb{R}^3 \to \mathbb{R}$ of the patial pexiderized functional equation

$$f(ux+vy, uy-vx, wz) = f(x,y,z) + f(v,u,w) + g(x,y,z)g(u,v,w), \tag{4.9}$$

are given by

$$\begin{cases} f(x,y,z) = \delta^2[M_1(\sqrt{x^2+y^2})M_2(z)-1] \\ g(x,y,z) = \delta[M_1(\sqrt{x^2+y^2})M_2(z)-1], \end{cases}$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and δ is an arbitrary constant.

It would be of interest to investigate whether the each of method of the proof can be extended to treat the general n-dimensions.

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