BEST PROXIMITY POINT THEOREMS FOR NONLINEAR MAPPINGS IN PARTIAL METRIC SPACES

BY

MR. APHINAT NINSRI

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (MATHEMATICS)

DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
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THESIS

BY

MR. APHINAT NINSRI

ENTITIED

BEST PROXIMITY POINT THEOREMS FOR NONLINEAR MAPPINGS IN PARTIAL METRIC SPACES

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ABSTRACT

Fixed point theory is the powerful tool for solving many real-world problems since many problems can be transformed to the fixed point problem. A fundamental theorem in fixed point theory is the Banach contraction mapping principle. This principle has many applications in several branches and so it was extended in many directions. However, almost all such results dilate upon the existence and uniqueness of a fixed point for self-mappings on some appropriate space such as a metric space, a norm space, an inner product space, and etc. In the case of nonself-mappings, the fixed point problem might have no solution and hence the concept of a best proximity point is introduced for approximating the best solution. This concept is also an important tool for investigating the global optimization problems. In this thesis, we introduce several various new types of generalized contraction mappings covering many types in the literature and give the idea of several tools for proving the best proximity point results. Based on the new tools, we establish the best proximity point results for the purposed generalized contraction mappings in partial metric spaces by using two methods including the fixed point method and the direct method. Our results improve the main results of Su and Yao [Su Y. and Yao, J. C. (2015). Further generalized contraction mapping principle and best proximity theorem on metric spaces. *Fixed Point Theory Appl.*, 2015:120.], Azizi *et al.* [Azizi, A., Moosaei, M., and Zarei, G. (2016). Fixed point theorems for almost generalized $\mathcal{C}$-contractive mappings in ordered complete metric spaces. *Fixed Point Theory and Appl.*, 2016:80.], and Nashine *et al.* [Nashine, H. K., Kadelburg, Z., Radenović, S., and Kim, J. K. (2012). Fixed point theorems under Hardy-Rogers contractive conditions on 0-complete ordered partial metric spaces. *Fixed Point Theory and Appl.*, 2012:180.] and many results in the literature. Moreover, we will give some example for supporting our results while many results in the literature can not be applied in such example. This guarantees the proper real generalization of our results.

Keywords: IC_p-property, 0-continuity, almost generalized $\mathcal{PC}$-contractions, weak ψ-ϕ-contractions, generalized p-Hardy-Rogers contractions

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Mr. Aphinat Ninsri

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CHAPTER 1

INTRODUCTION

In this chapter, we separate to two sections. First, we describe the history of fixed point and best proximity point results, and we give some works which are the inspiration of this thesis. Second, we give the overview described the objectives and the content of this thesis.

1.1 Literature review

Fixed point theory is an important tool for solving many problems, and it has many applications in several areas. Several problems can be changed as an equation of the form

$$Tx = x, \tag{1.1.1}$$

where T is a self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. A point x satisfying (1.1.1) is called a fixed point of T (see some example in Figure 1.1).

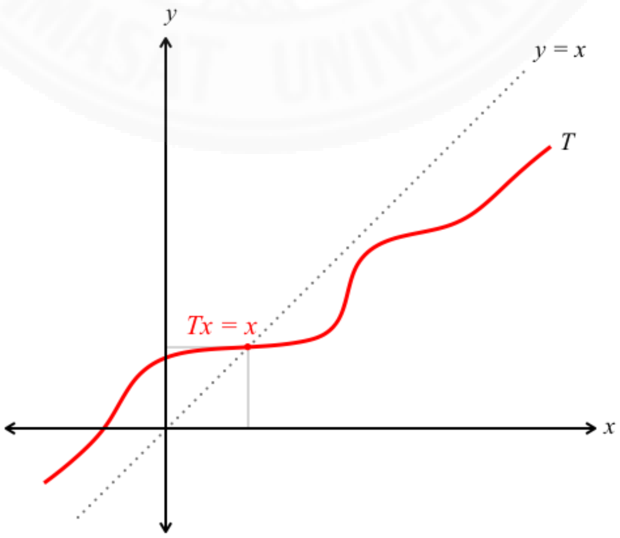

Figure 1.1: A fixed point for a mapping $T : \mathbb{R} \to \mathbb{R}$

The first important metric fixed point result, that is, the Banach contraction mapping principle, was celebrated by Banach in his thesis in 1922. This principle concerns with the Banach contractive condition of self mappings on complete metric spaces as follows:

Theorem 1.1.1 (The Banach contraction mapping principle [2]). Let (X,d) be a complete metric space and $T : X \to X$ be a **Banach contraction mapping**, that is, there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y) \tag{1.1.2}$$

for all $x,y \in X$. Then T has a unique fixed point.

The important theoretical applications of the Banach contraction mapping principle are the proof of the existence and uniqueness of solutions of ordinary differential equations, partial differential equations, integral equations, and linear algebraic equations. The process for finding the solutions of mentioned problems can be converted to the form of the fixed point problem for some Banach contraction mappings in metric spaces. Based on the above applications, this principle was expanded and developed in many ways and then many fixed point results were made. Some interested ways of extending and improving the Banach contraction mapping principle are

(1) to extend the contractive condition (1.1.2) to more general contractive conditions,

(2) to replace the metric space (X,d) by certain generalized metric spaces,

(3) to extend the self-mapping T to more general nonself-mappings.

In the first mentioned direction, the first results were due to Kannan [12], Chaterjea [5], Zamfirescu [24], and many others. In addition, many investigations extend the Banach contractive condition by using several control functions such as works of Geraghty [9], Hardy and Rogers [10], Berinde [4], Suzuki [20] and many authors. In recently, Yan [22] introduced new contraction mappings with control functions and then Su and Yao [21] extended the idea in [22] by given the improved contraction mappings and established the fixed point theorem for such new generalized contraction mappings in metric spaces. Most recently, Azizi *et al.* [1] introduced the concept of an almost generalized C-contraction mapping by using two control functions which is more general than the weak contraction mapping due to Berinde [4]. Moreover, they established the existence of fixed point theorems in metric spaces.

In the second mentioned direction, many mathematicians introduced new generalized metric spaces. For instant, Matthews [14] introduced the concept of a partial metric space as a one of generalizations of the concept of a metric space. The notion of a partial metric space as a part of the study of denotational semantics of data-flow networks. Moreover, he established new fixed point results for some contractions in partial metric spaces. Afterward, several authors have focused on fixed point theorems in partial metric spaces such as Heckmann [11], Kopperman *et al.* [13], Romaguera [17], and Rus [18]. In 2012, Nashine *et al.* [15] established a fixed point theorem for some contraction mappings in 0-complete order partial metric spaces.

In the last mentioned direction, the motivation of this derection based on the fact that some problem cannot change to the form of a self-mapping. So we will consider in the sense of a nonself-mapping $T : A \to B$, where A and B are two nonempty subsets of a metric space (X,d). If $A \cap B = \emptyset$, then the equation $Tx = x$ might have no solution. Under this circumstance, it is meaningful to find a point $x \in A$ such that $d(x,Tx)$ is minimum. If

$$d(x,Tx) = d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}, \tag{1.1.3}$$

then $d(x,Tx)$ is the global minimum value $d(A,B)$ and so x is an approximate solution of the equation $Tx = x$ with the least possible error. Such a solution is known as a **best proximity point** of the mapping T and thus a point $x \in A$ is called the best proximity point of T if $d(x,Tx) = d(A,B)$ (see the idea of this point in Figure 1.2). On the another view, a best proximity point of T is a point $x \in A$ such that $d(x,Tx)$ is a global minimizing of the following problem:

$$\boxed{\min d(x, Tx) \text{ subject to } x \in A}$$

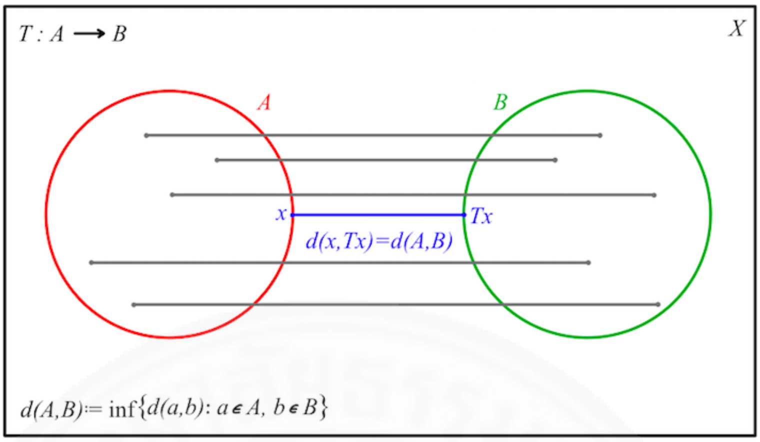

Figure 1.2: The idea of a best proximity point

In 1969, Fan [6] initiated the notion of the best proximity point and established a classical best approximation theorem. Afterwards, several authors prove the existence and the uniqueness of best proximity point results in some distance spaces. In recent time, many mathematician investigated best proximity point results in metric spaces and partial metric spaces by using the fixed point method and the directed method.

The aim of this thesis is to prove the best proximity point theorems in partial metric spaces which are the extending and improving of the fixed point results of the following mathematicians:

(1) Su and Yao [21];

(2) Azizi *et al.* [1];

(3) Nashine *et al.* [15].

The useful tools for proving the main results in this thesis consists two methods, that is, the fixed point method and the direct method. Moreover, we illustrate to support our results by showing the example which cannot be applied by the results in the literature.

1.2 Overview

The objectives of this thesis are

(1) to extend fixed point results of Su and Yao [21] to best proximity point results in partial metric spaces;

(2) to extend fixed point results of Azizi *et al.* [1] to best proximity point results in partial metric spaces;

(3) to extend fixed point results of Nashine *et al.* [15] to best proximity point results in partial metric spaces.

In the first mentioned topic, we introduce the new type of mappings which is called a weak ψ-ϕ-contraction mapping and then we establish the fixed point theorems for such mappings in partial metric spaces. Next, we present some example and numerical result for the main result. By providing this example, we show that our main result is a real generalization of the fixed point results of several mathematicians in the literatures. Moreover, we apply the fixed point result to prove the existence theorems of best proximity points results for the nonself-mappings in partial metric spaces. All of these results are the improved work of Su and Yao [21].

In the second mentioned topic, we introduce the new generalized contraction mapping which is called the almost generalized $\mathcal{PC}$-contraction mapping and then we establish some common fixed point theorem for such mappings in partial metric spaces. Moreover, we use the common fixed point result to prove the extence theorems of common best proximity point results for the nonself-mappings in partial metric spaces. All of these results are the improved work of Azizi *et al.* [1].

In the last mentioned topic, we define the concept of a generalized p-Hardy-Rogers contraction mapping in the framework of partial metric spaces. Also, we introduce the new concept of continuity is called 0-continuity in partial metric spaces and establish the existence of best proximity points result for p-Hardy-Rogers contraction mappings in 0-complete partially ordered partial metric spaces by using the purposed continuity. All of these results are the improved work of Nashine *et al.* [15].

Next, we are going to clarify the content of this thesis.

In Chapter 1, we describe about history of fixed point and best proximity point results which are the motivation of this thesis, and give the overviews of this thesis.

In Chapter 2, we describe about all of notations, definitions, theorems, and useful tools for using in next chapter.

In Chapters 3,4,5, we describe about main results of this thesis. We introduce new generalized contraction types and establish best proximity point results in partial metric spaces by using two methods including the fixed point method and the direct method. Our results improve the main results of Su and Yao [21], Azizi *et al.* [1], and Nashine *et al.* [15] and the obtained results can be applied in the global optimization problems. In addition, we will give some example for supporting our result which is a real generalization of several mathematicians in the literatures.

In Chapter 6, we describe about some conclusions and advantages of the main results of this thesis.

CHAPTER 2

PRELIMINARIES

In this chapter, we give some notations, definitions, properties, examples and other useful tools for using in this thesis. First of all, we denote some notations as follows:

- $\mathbb{Z}$ denotes the set of integers,
- $\mathbb{N}$ denotes the set of positive integers,
- $\mathbb{R}$ denotes the set of real numbers,
- $\mathbb{R}^-$ denotes the set of negative real numbers,
- $\mathbb{R}^+$ denotes the set of positive real numbers,
- $\mathbb{R}_+$ denotes the set of nonnegative real numbers,
- $\mathbb{Q}$ denotes the set of rational numbers.

2.1 Partially order sets

In this section, we give the definition and examples of partially order sets as follows:

Definition 2.1.1. Let X be a nonempty set. A binary relation $\preceq$ on X is called a **partial order** in X if it satisfying the following conditions for all $x, y, z \in X$:

(a) $x \preceq x$ (*reflexivity*);

(b) if $x \preceq y$ and $y \preceq x$, then $x = y$ (*antisymmetry*);

(c) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (*transitivity*).

A nonempty set X with a partial order defined on X is called a **partially ordered set**. Two elements x and y in a partially ordered set are called **comparable** if either $x \preceq y$ or $y \preceq x$ holds. A subset K of X is said to be **well ordered** if every two elements of K are comparable. Moreover, the symbol $x \prec y$ means that $x \preceq y$ but $x \neq y$.

Example 2.1.2. (1) The ordered pair $(\mathbb{R}, \leq)$ is a partially ordered set.

(2) Let X be a set. Then $(P(X), \subseteq)$ is a partially ordered set.

(3) For $a, b \in \mathbb{Z}$, we let $a|b$ means a divides b. Then $(\mathbb{Z}, |)$ is a partially ordered set.

Definition 2.1.3. Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two partially ordered sets. A function $f : X \to Y$ is called

(1) **increasing** if and only if $x \prec_X y \Longrightarrow f(x) \preceq_Y f(y)$;

(2) **decreasing** if and only if $x \prec_X y \Longrightarrow f(y) \preceq_Y f(x)$;

(3) **strictly increasing** if and only if $x \prec_X y \Longrightarrow f(x) \prec_Y f(y)$;

(4) **strictly decreasing** if and only if $x \prec_X y \Longrightarrow f(y) \prec_Y f(x)$.

We give examples of the above definition as follows:

Example 2.1.4. Let $(\mathbb{R}, \leq)$ be a partially ordered set and $f : (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ be defined by

$$f(x) = x^3$$

for all $x \in \mathbb{R}$. Then f is an increasing function (see in Figure 2.1).

Example 2.1.5. Let $(\mathbb{R}, \leq)$ be a partially ordered set and $f : (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ be defined by

$$f(x) = -x^3$$

for all $x \in \mathbb{R}$. Then f is a decreasing function (see in Figure 2.2).

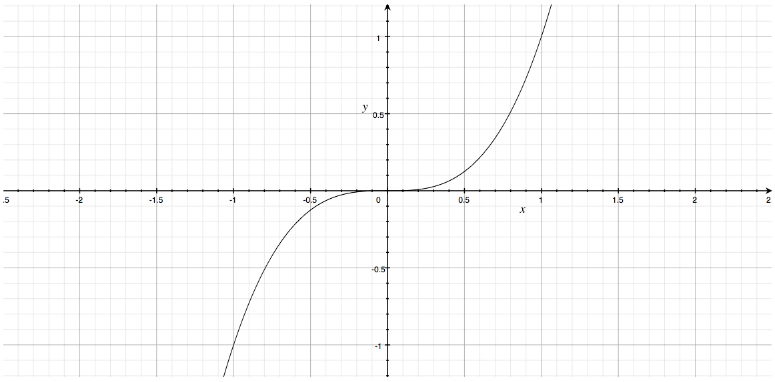

Figure 2.1: An example of an increasing function

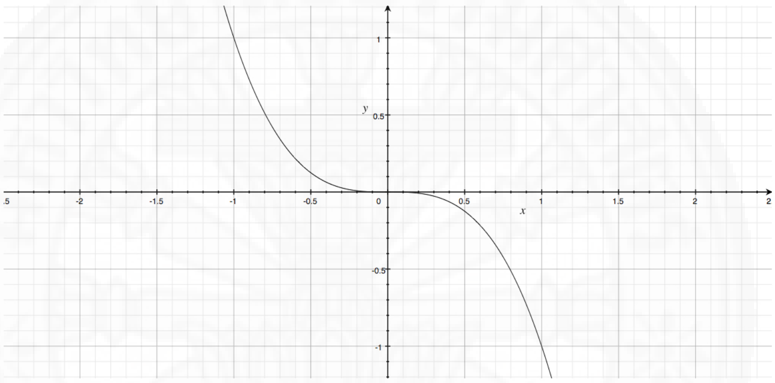

Figure 2.2: An example of a decreasing function

Definition 2.1.6. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subseteq X$ is called

(1) **increasing** (or **nondecreasing**) if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$;

(2) **decreasing** (or **nonincreasing**) if $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$;

(3) **strictly increasing** if $x_n \prec x_{n+1}$ for all $n \in \mathbb{N}$;

(4) **strictly decreasing** if $x_{n+1} \prec x_n$ for all $n \in \mathbb{N}$;

(5) **monotone** if it is either increasing or decreasing.

We give examples of the above definition as follows:

Example 2.1.7. Let $\{x_n\}$ be a sequence in a partially order set $(\mathbb{R}, \leq)$ defining by

$$x_n = n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is an increasing sequence.

Example 2.1.8. Let $\{x_n\}$ be a sequence in a partially order set $(\mathbb{R}, \leq)$ defining by

$$x_n = 1 - n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a decreasing sequence.

Example 2.1.9. Let $\{x_n\}$ be a sequence in a partially order set $(\mathbb{R}, \leq)$ defining by

$$x_n = (-1)^n \frac{1}{n}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is not a monotone sequence.

2.2 Metric spaces

In 1906, the French mathematician Fréchet [7] introduced the concept of a metric space which is the center of several research activities. Here, we give the definition of a metric space as follows:

Definition 2.2.1 ([7])**.** Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is called a **metric** on X if the following conditions hold for all $x, y, z \in X$:

(M1) $d(x,y) \geq 0$ (*non-negativity*);

(M2) $d(x,y) = 0 \iff x = y$ (*identity of indiscernibles*);

(M3) $d(x,y) = d(y,x)$ (*symmetry*);

(M4) $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

The set X together with a metric d is called a **metric space**, which is denoted by (X,d).

Now, we give some examples of a metric space as follows:

Example 2.2.2. Let X be nonempty set and $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then d is a metric on $\mathbb{R}$, which is called **discrete metric**, and $(\mathbb{R}, d)$ is called a **discrete metric space**.

Example 2.2.3. Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) = |x - y|$$

for all $x, y \in \mathbb{R}$. Then d is a metric on $\mathbb{R}$, which is called **usual metric**, and $(\mathbb{R}, d)$ is called a **usual metric space**.

Example 2.2.4. The n-dimensional set $\mathbb{R}^n$ is a metric space with respect to the mapping $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

for all $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The function d is called a **Euclidean metric** and $(\mathbb{R}^n, d)$ is called a **Euclidean metric space**.

Example 2.2.5. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

for all $x, y \in \mathbb{R}^2$. Then d is a metric on $\mathbb{R}^2$, which is called the $\boldsymbol{l^1}$**-metric**. It's also referred to informally as the **taxicab metric** because it's the distance one would travel by taxi on a rectangular grid of streets.

Example 2.2.6. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

for all $x, y \in \mathbb{R}^2$. Then d is a metric on $\mathbb{R}^2$, which is called the $\boldsymbol{l^\infty}$**-metric**, or the **maximum metric**.

Example 2.2.7. Let $C(K)$ be the set of continuous functions $f : K \to \mathbb{R}$, where $K \subseteq \mathbb{R}$ is compact; for example we could take $K = [a,b]$ to be a closed, bounded interval. For all $f,g \in C(K)$, define

$$d(f,g) = \sup_{x \in K} |f(x) - g(x)|.$$

The function $d : C(K) \times C(K) \to \mathbb{R}$ is well-defined, since a continuous function on a compact set is bounded; in fact, such a function attains it maximum value, so we could also write

$$d(f,g) = \max_{x \in K} |f(x) - g(x)|.$$

Then d is a metric on $C(K)$, called **Chebyshev distance** or **maximum metric**.

Definition 2.2.8. Let (X,d) be a metric space, $a \in X$ and $r > 0$. Then the set

$$B_r(a) := \{x \in X : d(x,a) < r\}$$

is called a **neighborhood** (or an open ball) with centre a and radius r.

Definition 2.2.9. Let (X,d) be a metric space. A set $G \subseteq X$ is said to be **open** if, for each $x \in G$, there exists an $r > 0$ such that $B_r(a) \subseteq G$.

Definition 2.2.10. Let (X,d) be a metric space. A set $G \subseteq X$ is **closed** if it is the complement of an open set.

Definition 2.2.11. Let (X,d) be a metric space and a set $G \subseteq X$.

(1) The **interior** of G is the union of all open subsets of G and it is denoted by $int(G)$ or G°.

(2) The **closure** of G is the intersection of all closed sets that contain G and it is denoted by $cl(G)$ or $\overline{G}$.

Definition 2.2.12. Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X.

(1) A sequence $\{x_n\}$ is called **Cauchy sequence** if for every $\epsilon > 0$ there exists a positive integer N such that

$$d(x_n, x_m) < \epsilon$$

for all $n,m \geq N$.

(2) A sequence $\{x_n\}$ is called **convergent** to a point x in X if for every $\epsilon > 0$ there exists a positive integer N such that

$$d(x_n, x) < \epsilon$$

for all $n \geq N$. Denoted by $x_n \to x$ as $n \to \infty$ or

$$\lim_{n\to\infty} x_n = x.$$

(3) A metric space (X, d) is said to be **complete** if every Cauchy sequences in X converges to an element of it.

Lemma 2.2.13. A sequence $\{x_n\}$ in a metric space (X, d) converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} d(x_n, x) = 0$.

Lemma 2.2.14. A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$ or $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

We will show some examples of a complete metric space as follows:

Example 2.2.15. (1) The usual metric spaces $\mathbb{R}$ and $\mathbb{C}$ are complete.

(2) The usual metric space $\mathbb{Q}$ is not complete.

(3) The Chebyshev metric space is complete.

(4) The discrete metic space is complete.

Definition 2.2.16. Let (X, d) be a metric space. A **limit point** of a set G in X is an element $\overline{x} \in X$ for which there is a sequence in G that converges to $\overline{x}$.

Definition 2.2.17. Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $T : X \to Y$ is called **continuous** if, for every $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$y \in X \text{ with } d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(Tx, Ty) < \epsilon.$$

Remark 2.2.18. A mapping T is continuous if and only if a mapping T is sequentially continuous, that is, whenever $\{x_n\}$ is convergent to x, $\{Tx_n\}$ is a convergent to Tx.

Definition 2.2.19. Let (X,d) be a metric space. A mapping $T : X \to \mathbb{R}$ is said to be **lower semi-continuous** at $x_0 \in X$ if for each sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ as $n \to \infty$, we have

$$Tx_0 \leq \liminf_{n\to\infty} Tx_n.$$

Example 2.2.20. Let $(\mathbb{R},d)$ be a metric space and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then T is a lower semi-continuous at $x = 0$.

Definition 2.2.21. Let (X,d) be a metric space. A mapping $T : X \to \mathbb{R}$ is said to be **upper semi-continuous** at $x_0 \in X$ if for each sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} Tx_n \leq Tx_0.$$

Example 2.2.22. Let $(\mathbb{R},d)$ be a metric space and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then T is a upper semi-continuous at $x = 0$.

Definition 2.2.23. The partially order metric space $(X,d,\preceq)$[1] is called **regular** if it has the following properties:

(a) if $\{x_n\}$ is any nondecreasing sequence in X converging to x, then $x_n \preceq x$ for any $n \in \mathbb{N}$;

(b) if $\{x_n\}$ is any nonincreasing sequence in X converging to x, then $x_n \succeq x$ for any $n \in \mathbb{N}$.

[1] (X,d) is a metric space and $\preceq$ is a partially order set

2.3 Partial metric spaces

In 1994, Matthews [14] introduced the concept of a partial metric space as a one of generalizations of the concept of a metric space as follows:

Definition 2.3.1 ([14])**.** Let X be a nonempty set. A mapping $p: X \times X \to [0,\infty)$ is called a **partial metric** on X if the following conditions hold for all $x,y,z \in X$:

(P1) $p(x,x) \leq p(x,y)$;

(P2) $x = y \iff p(x,x) = p(x,y) = p(y,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a **partial metric space**.

Remark 2.3.2. From (P1), (P2) and (P3), if $p(x,y) = 0$, then $x = y$. But the converse need not be true.

Now, we give some examples of a partial metric space as follows:

Example 2.3.3. Let $X = [0,\infty)$ and $p: X \times X \to [0,\infty)$ be defined by

$$p(x,y) = \max\{x,y\}$$

for all $x,y \in X$. Then (X,p) is a partial metric space.

Example 2.3.4. Let $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$ and $p: X \times X \to [0,\infty)$ be defined by

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$$

for all $[a,b],[c,d] \in X$. Then (X,p) is a partial metric space.

Each partial metric p on a nonempty set X generates a T_0-topology τ_p on X which has the family of open p-balls $\{B_p(x,\epsilon) : x \in X : \epsilon > 0\}$, where

$$B_p(x,\epsilon) := \{y \in X : p(x,y) < p(x,x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$, forms a base of τ_p. The concepts of the closeness and the closure of a set in partial metric spaces are taken from topological space (X, τ_p).

Definition 2.3.5 ([14]). Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X.

(1) A sequence $\{x_n\}$ is called **convergent** to a point $x \in X$ if and only if $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$, which is denoted by $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ is called a **Cauchy sequence** if and only if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(3) The partial metric space (X,p) is said to be **complete** if and only if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Remark 2.3.6. A limit of a sequence $\{x_n\}$ in a partial metric space (X,p) need not be unique. Moreover, if $\{x_n\}$ and $\{y_n\}$ are sequences in a partial metric space (X,p) such that $x_n \to x \in X$ and $y_n \to y \in X$, then $p(x_n,y_n)$ need not be converges to $p(x,y)$, that is, p need not be continuous.

Next, we give an example which supports the above remark.

Example 2.3.7. Let $X = [0,\infty)$ and $p : X \times X \to [0,\infty)$ be defined by

$$p(x,y) = \max\{x,y\}$$

for $x,y \in X$. Then (X,p) is a partial metric space. Define the sequence $\{x_n\}$ in X by $x_n = 2$ for all $n \in \mathbb{N}$. For each $x \geq 2$, we obtain

$$p(x_n,x) = p(x,x)$$

for all $n \in \mathbb{N}$. It implies that

$$\lim_{n\to\infty} p(x_n,x) = p(x,x)$$

for all $x \geq 2$, that is, $\{x_n\}$ converges to a point $x \geq 2$. Thus the limit of $\{x_n\}$ need not be unique. Also, if $x_n \to 4$ as $n \to \infty$, then

$$\lim_{n\to\infty} p(x_n, x_n) = 2 \neq p(4,4),$$

which yields that p need not be continuous.

If p is a partial metric on a nonempty set X, then the mapping $d_p : X \times X \to [0,\infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2.3.1}$$

is a metric on X. Furthermore, a sequence $\{x_n\}$ in X converges to $x \in X$ in the sense of a metric space (X, d_p) if and only if

$$p(x,x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.3.2}$$

Example 2.3.8. Let $X = [0,\infty)$ and $p : X \times X \to [0,\infty)$ be defined by

$$p(x,y) = \max\{x,y\}$$

for $x, y \in X$. Then (X,p) is a partial metric space. The corresponding metric $d_p : X \times X \to [0,\infty)$ is defined by

$$d_p(x,y) = 2\max\{x,y\} - x - y = |x-y|$$

for all $x, y \in X$.

Example 2.3.9. Let (X,d) be a metric space and $c \geq 0$. Then a mapping $p : X \times X \to [0,\infty)$ which is defined by

$$p(x,y) = d(x,y) + c$$

for all $x, y \in X$, is a partial metric on X. So the corresponding metric $d_p : X \times X \to [0,\infty)$ is defined by

$$d_p(x,y) = 2d(x,y)$$

for all $x, y \in X$.

Here, we give the relations between the concepts of a Cauchy sequence (the completeness) in a partial metric space (X,p) and a Cauchy sequence (and the completeness) in the corresponding metric space (X,d_p).

Lemma 2.3.10 ([14])**.** Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}$ in X is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).

(2) A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.

Lemma 2.3.11 ([23])**.** Let (X,p) be a partial metric space and d_p the induced metric. If $\{x_n\},\{y_n\} \subseteq X$ converge to $x \in X$, and $y \in X$, with respect to d_p, then $\{p(x_n,y_n)\}$ converge to $p(x,y)$ as $n \to \infty$.

Definition 2.3.12 ([17])**.** Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}$ in (X,p) is called a **0-Cauchy sequence** if and only if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$.

(2) The partial metric space (X,p) is said to be **0-complete** if and only if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p to a point $x \in X$ such that

$$p(x,x) = 0.$$

Remark 2.3.13. It's easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

Lemma 2.3.14 ([17])**.** Let (X,p) be a partial metric space.

(1) Every 0-Cauchy sequence in (X,p) is Cauchy sequence in (X,d_p).

(2) If (X,p) is complete, then it is 0-complete.

Example 2.3.15 ([17]). The set $X = [0,\infty) \cap \mathbb{Q}$ with the mapping $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = \max\{x,y\}$$

for all $x,y \in X$, is a 0-complete partial metric space, but it is not complete because $d_p(x,y) = |x-y|$ and (X,d_p) is not complete. Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for all $n \in \mathbb{N}$ is a Cauchy sequence in (X,p), but it is not a 0-Cauchy sequence.

Definition 2.3.16. The partially order partial metric space $(X,p,\preceq)$[2] is called **regular** if it has the following properties:

(a) if $\{x_n\}$ is any nondecreasing sequence in X converging to x, then $x_n \preceq x$ for any $n \in \mathbb{N}$;

(b) if $\{x_n\}$ is any nonincreasing sequence in X converging to x, then $x_n \succeq x$ for any $n \in \mathbb{N}$.

2.4 Fixed point and common fixed point basics

In this section, we give some definitions, notations, and examples of fixed points and common fixed points as follows:

Definition 2.4.1. Let X,Y be a nonempty sets and $T : X \to Y$ be a mapping. A point $x \in X$ is called a **fixed point** of T if

$$Tx = x.$$

Example 2.4.2. Let $X = \mathbb{R}$ and a mapping $T : X \to X$ be defined by

$$Tx = x^3$$

for all $x \in X$. Therefore, the fixed points of T are $-1,0,1$.

[2] (X,p) is a partial metric space and $\preceq$ is a partially order set

Example 2.4.3. Let $X = \mathbb{R}$ and a mapping $T : X \to X$ be defined by

$$Tx = x^2 + x + 1$$

for all $x \in X$. Therefore, T has no fixed points.

Example 2.4.4. Let $X = \mathbb{R}$ and a mapping $T : X \to X$ be defined by

$$Tx = \cos(x)$$

for all $x \in X$. Therefore, T has a fixed point, as one can see by looking at the Figure 2.3.

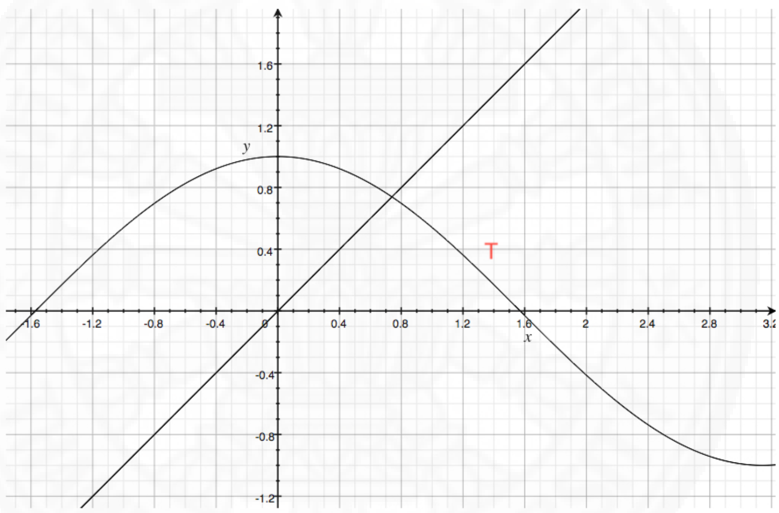

Figure 2.3: The function $Tx = \cos(x)$

Next, we will give the definition of a common fixed point as follows:

Definition 2.4.5. Let X be a nonempty set and $S, T : X \to X$ be two mappings. A point $x \in X$ is called a **common fixed point** of S and T if

$$Sx = Tx = x.$$

Example 2.4.6. Let $X = [0,1]$ and $S, T : X \to X$ be two mappings defined by

$$Sx = \frac{x}{2} - \frac{x^2}{8}$$

for all $x \in X$ and

$$Tx = \frac{x}{2}$$

for all $x \in X$. Therefore, the common fixed point of S and T is 0.

Example 2.4.7. Let $X = [0,\infty)$ and $S,T : X \to X$ be two mappings defined by

$$Sx = \begin{cases} 4 & \text{if } 0 \leq x < 1, \\ x^4 & \text{if } 1 \leq x < \infty \end{cases}$$

and

$$Tx = \begin{cases} 3 & \text{if } 0 \leq x < 1, \\ 2 - \frac{1}{x^3} & \text{if } 1 \leq x < \infty. \end{cases}$$

Therefore, the common fixed point of S and T is 1.

2.5 Best proximity point and common best proximity point basics

In this section, we give some definitions, notations, and examples fo best proximity points and common best proximity points as follows:

Definition 2.5.1. Let A, B be two subsets of a metric space (X,d) and $T : A \to B$ be a mapping. A point $x \in A$ is called a **best proximity point** of T if

$$d(x,Tx) = d(A,B),$$

where $d(A,B) := \inf\{d(a,b) : a \in A \text{ and } b \in B\}$.

Remark 2.5.2. In the case of A and B are equal, the best proximity points reduce to the fixed points.

Example 2.5.3. Let $A = [-1,0]$ and $B = [0,1]$ be two subsets of a usual metric space $(\mathbb{R},d)$. Define a mapping $T : A \to B$ by

$$Tx = \begin{cases} -\frac{x}{2} & \text{if } x \in A, \\ -x & \text{if } x \in B. \end{cases}$$

Therefore, the best proximity point of T is 0. In this case, 0 is also a fixed point of T and $0 \in A \cap B$.

Example 2.5.4. Let $X = \mathbb{R}^2$ and d be a metric on X defined as

$$d(x,y) = |x_1 - x_2| + |y_1 - y_2|$$

for $x = (x_1, y_1)$, $y = (x_2, y_2) \in X$. Let

$$\begin{aligned} A &= \{(x,1) : 0 \leq x \leq 1\}, \\ B &= \{(x,-1) : 0 \leq x \leq 1\}, \end{aligned}$$

and $T : A \to B$ be defined as follows:

$$T\Big((x,1)\Big) = \Big(\frac{x}{2}, -1\Big)$$

for all $(x,1) \in A$. Therefore, the best proximity point of T is a point $(0,1) \in A$.

Definition 2.5.5. Let A, B be two subsets of a metric space (X,d) and $S, T : A \to B$ be two mappings. A point $x \in A$ is called a **common best proximity point** of S and T if

$$d(x, Sx) = d(x, Tx) = d(A,B),$$

where $d(A,B) := \inf\{d(a,b) : a \in A \text{ and } b \in B\}$.

Remark 2.5.6. In the case of S and T are equal, the common best proximity points reduce to the best proximity points.

Example 2.5.7. Consider the space $\mathbb{R}$ with the usual metric and let $A = \{-4, 0, 4\}$, $B = \{-2, -1, 2\}$. Define two mappings $S, T : A \to B$ by

$$S(0) = -1, \quad S(4) = 2, \quad S(-4) = -2 \quad \text{ and } \quad T(x) = -1$$

for all $x \in A$, respectively. Then the common best proximity point of S and T is 0.

Example 2.5.8. Consider the space $\mathbb{R}$ with the usual metric and let $A = [1, \infty)$, $B = (\infty, -1]$. Define two mappings $S, T : A \to B$ by

$$Sx = -1, \qquad Tx = -x$$

for all $x \in A$ and define the mappings $F, G : B \to A$ by

$$Fy = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ 2 & \text{otherwise,} \end{cases} \qquad Gy = -y$$

for all $y \in B$. Therefore, the common best proximity point of S and T is 1. Also, the common best proximity point of F and G is -1.

Example 2.5.9. Let $X = [0,1] \times [0,1]$ and d be the Euclidean metric. Let

$$A = \{(0,y) : 0 \leq y \leq 1\},$$
$$B = \{(1,y) : 0 \leq y \leq 1\},$$

and $S, T : A \to B$ be defined as follows:

$$S(0,y) = (1,y) \text{ for all } y \in A \quad \text{and} \quad T(0,y) = \left(1, \frac{y}{4}\right) \text{ for all } y \in A.$$

Therefore, the common best proximity point of S and T is a point $(0,0)$.

In partial metric spaces, the concepts of a best proximity point and a common best proximity point has the same sense in metric spaces. Next, we give the details of these concepts.

Definition 2.5.10. Let A, B be two subsets of a partial metric space (X, p) and $T : A \to B$ be a mapping. A point $x \in A$ is called a **best proximity point** of T if

$$p(x, Tx) = p(A, B),$$

where $p(A,B) := \inf\{p(a,b) : a \in A \text{ and } b \in B\}$.

Lemma 2.5.11. Let A, B be two subsets of a partial metric space (X, p) and $T : A \to B$ be a mapping. If $A = B$ and a point $x \in A$ is a best proximity point of T, then x is a fixed point of T.

Proof. Since $A = B$ and a point $x \in A$ is a best proximity point of T, we get

$$p(x, Tx) = p(A, B) = p(A, A).$$

By using (P1) and the definition of $p(A,A) = \inf\{p(a,a) : a \in A\}$, we get

$$p(x,Tx) \leq p(x,x) \leq p(x,Tx) \tag{2.5.1}$$

and

$$p(x,Tx) \leq p(Tx,Tx) \leq p(x,Tx). \tag{2.5.2}$$

It follows from (2.5.1) and (2.5.2) that

$$p(x,Tx) = p(x,x) = p(Tx,Tx).$$

By using (P2), we obtain $Tx = x$. This completes the proof. □

Definition 2.5.12. Let A,B be two subsets of a partial metric space (X,p) and $S,T : A \to B$ be two mappings. A point $x \in A$ is called a **common best proximity point** of S and T if

$$p(x,Sx) = p(x,Tx) = p(A,B),$$

where $p(A,B) := \inf\{p(a,b) : a \in A \text{ and } b \in B\}$.

2.6 Contractive conditions with fixed point results

2.6.1 The classical results

The fundamental contractions are given by many researchers such as Banach [2], Kannan [12], Chatterjea [5], Zamfirescu [24], Hardy and Rogers [10], Berinde [4], and Geraghty [9]. The first importance contractive condition was introduced by Banach [2] as follows:

Definition 2.6.1 ([2])**.** The self-mapping T on a metric space (X,d) is called a **Banach contraction mapping** if there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y) \tag{2.6.1}$$

for all $x,y \in X$.

Moreover, he established a unique fixed point theorem as follows:

Theorem 2.6.2 ([2]). Let (X,d) be a complete metric space and $T : X \to X$ be a Banach contraction mapping. Then T has a unique fixed point.

Example 2.6.3. Let $(\mathbb{R},d)$ be a usual metric space and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \frac{x}{2} + 5$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R},d)$ is a complete metric space and T is a Banach contraction mapping. By Theorem 2.6.2, T has a unique fixed point.

Example 2.6.4. Let $X = [1,2]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = (1+x)^{\frac{1}{3}}$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R},d)$ is a complete metric space and T is a Banach contraction mapping. By Theorem 2.6.2, T has a unique fixed point.

Example 2.6.5. Let $X = [0,1]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \cos(x)$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R},d)$ is a complete metric space and T is a Banach contraction mapping. By Theorem 2.6.2, T has a unique fixed point.

In 1969, Kannan [12] introduced new contraction mappings and established a unique fixed point theorem for such mappings in complete metric spaces as follows:

Definition 2.6.6 ([12]). The self-mapping T on a metric space (X,d) is called a **Kannan contraction mapping** if there exists $k \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \tag{2.6.2}$$

for all $x,y \in X$.

Theorem 2.6.7 ([12])**.** Let (X,d) be a complete metric space and $T: X \to X$ be a Kannan contraction mapping. Then T has a unique fixed point.

Example 2.6.8. Let $X = [0,1]$, d be a usual metric on X and $T: X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, \frac{1}{2}), \\ \frac{x}{5} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then $(\mathbb{R}, d)$ is a complete metric space and T is a Kannan contraction mapping. By Theorem 2.6.7, T has a unique fixed point.

The Kannan fixed point result is not an extension of the Banach contraction mapping principle. In 1972, Chatterjea [5] introduced a new contraction mapping which is not an extension of a Banach contraction mapping and a Kannan contraction mapping as follows:

Definition 2.6.9 ([5])**.** The self-mapping T on a metric space (X,d) is called a **Chatterjea contraction mapping** if there exists $k \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq k[d(x,Ty) + d(y,Tx)] \tag{2.6.3}$$

for all $x, y \in X$.

Also, he established a unique fixed point result for Chatterjea contraction mappings in complete metric spaces as follows:

Theorem 2.6.10 ([5])**.** Let (X,d) be a complete metric space and $T: X \to X$ be a Chatterjea contraction mapping. Then T has a unique fixed point.

Example 2.6.11. Let $X = [0,1]$, d be a usual metric on X and $T: X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{5} & \text{if } x \in [0, \frac{8}{15}), \\ \frac{1}{3}, & \text{if } x \in [\frac{8}{15}, 1]. \end{cases}$$

Then $(\mathbb{R}, d)$ is a complete metric space and T is a Chatterjea contraction mapping. By Theorem 2.6.10, T has a unique fixed point.

Afterward, Zamfirescu [24] extended the contractive conditions of Banach [2], Kannan [12], and Chatterjea [5] to a new generalized contractive condition as follows:

Definition 2.6.12 ([24])**.** The self-mapping T on a metric space (X,d) is called a **Zamfirescu contraction mapping** if there exists $\xi \in [0,1)$ such that

$$d(Tx,Ty) \leq \xi \max\left\{d(x,y), \frac{1}{2}[d(x,Tx)+d(y,Ty)], \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}$$

for all $x,y \in X$.

Remark 2.6.13. It is easy to see that the Zamfirescu contraction mapping improve the following mappings:

(1) the Banach contraction mapping in [2];

(2) the Kannan contraction mapping in [12];

(3) the Chatterjea contraction mapping in [5].

Then he established the new results for Zamfirescu contraction mappings in complete metric spaces as follows:

Theorem 2.6.14 ([24])**.** Let (X,d) be a complete metric space and $T : X \to X$ be a Zamfirescu contraction mappings. Then T has a unique fixed point.

Example 2.6.15. Let $X = [0,1]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{2}{3} & \text{if } x \in [0,1), \\ 0 & \text{if } x = 1. \end{cases}$$

Then $(\mathbb{R},d)$ is a complete metric space and T is a Zamfirescu contraction mapping. By Theorem 2.6.14, T has a unique fixed point.

In 1973, Hardy and Rogers [10] introduced the new contractive condition covering the contractive conditions of Banach [2], Kannan [12], and Chatterjea [5] and established the fixed point theorem as follows:

Theorem 2.6.16 ([10]). Let (X,d) be a complete metric space and T be a self-mapping of X satisfying the following condition:

$$d(Tx,Ty) \leq ad(x,Tx)+bd(y,Ty)+cd(x,Ty)+ed(y,Tx)+fd(x,y) \qquad (2.6.4)$$

for all $x,y \in X$, where a,b,c,e,f are nonnegative real numbers with $a+b+c+e+f<1$. Then T has a unique fixed point.

In the same year, Geraghty [9] introduced the new generalized contractive condition which is called Geraghty-contractive condition and established the famous fixed point theorem for mappings satisfying such contractive condition in metric spaces as follows:

Definition 2.6.17 ([9]). Let (X,d) be a metric space and $\beta:[0,\infty)\to[0,1)$ be a function satisfying the following condition:

$$\beta(t_n)\to 1 \text{ as } n\to\infty \quad \Longrightarrow \quad t_n\to 0 \text{ as } n\to\infty.$$

The mapping $T:X\to X$ is called a **Geraghty-contraction mapping** if and only if

$$d(Tx,Ty)\leq \beta(d(x,y))d(x,y)$$

for all $x,y\in X$.

Remark 2.6.18. If we take $\beta(t)=k\in[0,1)$ in the above definition, it reduces to the concept of the Banach contraction mapping.

Theorem 2.6.19 ([9]). Let (X,d) be a complete metric space and $T:X\to X$ be a Geraghty-contraction mapping. Then T has a unique fixed point.

Example 2.6.20 ([3]). Let $X=[0,\infty)$, d be a usual metric on X and $T:X\to X$ be defined by

$$Tx=\frac{x}{1+x}$$

for all $x\in X$ and $\beta:[0,\infty)\to[0,1)$ by

$$\beta(t)=\begin{cases}\frac{2}{2+t} & \text{if } t>0,\\ 0 & \text{if } t=0.\end{cases}$$

Then $(\mathbb{R},d)$ is a complete metric space and T is a Geraghty-contraction mapping. By Theorem 2.6.19, T has a unique fixed point.

In 2004, Berinde [4] extended the contractive conditions due to Banach [2], Kannan [12], Chatterjea [5], and Zamfirescu [24] and established the fixed point results for mappings satisfying such condition as follows:

Definition 2.6.21 ([4])**.** The self-mapping T on a metric space (X,d) is called a **weak contraction mapping** if there exist a constant $\delta \in [0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx) \tag{2.6.5}$$

for all $x,y \in X$.

Theorem 2.6.22 ([4])**.** Let (X,d) be a complete metric space and $T: X \to X$ be a weak contraction mapping. Then T has a fixed point.

In addition, he imposed an additional contractive condition for proving the uniqueness of the fixed point of a weak contraction mapping as follows:

Theorem 2.6.23 ([4])**.** Let (X,d) be a complete metric space and $T: X \to X$ be a weak contraction mapping for which there exist $\theta \in [0,1)$ and some $L_1 \geq 0$ such that

$$d(Tx,Ty) \leq \theta d(x,y) + L_1 d(x,Tx)$$

for all $x,y \in X$. Then T has a unique fixed point.

Example 2.6.24 ([4])**.** Let $X = [0,1]$, d be a usual metric on X and $T: X \to X$ be identity mappings, i.e.,

$$Tx = x$$

for all $x \in X$. Then $(\mathbb{R},d)$ is a complete metric space and T is a Geraghty-contraction mapping. By Theorem 2.6.22, T has a fixed point.

In 2008, Suzuki [20] proved a fixed point theorem that is a generalization of the Banach contraction mapping principle [2] as follows:

Theorem 2.6.25 ([20])**.** Let (X,d) be a complete metric space and T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ onto $(\frac{1}{2},1]$ by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \leq r \leq 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \leq d(x,y) \quad \Longrightarrow \quad d(Tx,Ty) \leq rd(x,y) \tag{2.6.6}$$

for all $x,y \in X$. Then T has a unique fixed point $z \in X$.

Example 2.6.26 ([20])**.** Let $X = \{(0,0),(4,0),(0,4),(4,5),(5,4)\}$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for all $(x_1,x_2),(y_1,y_2) \in X$. Then (X,d) is a complete metric space. Define a mapping T by

$$T(x_1,x_2) = \begin{cases} (x_1,0) & \text{if } x_1 \leq x_2, \\ (0,x_2) & \text{if } x_1 > x_2. \end{cases}$$

Then T satisfies the condition (2.6.6) in Theorem 2.6.25. Therefore, T has a unique fixed point.

2.6.2 The results of Su and Yao

First, we give needed notations about the class of some useful control functions. Throughout this thesis, unless otherwise specified, Γ denotes the class of all functions $\gamma : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) γ is continuous and nondecreasing;

(b) $\gamma(t) = 0$ if and only if $t = 0$.

In 2012, Yan *et al.* [22] established a new fixed point theorem by extending the Banach contraction mapping principle and using some control functions as follows:

Theorem 2.6.27 ([22])**.** Let $(X, \preceq)$ be a partially order sets and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that $T : X \to X$ is a continuous and nondecreasing mapping such that

$$\gamma(d(Tx,Ty)) \leq \phi(d(x,y)) \tag{2.6.7}$$

for all $x,y \in X$ with $x \succeq y$, where $\gamma \in \Gamma$ and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\gamma(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

In 2015, Su and Yao [21] extended the idea in [22] by given the idea of improved contraction mappings and established the fixed point theorem for such new generalized contraction mappings in metric spaces as follows:

Theorem 2.6.28 ([21])**.** Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)) \tag{2.6.8}$$

for all $x,y \in X$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions satisfying the following conditions:

(a) $\psi(a) \leq \phi(b) \quad \Longrightarrow \quad a \leq b$;

(b) $\begin{cases} \psi(a_n) \leq \phi(b_n) \\ a_n \to \epsilon,\, b_n \to \epsilon \end{cases} \quad \Longrightarrow \quad \epsilon = 0.$

Then T has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to this fixed point.

Example 2.6.29 ([21])**.** The following functions satisfy the conditions (a) and (b) in Theorem 2.6.28:

(1) $\begin{cases} \psi_1(t) = t, \\ \phi_1(t) = \alpha t, \end{cases}$ where $0 < \alpha < 1$ is a constant;

(2) $\begin{cases} \psi_2(t) = t^2, \\ \phi_2(t) = \ln(t^2+1); \end{cases}$

(3) $\begin{cases} \psi_3(t) = t, \\ \phi_3(t) = \begin{cases} t^2 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ t - \frac{3}{8} & \text{if } \frac{1}{2} < t < \infty; \end{cases} \end{cases}$

(4) $\begin{cases} \psi_4(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1, \\ t - \frac{1}{2} & \text{if } 1 < t < \infty, \end{cases} \\ \phi_4(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \leq t \leq 1, \\ t - \frac{4}{5} & \text{if } 1 < t < \infty; \end{cases} \end{cases}$

(5) $\begin{cases} \psi_5(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ \alpha t^2 & \text{if } 1 \leq t < \infty, \end{cases} \\ \phi_5(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 1, \\ \beta t & \text{if } 1 \leq t < \infty, \end{cases} \end{cases}$ where $0 < \beta < \alpha$ are constants.

2.6.3 The results of Azizi *et al.*

First, we give needed notations about the class of some useful control functions. Throughout this thesis, unless otherwise specified, Λ denotes the class of all functions $\lambda : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) λ is lower semi-continuous and nondecreasing with respect to both of its components;

(b) $\lambda(s,t) = 0$ if and only if $s = t = 0$.

In 2016, Azizi *et al.* [1] extended the idea of Berinde [4] by given the idea of almost generalized C-contractive mappings and established common fixed point theorems for such new generalized contraction mappings in partially order complete metric spaces as follows:

Definition 2.6.30 ([1])**.** Let $(X,\preceq,d)$ be a partially order metric space, and let f,g be two self-mappings of X. The mapping f is said to be **almost generalized $\mathcal{C}$-contraction with respect to g** if there exist $\xi \geq 0$ and $(\gamma,\lambda) \in \Gamma \times \Lambda$ such that

$$\gamma(d(fx,gx)) \leq \gamma(M(x,y)) - \lambda(M'(x,y),M''(x,y)) + \xi\gamma(N(x,y)) \tag{2.6.9}$$

for all $x,y \in X$ with $x \preceq y$, where

$$\begin{aligned} &M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \tfrac{d(x,gy)+d(y,fx)}{2}\right\}, \\ &M'(x,y) = \max\{d(x,y), d(x,fx), d(x,gy)\}, \\ &M''(x,y) = \max\{d(x,y), d(y,gy), d(fx,y)\}, \quad \text{and} \\ &N(x,y) = \min\{d(x,fx), d(y,fx), d(x,gy)\}. \end{aligned}$$

Theorem 2.6.31 ([1])**.** Let $(X,\preceq,d)$ be an ordered complete metric space and $f,g : X \to X$ be two weakly increasing mappings[3] which f is an almost generalized $\mathcal{C}$-contraction mapping with respect to g. If either f or g is continuous, then f and g have a common fixed point.

Example 2.6.32 ([1])**.** Let $X = [1,\infty)$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ x+y & \text{if } x \neq y. \end{cases}$$

Then (X,d) is a complete metric space. Define the mappings $f,g : X \to X$ by

$$fx = \begin{cases} 1 & \text{if } 1 \leq x \leq 3, \\ x-2 & \text{if } 3 < x \end{cases}$$

[3] $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$

and

$$gx = \begin{cases} 1 & \text{if } 1 \leq x \leq 4, \\ x-3 & \text{if } 4 < x. \end{cases}$$

Also, define $\gamma : X \to X$ and $\lambda : X \times X \to X$ by $\gamma(t) = t^2$ and $\lambda(s,t) = \frac{s+t}{2}$ for all $s,t \in X$, respectively. Consider a relation $\preceq$ on X by

$$x \preceq y \iff y \leq x$$

for all $x,y \in X$. Then f and g satisfy all of hypotheses in Theorem 2.6.31. Therefore, f and g have common fixed points.

2.6.4 The results of Nashine *et al.*

In 2012, Nashine *et al.* [15] established a fixed point theorem in 0-complete ordered partial metric spaces which is more general than the result of Hardy and Rogers [10] as follows:

Theorem 2.6.33 ([15])**.** Let $(X,p,\preceq)$ be a 0-complete ordered partial metric space and $T : X \to X$ be a nondecreasing (nonincreasing) mapping such that

$$p(Tx,Ty) \leq M(x,y) \tag{2.6.10}$$

for all comparable $x,y \in X$, where

$$M(x,y) = Ap(x,y) + Bp(x,Tx) + Cp(y,Ty) + Dp(y,Tx) + Ep(x,Ty),$$

$A,B,C,D,E \geq 0$ and $A+B+C+D+E < 1$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ (resp. $x_0 \succeq Tx_0$) and

(a) T is continuous or

(b) X is regular.

Then T has a fixed point z and $p(Tz,Tz) = 0 = p(z,z)$.

Example 2.6.34 ([15])**.** Let $X = [0,\infty) \cap \mathbb{Q}$ and $p : X \times X \to [0,\infty)$ defined by $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then (X,p) is a 0-complete partial metric space. We endow X with the partial order $\preceq$ by

$$x \preceq y \iff x = y \quad \text{or} \quad (x,y \in [0,1] \text{ with } x \leq y).$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x^2}{1+x} & \text{if } x \in [0,1], \\ \frac{x}{2} & \text{if } x > 1. \end{cases}$$

Then T satisfies all of hypotheses of Theorem 2.6.33. Therefore, T has a fixed point.

CHAPTER 3

BEST PROXIMITY POINT RESULTS FOR WEAK ψ-ϕ-CONTRACTION MAPPINGS

In this chapter, we improve the results of Su and Yao [21] by defining the new type of contraction mappings and extend fixed point results to best proximity point results in partial metric spaces. That is, we introduce the new type of mappings which is called a weak ψ-ϕ-contraction mapping and establish fixed point theorems for such mappings in partial metric spaces. We present some example and numerical result for supporting the main result. By providing this example, we show that our main result is a real generalization of the fixed point results of several mathematicians in the literatures. Moreover, we apply the fixed point result to prove the existence theorems of best proximity points for the nonself-mappings in partial metric spaces.

3.1 The weak ψ-ϕ-contraction mappings with fixed point results

In this section, we introduce the concept of a weak ψ-ϕ-contraction mapping in the setting of partial metric spaces and prove the existence and convergence theorems of fixed points for such mappings. Moreover, we present some example and numerical result for supporting the main result.

Definition 3.1.1. Let (X,p) be a partial metric space. The mapping $T : X \to X$ is called a **weak ψ-ϕ-contraction mapping** if

$$\psi(p(Tx,Ty)) \leq \phi(p(x,y)) \tag{3.1.1}$$

for all $x,y \in X$, where $\psi,\phi : [0,\infty) \to [0,\infty)$ are two functions satisfying the following conditions:

(a) $a,b \in [0,\infty)$ with $\psi(a) \leq \phi(b) \implies a \leq b$;

(b) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in X such that $\psi(p(a_n,b_n)) \leq \phi(p(c_n,d_n))$ and $p(a_n,b_n) \to \epsilon$, $p(c_n,d_n) \to \epsilon$ as $n \to \infty \implies \epsilon = 0$.

Remark 3.1.2. From Definition 3.1.1, if T is a weak ψ-ϕ-contraction mapping, then we get

$$p(Tx,Ty) \leq p(x,y) \tag{3.1.2}$$

for all $x,y \in X$, that is, T is a nonexpensive mapping in the sense of partial metric spaces.

Theorem 3.1.3. Let (X,p) be a complete partial metric space and $T : X \to X$ be a weak ψ-ϕ-contraction mapping. Then T has a unique fixed point. Moreover, for any given $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to the fixed point of T.

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ in X by

$$x_n = T^n x_0 = Tx_{n-1} \tag{3.1.3}$$

for all $n \in \mathbb{N}$. From (3.1.1) and (3.1.3) we have

$$\psi(p(x_{n+1},x_n)) = \psi(p(Tx_n,Tx_{n-1})) \leq \phi(p(x_n,x_{n-1})) \tag{3.1.4}$$

for all $n \in \mathbb{N}$. From Remark 3.1.2, we have

$$p(x_{n+1},x_n) \leq p(x_n,x_{n-1})$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{p(x_{n+1},x_n)\}$ is nonincreasing and so there exists $\delta \geq 0$ such that

$$p(x_{n+1},x_n) \to \delta \tag{3.1.5}$$

as $n \to \infty$. By using the condition (b) in Definition 3.1.1 with (3.1.4) and (3.1.5), we have $\delta = 0$. From (P1), we have

$$p(x_n,x_n) \leq p(x_n,x_{n+1})$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequlity, we get

$$p(x_n,x_n) \to 0. \tag{3.1.6}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in partial metric spaces, by using Lemma 2.3.10. Assume that $\{x_n\}$ is not a Cauchy sequence in metric space (X, d_p). It follows that there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ with $n_k > m_k > k$ such that

$$d_p(x_{n_k}, x_{m_k}) \geq \epsilon \tag{3.1.7}$$

for all $k \in \mathbb{N}$. Further, corresponding to m_k we can choose n_k in such a way that it is the smallest inter with $n_k > m_k$ satisfying (3.1.7). Then

$$d_p(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{3.1.8}$$

From (3.1.7) and (3.1.8), we have

$$\begin{aligned}
\epsilon &\leq d_p(x_{n_k}, x_{m_k}) \\
&\leq d_p(x_{n_k}, x_{n_k-1}) + d_p(x_{n_k-1}, x_{m_k}) \\
&< d_p(x_{n_k}, x_{n_k-1}) + \epsilon \\
&\leq 2p(x_{n_k}, x_{n_k-1}) + \epsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequlity, we get

$$\lim_{k\to\infty} d_p(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.1.9}$$

By using the triangular inequality, we have

$$\begin{aligned}
d_p(x_{n_k}, x_{m_k}) &\leq d_p(x_{n_k}, x_{n_k-1}) + d_p(x_{n_k-1}, x_{m_k-1}) + d_p(x_{m_k-1}, x_{m_k}) \\
&\leq 2p(x_{n_k}, x_{n_k-1}) + d_p(x_{n_k-1}, x_{m_k-1}) + 2p(x_{m_k-1}, x_{m_k})
\end{aligned}$$

and

$$\begin{aligned}
d_p(x_{n_k-1}, x_{m_k-1}) &\leq d_p(x_{n_k-1}, x_{n_k}) + d_p(x_{n_k}, x_{m_k}) + d_p(x_{m_k}, x_{m_k-1}) \\
&\leq 2p(x_{n_k-1}, x_{n_k}) + d_p(x_{n_k}, x_{m_k}) + 2p(x_{m_k}, x_{m_k-1}).
\end{aligned}$$

Letting $k \to \infty$ in the above two inequalities and applying (3.1.9), we have

$$\lim_{k\to\infty} d_p(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.1.10}$$

From the definition of d_p, (3.1.6), (3.1.9) and (3.1.10), we get

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \frac{\epsilon}{2} \tag{3.1.11}$$

and

$$\lim_{k\to\infty} p(x_{n_k-1}, x_{m_k-1}) = \frac{\epsilon}{2}, \tag{3.1.12}$$

respectively. From (3.1.1), we obtain

$$\psi(p(x_{n_k}, x_{m_k})) \leq \phi(p(x_{n_k-1}, x_{m_k-1})). \tag{3.1.13}$$

It follows from the condition (b) in Definition 3.1.1 with (3.1.11), (3.1.12) and (3.1.13), we get $\epsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence in metric space (X, d_p).

From Lemma 2.3.10, (X, d_p) is complete and so the sequence $\{x_n\}$ is a convergent sequence in the metric space (X, d_p). It follows that $\lim_{n\to\infty} d_p(x_n, z) = 0$ for some $z \in X$. From the fact in (2.3.2), we have

$$p(z,z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.1.14}$$

Since $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p), we have

$$\lim_{n,m\to\infty} d_p(x_n, x_m) = 0.$$

From the definition of d_p and (3.1.6), we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Therefore, (3.1.14) implies that

$$\lim_{n\to\infty} p(x_n, z) = 0. \tag{3.1.15}$$

Due to (P4) and Remark 3.1.2, we have

$$\begin{aligned} p(Tz, z) &\leq p(Tz, Tx_n) + p(Tx_n, z) - p(Tx_n, Tx_n) \\ &\leq p(z, x_n) + p(Tx_n, z) \\ &= p(z, x_n) + p(x_{n+1}, z). \end{aligned} \tag{3.1.16}$$

Letting $n \to \infty$ in (3.1.16), we get $p(Tz, z) = 0$ and hence $Tz = z$.

Finally, we will show that z is unique fixed point of T. Assume that there exists $u \in X$ such that $z \neq u$ and $Tu = u$. From (3.1.1), we obtain

$$\psi(p(z,u)) = \psi(p(Tz, Tu)) \leq \phi(p(z,u)).$$

By using the condition (b) in Definition 3.1.1, we get $p(z,u)=0$ and hence $z=u$, which is a contradiction. Therefore, z is a unique fixed point of T. This completes the proof. $\square$

The following example shows that Theorem 3.1.3 properly generalizes Theorem 2.1 of Su and Yao [21].

Example 3.1.4. Let $X=[0,1]$ endowed with the partial metric $p: X\times X \to [0,\infty)$ defined by $p(x,y)=\max\{x,y\}$ for all $x,y\in X$. Therefore, the partial metric space (X,p) is complete because (X,d_p) is complete. Indeed, for any $x,y\in X$,

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y)=2\max\{x,y\}-x-y=|x-y|,$$

thus, $(X,d_p)=([0,1],|\cdot|)$ is a complete metric space.

Define a mapping $T: X\to X$ and functions $\psi,\phi:[0,\infty)\to[0,\infty)$ by

$Tx=\dfrac{2x^2}{3}$ for all $x\in X$, $\psi(t)=t$ for all $t\in[0,\infty)$ and $\phi(t)=\dfrac{5t}{6}$ for all $t\in[0,\infty)$.

Now, we will show that conditions (a) and (b) in Definition 3.1.1 hold. First, we assume that $a,b\in[0,\infty)$ with $\psi(a)\le\phi(b)$. Then $a=\psi(a)\le\phi(b)=\dfrac{5b}{6}\le b$. Next, we assume that $\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ are sequences in $[0,\infty)$ with $\psi(p(a_n,b_n))\le\phi(p(c_n,d_n))$ and $p(a_n,b_n)\to\epsilon$, $p(c_n,d_n)\to\epsilon$ as $n\to\infty$. For each $n\in\mathbb{N}$, we get

$$p(a_n,b_n)=\psi(p(a_n,b_n))\le\phi(p(c_n,d_n))=\frac{5p(c_n,d_n)}{6}. \tag{3.1.17}$$

Taking limit as $n\to\infty$ in (3.1.17), we have $\epsilon\le\dfrac{5\epsilon}{6}$ and so $\epsilon=0$.

Next, we will show that T satisfies the condition (3.1.1). Let $x,y\in X$. Without loss of generality, we may assume that $x>y$. Thus, $p(x,y)=\max\{x,y\}=x$ and $p(Tx,Ty)=\max\left\{\dfrac{2x^2}{3},\dfrac{2y^2}{3}\right\}=\dfrac{2x^2}{3}$. Also, we have

$$\psi(p(Tx,Ty))=\psi\left(\frac{2x^2}{3}\right)=\frac{2x^2}{3}<\frac{5x}{6}=\phi(x)=\phi(p(x,y)).$$

Therefore, T is a weak ψ-ϕ-contraction mapping. By using Theorem 3.1.3, we can conclude that T has a unique fixed point.

In this case, $x = 0$ is a unique fixed point of T. For initial points $x_0 = 0.2, 0.3, 0.6, 0.7$ and 0.9, the value of x_n which is defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ and the behavior of these iterations appear in Table 1 and Figure 3.1.4, respectively.

Table 1 Comparative results of Example 3.1.4

Step	$x_0 = 0.2$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.7$	$x_0 = 0.9$
1	0.026666666667	0.060000000000	0.240000000000	0.326666666667	0.540000000000
2	0.000474074074	0.002400000000	0.038400000000	0.071140740741	0.194400000000
3	0.000000149831	0.000003840000	0.000983040000	0.003374003329	0.025194240000
4	0.000000000000	0.000000000010	0.000000644245	0.000007589266	0.000423166486
5	0.000000000000	0.000000000000	0.000000000000	0.000000000038	0.000000119380
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

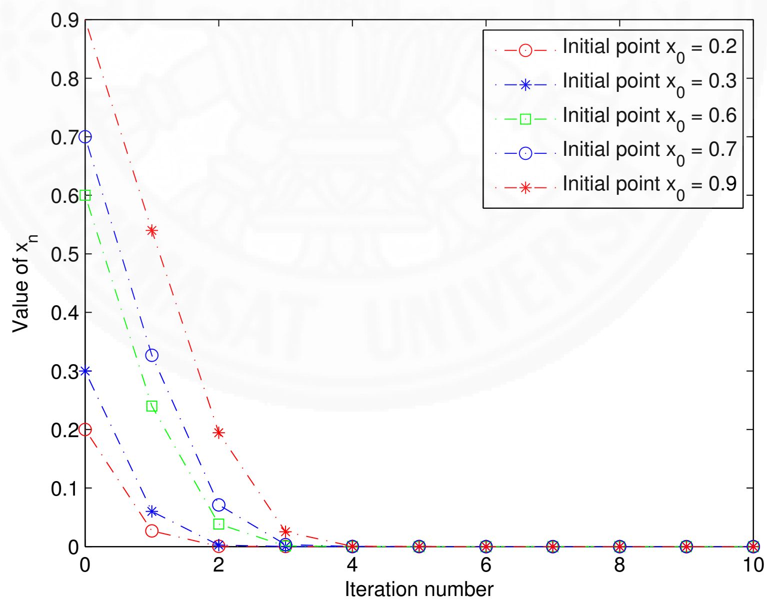

Figure 3.1: Behavior of the iteration processes with initial points $x_0 = 0.2$, 0.3, 0.6, 0.7, 0.9 in Example 3.1.4.

Remark 3.1.5. Under the definition of T in Example 3.1.4 and the usual metric d on X, the contractive condition in Theorem 2.6.28 is not true for every mappings

ψ and ϕ. Indeed, if the contractive condition (2.6.8) holds, we get

$$\psi(d(T(1),T(0.7))) \le \phi(d(1,0.7)) \quad \Longrightarrow \quad \psi(0.34) \le \phi(0.3) \quad \Longrightarrow \quad 0.34 \le 0.3,$$

which is a contradiction. Therefore, the main result of Su and Yao [21] can not be applied for this case.

By using the similar method in the proof of Theorem 3.1.3, we get the following result.

Theorem 3.1.6. Let (X,d) be a complete partial metric space and $T : X \to X$ be a mapping such that

$$\psi(p(Tx,Ty)) \le \phi(p(x,y)) \tag{3.1.18}$$

for all $x,y \in X$, where $\psi,\phi : [0,\infty) \to [0,\infty)$ are two functions satisfying the following conditions:

(a) $a,b \in [0,\infty)$ with $\psi(a) \le \phi(b) \quad \Longrightarrow \quad a \le b$;

(b*) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in X such that $\psi(p(a_n,b_n)) \le \phi(p(c_n,d_n))$ and $d_p(a_n,b_n) \to \epsilon$, $d_p(c_n,d_n) \to \epsilon$ as $n \to \infty \quad \Longrightarrow \quad \epsilon = 0$.

Then T has a unique fixed point. Moreover for any given $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to this fixed point.

Remark 3.1.7. We can prove the similar fixed point results with Theorem 3.1.3 by replacing condition (b) in Definition 3.1.1 and condition (b*) in Theorem 3.1.6 by the following condition:

(b**): $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,\infty)$ such that $\psi(a_n) \le \phi(b_n)$ and $a_n \to \epsilon$, $b_n \to \epsilon$ as $n \to \infty \quad \Longrightarrow \quad \epsilon = 0$.

Example 3.1.8. Consider the functions $\psi_i,\phi_i : [0,\infty) \to [0,\infty)$, where $i = 1,2,3,4,5$, defined by

$\psi_1(t) = t$ and $\phi_1(t) = \alpha t$, where $0 \le \alpha < 1$;

$\psi_2(t) = t^2$ and $\phi_2(t) = \ln(t^2+1)$;

$$\psi_3(t) = t \text{ and } \phi_3(t) = \begin{cases} t^2 & \text{if } x \in [0,1/2], \\ t - \frac{3}{8} & \text{if } x \in (1/2,\infty); \end{cases}$$

$$\psi_4(t) = \begin{cases} t & \text{if } x \in [0,1], \\ t - \frac{1}{2} & \text{if } x \in (1,\infty) \end{cases} \text{ and } \phi_4(t) = \begin{cases} \frac{t}{2} & \text{if } x \in [0,1], \\ t - \frac{4}{5} & \text{if } x \in (1,\infty); \end{cases}$$

$$\psi_5(t) = \begin{cases} t & \text{if } x \in [0,1), \\ \alpha t^2 & \text{if } x \in [1,\infty) \end{cases} \text{ and}$$

$$\phi_5(t) = \begin{cases} t^2 & \text{if } x \in [0,1), \\ \beta t & \text{if } x \in [1,\infty), \end{cases} \text{ where } 0 < \beta < 1 < \alpha;$$

Then mappings ψ_i and ϕ_i satisfy condition (b**) for all $i = 1,2,3,4,5$.

Example 3.1.9. If $\psi,\phi : [0,\infty) \to [0,\infty)$ are two functions satisfying the following conditions:

(a) $\psi(0) = \phi(0) = 0$;

(b) $\psi(t) > \phi(t)$ for all $t > 0$;

(c) ψ is nondecreasing lower semi-continuous and ϕ is upper semi-continuous.

Then ψ and ϕ satisfy condition (b**).

Remark 3.1.10. It has been pointed out in some studies that the several fixed point results can be concluded from our result under some suitable mappings in Examples 3.1.8 and 3.1.9. Hence, our results generalize and complement the many fixed point results in partial metric spaces.

From the fact that each metric space is a partial metric space, we get the following result.

Corollary 3.1.11. Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)) \tag{3.1.19}$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are two functions satisfying the following conditions:

(a) $a, b \in [0, \infty)$ with $\psi(a) \leq \phi(b) \quad \Longrightarrow \quad a \leq b$;

(b) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in X such that $\psi(d(a_n, b_n)) \leq \phi(d(c_n, d_n))$ and $d(a_n, b_n) \to \epsilon$, $d(c_n, d_n) \to \epsilon$ as $n \to \infty \quad \Longrightarrow \quad \epsilon = 0$.

Then T has a unique fixed point. Moreover, for any given $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to the fixed point of T.

From Remark 3.1.7, we obtain the main result of Su and Yao [21].

Corollary 3.1.12 ([21]). Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)) \tag{3.1.20}$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are two functions satisfying the following conditions:

(a) $a, b \in [0, \infty)$ with $\psi(a) \leq \phi(b) \quad \Longrightarrow \quad a \leq b$;

(b) $\{a_n\}$ and $\{b_n\}$ are sequences in $[0, \infty)$ such that $\psi(a_n) \leq \phi(b_n)$ and $a_n \to \epsilon$, $b_n \to \epsilon$ as $n \to \infty \quad \Longrightarrow \quad \epsilon = 0$.

Then T has a unique fixed point. Moreover, for any given $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to the fixed point of T.

3.2 Best proximity point results for weak ψ-ϕ-contraction mappings by using the fixed point method

In this section, we establish the existence and uniqueness results of the best proximity point by using the fixed point results in Section 3.1. We first recollect basic notions, definitions and fundamental results. Throughout this thesis, unless otherwise specified.

Let A, B be two nonempty subsets of a partial metric space (X, p). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : p(x, y) = p(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : p(x, y) = p(A, B) \text{ for some } x \in A\},$$

where $p(A, B) := \inf\{p(a, b) : a \in A \text{ and } b \in B\}$.

Definition 3.2.1. Let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two given functions and (A, B) be a pair of nonempty subsets of a partial metric space (X, p) with $A_0 \neq \emptyset$. The pair (A, B) has the **weak ψ-φ-P-property** if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$p(x_1, y_1) = p(A, B) \text{ and } p(x_2, y_2) = p(A, B) \implies \psi(p(x_1, x_2)) \leq \varphi(p(y_1, y_2)).$$

Lemma 3.2.2. Let (A, B) be a pair of nonempty closed subsets of a complete partial metric space (X, p) such that p is continuous, $A_0 \neq \emptyset$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two functions satisfying the following conditions:

(a) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} \psi(p(a_n, b_m)) = 0$, then $\lim_{n,m\to\infty} p(a_n, b_m) = 0$;

(b) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} p(a_n, b_m) = p(x, x)$ for some $x \in B$, then $\lim_{n,m\to\infty} \varphi(p(a_n, b_m)) = 0$.

Suppose that the pair (A, B) has the weak ψ-φ-P-property. If $\{y_n\}$ is a sequence in B_0 such that $y_n \to b \in B$, then $b \in B_0$.

Proof. Since $y_n \in B_0$ for all $n \in \mathbb{N}$, there is $x_n \in A_0$ such that

$$p(x_n, y_n) = p(A, B) \tag{3.2.1}$$

for all $n \in \mathbb{N}$. It follows from the weak ψ-φ-P-property that

$$\psi(p(x_n, x_m)) \leq \varphi(p(y_n, y_m)) \tag{3.2.2}$$

for all $n,m \in \mathbb{N}$. Since p is continuous, we get $\lim_{n,m\to\infty} p(y_n,y_m) = p(b,b)$. From (b), we obtain

$$\lim_{n,m\to\infty} \varphi(p(y_n,y_m)) = 0. \tag{3.2.3}$$

By using (3.2.3) in (3.2.2), we have

$$\lim_{n,m\to\infty} \psi(p(x_n,x_m)) = 0. \tag{3.2.4}$$

It follows from (a) and (3.2.4) that

$$\lim_{n,m\to\infty} p(x_n,x_m) = 0 \tag{3.2.5}$$

and hence $\{x_n\}$ is a Cauchy sequence in (X,p). By the completeness of X, there is a point $a \in A$ such that $x_n \to a$ as $n \to \infty$. By taking the limit as $n \to \infty$ in (3.2.1), we get $p(a,b) = p(A,B)$. It yields that $b \in B_0$. This completes the proof. $\square$

Theorem 3.2.3. Let (A,B) be a pair of nonempty closed subsets of a complete partial metric space (X,p) such that $A_0 \neq \emptyset$ and $\psi,\varphi,\phi : [0,\infty) \to [0,\infty)$ be three functions satisfying the following conditions:

(a) if $a,b \in [0,\infty)$ with $\psi(a) \leq \phi(b)$, then $a \leq b$;

(b) if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in X such that $\psi(p(a_n,b_n)) \leq \phi(p(c_n,d_n))$ and $p(a_n,b_n) \to \epsilon$, $p(c_n,d_n) \to \epsilon$ as $n \to \infty$, then $\epsilon = 0$;

(c) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} \psi(p(a_n,b_m)) = 0$, then $\lim_{n,m\to\infty} p(a_n,b_m) = 0$;

(d) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} p(a_n,b_m) = p(x,x)$ for some $x \in B$, then $\lim_{n,m\to\infty} \varphi(p(a_n,b_m)) = 0$;

(e) if $a,b \in [0,\infty)$ with $\varphi(a) \leq \phi(b)$, then $a \leq b$.

Suppose that p is continuous and $T : A \to B$ is a sequentially continuous mapping such that

$$\varphi(p(Tx,Ty)) \leq \phi(p(x,y)) \tag{3.2.6}$$

for all $x,y \in A$. If the pair (A,B) has the weak ψ-φ-P-property and $T(A_0) \subseteq B_0$, then there exists a unique point $x^* \in A$ such that $p(x^*,Tx^*) = p(A,B)$.

Proof. First, let $\overline{A}_0$ be the closure of A_0. We claim that $T(\overline{A}_0) \subseteq B_0$. Indeed, if $x \in A_0$, then we are done. If $x \in \overline{A}_0 \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$ as $n \to \infty$. By the sequentially continuity of T, we get a sequence $\{Tx_n\} \subseteq B_0$ converges to a point $Tx \in B$. From Lemma 3.2.2, we have $Tx \in B_0$. Therefore, $T(\overline{A}_0) \subseteq B_0$.

Since $T(\overline{A}_0) \subseteq B_0$, we can define the operator $P_{A_0} : T(\overline{A}_0) \to A_0$ by $P_{A_0}y = x$ for all $y \in T(\overline{A}_0)$, where $x \in A_0$ and $p(x,y) = p(A,B)$. It follows from the weak ψ-φ-P-property and the condition (3.2.6) that

$$\begin{aligned} \psi(p(P_{A_0}Tx_1, P_{A_0}Tx_2)) &\leq \varphi(p(Tx_1, Tx_2)) \\ &\leq \phi(p(x_1, x_2)) \end{aligned}$$

for all $x_1, x_2 \in \overline{A}_0$. This shows that a mapping $P_{A_0} \circ T : \overline{A}_0 \to \overline{A}_0$ is a weak ψ-φ-P-property mapping. By using Theorem 3.1.3 we get $P_{A_0}T$ has a unique fixed point, that is, $P_{A_0}Tx^* = x^* \in A_0$. It implies that

$$p(x^*, Tx^*) = p(A,B).$$

Therefore, x^* is the unique element in A_0 such that $p(x^*, Tx^*) = p(A,B)$. It follows that x^* is also the unique point in A such that $p(x^*, Tx^*) = p(A,B)$. This completes the proof. $\square$

By using the same technique in the proof of Theorem 3.2.3 with Theorem 3.1.6, we get the following result.

Theorem 3.2.4. Let (A,B) be a pair of nonempty closed subsets of a complete partial metric space (X,p) such that $A_0 \neq \emptyset$ and $\psi, \varphi, \phi : [0,\infty) \to [0,\infty)$ are three functions satisfying the following conditions:

(a) if $a,b \in [0,\infty)$ with $\psi(a) \leq \phi(b)$, then $a \leq b$;

(b) if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in X such that $\psi(p(a_n,b_n)) \leq \phi(p(c_n,d_n))$ and $d_p(a_n,b_n) \to \epsilon$, $d_p(c_n,d_n) \to \epsilon$ as $n \to \infty$, then $\epsilon = 0$;

(c) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} \psi(p(a_n,b_m)) = 0$, then $\lim_{n,m\to\infty} p(a_n,b_m) = 0$;

(d) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} p(a_n,b_m)=p(x,x)$ for some $x\in B$, then $\lim_{n,m\to\infty}\varphi(p(a_n,b_m))=0$;

(e) if $a,b\in[0,\infty)$ with $\varphi(a)\le\phi(b)$, then $a\le b$.

Suppose that p is continuous and $T:A\to B$ is a sequentially continuous mapping such that

$$\varphi(p(Tx,Ty))\le\phi(p(x,y)) \tag{3.2.7}$$

for all $x,y\in A$. If the pair (A,B) has the weak ψ-φ-P-property and $T(A_0)\subseteq B_0$, then there exists a unique point $x^*\in A$ such that $p(x^*,Tx^*)=p(A,B)$.

Since each metric space is a partial metric space, we obtain the next results.

Corollary 3.2.5. Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0\neq\emptyset$ and $\psi,\varphi,\phi:[0,\infty)\to[0,\infty)$ are three functions satisfying the following conditions:

(a) if $a,b\in[0,\infty)$ with $\psi(a)\le\phi(b)$, then $a\le b$;

(b) if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in $[0,\infty)$ such that $\psi(d(a_n,b_n))\le\phi(d(c_n,d_n))$ and $d(a_n,b_n)\to\epsilon$, $d(c_n,d_n)\to\epsilon$ as $n\to\infty$, then $\epsilon=0$;

(c) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty}\psi(d(a_n,b_m))=0$, then $\lim_{n,m\to\infty} d(a_n,b_m)=0$;

(d) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} d(a_n,b_m)=0$, then $\lim_{n,m\to\infty}\varphi(d(a_n,b_m))=0$;

(e) if $a,b\in[0,\infty)$ with $\varphi(a)\le\phi(b)$, then $a\le b$.

Suppose that $T:A\to B$ is a sequentially continuous mapping such that

$$\varphi(d(Tx,Ty))\le\phi(d(x,y)) \tag{3.2.8}$$

for all $x,y\in A$. If the pair (A,B) has the weak ψ-φ-P-property and $T(A_0)\subseteq B_0$, then there exists a unique point $x^*\in A$ such that $d(x^*,Tx^*)=d(A,B)$.

Corollary 3.2.6. Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$ and $\psi, \varphi, \phi : [0,\infty) \to [0,\infty)$ be three functions satisfying the following conditions:

(a) if $a, b \in [0,\infty)$ with $\psi(a) \leq \phi(b)$, then $a \leq b$;

(b) if $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,\infty)$ such that $\psi(a_n) \leq \phi(b_n)$ and $a_n \to \epsilon$, $b_n \to \epsilon$ as $n \to \infty$, then $\epsilon = 0$;

(c) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} \psi(d(a_n, b_m)) = 0$, then $\lim_{n,m\to\infty} d(a_n, b_m) = 0$;

(d) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} d(a_n, b_m) = 0$, then $\lim_{n,m\to\infty} \varphi(d(a_n, b_m)) = 0$;

(e) if $a, b \in [0,\infty)$ with $\varphi(a) \leq \phi(b)$, then $a \leq b$.

Suppose that $T : A \to B$ is a sequentially continuous mapping such that

$$\varphi(d(Tx, Ty)) \leq \phi(d(x,y)) \tag{3.2.9}$$

for all $x, y \in A$. If the pair (A,B) has the (ψ,φ)-P-property and $T(A_0) \subseteq B_0$, then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.

CHAPTER 4

COMMON BEST PROXIMITY POINT RESULTS FOR ALMOST GENERALIZED $\mathcal{PC}$-CONTRACTION MAPPINGS

In this chapter, we improve the results of Azizi *et al.* [1] by defining new generalized contraction mappings and extend common fixed point results to common best proximity point results in partial metric spaces. That is, we introduce the new generalized contraction mapping which is called an almost generalized $\mathcal{PC}$-contraction mapping and then we establish some common fixed point theorem for such mappings in partial metric spaces. Moreover, we use the common fixed point result to prove the common best proximity point results for nonself-mappings in partial metric spaces.

4.1 The almost generalized $\mathcal{PC}$-contraction mappings with fixed point results

In this section, we introduce the new generalized contraction mappings and establish common fixed point for such mappings in partial metric spaces. First, we give the definition of the new generalized contraction mappings in partial metric spaces as follows:

Definition 4.1.1. Let (X,p) be a partial metric space and S,T be two self-mappings on X.

(1) The mapping S is said to be an **almost generalized $\mathcal{PC}$-contraction mapping with respect to $\boldsymbol{T}$** if there exist $\xi \geq 0$ and $(\gamma,\lambda) \in \Gamma \times \Lambda$ such that

$$\gamma(p(Sx,Tx)) \leq \gamma\Big(M(x,y)\Big) - \lambda\Big(M'(x,y), M''(x,y)\Big) + \xi\gamma\Big(N(x,y)\Big) \quad (4.1.1)$$

for all $x, y \in X$, where

$$\begin{aligned}
M(x,y) &= \max\left\{p(x,y), p(x,Sx), p(y,Ty), \tfrac{p(x,Ty)+p(y,Sx)}{2}\right\},\\
M'(x,y) &= \max\{p(x,y), p(x,Sx), p(x,Ty)\},\\
M''(x,y) &= \max\{p(x,y), p(y,Ty), p(Sx,y)\},\\
N(x,y) &= \min\{p(x,Sx), p(y,Sx) - p(y,y), p(x,Ty)\}.
\end{aligned}$$

(2) The mapping S is said to be an **almost generalized $\mathcal{PC}$-contraction mapping** if the mappings S is said to be almost generalized $\mathcal{PC}$-contraction mapping with respect to S.

Next, we will give the useful lemma for proving the main theorem as follows:

Lemma 4.1.2. Let (X, p) be a partial metric space, and S, T be two self-mappings of X which S is an almost generalized $\mathcal{PC}$-contractive mapping with respect to T. Fix $x_1 \in X$ and define a sequence $\{x_n\}$ by $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ and the sequence $\{x_n\}$ is nondecreasing, then $\{x_n\}$ is a Cauchy sequence.

Proof. Since S is an almost generalized $\mathcal{PC}$-contraction mapping with respect to T, there exists $(\gamma, \lambda, \xi) \in \Gamma \times \Lambda \times [0, \infty)$ such that

$$\gamma(p(Sx,Ty)) \leq \gamma(M(x,y)) - \lambda(M'(x,y), M''(x,y)) + \xi\gamma(N(x,y)) \tag{4.1.2}$$

for all $x, y \in X$, where

$$\begin{aligned}
M(x,y) &= \max\left\{p(x,y), p(x,Sx), p(y,Ty), \tfrac{p(x,Ty)+p(y,Sx)}{2}\right\},\\
M'(x,y) &= \max\{p(x,y), p(x,Sx), p(x,Ty)\},\\
M''(x,y) &= \max\{p(x,y), p(y,Ty), p(Sx,y)\},\\
N(x,y) &= \min\{p(x,Sx), p(y,Sx) - p(y,y), p(x,Ty)\}.
\end{aligned}$$

Now, we will show that the sequence $\{x_n\}$ is a Cauchy sequence in a partial metric space (X, p). It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. By using Lemma 2.3.10, we will assume that $\{x_{2n}\}$ is not a Cauchy sequence in metric space

(X, d_p). Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of the sequence $\{x_n\}$ with $2n_k > 2m_k > k$ such that

$$d_p(x_{2m_k}, x_{2n_k}) \geq \epsilon \tag{4.1.3}$$

for all $k \in \mathbb{N}$. Further, corresponding to m_k we can choose n_k in such a way that it is the smallest interger with $n_k > m_k$ satisfied (4.1.3). Then

$$d_p(x_{2m_k}, x_{2n_k-2}) < \epsilon. \tag{4.1.4}$$

From (4.1.3), (4.1.4) and the triangle inequality implies that

$$\begin{aligned}
\epsilon &\leq d_p(x_{2m_k}, x_{2n_k}) \\
&\leq d_p(x_{2m_k}, x_{2n_k-2}) + d_p(x_{2n_k-2}, x_{2n_k-1}) + d_p(x_{2n_k-1}, x_{2n_k}) \\
&< \epsilon + d_p(x_{2n_k-2}, x_{2n_k-1}) + d_p(x_{2n_k-1}, x_{2n_k}) \\
&\leq \epsilon + 2p(x_{2n_k-2}, x_{2n_k-1}) + 2p(x_{2n_k-1}, x_{2n_k}).
\end{aligned}$$

Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ and letting $k \to \infty$ in the above inequalities, we get

$$\lim_{k\to\infty} d_p(x_{2m_k}, x_{2n_k}) = \epsilon, \tag{4.1.5}$$

that is,

$$\lim_{k\to\infty} p(x_{2m_k}, x_{2n_k}) = \frac{\epsilon}{2}. \tag{4.1.6}$$

By using the triangle inequality, we have

$$\begin{aligned}
d_p(x_{2m_k}, x_{2n_k}) &\leq d_p(x_{2m_k}, x_{2m_{k+1}}) + d_p(x_{2m_{k+1}}, x_{2n_{k+1}}) + d_p(x_{2n_{k+1}}, x_{2n_k}) \\
&\leq 2p(x_{2m_k}, x_{2m_{k+1}}) + d_p(x_{2m_{k+1}}, x_{2n_{k+1}}) + 2p(x_{2n_{k+1}}, x_{2n_k})
\end{aligned}$$

and

$$\begin{aligned}
d_p(x_{2m_{k+1}}, x_{2n_{k+1}}) &\leq d_p(x_{2m_{k+1}}, x_{2m_k}) + d_p(x_{2m_k}, x_{2n_k}) + d_p(x_{2n_k}, x_{2n_{k+1}}) \\
&\leq 2p(x_{2m_{k+1}}, x_{2m_k}) + d_p(x_{2m_k}, x_{2n_k}) + 2p(x_{2n_k}, x_{2n_{k+1}}).
\end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequalities and applying (4.1.5), we get

$$\lim_{k\to\infty} d_p(x_{2m_{k+1}}, x_{2n_{k+1}}) = \epsilon.$$

Similarly, we obtain

$$\begin{aligned}\lim_{k\to\infty} d_p(x_{2m_{k+1}}, x_{2n_k}) &= \lim_{k\to\infty} d_p(x_{2m_k+2}, x_{2n_k})\\ &= \lim_{k\to\infty} d_p(x_{2m_{k+1}}, x_{2n_{k+1}})\\ &= \epsilon. \qquad (4.1.7)\end{aligned}$$

Then

$$\begin{aligned}\lim_{k\to\infty} p(x_{2m_{k+1}}, x_{2n_k}) &= \lim_{k\to\infty} p(x_{2m_k+2}, x_{2n_k})\\ &= \lim_{k\to\infty} p(x_{2m_{k+1}}, x_{2n_{k+1}})\\ &= \frac{\epsilon}{2}. \qquad (4.1.8)\end{aligned}$$

We substitute x with $x_{2m_{k+1}}$ and y with x_{2n_k} in inequality (4.1.2), it follows that

$$\begin{aligned}\gamma(p(x_{2m_k+2}, x_{2n_{k+1}})) &= \gamma(p(Sx_{2m_{k+1}}, Tx_{2n_k}))\\ &\leq \gamma(M(x_{2m_{k+1}}, x_{2n_k})) - \lambda(M'(x_{2m_{k+1}}, x_{2n_k}), M''(x_{2m_{k+1}}, x_{2n_k}))\\ &\quad +\xi\gamma(N(x_{2m_{k+1}}, x_{2n_k})), \qquad (4.1.9)\end{aligned}$$

where

$$\begin{aligned}M(x_{2m_{k+1}}, x_{2n_k}) &= \max\Big\{p(x_{2m_{k+1}}, x_{2n_k}), p(x_{2m_{k+1}}, Sx_{2m_{k+1}}), p(x_{2n_k}, Tx_{2n_k}),\\ &\qquad \frac{p(x_{2m_{k+1}}, Tx_{2n_k}) + p(x_{2n_k}, Sx_{2m_{k+1}})}{2}\Big\}\\ &= \max\Big\{p(x_{2m_{k+1}}, x_{2n_k}), p(x_{2m_{k+1}}, x_{2m_k+2}), p(x_{2n_k}, x_{2n_{k+1}}),\\ &\qquad \frac{p(x_{2m_{k+1}}, x_{2n_{k+1}}) + p(x_{2n_k}, x_{2m_k+2})}{2}\Big\},\\ M'(x_{2m_{k+1}}, x_{2n_k}) &= \max\{p(x_{2m_{k+1}}, x_{2n_k}), p(x_{2m_{k+1}}, Sx_{2m_{k+1}}), p(x_{2m_{k+1}}, Tx_{2n_k})\}\\ &= \max\{p(x_{2m_{k+1}}, x_{2n_k}), p(x_{2m_{k+1}}, x_{2m_k+2}), p(x_{2m_{k+1}}, x_{2n_{k+1}})\},\\ M''(x_{2m_{k+1}}, x_{2n_k}) &= \max\{p(x_{2m_{k+1}}, x_{2n_k}), p(x_{2n_k}, Tx_{2n_k}), p(x_{2n_k}, Sx_{2m_{k+1}})\}\\ &= \max\{p(x_{2m_{k+1}}, x_{2n_k}), p(x_{2n_k}, x_{2n_{k+1}}), p(x_{2n_k}, x_{2m_k+2})\},\\ N(x_{2m_{k+1}}, x_{2n_k}) &= \min\{p(x_{2m_{k+1}}, Sx_{2m_{k+1}}), p(x_{2n_k}, Sx_{2m_{k+1}}) - p(x_{2n_k}, x_{2n_k}),\\ &\qquad p(x_{2m_{k+1}}, Tx_{2n_k})\}\\ &= \min\{p(x_{2m_{k+1}}, x_{2m_k+2}), p(x_{2n_k}, x_{2m_k+2}) - p(x_{2n_k}, x_{2n_k}),\\ &\qquad p(x_{2m_{k+1}}, x_{2n_{k+1}})\}.\end{aligned}$$

Letting $k \to \infty$ in the above equalities and applying (4.1.6), (4.1.8), we obtain

$$\lim_{k\to\infty} M(x_{2m_{k+1}}, x_{2n_k}) = \frac{\epsilon}{2}, \tag{4.1.10}$$

$$\lim_{k\to\infty} M'(x_{2m_{k+1}}, x_{2n_k}) = \frac{\epsilon}{2}, \tag{4.1.11}$$

$$\lim_{k\to\infty} M''(x_{2m_{k+1}}, x_{2n_k}) = \frac{\epsilon}{2}, \tag{4.1.12}$$

$$\lim_{k\to\infty} N(x_{2m_{k+1}}, x_{2n_k}) = 0. \tag{4.1.13}$$

Taking the limit as $k \to \infty$ in inequality (4.1.9) and using (4.1.10)-(4.1.13), the continuity of γ, and the lower semi-continuity of λ, we get

$$\gamma\left(\frac{\epsilon}{2}\right) \leq \gamma\left(\frac{\epsilon}{2}\right) - \lambda\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right),$$

which yields $\lambda(\frac{\epsilon}{2}, \frac{\epsilon}{2}) = 0$. Therefore, $\frac{\epsilon}{2} = 0$, that is, $\epsilon = 0$, which contradicts the positivity of ϵ. Hence, $\{x_n\}$ is a Cauchy sequence. $\square$

Next, we present the main result of this section.

Theorem 4.1.3. Let (X, p) be a complete partial metric space and $S, T : X \to X$ be two mappings such that S is an almost generalized $\mathcal{PC}$-contraction mapping with respect to T. If either S or T is continuous, then S and T have a unique common fixed point.

Proof. First, we will show the following claim:

$$z \text{ is a fixed point of } S \quad \Longleftrightarrow \quad z \text{ is a fixed point of } T. \tag{4.1.14}$$

$(\Longrightarrow)$ Suppose that z is a fixed point of S. Inequality (4.1.1) implies that

$$\gamma(p(z, gz)) = \gamma(p(fz, gz)) \leq \gamma(M(z, z)) - \lambda(M'(z, z), M''(z, z)) + \xi\gamma(N(z, z)),$$

where

$$\begin{aligned}
M(x,y) &= \max\left\{p(z,z), p(z,Sz), p(z,Tz), \frac{p(z,Tz)+p(z,Sz)}{2}\right\} \\
&= \max\left\{p(z,z), p(z,Tz), \frac{p(z,Tz)+p(z,z)}{2}\right\} \\
&= \max\{p(z,z), p(z,Tz)\} \\
&= p(z,Tz), \\
M'(x,y) &= \max\{p(z,z), p(z,Sz), p(z,Tz)\} \\
&= \max\{p(z,z), p(z,Tz)\} \\
&= p(z,Tz), \\
M''(x,y) &= \max\{p(z,z), p(z,Tz), p(z,Sz)\} \\
&= \max\{p(z,z), p(z,Tz)\} \\
&= p(z,Tz), \\
N(x,y) &= \min\{p(z,Sz), p(z,Sz)-p(z,z), p(z,Tz)\} \\
&= \min\{p(z,z), p(z,Sz)-p(z,z), p(z,Tz)\} \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\gamma(p(z,Tz)) \leq \gamma(p(z,Tz)) - \lambda(p(z,Tz), p(z,Tz)).$$

It yields $\lambda(p(z,Tz), p(z,Tz)) = 0$. Since $\lambda \in \Lambda$, we get $p(z,Tz) = 0$ and so $z = Tz$. Therefore, z is a fixed point of T.

($\Longleftarrow$) By the similarly process, we can show this claim.

Next, we will show the existence of a common fixed point of S and T. Let x_1 be an arbitrary element in X. Define a sequence $\{x_n\}$ by

$$x_{2n} = Sx_{2n-1} \text{ and } x_{2n+1} = Tx_{2n}$$

for all $n \in \mathbb{N}$. If $x_{2m} = x_{2m-1}$ for some $m \in \mathbb{N}$, then

$$Sx_{2m-1} = x_{2m} = x_{2m-1}.$$

So x_{2m-1} is a fixed point of S. From (4.1.14), x_{2m-1} is also a common fixed point of S and T. If $x_{2m+1} = x_{2m}$ for some $m \in \mathbb{N}$, then

$$Tx_{2m} = x_{2m+1} = x_{2m}.$$

Hence, x_{2m} is a fixed point of T. From (4.1.14), x_{2m} is also a common fixed point of S and T. Therefore, we may suppose that $x_n \neq x_{n+1}$ for any $x \in \mathbb{N}$. From (4.1.1), we have

$$\begin{aligned} \gamma(p(x_{2n}, x_{2n+1})) &= \gamma(p(Sx_{2n-1}, Tx_{2n})) \\ &\leq \gamma(M(x_{2n-1}, x_{2n})) - \lambda(M'(x_{2n-1}, x_{2n}), M''(x_{2n-1}, x_{2n})) \\ &\quad +\xi\gamma(N(x_{2n-1}, x_{2n})), \end{aligned} \tag{4.1.15}$$

where

$$\begin{aligned} M(x_{2n-1}, y_{2n}) &= \max\Big\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, Sx_{2n-1}), p(x_{2n}, Tx_{2n}), \\ &\quad \frac{p(x_{2n-1}, Tx_{2n}) + p(Sx_{2n-1}, x_{2n})}{2}\Big\} \\ &= \max\Big\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \\ &\quad \frac{p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})}{2}\Big\} \\ &= \max\Big\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \frac{p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})}{2}\Big\} \\ &\leq \max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}, \end{aligned} \tag{4.1.16}$$

$$\begin{aligned} M'(x_{2n-1}, x_{2n}) &= \max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, Sx_{2n-1}), p(x_{2n-1}, Tx_{2n})\} \\ &= \max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n+1})\} \\ &= \max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n+1})\} \\ &\geq p(x_{2n-1}, x_{2n}), \end{aligned} \tag{4.1.17}$$

$$\begin{aligned} M''(x_{2n-1}, x_{2n}) &= \max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, Tx_{2n}), p(x_{2n}, Sx_{2n-1})\} \\ &= \max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n})\} \\ &\geq p(x_{2n-1}, x_{2n}), \end{aligned} \tag{4.1.18}$$

$$\begin{aligned} N(x_{2n-1},x_{2n}) &= \min\{p(x_{2n-1},Sx_{2n-1}),p(x_{2n},Sx_{2n-1}),p(x_{2n-1},Tx_{2n})\} \\ &= \min\{p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n})-p(x_{2n},x_{2n}),p(x_{2n-1},x_{2n+1})\} \\ &= 0. \end{aligned} \tag{4.1.19}$$

Thus, from inequality (4.1.15)-(4.1.19), we get

$$\begin{aligned} \gamma(p(x_{2n},x_{2n+1})) &\leq \gamma(\max\left\{p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\right\}) \\ &\quad -\lambda(p(x_{2n-1},x_{2n}),p(x_{2n-1},x_{2n})). \end{aligned} \tag{4.1.20}$$

Since $\lambda(p(x_{2n-1},x_{2n}),p(x_{2n-1},x_{2n}))>0$, we get

$$\gamma(p(x_{2n},x_{2n+1}))<\gamma(\max\{p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\}). \tag{4.1.21}$$

It follows from γ is nondecreasing that

$$p(x_{2n},x_{2n+1})<\max\{p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n+1})\}$$

and so

$$p(x_{2n},x_{2n+1})<p(x_{2n-1},x_{2n}). \tag{4.1.22}$$

Hence, the inequality (4.1.20) becomes

$$\gamma(p(x_{2n},x_{2n+1}))\leq\gamma(p(x_{2n-1},x_{2n}))-\lambda(p(x_{2n-1},x_{2n}),p(x_{2n-1},x_{2n})). \tag{4.1.23}$$

Similarly process, we obtain

$$p(x_{2n+1},x_{2n+2})<p(x_{2n+1},x_{2n}). \tag{4.1.24}$$

From the inequalities (4.1.22) and (4.1.24), we get $\{p(x_n,x_{n+1})\}$ is strictly decreasing and bounded below. Then $\{p(x_n,x_{n+1})\}$ is convergent, that is, $\lim_{n\to\infty}p(x_n,x_{n+1})=a\in[0,\infty)$. Taking the limit superior as $n\to\infty$ in (4.1.23), we get

$$\begin{aligned} \limsup_{n\to\infty}\gamma(p(x_{2n+1},x_{2n})) &\leq \limsup_{n\to\infty}\gamma(p(x_{2n},x_{2n-1})) \\ &\quad -\liminf_{n\to\infty}\lambda(p(x_{2n},x_{2n+1}),p(x_{2n},x_{2n-1})). \end{aligned} \tag{4.1.25}$$

By the continuity of γ and the lower semi-continuity of λ, we get

$$\gamma(a)\leq\gamma(a)-\lambda(a,a).$$

Thus, $\lambda(a,a)=0$ and so $a=0$. Therefore,

$$\lim_{n\to\infty} p(x_n,x_{n+1})=0. \tag{4.1.26}$$

Since the sequence $\{x_n\}$ is nondecreasing and $\lim_{n\to\infty} p(x_n,x_{n+1})=0$, Lemma 4.1.2 implies that $\{x_n\}$ is a Cauchy sequence in (X,p). From Lemma 2.3.10, since (X,d_p) is complete and so the sequence $\{x_n\}$ is a convergent sequence in the metric space (X,d_p) and so

$$\lim_{n\to\infty} d_p(x_n,x^*)=0 \tag{4.1.27}$$

for some $x^*\in X$. From (2.3.2), we obtain

$$p(x^*,x^*)=\lim_{n\to\infty} p(x_n,x^*)=\lim_{n,m\to\infty} p(x_n,x_m). \tag{4.1.28}$$

From (4.1.28) and (4.1.27), we obtain

$$\lim_{n\to\infty} p(x_n,x^*)=0. \tag{4.1.29}$$

Without loss of generality we may assume that S is continuous. Since $x_{2n-1}\to x^*$ as $n\to\infty$, by the continuity of S we get $x_{2n}=Sx_{2n-1}\to Sx^*$ as $n\to\infty$. That is, $\lim_{n\to\infty} p(x_{2n},Sx^*)=p(Sx^*,Sx^*)$. From Definition 4.1.1 we obtain

$$\begin{aligned}\gamma(p(Sx^*,Tx_{2n})) &\le \gamma(M(x^*,x_{2n}))-\lambda(M'(x^*,x_{2n}),M''(x^*,x_{2n}))\\ &\quad +\xi\gamma(N(x^*,x_{2n})),\end{aligned} \tag{4.1.30}$$

where

$$\begin{aligned}
&M(x^*,x_{2n})=\max\{p(x^*,x_{2n}),p(x^*,Sx^*),p(x_{2n},Tx_{2n}),\tfrac{p(x^*,Tx_{2n})+p(x_{2n},Sx^*)}{2}\},\\
&M'(x^*,x_{2n})=\max\{p(x^*,x_{2n}),p(x^*,Sx^*),p(x^*,Tx_{2n})\},\\
&M''(x^*,x_{2n})=\max\{p(x^*,x_{2n}),p(x_{2n},Tx_{2n}),p(x_{2n},Sx^*)\},\\
&N(x^*,x_{2n})=\min\{p(x^*,Sx^*),p(x_{2n},Sx^*),p(x^*,Tx_{2n})\}.
\end{aligned}$$

Taking limit as $n\to\infty$ in above equalities, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(x^*,x_{2n})=p(x^*,Sx^*),\\
&\lim_{n\to\infty} M'(x^*,x_{2n})=p(x^*,Sx^*),\\
&\lim_{n\to\infty} M''(x^*,x_{2n})=p(Sx^*,Sx^*),\\
&\lim_{n\to\infty} N(x^*,x_{2n})=0.
\end{aligned} \tag{4.1.31}$$

Taking limit superior as $k \to \infty$ in inequality (4.1.30) and using the continuity of γ and the lower semi-continuity of λ, we get

$$\begin{aligned}\limsup_{n\to\infty} \gamma(p(Sx^*, Tx_{2n})) &\leq \gamma(p(x^*, Sx^*)) - \lambda(p(x^*, Sx^*), p(Sx^*, Sx^*)) + \xi\gamma(0) \\ &= \gamma(p(x^*, Sx^*)) - \lambda(p(x^*, Sx^*), p(Sx^*, Sx^*))\end{aligned}$$

and so

$$\begin{aligned}\limsup_{n\to\infty} \gamma(p(Sx^*, x_{2n+1})) &\leq \gamma(p(x^*, Sx^*)) - \lambda(p(x^*, Sx^*), p(Sx^*, Sx^*)) \\ &\leq \gamma(p(Sx^*, x_{2n+1}) + p(x_{2n+1}, x^*)) - \lambda(p(x^*, Sx^*), p(Sx^*, Sx^*))\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit superior as $n \to \infty$ again, we get

$$\begin{aligned}\limsup_{n\to\infty} \gamma(p(Sx^*, x_{2n+1})) &\leq \limsup_{n\to\infty} \gamma(p(Sx^*, x_{2n+1}) + p(x_{2n+1}, x^*)) \\ &\quad -\lambda(p(x^*, Sx^*), p(Sx^*, Sx^*)).\end{aligned}$$

Thus $\lambda(p(x^*, Sx^*), p(Sx^*, Sx^*)) = 0$ and so $p(x^*, Sx^*) = 0$. Therefore $Sx^* = x^*$. From (4.1.14), we obtain x^* is also a common fixed point of T and thus x^* is a common fixed point of S and T.

Finally, we will show that x^* is a unique common fixed point of S and T. Suppose that y^* is a common fixed point of S and T. By inequality (4.1.1), we get

$$\begin{aligned}\gamma(p(x^*, y^*)) &\leq \gamma(M(x^*, y^*)) - \lambda(M'(x^*, y^*), M''(x^*, y^*)) + \xi\gamma(N(x^*, y^*)) \\ &= \gamma(p(x^*, y^*)) - \lambda(p(x^*, y^*), p(x^*, y^*)).\end{aligned}$$

Thus, $\lambda(p(x^*, y^*), p(x^*, y^*)) = 0$. Since $\lambda \in \Lambda$, we get $p(x^*, y^*) = 0$ and hence $x^* = y^*$. The proof is complete. $\square$

Corollary 4.1.4. Let (X, p) be a complete partial metric space and $S : X \to X$ be a continuous almost generalized $\mathcal{PC}$-contraction mapping. Then S has a unique fixed point.

4.2 Common best proximity point results for almost generalized $\mathcal{PC}$-contraction mappings by using the common fixed point method

In this section, we apply the results from Section 4.1 to solve the common best proximity point results related the global optimization of multi-objective functions.

Frist, we introduce new property which is an important tool for proving the common best proximity point results.

Definition 4.2.1. Let (A,B) be a pair of nonempty closed subsets of a partial metric space (X,p) such that $A_0 \neq \emptyset$ and $S : A \to B$ be a mapping. The pair (A,B) has the ***IC_p*-property with respect to *S*** if every mapping $\xi : S(\overline{A}_0) \to A_0$ satisfying $p(\xi(w), w) = p(A,B)$ for all $w \in S(\overline{A}_0)$ is continuous.

Example 4.2.2. Let (A,B) be a pair of nonempty closed subsets of a partial metric space (X,p) such that $A_0 \neq \emptyset$ and $S : A \to B$ be a mapping. If $A = B$, then the pair (A,B) has the IC_p-property with respect to S.

Next, we establish the new common best proximity point result in partial metric spaces via the common fixed point process in Section 4.1.

Theorem 4.2.3. Let (A,B) be a pair of nonempty closed subsets of a complete partial metric space (X,p) such that $A_0 \neq \emptyset$ and $\gamma, \varphi, \lambda : [0,\infty) \to [0,\infty)$ are three functions satisfying the following conditions:

(a) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} \gamma(p(a_n, b_m)) = 0$, then $\lim_{n,m\to\infty} p(a_n, b_m) = 0$;

(b) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} p(a_n, b_m) = p(x,x)$ for some $x \in B$, then $\lim_{n,m\to\infty} \varphi(p(a_n, b_m)) = 0$.

Suppose that p is continuous, $\xi \geq 0$ and $S,T : A \to B$ are sequentially continuous mappings such that $f(A_0) \subseteq B_0$, $g(A_0) \subseteq B_0$ and S,T satisfy the following condition:

$$\gamma(p(Sx,Ty)) \leq \gamma(M(x,y)) - \lambda(M'(x,y), M''(x,y)) + \xi\gamma(N(x,y)) \qquad (4.2.1)$$

for all $x, y \in X$, where

$$
\begin{aligned}
M(x,y) &= \max\left\{p(x,y), p(x,Sx), p(y,Ty), \tfrac{p(x,Ty)+p(y,Sx)}{2}\right\},\\
M'(x,y) &= \max\{p(x,y), p(x,Sx), p(x,Ty)\},\\
M''(x,y) &= \max\{p(x,y), p(y,Ty), p(Sx,y)\},\\
N(x,y) &= \min\{p(x,Sx), p(y,Sx) - p(y,y), p(x,Ty)\}.
\end{aligned}
$$

If the pair (A,B) has the IC_p-property with respect to S (or the IC_p-property with respect to T) and (A,B) has the weak γ-φ-P-property, then S and T has a unique common best proximity point $x^* \in A$, that is, $p(x^*, Sx^*) = p(x^*, Tx^*) = p(A,B)$.

Proof. Let $\overline{A}_0$ be the closure of A_0. First, we want to show that $S(\overline{A}_0) \subseteq B_0$. Let $x \in \overline{A}_0$. If $x \in A_0$, then $Sx \in B_0$. If $x \in \overline{A}_0 \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$ as $n \to \infty$. By the sequentially continuity of S, we get a sequence $\{Sx_n\} \subseteq B_0$ converges to a point $Sx \in B$. From Lemma 3.2.2 we have $Sx \in B_0$. Therefore, $S(\overline{A}_0) \subseteq B_0$. Similarly, we can prove that $T(\overline{A}_0) \subseteq B_0$.

Since $S(\overline{A}_0) \subseteq B_0$ and $T(\overline{A}_0) \subseteq B_0$, we can define the operators $P_{A_0} : S(\overline{A}_0) \to A_0$ and $Q_{A_0} : T(\overline{A}_0) \to A_0$ by

$$P_{A_0}y = x \text{ for all } y \in S(\overline{A}_0), \text{ where } x \in A_0 \text{ such that } p(x,y) = p(A,B)$$

and

$$Q_{A_0}v = u \text{ for all } v \in T(\overline{A}_0), \text{ where } u \in A_0 \text{ such that } p(u,v) = p(A,B).$$

As the pair (A,B) has the IC_p-property with respect to S (or IC_p-property with respect to T), we obtain $P_{A_0} \circ S$ is continuous (or $P_{A_0} \circ T$ is continuous). It follows from the weak γ-φ-P-property and the condition (4.2.1) that

$$
\begin{aligned}
\gamma(p(P_{A_0}Sx_1, Q_{A_0}Tx_2)) &\le \varphi(p(Sx_1, Tx_2))\\
&\le \gamma(M(x_1,x_2)) - \lambda(M'(x,y), M''(x,y)) + \xi\gamma(N(x,y))
\end{aligned}
$$

for all $x_1, x_2 \in \overline{A}_0$. This shows that a mapping $P_{A_0} \circ S : \overline{A}_0 \to \overline{A}_0$ is a weak γ-φ-P-property mapping with respect to $Q_{A_0} \circ T : \overline{A}_0 \to \overline{A}_0$. By using Theorem 4.1.3, we obtain $P_{A_0}S$ and $P_{A_0}T$ has a unique common fixed point, that is, $P_{A_0}Sx^* = Q_{A_0}Tx^* = x^* \in A_0$. It implies that

$$p(x^*, Sx^*) = p(x^*, Tx^*) = p(A, B).$$

It follows that x^* is also the unique common best proximity point of S and T. This completes the proof. $\square$

Corollary 4.2.4. Let (A, B) be a pair of nonempty closed subsets of a complete partial metric space (X, p) such that $A_0 \neq \emptyset$ and $\gamma, \varphi, \lambda : [0, \infty) \to [0, \infty)$ are three functions satisfying the following conditions:

(a) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} \gamma(p(a_n, b_m)) = 0$, then $\lim_{n,m\to\infty} p(a_n, b_m) = 0$;

(b) if $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\lim_{n,m\to\infty} p(a_n, b_m) = p(x, x)$ for some $x \in B$, then $\lim_{n,m\to\infty} \varphi(p(a_n, b_m)) = 0$.

Suppose that p is continuous, $\xi \geq 0$ and $S : A \to B$ is a sequentially continuous mapping such that $S(A_0) \subseteq B_0$ and S satisfies the following condition:

$$\varphi(p(Sx, Ty)) \leq \gamma(M(x, y)) - \lambda(M'(x, y), M''(x, y)) + \xi\gamma(N(x, y)) \tag{4.2.2}$$

for all $x, y \in X$, where

$$\begin{aligned}
M(x, y) &= \max\left\{p(x, y), p(x, Sx), p(y, Sy), \tfrac{p(x,Sy)+p(y,Sx)}{2}\right\},\\
M'(x, y) &= \max\{p(x, y), p(x, Sx), p(x, Sy)\},\\
M''(x, y) &= \max\{p(x, y), p(y, Sy), p(Sx, y)\},\\
N(x, y) &= \min\{p(x, Sx), p(y, Sx) - p(y, y), p(x, Sy)\}.
\end{aligned}$$

If the pair (A, B) has the weak γ-φ-P-property, then S has a unique best proximity point $x^* \in A$, that is, $p(x^*, Sx^*) = p(A, B)$.

CHAPTER 5

BEST PROXIMITY POINT RESULTS FOR GENERALIZED *P*-HARDY-ROGERS CONTRACTION MAPPINGS

In this chapter, we improve the results of Nashine *et al.* [15] to best proximity point results in partial metric spaces. That is, we define the concepts of a generalized p-Hardy-Rogers contraction mapping and the new type of the continuity in the framework of partial metric spaces. We use these concepts to establish new best proximity point theorems in the sense of 0-complete partially ordered partial metric spaces.

5.1 The generalized *p*-Hardy-Rogers contraction mappings

In this section, we introduce the new idea of a generalized p-Harday-Rogers contraction mapping in the framework of partial metric spaces as follows:

Definition 5.1.1. Let A, B be two subsets of a partially ordered partial metric space $(X, p, \preceq)$. A mapping $T : A \to B$ is called a **generalized *p*-Hardy-Rogers contraction mapping** if there are $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that

$$\left.\begin{array}{l} x, y, u, v \in A, \\ x, y \text{ are comparable}, \\ p(u, Tx) = p(A, B), \\ p(v, Ty) = p(A, B), \end{array}\right\} \implies p(u, v) \leq M(x, y, u, v), \tag{5.1.1}$$

where

$$M(x, y, u, v) := a_1 p(x, y) + a_2 p(x, u) + a_3 p(y, v) + a_4 p(y, u) + a_5 p(x, v).$$

Now, we introduce the new concept which is called **0-continuity** as follows:

Definition 5.1.2. Let (X,p) be a partial metric space. A mapping $T : A \to B$ is called

(1) **Picard sequentially 0-continuous with respect to (A,B)** if $\{x_n\}$ is a sequence in X such that $p(x_{n+1}, Tx_n) = p(A,B)$, $x_n \to x^*$ as $n \to \infty$ and $p(x^*, x^*) = 0$, then $p(Tx_n, Tx^*) \to p(Tx^*, Tx^*) = 0$ as $n \to \infty$;

(2) **sequentially 0-continuous with respect to (A,B)** if $\{x_n\}$ is a sequence in X such that $x_n \to x^*$ as $n \to \infty$ and $p(x^*, x^*) = 0$, then $p(Tx_n, Tx^*) \to p(Tx^*, Tx^*) = 0$ as $n \to \infty$.

Remark 5.1.3. From Definition 5.1.2, if T is sequentially 0-continuous with respect to (A,B), then T is Picard sequentially 0-continuous with respect to (A,B). But the converse is not true.

Figure 5.1 shows the relation between several types of the continuity in partial metric spaces and metric spaces.

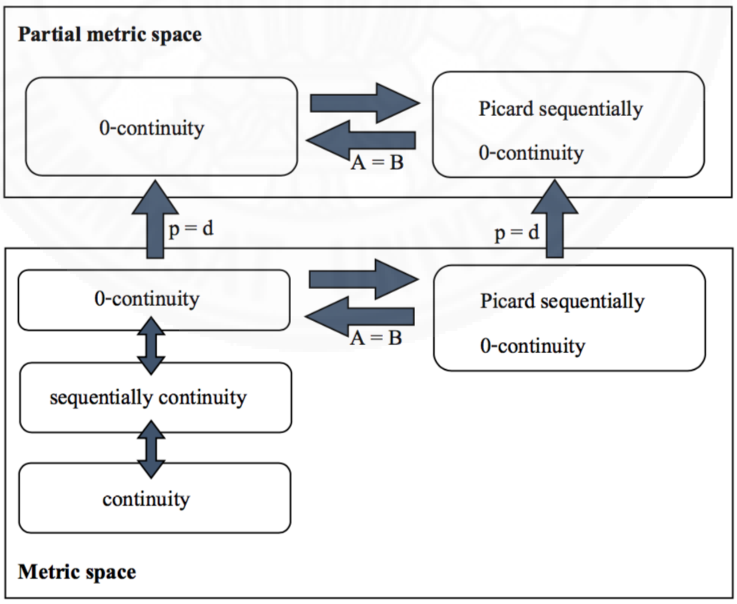

Figure 5.1: The relation between several types of the continuity in partial metric spaces and metric spaces.

Next, we give useful some lemma which is one illustrate for supporting Definition 5.1.2 as follows:

Lemma 5.1.4. Let A be a closed subset of a 0-complete partially ordered partial metric space $(X,p,\preceq)$ and $T:A\to A$ be a nondecreasing (nonincreasing) mapping and satisfies the following condition holds: there are $a_1,a_2,a_3,a_4,a_5\geq 0$ with $a_1+a_2+a_3+a_4+a_5<1$ such that

$$p(Tx,Ty)\leq M(x,y) \tag{5.1.2}$$

for all all comparable $x,y\in X$, where

$$M(x,y):=a_1p(x,y)+a_2p(x,Tx)+a_3p(y,Ty)+a_4p(y,Tx)+a_5p(x,Ty).$$

Suppose that there exists $x_0\in X$ with $x_0\preceq Tx_0$ (resp. $x_0\succeq Tx_0$) and T is continuous mapping. Then T is a Picard sequentially 0-continuous with respect to (A,A).

Proof. Suppose that $\{x_n\}$ is a sequence in X such that $p(x_{n+1},Tx_n)=p(A,A)$, $x_n\to x^*$ as $n\to\infty$ and $p(x^*,x^*)=0$. For each $n\in\mathbb{N}$, we get

$$p(x^*,Tx^*)\leq p(x^*,x_{n+1})+p(x_{n+1},Tx^*)-p(x_{n+1},x_{n+1})\leq p(x^*,x_{n+1})+p(x_{n+1},Tx^*).$$

Letting $n\to\infty$ in the above inequality and using the fact that $p(x_{n+1},Tx_n)=p(A,A)$, $x_n\to x^*$ as $n\to\infty$ and $p(x^*,x^*)=0$, we get

$$p(x^*,Tx^*)\leq p(x^*,x^*)+p(Tx^*,Tx^*)=p(Tx^*,Tx^*).$$

From (P1), we obtain $p(x^*,Tx^*)=p(Tx^*,Tx^*)$. Assume that $p(x^*,Tx^*)>0$. Since $x^*\preceq x^*$ and T satisfies (5.1.2), we get

$$p(x^*,Tx^*)=p(Tx^*,Tx^*)\leq M(x^*,x^*)=(a_1+a_2+a_3+a_4+a_5)p(x^*,Tx^*)<p(x^*,Tx^*),$$

which is a contradiction. Hence, $p(x^*,Tx^*)=0$ and hence $p(Tx^*,Tx^*)=0$, that is, T is a Picard sequentially 0-continuous with respect to (A,A). $\square$

We give example for supporting by above the lemma.

Example 5.1.5. Let $X = [0,\infty) \cap \mathbb{Q}$ and $p : X \times X \to [0,\infty)$ be defined by

$$p(x,y) = \max\{x,y\}$$

for all $x,y \in X$. We endow X with the partial order

$$x \preceq y \iff x = y \quad \text{or} \quad (x,y \in [0,1] \text{ with } x \leq y).$$

Then $(X,p,\preceq)$ is a 0-complete partially ordered partial metric space. Define $T : X \to X$ as follows:

$$Tx = \begin{cases} \frac{x^2}{1+x} & \text{if } x \in [0,1], \\ \frac{x}{2} & \text{if } x > 1. \end{cases}$$

Under all these settings, T is Picard sequentially 0-continuous with respect to (X,X). The readers can be seen the concept of this proof in Example 2.4 in [15].

5.2 Best proximity point results for generalized p-Hardy-Rogers contraction mappings by using the direct method

In this section, we establish the unique best proximity point result for a generalized p-Hardy-Rogers contraction mapping in 0-complete partially ordered partial metric space as follows:

Theorem 5.2.1. Let A,B be two nonempty subsets of a 0-complete partially ordered partial metric space $(X,p,\preceq)$ such that A_0 is nonempty and closed and $T : A \to B$ be a generalized p-Hardy-Rogers contraction mapping with the properties that $T(A_0) \subseteq B_0$ and T is proximally nondecreasing on A_0 (proximally nondecreasing on A_0). Assume that

(a) either T is a Picard sequentially 0-continuous with respect to (A,B) or X is regular and

(b) there exist elements $x_0, x_1 \in A_0$ for which $p(x_1,Tx_0) = p(A,B)$ and $x_0 \preceq x_1$ $(x_1 \preceq x_0)$.

Then T has a best proximity point in A_0. Moreover, the set $B_{est}(T)$ of best proximity points of T is well ordered if and only if it is a singleton.

Proof. Starting wth $x_0, x_1 \in A_0$ in the hypothesis, we get $x_0 \preceq x_1$ and

$$p(x_1, Tx_0) = p(A, B). \tag{5.2.1}$$

Since $x_1 \in A_0$ and $T(A_0) \subseteq B_0$, there exists a point $x_2 \in A_0$ such that

$$p(x_2, Tx_1) = p(A, B). \tag{5.2.2}$$

As T is proximally increasing on A_0, we get $x_1 \preceq x_2$. By similarly process, we obtain a sequence $\{x_n\}$ in A_0 such that

$$x_n \preceq x_{n+1} \tag{5.2.3}$$

and

$$p(x_{n+1}, Tx_n) = p(A, B) \tag{5.2.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (5.1.1), we obtain

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq M(x_n, x_{n+1}) \\ &= a_1 p(x_n, x_{n+1}) + a_2 p(x_n, x_{n+1}) + a_3 p(x_{n+1}, x_{n+2}) \\ &\quad + a_4 p(x_{n+1}, x_{n+1}) + a_5 p(x_n, x_{n+2}) \\ &\leq a_1 p(x_n, x_{n+1}) + a_2 p(x_n, x_{n+1}) + a_3 p(x_{n+1}, x_{n+2}) + a_4 p(x_{n+1}, x_{n+1}) \\ &\quad + a_5 p(x_n, x_{n+1}) + a_5 p(x_{n+1}, x_{n+2}) - a_5 p(x_{n+1}, x_{n+1}) \\ &= (a_1 + a_2 + a_5) p(x_n, x_{n+1}) + (a_3 + a_5) p(x_{n+1}, x_{n+2}) \\ &\quad + (a_4 - a_5) p(x_{n+1}, x_{n+1}). \end{aligned} \tag{5.2.5}$$

Using (5.1.1), (5.2.3) and (5.2.4), we obtain

$$\begin{aligned} p(x_{n+2}, x_{n+1}) &\leq M(x_{n+1}, x_n) \\ &= a_1 p(x_{n+1}, x_n) + a_2 p(x_{n+1}, x_{n+2}) + a_3 p(x_n, x_{n+1}) a_4 p(x_n, x_{n+2}) \\ &\quad + a_5 p(x_{n+1}, x_{n+1}) \\ &\leq (a_1 + a_3 + a_4) p(x_n, x_{n+1}) + (a_2 + a_4) p(x_{n+1}, x_{n+2}) \\ &\quad + (a_5 - a_4) p(x_{n+1}, x_{n+1}). \end{aligned} \tag{5.2.6}$$

Adding up (5.2.5) and (5.2.6), we have

$$p(x_{n+1}, x_{n+2}) \leq \rho p(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$0 \leq \rho := \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - (a_2 + a_3 + a_4 + a_5)} < 1.$$

It implies that

$$p(x_n, x_{n+1}) \leq \rho^n p(x_0, x_1)$$

for all $n \in \mathbb{N}$ and hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{5.2.7}$$

For each $m, n \in \mathbb{N}$ with $n > m$, we get

$$p(x_n, x_m) \leq (\rho^m + \cdots + \rho^{n-1}) p(x_0, x_1) \leq \frac{\rho^m}{1-\rho} p(x_0, x_1)$$

and so

$$\lim_{m,n\to\infty} p(x_n, x_m) = 0. \tag{5.2.8}$$

Hence, $\{x_n\}$ is a 0-Cauchy sequence in A_0. Since A_0 is a closed subset of a 0-complete partial metric space (X, p), there exists $x^* \in A_0$ such that $x_n \to x^*$ in (X, p) and $p(x^*, x^*) = 0$ and hence

$$\lim_{n\to\infty} p(x_n, x^*) = p(x^*, x^*) = 0. \tag{5.2.9}$$

Now, we will show that $p(x^*, Tx^*) = p(A, B)$.

(1) Assume that T is Picard sequentially 0-continuous with respect to (A, B).

Using (P4) we get

$$\begin{aligned} p(x^*, Tx^*) &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) \\ &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx_n) + p(Tx_n, Tx^*) - p(Tx_n, Tx_n) \\ &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx_n) + p(Tx_n, Tx^*) \end{aligned}$$

for all $n \in \mathbb{N}$. From (5.2.4) we obtain

$$p(x^*, Tx^*) \leq p(x^*, x_{n+1}) + p(A, B) + p(Tx_n, Tx^*).$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, applying (5.2.9) and the Picard sequentially 0-continuity with respect to (A, B) of T, we get

$$p(x^*, Tx^*) \leq p(x^*, x^*) + p(A, B) + p(Tx^*, Tx^*) = p(A, B).$$

Therefore, $p(x^*, Tx^*) = p(A, B)$, that is, x^* is a best proximity point of T.

(2) Assume that X is regular.

By (5.2.3) and (5.2.9), we get

$$x_n \preceq x^* \tag{5.2.10}$$

for all $n \in \mathbb{N}$. Since $T(A_0) \subseteq B_0$ and $x^* \in A_0$, there exists a point $z \in A_0$ such that

$$p(z, Tx^*) = p(A, B). \tag{5.2.11}$$

By (5.2.4), (5.2.10) and (5.2.11), we get

$$\begin{aligned} p(x_{n+1}, z) &\leq M(x_n, x^*) \\ &= a_1 p(x_n, x^*) + a_2 p(x_n, x_{n+1}) + a_3 p(x^*, z) \\ &\quad + a_4 p(x^*, x_{n+1}) + a_5 p(x_n, z). \end{aligned} \tag{5.2.12}$$

Taking limit as $n \to \infty$ in the above inequality and using Lemma 2.3.11, we get

$$p(x^*, z) \leq (a_3 + a_5) p(x^*, z).$$

This implies that $p(x^*, z) = 0$. Thus $x^* = z$ and so $p(x^*, Tx^*) = p(A, B)$. Therefore, x^* is a best proximity point of T.

In addition, we will show that the best proximity point of T is unique if and only if $B_{est}(T)$ is well ordered. Now we suppose that the set of best proximity point T is well ordered and we will claim that the best proximity point of T is unique. So, assume by contrary that there are $\omega_1, \omega_2 \in A$ with $p(\omega_1, T\omega_1) = p(A, B)$

and $p(\omega_2, T\omega_2) = p(A,B)$ such that $\omega_1 \neq \omega_2$. Then ω_1 and ω_2 are comparable. We can replace $x = u = \omega_1$ and $y = v = \omega_2$ in (5.1.1) and so

$$\begin{aligned}
p(\omega_1, \omega_2) &\leq M(\omega_1, \omega_2) \\
&= a_1 p(\omega_1, \omega_2) + a_2 p(\omega_1, \omega_1) + a_3 p(\omega_2, \omega_2) + a_4 p(\omega_2, \omega_1) + \omega_5 p(\omega_1, \omega_2) \\
&\leq (a_1 + a_4 + a_5) p(\omega_1, \omega_2) + a_2 p(\omega_1, \omega_2) + a_3 p(\omega_1, \omega_2) \\
&= (a_1 + a_2 + a_3 + a_4 + a_5) p(\omega_1, \omega_2) \\
&< p(\omega_1, \omega_2),
\end{aligned}$$

which is a contradiction. The converse is easy to proof. This completes the proof. □

Definition 5.2.2. Let A, B be two nonempty subsets of a partially ordered partial metric space $(X, p, \preceq)$ with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the **P-property** if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} p(x_1, y_1) = p(A,B), \\ p(x_2, y_2) = p(A,B) \end{aligned}\right\} \implies p(x_1, x_2) = p(y_1, y_2).$$

Remark 5.2.3. The concept of P-property in the case of metric spaces was first introduced by [19].

Lemma 5.2.4 ([8])**.** Let (A,B) be a pair of nonempty closed subsets of a complete partial metric space (X, p) such that A_0 is nonempty and (A,B) has the P-property. Then (A_0, B_0) is a closed pair of subsets of X.

Remark 5.2.5. The reader can be seen Lemma 5.2.4 in the version of metric spaces in [8].

Theorem 5.2.6. Let A, B be two nonempty closed subsets of a 0-complete partially ordered partial metric space $(X, p, \preceq)$ such that A_0 is nonempty and (A,B) satisfies the P-property and $T : A \to B$ be proximally nondecreasing on A_0 (proximally nondecreasing on A_0) with the properties that $T(A_0) \subseteq B_0$ and satisfies the following condition holds: there are $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that

$$p(u, v) \leq M(x, y, u, v) \tag{5.2.13}$$

for all $x,y,u,v \in A$, where

$$M(x,y,u,v) := a_1p(x,y) + a_2p(x,u) + a_3p(y,v) + a_4p(y,u) + a_5p(x,v).$$

Assume either T is a Picard sequentially 0-continuous with respect to (A,B) or X is regular and there exist elements $x_0, x_1 \in A_0$ for which $p(x_1, Tx_0) = p(A,B)$ and $x_0 \preceq x_1$ $(x_1 \preceq x_0)$. Then T has a best proximity point in A_0. Moreover, the set $B_{est}(T)$ of best proximity points of T is well ordered if and only if it is a singleton.

Proof. By using Lemma 5.2.4 and applying Theorem 5.2.1, we get this result. □

Remark 5.2.7. Note that for $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, the generalized p-Hardy-Rogers contraction mapping reduce into the following mappings:

(1) the Banach contraction mapping if $a_2 = a_3 = a_4 = a_5 = 0$;

(2) the Kannan contraction mapping if $a_1 = a_4 = a_5 = 0$;

(3) the Chaterjia contraction mapping if $a_1 = a_2 = a_3 = 0$;

(4) the Reich Contraction mapping if $a_2 = a_3 = 0$.

Therefore among all above defnitions, the generalized p-Hardy-Rogers contraction mapping is the most general contraction mapping.

Corollary 5.2.8. Let A be a closed subset of a 0-complete ordered partial metric space $(X, p, \preceq)$ such that A_0 is nonempty and closed and $T : A \to A$ be a nondecreasing (nonincreasing) mapping satisfying the following condition holds: there are $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that

$$p(Tx, Ty) \leq M(x,y,u,v)$$

for all comparable $x,y \in X$, where

$$M(x,y,u,v) := a_1p(x,y) + a_2p(x,Tx) + a_3p(y,Ty) + a_4p(y,Tx) + a_5p(x,Ty).$$

Suppose that there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ (resp. $x_0 \succeq Tx_0$) and the following conditions hold:

(a) T is continuous or

(b) X is regular.

Then T has a fixed point z and $p(Tz,Tz) = 0 = p(z,z)$. Moreover, the set of all fixed points of T is well ordered if and only if it is a singleton set.

Proof. By applying Lemma 5.1.4 and using Theorem 5.2.1, then T has a unique best proximity point. □

Corollary 5.2.9 ([15])**.** Let $(X, p, \preceq)$ be a 0-complete ordered partial metric space and $T : X \to X$ be a nondecreasing (nonincreasing) mapping satisfying the following condition holds: there are $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that

$$p(Tx,Ty) \leq M(x,y,Tx,Ty)$$

for all comparable $x, y \in X$, where

$$M(x,y,Tx,Ty) := a_1 p(x,y) + a_2 p(x,Tx) + a_3 p(y,Ty) + a_4 p(y,Tx) + a_5 p(x,Ty).$$

Suppose that there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ (resp. $x_0 \succeq Tx_0$) and the following conditions hold:

(a) T is continuous or

(b) X is regular.

Then T has a fixed point z and $p(Tz,Tz) = 0 = p(z,z)$. Moreover, the set of all fixed points of T is well ordered if and only if it is a singleton set.

CHAPTER 6

CONCLUSIONS

In summary, we introduced new type of contractions and established new best proximity point results based on two methods including the fixed point method and the direct method. We are going to summarize all of main results in this thesis as follows.

In Chapter 3, we extended some contraction mappings and proved the fixed point results for such mappings in partial metric spaces (see in Theorem 3.1.3 and 3.1.6). These results extend and improve various fixed point results in partial metric spaces and generalize many fixed point results in metric space. In addition, it has been pointed out that the existence of best proximity point results can be concluded from our fixed point result (see in Theorem 3.2.3 and Theorem 3.2.4). Our results extend and improve the main results of Su and Yao [21].

In Chapter 4, we generalized some contraction mappings and prove the common fixed point results in partial metric spaces (see in Theorem 4.1.3). These results improve and generalize many common fixed point and fixed point results in partial metric spaces and metric spaces. Moreover, we introduced new useful property (see in Definition 4.2.1) and establish the existence of common best proximity point results by using such property together with the common fixed point results (see in Theorem 4.2.3). Our results extend and improve the main results of Azizi *et al.* [1].

In Chapter 5, we defined the new contraction for nonself-mappings in partially ordered partial metric spaces. Furthermore, we introduced the new type of the continuity in such spaces (see in Definition 5.1.2). On such new two ideas, we proved the best proximity point results in 0-complete partially ordered partial metric spaces (see in Theorem 5.2.1). Some particular cases are presented to confirm the significance and unifying power of obtained generalizations. Our results extend and improve the main results of Nashine *et al.* [15].

Finally, we give the advantages of main results in the thesis. Based in the fact that the best proximity point results has the wider applications than the fixed point results, our new best proximity point results from Chapters 3,4,5 can be solved some real-world problems which are not applied by the fixed point results such as the global optimization problems.

REFERENCES

[1] Azizi, A., Moosaei, M., & Zarei, G. (2016). Fixed point theorems for almost generalized $\mathcal{C}$-contractive mappings in ordered complete metric spaces. *Fixed Point Theory and Appl.*, 2016:80.

[2] Banach, S. (1922) Sur les opérations dans lesensembles abstraits et leurs applications aux équations intégrales. *Fund. Math.*, 3, 133–181.

[3] Babu, G. V. R, & Subhashini, P. (2012) Coupled fixed point theorems in partially ordered metric spaces by altering distances via Geraghty contraction. *Journal of Advanced Research in Applied Mathematics*, 4 (4), 78–95.

[4] Berinde, V. (2004). Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum*, 9 (1), 43–53.

[5] Chatterjea, S. K. (1972). Fixed point theorems. *C. R. Acad. Bulgare Sci.*, 25, 727–730.

[6] Fan, K. (1969). Extensions of two fixed point theorems of F. E. Broder. *Math. Z.*, 112, 234–240.

[7] Fréchet, M. (1906). Sur quelques points du calcul fonctionnel. *Rend. Circ. Mat. Palermo*, 22, 1–74.

[8] Gabeleh, M. (2013). Proximal weakly contractive and proximal nonexpansive non-self-mappings in metric and Banach spaces. *J. Optim. Theory Appl.*, 158, 615–625.

[9] Geraghty, M. (1973). On contractive mappings. *Proc. Am. Math. Soc.*, 40, 604–608.

[10] Hardy, G. E. & Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. *Canad. Math. Bull.*, 16, 201–206.

[11] Heckmann, R. (1999) Approximation of metric spaces by partial metric spaces. *Appl. Categ. Struct.*, 7, 71–83.

[12] Kannan, R. (1969) Some results on fixed points II. *Amer. Math. Monthly*, 76, 405–408.

[13] Kopperman, R., Matthews, S., & Pajoohesh, H. (2004). Partial metrizability in value quantales. *Appl. Gen. Topol.*, 5(1), 115–127.

[14] Matthews, S. G. (1994). Partial metric topology. *in: Proc. 8th Summer Conference on General Topology and Applications*, Ann. New York Acad. Sci. 728, 183–197.

[15] Nashine, H. K., Kadelburg, Z., Radenović, S., & Kim, J. K. (2012). Fixed point theorems under Hardy-Rogers contractive conditions on 0-complete ordered partial metric spaces. *Fixed Point Theory and Appl.*, 2012:180.

[16] Ninsri, A., & Sintunavarat, W. (2018). Towards a generalized contractivity in partial metric spaces with the existence results of fixed/best proximity points. *Fixed Point Theory and Appl.* 20, 1.

[17] Romaguera, S. (2010). A Kirk type characterization of completeness for partial metric spaces. *Fixed Point Theory Appl.*, 2010, 1–6.

[18] Rus, I. A. (2008). Fixed point theory in partial metric spaces. *An. Univ. Vest. Timis., Ser. Mat. Inform.* 46(2), 141–160.

[19] Sankar Raj, V. (2011). A best proximity point theorem for weakly contractive non-self-mappings. *Nonlinear Anal., Theory Methods Appl.*, 74, 4804–4808.

[20] Suzuki, T. (2008). A generalized Banach contraction principle that characterizes metric completeness. *Proceedings of the American Math. Soc.*, 136 (5), 1861–1869.

[21] Su Y. & Yao, J. C. (2015). Further generalized contraction mapping principle and best proximity theorem on metric spaces. *Fixed Point Theory Appl.*, 2015:120.

[22] Yan, F., Su, Y., & Feng, Q. (2012). A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations. *Fixed Point Theory Appl.*, 2012:152.

[23] Zhao, H., Lu, Y., Sridarat, P. Suantai, S., & Cho, Y. J. (2017). Common fixed point theorems in non-Archimedean fuzzy metric-like spaces with applications. *J. Nonlinear Sci. Appl.*, 10, 3708–3718.

[24] Zamfirescu, T. (1972). Fixed point theorems in metric spaces. *Arch. Math.*, 23, 292–298.

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Publications

1. Ninsri, A., & Sintunavarat, W. (2018). Toward a generalized contractive condition in partial metric spaces with the existence results of fixed points and best proximity points. Journal of Fixed Point Theory and Applications, 20:13, 1–15.

2. Ninsri, A., & Sintunavarat, W. (2018). A generalized Hardy-Rogers type with φ-best proximity point result. International Journal of Pure Mathematics, 5, 33–36.

Oral Presentations

1. Ninsri, A., & Sintunavarat, W. (2015). A fixed point theorems in generalized metric spaces for mappings satisfying a new contractive condition. Fixed Point Theory for Nonlinear Mappings and Applications to Equilibrium and Optimization Problems, 31 August - 1 September 2015, Chiang Mai University, Chiang Mai, Thailand.

2. Ninsri, A., & Sintunavarat, W. (2015). NS-convex contraction mappings in metric spaces with approximate fixed point theorems. The International Conference on Science and Technology 2015, 4-6 November 2015, Rajamangala University of Technology Thanyaburi, Pathumthani, Thailand.

3. Ninsri, A., & Sintunavarat, W. (2015). Further generalized contraction mapping principle in partial metric spaces. The 11^{th} IMT-GT International Conference on Mathematics, Statistics and Its Applications 2015 (ICMSA 2015), 23-25 November 2015, Ambassador City Jomtien Hotel, Pattaya, Thailand.

4. Ninsri, A., & Sintunavarat, W. (2016). Fixed point theorems for ψ-ϕ-partial mappings and best proximity point theorems in partial metric spaces. The 9^{th} Asian Conference on Fixed Point Theory and Optimization (ACFPTO2016), 18-20 May 2016, King Mongkut's University of Technology Thonburi, Bangkok, Thailand.

5. Ninsri, A., & Sintunavarat, W. (2016). Approximate fixed point theorem for (α,β)-generalized convex contraction mappings. The 8^{th} National Science Research Conference, 30-31 May 2016, University of Phayao, Phayao, Thailand.

6. Ninsri, A., & Sintunavarat, W. (2016). Best proximity points for $\mathcal{L}$-cyclic generalized contractions with property UC* in metric spaces. International Conference on Applied Statistics 2016 (ICAS2016), 13-15 July 2016, Phuket, Thailand.

7. Ninsri, A., & Sintunavarat, W. (2016). Further generalized contraction mappings in partial metric spaces with best proximity point theorems. The 42^{nd} Congress on Science and Technology of Thailand (STT 42), 30 November - 2 December 2016, Centara Grand, Bangkok, Thailand.

8. Ninsri, A., & Sintunavarat, W. (2017). A best proximity point theorem for Suzuki's mappings in specific regular spaces. The 22^{nd} Annual Meeting in Mathematics (AMM 2017), 2-4 June 2017, Lotus Pang Suan Kaew Hotel, Chiang Mai, Thailand.

9. Ninsri, A., & Sintunavarat, W. (2017). Best proximity point theorems of generalized weakly contraction mappings in partially order metric spaces.

12th International Conference on Fixed Point Theory and Its Applications (ICFPTA 2017), 24-28 July 2017, Newcastle, New South Wales, Australia.

10. Ninsri, A., & Sintunavarat, W. (2018). Fixed point theorems approach to the generalization of best proximity point theorems in partial metric spaces. 10th Nation Science Research Conference (SRC10), 24-25 May 2018, Mahasarakham University, Mahasarakham, Thailand.

11. Ninsri, A., & Sintunavarat, W. (2018). On best proximity point theorems via modified the Hardy-Rogers contraction mappings in 0-complete partial ordered partial metric spaces. The 23rd Annual Meeting in Mathematics (AMM2018), 3-5 May 2018, Mandarin Hotel, Bangkok, Thailand.

12. Ninsri, A., & Sintunavarat, W. (2018). A generalized Hardy-Rogers type with φ-best proximity point result. 2nd International Conference on Applied Mathematics and Computer Science (AMACS2018), 26-28 May 2018, The Church Palace, Rome, Italy.

13. Ninsri, A., & Sintunavarat, W. (2018). Common best proximity point theorems for mappings concerning some control functions in non-Archimedean fuzzy metric-like spaces. The 10th Asian Conference on Fixed Point Theory and Optimizations (ACFPTO2018), 16-18 July 2018, Chiang Mai University, Chiang Mai, Thailand.

Poster Presentations

1. Ninsri, A., & Sintunavarat, W. (2017). Best proximity point theorem for some generalized contraction mappings in partial metric spaces. National Science Research Conference 9th, 25-26 May 2017, Burapha University, Bangsaen, Chonburi, Thailand.

2. Ninsri, A., & Sintunavarat, W. (2018). Common fixed point and best proximity point theorems for an almost generalized $\mathcal{PC}$-contraction mappings in partial metric spaces. 10th Nation Science Research Conference (SRC10), 24-25 May 2018, Mahasarakham University, Mahasarakham, Thailand.