WIENER INDEX AND LAPLACIAN MATRIX
OF ZERO DIVISOR GRAPHS OF $\mathbb{Z}_{p^n}$ AND $\mathbb{Z}_{pqr}$

BY

MISS BENCHAMAT TAKHWAN

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
ACADEMIC YEAR 2017
COPYRIGHT OF THAMMASAT UNIVERSITY
WIENER INDEX AND LAPLACIAN MATRIX
OF ZERO DIVISOR GRAPHS OF $\mathbb{Z}_{pn}$ AND $\mathbb{Z}_{pqr}$

BY

MISS BENCHAMAT TAKHWAN

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
ACADEMIC YEAR 2017
COPYRIGHT OF THAMMASAT UNIVERSITY
THAMMASAT UNIVERSITY
FACULTY OF SCIENCE AND TECHNOLOGY

THESIS

BY

MISS BENCHAMAT TAKHWAN

ENTITLED

WIENER INDEX AND LAPLACIAN MATRIX OF ZERO DIVISOR GRAPHS OF $\mathbb{Z}_{pq}$ AND $\mathbb{Z}_{pqr}$

was approved as partial fulfillment of the requirements for
the degree of Master of Science (Mathematics)
on July 5, 2018

Chairman

Member and Advisor

Member and Co-Advisor

Member

Member

Dean

(Assistant Professor Archara Pacheenburawana, Ph.D.)

(Assistant Professor Khjee Jantarakhajorn, Ph.D.)

(Borworn Khuhirun, Ph.D.)

(Professor Yotsanan Meemark, Ph.D.)

(Nantapath Trakultraipruk, Ph.D.)

(Associate Professor Somchai Chakhatrakan, Ph.D.)
ABSTRACT

Let \( R \) be a commutative ring with identity and let \( Z[R] \) be the set of all zero divisors of \( R \). The zero divisor graph of \( R \), denoted by \( \Gamma[R] \), is an undirected graph with vertex set \( Z[R] \) such that two distinct vertices \( x, y \in Z[R] \) are adjacent if \( xy = 0 \).

Let \( p, q \) and \( r \) be prime numbers such that \( p < q < r \). We study the Wiener index of the zero divisor graph of \( \mathbb{Z}_{p^4} \). In addition, we generalize the Wiener index of the zero divisor graph of \( \mathbb{Z}_{p^n} \) for any integer \( n \geq 4 \) and \( \mathbb{Z}_{pqr} \).

Furthermore, we determine the Laplacian matrix of the zero divisor graph of \( \mathbb{Z}_{p^n} \) for \( n \in \{2, 3\} \). Then, we generalize the Laplacian matrix of the zero divisor graph of \( \mathbb{Z}_{p^n} \) for any integer \( n \geq 4 \) and \( \mathbb{Z}_{pqr} \).

**Keywords:** Zero divisor graph, Wiener index, Laplacian matrix
ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my advisors Assistant Professor Dr. Khajee Jantarakhajorn and Dr. Borworn Khuhirun, for giving me an opportunity to do this work and for their helpful advice, immense knowledge and constant support. Their guidance helped me throughout the research and writing of this thesis.

I would also like to sincerely thank the thesis committee members: Professor Dr. Yotsanan Meemark, Assistant Professor Dr. Archara Pacheenburawana and Dr. Nantapath Trakultraipruk, for their helpful advice and feedback.

Finally, I would like to thank my parents, for supporting me through everything and my friends, for listening and supporting me through this research.

The work was supported by a Graduate Scholarship from the Faculty of Science and Technology (Thammasat University), for which I am grateful.

Miss Benchamat Takhwan
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td></td>
<td>(2)</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td></td>
<td>(4)</td>
</tr>
<tr>
<td>CHAPTER 1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER 2</td>
<td>PRELIMINARIES</td>
<td>3</td>
</tr>
<tr>
<td>2.1</td>
<td>Graph</td>
<td>3</td>
</tr>
<tr>
<td>2.2</td>
<td>Ring</td>
<td>5</td>
</tr>
<tr>
<td>2.3</td>
<td>Zero divisor graph and Wiener index</td>
<td>6</td>
</tr>
<tr>
<td>CHAPTER 3</td>
<td>WIENER INDEX OF SOME ZERO DIVISOR GRAPHS</td>
<td>9</td>
</tr>
<tr>
<td>3.1</td>
<td>Wiener index of the zero divisor graphs of $\mathbb{Z}_{p^n}$ where $n \geq 4$</td>
<td>9</td>
</tr>
<tr>
<td>3.2</td>
<td>Wiener index of the zero divisor graphs of $\mathbb{Z}_{pqr}$</td>
<td>20</td>
</tr>
<tr>
<td>CHAPTER 4</td>
<td>LAPLACIAN MATRIX OF SOME ZERO DIVISOR GRAPHS</td>
<td>31</td>
</tr>
<tr>
<td>4.1</td>
<td>Laplacian matrix of the zero divisor graphs of $\mathbb{Z}<em>{p^2}$ and $\mathbb{Z}</em>{p^3}$</td>
<td>32</td>
</tr>
<tr>
<td>4.2</td>
<td>Laplacian matrix of the zero divisor graphs of $\mathbb{Z}_{p^n}$ where $n \geq 4$</td>
<td>35</td>
</tr>
<tr>
<td>4.3</td>
<td>Laplacian matrix of the zero divisor graphs of $\mathbb{Z}_{pqr}$</td>
<td>40</td>
</tr>
<tr>
<td>CHAPTER 5</td>
<td>CONCLUSIONS AND FUTURE WORK</td>
<td>47</td>
</tr>
<tr>
<td>5.1</td>
<td>Conclusions</td>
<td>47</td>
</tr>
<tr>
<td>5.2</td>
<td>Future work</td>
<td>47</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
<td>48</td>
</tr>
<tr>
<td>BIOGRAPHY</td>
<td></td>
<td>49</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figures</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>7</td>
</tr>
<tr>
<td>2.2</td>
<td>7</td>
</tr>
<tr>
<td>3.1</td>
<td>13</td>
</tr>
<tr>
<td>3.2</td>
<td>14</td>
</tr>
<tr>
<td>3.3</td>
<td>19</td>
</tr>
<tr>
<td>3.4</td>
<td>25</td>
</tr>
<tr>
<td>3.5</td>
<td>29</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

In the past several years, many researchers have established connections between graph theory and ring theory, such as the commuting graph and the zero divisor graph \([2, 3, 7]\). The zero divisor graph was first introduced in 1988 by Beck [3]. He let all elements of a commutative ring be the vertices in a graph such that two distinct vertices \(x\) and \(y\) are adjacent if \(xy = 0\). The definition of vertex set was later refined by Anderson and Livingston [2], so that vertices are the zero divisors of a commutative ring.

Let \(R\) be a commutative ring with identity and let \(Z[R]\) be the set of all zero divisors of \(R\). The zero divisor graph of \(R\), denoted by \(\Gamma[R]\), is an undirected graph with vertex set \(Z[R]\) such that two distinct vertices \(x, y \in Z[R]\) are adjacent if \(xy = 0\). Zero divisor graphs have been studied extensively \([2, 3, 5, 10]\). These have investigated graph properties such as diameter and clique number.

The Wiener index of a graph was first introduced by Wiener [11]. The Wiener index of a graph \(G\), denoted by \(W(G)\), is the sum of all distances between all pairs of vertices in \(G\).

The concept of the energy and Wiener index of zero divisor graphs of \(\mathbb{Z}_n\) was introduced by Ahmadi and Jahani-Nezhad [1], who proved the energy and Wiener index of zero divisor graph of rings \(\mathbb{Z}_n\) for \(n = p^2\) and \(n = pq\). Reddy, Jain and Laxmikanth [8] extended the results of Ahmadi and Jahani-Nezhad. They studied the adjacency matrix, energy and Wiener index of zero divisor graph of rings \(\mathbb{Z}_n\), where \(n = p^3\) and \(n = p^2q\). However, the Wiener index of zero divisor graph of \(\mathbb{Z}_{pq}\) for any \(n \geq 4\), including \(\mathbb{Z}_{pqr}\) are still unknown, where \(p, q\) and \(r\) are prime numbers such that \(p < q < r\).

In graph theory, several matrices are associated with a graph, such as the
adjacency matrix and Laplacian matrix.

Let \( G = (V(G), E(G)) \) be a simple graph of order \( n \) with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). The Laplacian matrix of \( G \), denoted \( L(G) \), is defined to be an \( n \times n \) matrix \( L(G) = [l_{ij}]_{n \times n} \) such that

\[
l_{i,j} = \begin{cases} 
\deg(v_i) & \text{if } i = j, \\
-1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \deg(v_i) \) is the degree of the vertex \( v_i \).

The Laplacian matrix of a graph has been studied in many areas of mathematical research [9]. However, the Laplacian matrix of zero divisor graphs of \( \mathbb{Z}_{p^n} \) for any \( n \geq 2 \), including \( \mathbb{Z}_{pqr} \) are still unknown, where \( p, q \) and \( r \) are prime numbers such that \( p < q < r \).

Next, we describe the content of the thesis. In Chapter 2, we introduce the bases of graph, ring and the zero divisor graph, including the Wiener index.

Chapter 3 is divided into two sections. In section 3.1, we introduce the Wiener index of the zero divisor graph of \( \mathbb{Z}_{p^4} \). Next, we determine the generalization of the Wiener index of the zero divisor graph of \( \mathbb{Z}_{p^n} \) for any integer \( n \geq 4 \). The Wiener index of the zero divisor graph of \( \mathbb{Z}_{pqr} \) is discussed in section 3.2.

Chapter 4 is divided into three sections. In section 4.1, we determine the Laplacian matrix of the zero divisor graph of \( \mathbb{Z}_{p^n} \) for \( n \in \{2, 3\} \). The Laplacian matrix of the zero divisor graph of \( \mathbb{Z}_{p^n} \) for any integer \( n \geq 4 \) is introduced in section 4.2 and the Laplacian matrix of the zero divisor graph of \( \mathbb{Z}_{pqr} \) in section 4.3.

Finally, Chapter 5 gives our conclusions and suggests future work.
CHAPTER 2
PRELIMINARIES

We begin this chapter by giving the basic knowledge of graph, ring, the zero divisor graph and the Wiener index.

2.1 Graph

We use the graph terminology from [4]. Throughout this thesis, the graphs we discuss are assumed to be simple and finite.

A simple graph $G$ is an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set, called the vertex set of $G$ and $E(G)$ is a set of 2–element subsets of $V(G)$, called the edge set of $G$. Any two distinct vertices $u, v$ in $G$ are adjacent if \{u, v\} $\in$ $E(G)$. If $e = \{u, v\}$ is an edge of $G$, the vertex $u$ and the edge $e$ (as well as $v$ and $e$) are said to be incident with each other.

The number of vertices of $G$ is called the order of $G$ and denoted by $|V(G)|$. The number of edges of $G$ is called the size of $G$ and denoted by $|E(G)|$.

A graph with exactly one vertex is called the trivial graph. All other graphs are called nontrivial. The degree of a vertex $v$ of $G$ is the number of edges incident with $v$, written as $\deg(v)$.

A graph $G$ is a complete graph if every two distinct vertices of $G$ are adjacent. The complete graph of order $n$ is denoted by $K_n$. Since every two distinct vertices of $K_n$ are adjacent, the size of $K_n$ is $\binom{n}{2} = \frac{n(n-1)}{2}$.

A $u$–$v$ walk in a graph $G$ is a finite sequence of vertices in $G$, beginning with $u$ and ending at $v$ such that consecutive vertices in the sequence are adjacent, that
is, we can express this $u - v$ walk as

$$W : u = v_0, v_1, \ldots, v_k = v$$

where $k \geq 0$ and $v_i$ and $v_{i+1}$ are adjacent for $i = 0, 1, 2, \ldots, k - 1$. Each vertex $v_i$ ($0 \leq i \leq k$) and each edge $v_i v_{i+1}$ ($0 \leq i \leq k - 1$) is said to lie on or belong to $W$. If $u = v$, then a walk $W$ is **closed**, while if $u \neq v$, then $W$ is **open**. The number of edges in a walk is called the **length** of the walk. A walk in which all the edges are distinct is a **trail**. A trail in which all the vertices are distinct is a **path**. We usually denote such a path by $v_0, v_1, \ldots, v_k$.

If $G$ contains a $u - v$ path, then $u$ and $v$ are said to be **connected** in $G$. A graph $G$ is connected if every two vertices of $G$ are connected, that is, if $G$ contains a $u - v$ path for every pair of distinct vertices $u$ and $v$ in $G$.

Let $u$ and $v$ be vertices in a connected graph $G$. The **distance** $d(u, v)$ from $u$ to $v$ is the length of the shortest path from $u$ to $v$. The greatest distance between any two vertices of $G$ is called the **diameter** of $G$, denoted by $\text{diam}(G)$.

Let $G$ be a graph of order $n$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of $G$ is an $n \times n$ matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 
1 & \text{if } v_i \text{ is adjacent to } v_j, \\
0 & \text{otherwise}.
\end{cases}$$

The **degree matrix** of $G$ is an $n \times n$ diagonal matrix $D(G) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} 
\deg(v_i) & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}$$

The **Laplacian matrix** of $G$ is an $n \times n$ matrix $L(G) = [l_{ij}]$ defined by $L(G) = D(G) - \cdots$
A(G), that is

\[ l_{ij} = \begin{cases} 
\text{deg}(v_i) & \text{if } i = j, \\
-1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\
0 & \text{otherwise.}
\end{cases} \]

### 2.2 Ring

We next introduce some definitions and some useful theorems on ring theory from [6].

A **ring** \( (R, +, \cdot) \) is a set together with two binary operations \( + \) and \( \cdot \) (called addition and multiplication) satisfying the following axioms:

(i) \( (R, +) \) is an abelian group,

(ii) \( (R, \cdot) \) is a semigroup,

(iii) the distributive laws hold in \( R \) : for all \( a, b, c \in R \),

\[ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{and} \quad (a + b) \cdot c = (a \cdot c) + (b \cdot c). \]

We usually write \( ab \) rather than \( a \cdot b \) for \( a, b \in R \). The additive identity of \( R \) will always be denoted by 0.

A ring \( (R, +, \cdot) \) is said to be **commutative** if multiplication is commutative, that is,

\[ ab = ba \quad \text{for all } a, b \in R. \]

A nontrivial ring \( (R, +, \cdot) \) is a **ring with identity** 1, if \( R \) contains a multiplicative identity, denoted by 1, which satisfies

\[ 1a = a1 = a \quad \text{for all } a \in R. \]
Example 2.2.1. 1. \((\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot)\) and \((\mathbb{R}, +, \cdot)\) with usual addition and multiplication are commutative rings.

2. \(\mathbb{Z}_n\) with addition and multiplication modulo \(n\) is a commutative ring with identity.

3. \(M_2(\mathbb{Z})\), the set of all \(2 \times 2\) matrices with integer entries is a noncommutative ring with identity.

Let \(R\) be a commutative ring. A nonzero element \(a\) of \(R\) is called a zero divisor if there is a nonzero element \(b\) of \(R\) such that \(ab = 0\). The set of all zero divisors in \(R\) is denoted \(Z[R]\).

Example 2.2.2. The set of all zero divisors of \(\mathbb{Z}_8\) is \(\{2, 4, 6\}\) since \(2 \cdot 4\) and \(4 \cdot 6\) are 0 modulo 8.

Let \(R\) be a ring with identity 1. A nonzero element \(u\) of \(R\) is called a unit if there is an element \(v\) of \(R\) such that \(uv = vu = 1\).

Theorem 2.2.3. Let \(R\) be a finite commutative ring with identity. Then every element of \(R\) is either a unit or a zero divisor.

Theorem 2.2.4. Let \(n \in \mathbb{Z}^+\). Then any element \(a \in \mathbb{Z}_n\) is a zero divisor if and only if \(\gcd(a, n) \neq 1\).

2.3 Zero divisor graph and Wiener index

In this section, we provide the definitions of the zero divisor graph and the Wiener index and give examples.

Definition 2.3.1. [3] Let \(R\) be a finite commutative ring. The zero divisor graph of \(R\) is a simple graph whose set of vertices are all elements of \(R\) and two vertices \(x\) and \(y\) are adjacent if \(xy = 0\).
Example 2.3.2. For \( \mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\} \). The zero divisor graph of \( \mathbb{Z}_8 \) is given below.

![Zero Divisor Graph of \( \mathbb{Z}_8 \)](image)

Figure 2.1: The zero divisor graph of \( \mathbb{Z}_8 \) (Beck)

Throughout this thesis, we define the zero divisor graph following Anderson and Livingston [2].

Definition 2.3.3. [2] Let \( R \) be a finite commutative ring with identity and let \( Z[R] \) be the set of all zero divisors of \( R \). The zero divisor graph of \( R \), denoted by \( \Gamma[R] \), is an undirected graph with vertex set \( Z[R] \) such that any two distinct vertices \( x, y \in Z[R] \) are adjacent if \( xy = 0 \).

Example 2.3.4. For \( \mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\} \). Then \( Z[\mathbb{Z}_8] = \{2,4,6\} \) is the vertex set. Since \( 2 \cdot 4 = 0, 4 \cdot 6 = 0 \), \( \{2,4\} \) and \( \{4,6\} \) are edges in \( \Gamma[\mathbb{Z}_8] \). However \( \{2,6\} \) is not an edge in \( \Gamma[\mathbb{Z}_8] \) since \( 2 \cdot 6 \neq 0 \). The graph \( \Gamma[\mathbb{Z}_8] \) is given below.

![Zero Divisor Graph of \( \mathbb{Z}_8 \)](image)

Figure 2.2: The zero divisor graph of \( \mathbb{Z}_8 \) (Anderson and Livingston)
Definition 2.3.5. [11] Let $G$ be a graph with vertex set $V(G)$. The **Wiener index** of $G$, denoted by $W(G)$, is the sum of all distances between all pairs of vertices in $G$.

$$W(G) = \sum_{u,v \in V(G)} d(u,v).$$

Example 2.3.6. Let $\Gamma[Z_8]$ be the zero divisor graph of $Z_8$ shown in Figure 2.2. We get $d(2,4) = 1$, $d(4,6) = 1$ and $d(2,6) = 2$. Thus $W(\Gamma[Z_8]) = d(2,4) + d(4,6) + d(2,6) = 4$. 

\[\blacksquare\]
CHAPTER 3

WIENER INDEX OF SOME ZERO DIVISOR GRAPHS

In this chapter, we discuss the Wiener index of $\Gamma[Z_{p^4}]$ where $p$ is a prime number. Moreover, we generalize the Wiener index of the zero divisor graph of $Z_{p^n}$ for any $n \geq 4$ and $Z_{pqr}$ where $p, q$ and $r$ are prime numbers such that $p < q < r$.

3.1 Wiener index of the zero divisor graphs of $Z_{p^n}$ where $n \geq 4$

In this section, we first extend the Wiener index of $\Gamma[Z_{p^2}]$ and the Wiener index of $\Gamma[Z_{p^3}]$ into the Wiener index of $\Gamma[Z_{p^4}]$.

**Theorem 3.1.1.** Let $p$ be a prime number. Then the Wiener index of $\Gamma[Z_{p^4}]$ is

$$W(\Gamma[Z_{p^4}]) = p^6 - \frac{3}{2}p^4 - p^3 + \frac{1}{2}p^2 + 1.$$  

**Proof.** Let $p$ be a prime number and $\Gamma[Z_{p^4}]$ be the zero divisor graph of $Z_{p^4}$. Then $Z[Z_{p^4}] = \{p, 2p, 3p, \ldots, (p^3 - 1)p\}$ with cardinality $p^3 - 1$. We partition $Z[Z_{p^4}]$ into

$$Z[Z_{p^4}] = A_1 \cup A_2 \cup A_3$$

where

$$A_1 = \{k_1p : k_1 = 1, 2, 3, \ldots, p^3 - 1\} \text{ and } p \nmid k_1\}$$

$$A_2 = \{k_2p^2 : k_2 = 1, 2, 3, \ldots, p^2 - 1\} \text{ and } p \nmid k_2\}$$

$$A_3 = \{k_3p^3 : k_3 = 1, 2, 3, \ldots, p - 1\}$$
and \( A_i \cap A_j = \emptyset \) for any \( i \neq j \) with cardinalities

\[
|A_1| = p^2(p - 1), \\
|A_2| = p(p - 1), \\
|A_3| = (p - 1).
\]

Let \( x, y \in V(\Gamma[Z_{p^4}]) \). It is clear that \( d(x, y) = 0 \) if \( x = y \). Thus we may assume that \( x \) and \( y \) are distinct. We have the following six cases.

**Case 1 :** \( x, y \in A_1 \)

Then \( xy \neq 0 \), so \( d(x, y) \neq 1 \). Since every vertex in \( A_1 \) is adjacent to every vertex in \( A_3 \), we get \( d(x, y) = 2 \) and

\[
\sum_{x, y \in A_1} d(x, y) = 2 \binom{|A_1|}{2} = 2 \left[ \frac{|A_1|!}{2!([|A_1| - 2]!)} \right] = |A_1|(|A_1| - 1) = p^2(p - 1)[p^2(p - 1) - 1] = p^6 - 2p^5 + p^4 - p^3 + p^2.
\]

**Case 2 :** \( x, y \in A_2 \)

Then \( xy = 0 \). So we get \( d(x, y) = 1 \) and

\[
\sum_{x, y \in A_2} d(x, y) = \binom{|A_2|}{2} = \frac{|A_2|(|A_2| - 1)}{2} = \frac{p(p - 1)[p(p - 1) - 1]}{2} = p^4 - 2p^3 + p.
\]
Case 3: \( x, y \in A_3 \)

Then \( xy = 0 \). So we get \( d(x, y) = 1 \) and

\[
\sum_{x, y \in A_3} d(x, y) = \left( \frac{|A_3|}{2} \right) = \frac{|A_3|(|A_3| - 1)}{2} = \frac{(p - 1)((p - 1) - 1)}{2} = \frac{p^2 - 3p + 2}{2}.
\]

Case 4: \( x \in A_1 \) and \( y \in A_2 \)

Then \( xy \neq 0 \), so \( d(x, y) \neq 1 \). Since every vertex in \( A_1 \) is adjacent to every vertex in \( A_3 \) and every vertex in \( A_3 \) is adjacent to every vertex in \( A_2 \), we get \( d(x, y) = 2 \) and

\[
\sum_{x \in A_1, y \in A_2} d(x, y) = 2|A_1||A_2| = 2p^2(p - 1)[p(p - 1)] = 2p^5 - 4p^4 + 2p^3.
\]

Case 5: \( x \in A_1 \) and \( y \in A_3 \)

Then \( xy = 0 \). So we get \( d(x, y) = 1 \) and

\[
\sum_{x \in A_1, y \in A_3} d(x, y) = |A_1||A_3| = p^2(p - 1)(p - 1) = p^4 - 2p^3 + p^2.
\]

Case 6: \( x \in A_2 \) and \( y \in A_3 \)

Then \( xy = 0 \). So we get \( d(x, y) = 1 \) and

\[
\sum_{x \in A_2, y \in A_3} d(x, y) = |A_2||A_3| = p(p - 1)(p - 1) = p^3 - 2p^2 + p.
\]
Therefore, by all above cases, the Wiener index is

\[
W(\Gamma[\mathbb{Z}_p^4]) = p^6 - 2p^5 + p^4 - p^3 + p^2 + \frac{p^4 - 2p^3 + p}{2} + \frac{p^2 - 3p + 2}{2} \\
+ 2p^5 - 4p^4 + 2p^3 + p^4 - 2p^3 + p^2 + p^3 - 2p^2 + p \\
= p^6 - \frac{3}{2}p^4 - p^3 + \frac{1}{2}p^2 + 1.
\]

\[\Box\]
Example 3.1.2. Consider the ring $\mathbb{Z}_{16}$. The zero divisor graph of $\mathbb{Z}_{16}$ is given below.

![Zero Divisor Graph of $\mathbb{Z}_{16}$](image)

Figure 3.1: The zero divisor graph of $\mathbb{Z}_{16}$

Then we get

$$W(\Gamma[\mathbb{Z}_{16}]) = d(2, 4) + d(2, 6) + d(2, 8) + d(2, 10) + d(2, 12) + d(2, 14)$$
$$+ d(4, 6) + d(4, 8) + d(4, 10) + d(4, 12) + d(4, 14) + d(6, 8)$$
$$+ d(6, 10) + d(6, 12) + d(6, 14) + d(8, 10) + d(8, 12) + d(8, 14)$$
$$+ d(10, 12) + d(10, 14) + d(12, 14)$$
$$= 2 + 2 + 1 + 2 + 2 + 2 + 1 + 2 + 1 + 2 + 1 + 2 + 2 + 1 + 1 + 1$$
$$+ 2 + 2 + 2$$
$$= 35.$$

Since $16 = 2^4$, we have $p = 2$. By Theorem 3.1.1, the Wiener index of $\Gamma[\mathbb{Z}_{16}]$ is

$$p^6 - \frac{3}{2}p^4 - p^3 + \frac{1}{2}p^2 + 1 = (2^6) - \frac{3}{2}(2^4) - (2^3) + \frac{1}{2}(2^2) + 1$$
$$= 35.$$
Example 3.1.3. Consider the ring $\mathbb{Z}_{81}$. The zero divisor graph of $\mathbb{Z}_{81}$ is given below.

![Diagram](image)

Figure 3.2: The zero divisor graph of $\mathbb{Z}_{81}$

Since $81 = 3^4$, we have $p = 3$. By Theorem 3.1.1, the Wiener index of $\Gamma[\mathbb{Z}_{81}]$ is

$$p^6 - \frac{3}{2}p^4 - p^3 + \frac{1}{2}p^2 + 1 = (3^6) - \frac{3}{2}(3^4) - (3^3) + \frac{1}{2}(3^2) + 1 = 586.$$  

Next, we determine the generalization of Wiener index of $\Gamma[\mathbb{Z}_{p^n}]$ for any integer $n \geq 4$. For any zero divisor graph of $\mathbb{Z}_{p^n}$, we partition the vertex set of $\Gamma[\mathbb{Z}_{p^n}]$ into $n - 1$ sets,

$$V(\Gamma[\mathbb{Z}_{p^n}]) = \bigcup_{i=1}^{n-1} A_i$$

where $A_i = \{k_ip^i : k_i = 1, 2, 3, \ldots, p^{n-i} - 1 \text{ and } p \nmid k_i\}$. Then $|A_i| = p^{n-i-1}(p - 1)$ and $A_i \cap A_j = \emptyset$ for any $i, j \in \{1, 2, \ldots, n - 1\}$ such that $i \neq j$.

First, we start with some useful results.
Lemma 3.1.4. Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 2$. Let $i, j \in \{1, 2, \ldots, n-1\}$, $x \in A_i$ and $y \in A_j$. Then

$$d(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
1 & \text{if } x \neq y \text{ and } i + j \geq n, \\
2 & \text{if } x \neq y \text{ and } i + j < n.
\end{cases}$$

Proof. Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$. Let $i, j \in \{1, 2, \ldots, n-1\}$, $x \in A_i$ and $y \in A_j$. It is easy to see that if $x = y$, then $d(x, y) = 0$. Suppose that $x \neq y$. So $x = ap^i$ and $y = bp^j$ where $p \nmid a$ and $p \nmid b$. Therefore, $xy = abp^{i+j}$.

We have the following two cases.

Case 1 : $i + j \geq n$

Thus $xy = 0$, which implies that $\{x, y\}$ is edge in $\Gamma[\mathbb{Z}_{p^n}]$. Then $d(x, y) = 1$.

Case 2 : $i + j < n$

Thus $xy \neq 0$. Also, $p \nmid ab$, so $d(x, y) \neq 1$. Then there exists $z \in A_{n-1}$ such that $xz = 0$ and $yz = 0$. Clearly, $x, z, y$ is a path of length 2. Thus $d(x, y) = 2$.

This completes the proof. \hfill \square

We then define the following definition by employing Lemma 3.1.4.

Definition 3.1.5. Let $i, j \in \{1, 2, \ldots, n-1\}$. The distance between any two partition sets $A_i$ and $A_j$ is defined by

$$d(A_i, A_j) := d(x, y)$$

for any distinct $x \in A_i$ and $y \in A_j$.

Corollary 3.1.6. Let $p$ be a prime number and $n \geq 2$. Then $\text{diam } \Gamma[\mathbb{Z}_{p^n}] \leq 2$.

Proof. By Lemma 3.1.4, we have distance between any two distinct vertices in $\Gamma[\mathbb{Z}_{p^n}]$ are 0, 1 and 2. As a result, we get $\text{diam } \Gamma[\mathbb{Z}_{p^n}] \leq 2$. \hfill \square

Remark 3.1.7. $\mathbb{Z}_4$ is a ring whose zero divisor graph is the trivial graph. It is obvious that $W(\Gamma[\mathbb{Z}_4]) = 0$.
By utilizing Lemma 3.1.4 and Definition 3.1.5, we obtain the Wiener index of $\Gamma[Z_{pn}]$ where $n \geq 4$.

**Theorem 3.1.8.** Let $p$ be a prime number. Then the Wiener index of $\Gamma[Z_{pn}]$, where $n \geq 4$ is even, is

$$W(\Gamma[Z_{pn}]) = \sum_{i=1}^{n-1} \left[ |A_i|(|A_i| - 1) + 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right]$$

$$+ \sum_{i=\frac{n}{2}}^{n-1} \frac{|A_i|(|A_i| - 1)}{2} + \sum_{i=\frac{n}{2}}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|$$

where $|A_i| = p^{n-i-1}(p - 1)$.

**Proof.** Let $\Gamma[Z_{pn}]$ be the zero divisor graph of $Z_{pn}$. By Theorem 2.2.4, $Z[Z_{pn}] = \{p, 2p, 3p, \ldots, (p^{n-1} - 1)p\}$. Let $x \in A_i$ and $y \in A_j$ be any two distinct vertices. By Definition 3.1.5, it is clear that $d(A_i, A_j) = d(A_j, A_i)$. Without loss of generality, we consider $i \leq j$, so we get

$$W(\Gamma[Z_{pn}]) = \sum_{i=j} d(A_i, A_j) + \sum_{i<j} d(A_i, A_j).$$

Thus, we consider the following cases.

**Case 1** : $i = j$

If $i \leq \frac{n}{2} - 1$, then $i + j < n$. Also, if $i \geq \frac{n}{2}$, then $i + j \geq n$. By Lemma 3.1.4, we obtain

$$\sum_{i=j} d(A_i, A_j) = 2 \sum_{i=1}^{\frac{n}{2} - 1} \left( \frac{|A_i|}{2} \right) + \sum_{i=\frac{n}{2}}^{n-1} \left( \frac{|A_i|}{2} \right).$$

**Case 2** : $i < j$

**Subcase 2.1** : $i \leq \frac{n}{2} - 1$

If $j \leq n - i - 1$, then $i + j \leq n - 1$. Also, if $j \geq n - i$, then $i + j \geq n$. So we get

$$\sum_{i<j}^{i \leq \frac{n}{2} - 1} d(A_i, A_j) = \sum_{i=1}^{\frac{n}{2} - 1} \left[ 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right].$$

(3.1.1)

**Subcase 2.2** : $i \geq \frac{n}{2}$

Then $j > i \geq \frac{n}{2}$, so $i + j > n$. Consequently,

$$\sum_{i<j}^{i \geq \frac{n}{2}} d(A_i, A_j) = \sum_{i=\frac{n}{2}}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|.$$ 

(3.1.2)

Ref. code: 25605909031071FGX
By combining (3.1.1) and (3.1.2), we get the desired result when $i < j$ is

$$
\sum_{i<j} d(A_i, A_j) = \sum_{i=1}^{n-i} \frac{2}{2} \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j|
$$

$$
+ \sum_{i=\frac{n}{2}}^{n-2} \sum_{j=i+1}^{n-i-1} |A_i||A_j|
$$

By both cases above, we conclude that

$$
W(\Gamma[Z_{p^n}]) = \sum_{i<j} d(A_i, A_j) + \sum_{i<j} d(A_i, A_j)
$$

$$
= 2 \sum_{i=1}^{\frac{n}{2}-1} \left( \frac{|A_i|}{2} \right) + \sum_{i=\frac{n}{2}}^{n-i-1} \left( \frac{|A_i|}{2} \right) + \sum_{i=1}^{n-i-1} \left( \frac{2}{2} \sum_{j=i+1}^{n-i} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right)
$$

$$
+ \sum_{i=\frac{n}{2}}^{n-2} \sum_{j=i+1}^{n-i-1} |A_i||A_j|
$$

$$
= \sum_{i=1}^{\frac{n}{2}-1} |A_i|(|A_i| - 1) + 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j|
$$

$$
+ \sum_{i=\frac{n}{2}}^{n-1} |A_i|(|A_i| - 1) + \sum_{i=\frac{n}{2}}^{n-2} \sum_{j=i+1}^{n-i-1} |A_i||A_j|
$$

where $|A_i| = p^{n-i-1}(p - 1)$. □

**Theorem 3.1.9.** Let $p$ be a prime number. Then the Wiener index of $\Gamma[Z_{p^n}]$, where $n \geq 5$ is odd, is

$$
W(\Gamma[Z_{p^n}]) = \sum_{i=1}^{\frac{n}{2}} |A_i|(|A_i| - 1) + \sum_{i=\frac{n}{2}}^{n-i-1} \frac{|A_i|(|A_i| - 1)}{2}
$$

$$
+ \sum_{i=\frac{n}{2}}^{n-3} \left[ 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right]
$$

$$
+ \sum_{i=\frac{n}{2}}^{n-1} \sum_{j=i+1}^{n-i-1} |A_i||A_j|
$$

where $|A_i| = p^{n-i-1}(p - 1)$.

**Proof.** It follows from Theorem 3.1.8, we consider the following cases.

**Case 1 :** $i = j$
If \( i \leq \frac{n-1}{2} \), then \( i + j \leq n - 1 \). Also, if \( i \geq \frac{n+1}{2} \), then \( i + j \geq n \). By Lemma 3.1.4, we obtain
\[
\sum_{i=j} d(A_i, A_j) = 2 \sum_{i=1}^{n-1} \left( \frac{|A_i|}{2} \right) + \sum_{i=\frac{n+1}{2}}^{n-1} \left( \frac{|A_i|}{2} \right).
\]

Case 2: \( i < j \)

Subcase 2.1: \( i \leq \frac{n-3}{2} \)

If \( j \leq n - i - 1 \), then \( i + j \leq n - 1 \). Also, if \( j \geq n - i \), then \( i + j \geq n \). So we get
\[
\sum_{i<j, i \leq \frac{n-3}{2}} d(A_i, A_j) = \sum_{i=1}^{n-3} \left[ 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right]. \tag{3.1.3}
\]

Subcase 2.2: \( i \geq \frac{n-1}{2} \)

Since \( j > i \), we have \( j \geq \frac{n+1}{2} \). Thus \( i + j \geq n \). Consequently,
\[
\sum_{i<j, i \geq \frac{n-1}{2}} d(A_i, A_j) = \sum_{i=\frac{n-1}{2}}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|. \tag{3.1.4}
\]

By combining (3.1.3) and (3.1.4), we get the desired result when \( i < j \) is
\[
\sum_{i<j} d(A_i, A_j) = \sum_{i=1}^{n-3} \left[ 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right] + \sum_{i=\frac{n-1}{2}}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|.
\]

By both cases above, we conclude that
\[
W(\Gamma[Z_p^n]) = \sum_{i=j} d(A_i, A_j) + \sum_{i<j} d(A_i, A_j)
\]
\[
= 2 \sum_{i=1}^{n-1} \left( \frac{|A_i|}{2} \right) + \sum_{i=\frac{n+1}{2}}^{n-1} \left( \frac{|A_i|}{2} \right) + \sum_{i=1}^{n-3} \left[ 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right]
\]
\[
+ \sum_{i=\frac{n-1}{2}}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|.
\]

\[
= \sum_{i=1}^{n-1} |A_i||(A_i| - 1) + \sum_{i=\frac{n+1}{2}}^{n-1} \frac{|A_i||(A_i| - 1)}{2}
\]
\[
+ \sum_{i=1}^{n-3} \left[ 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j| + \sum_{j=n-i}^{n-1} |A_i||A_j| \right]
\]
\[
+ \sum_{i=\frac{n-1}{2}}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|.
\]
where \(|A_i| = p^{n-i-1}(p - 1)|. □

The following corollary is the Wiener index of zero divisor graph \(\mathbb{Z}_{p^n}\) for any integer \(n \geq 4\).

**Corollary 3.1.10.** Let \(p\) be a prime number. Then the Wiener index of \(\Gamma[\mathbb{Z}_{p^n}]\) where \(n \geq 4\) is

\[
W(\Gamma[\mathbb{Z}_{p^n}]) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} |A_i|(|A_i| - 1) + 2 \sum_{j=i+1}^{n-i-1} |A_i||A_j|
\]

\[
+ \sum_{j=n-i}^{n-1} |A_i||A_j| + \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n-1} |A_i|(|A_i| - 1)
\]

\[
+ \sum_{i=\lfloor \frac{n}{2} \rfloor}^{n-2} \sum_{j=i+1}^{n-1} |A_i||A_j|
\]

where \(|A_i| = p^{n-i-1}(p - 1)| for any \(i, j \in \{1, 2, \ldots, n - 1\}\).

**Proof.** It follows from Theorem 3.1.8 and 3.1.9. □

**Example 3.1.11.** Consider the ring \(\mathbb{Z}_{32}\). The zero divisor graph of \(\mathbb{Z}_{32}\) as shown below.

![Diagram of the zero divisor graph of \(\mathbb{Z}_{32}\)](image)

Figure 3.3: The zero divisor graph of \(\mathbb{Z}_{32}\)
Since \(32 = 2^5\) and By Theorem 3.1.9, we get the Wiener index of \(\Gamma[\mathbb{Z}_{32}]\) is

\[
W(\Gamma[\mathbb{Z}_{32}]) = \sum_{i=1}^{5-1} |A_i|(|A_i| - 1) + \sum_{i=5}^{5-1} \frac{|A_i|(|A_i| - 1)}{2} \\
+ \sum_{i=1}^{5-1} \left[ 2 \sum_{j=i+1}^{5-i-1} |A_i||A_j| + \sum_{j=5-i}^{5-1} |A_i||A_j| \right] \\
+ \sum_{i=2}^{5-1} \sum_{j=i+1}^{5-1} |A_i||A_j|
\]

\[
= |A_1|(|A_1| - 1) + |A_2|(|A_2| - 1) + \frac{|A_3|(|A_3| - 1)}{2} + \frac{|A_4|(|A_4| - 1)}{2} \\
+ 2 \left[ \sum_{j=2}^{3} |A_1||A_j| + \sum_{j=4}^{4} |A_1||A_j| \right] + \sum_{j=2+1}^{5-1} |A_2||A_j| + \sum_{j=3+1}^{5-1} |A_3||A_j|
\]

\[
= |A_1|(|A_1| - 1) + |A_2|(|A_2| - 1) + \frac{|A_3|(|A_3| - 1)}{2} + \frac{|A_4|(|A_4| - 1)}{2} \\
+ 2|A_1||A_2| + 2|A_1||A_3| + |A_1||A_4| + |A_2||A_3| + |A_2||A_4| + |A_3||A_4|
\]

Since \(p = 2\), so \(|A_1| = 8\), \(|A_2| = 4\), \(|A_3| = 2\) and \(|A_4| = 1\). Then we obtain

\[
W(\Gamma[\mathbb{Z}_{32}]) = 8(8 - 1) + 4(4 - 1) + \frac{2(2 - 1)}{2} + \frac{1(1 - 1)}{2} \\
+ 2(8)(4) + 2(8)(2) + (8)(1) + (4)(2) + (4)(1) + (2)(1) \\
= 187.
\]

3.2 Wiener index of the zero divisor graphs of \(\mathbb{Z}_{pqr}\)

In this section, we determine the Wiener index of the zero divisor graph of \(\mathbb{Z}_{pqr}\). We begin by giving the following definition.
Let $p, q$ and $r$ be prime numbers such that $p < q < r$. Let $\Gamma[\mathbb{Z}_{pqr}]$ be the zero divisor graph of $\mathbb{Z}_{pqr}$. We partition the vertex set of $\Gamma[\mathbb{Z}_{pqr}]$ into six sets,

$$V(\Gamma[\mathbb{Z}_{pqr}]) = \bigcup_{i=1}^{6} B_i$$

where

- $B_1 = \{k_1 p : k_1 = 1, 2, 3, \ldots, qr - 1 \text{ where } q \nmid k_1 \text{ and } r \nmid k_1\}$,
- $B_2 = \{k_2 q : k_2 = 1, 2, 3, \ldots, pr - 1 \text{ where } p \nmid k_2 \text{ and } r \nmid k_2\}$,
- $B_3 = \{k_3 r : k_3 = 1, 2, 3, \ldots, pq - 1 \text{ where } p \nmid k_3 \text{ and } q \nmid k_3\}$,
- $B_4 = \{k_4 pq : k_4 = 1, 2, 3, \ldots, r - 1\}$,
- $B_5 = \{k_5 pr : k_5 = 1, 2, 3, \ldots, q - 1\}$,
- $B_6 = \{k_6 qr : k_6 = 1, 2, 3, \ldots, p - 1\}$,

and $B_i \cap B_j = \emptyset$ for any distinct $i, j \in \{1, 2, \ldots, 6\}$. It is clear that $|B_4| = r - 1$, $|B_5| = q - 1$ and $|B_6| = p - 1$. Thus we determine cardinalities of $B_1, B_2$ and $B_3$. We consider the partition set

$$B_1 = \{k_1 p : k_1 = 1, 2, 3, \ldots, qr - 1 \text{ where } q \nmid k_1 \text{ and } r \nmid k_1\}.$$ 

Let

$$B_{1,q} = \{k_1 p : k_1 = 1, 2, 3, \ldots, qr - 1 \text{ and } q \mid k_1\}$$

and

$$B_{1,r} = \{k_1 p : k_1 = 1, 2, 3, \ldots, qr - 1 \text{ and } r \mid k_1\}.$$ 

Since $q$ and $r$ are distinct primes, $B_{1,q} \cap B_{1,r} = \emptyset$, so $|B_{1,q} \cap B_{1,r}| = 0$. We get $|B_{1,q}| = r - 1$ and $|B_{1,r}| = q - 1$. Then

$$|B_1| = (qr - 1) - (r - 1) - (q - 1)$$

$$= qr - r - q + 1$$

$$= (q - 1)r - (q - 1)$$

$$= (q - 1)(r - 1).$$

Thus $|B_1| = (q - 1)(r - 1)$.

Next, we consider the partition set

$$B_2 = \{k_2 q : k_2 = 1, 2, 3, \ldots, pr - 1 \text{ where } p \nmid k_2 \text{ and } r \nmid k_2\}.$$ 

Ref. code: 25605909031071FGX
Let \( B_{2,p} = \{ k_2q : k_2 = 1, 2, 3, \ldots, pr - 1 \text{ and } p \mid k_2 \} \)
and \( B_{2,r} = \{ k_2q : k_2 = 1, 2, 3, \ldots, pr - 1 \text{ and } r \mid k_2 \} \).
Since \( p \) and \( r \) are distinct primes, \( B_{2,p} \cap B_{2,r} = \emptyset \), so \( |B_{2,p} \cap B_{2,r}| = 0 \). We get \( |B_{2,p}| = r - 1 \) and \( |B_{2,r}| = p - 1 \). Then
\[
|B_2| = (pr - 1) - (r - 1) - (p - 1) \\
= pr - r - p + 1 \\
= (p - 1)r - (p - 1) \\
= (p - 1)(r - 1).
\]
Thus \( |B_2| = (p - 1)(r - 1) \).

Finally, we consider the partition set
\[ B_3 = \{ k_3r : k_3 = 1, 2, 3, \ldots, pq - 1 \text{ where } p \nmid k_3 \text{ and } q \mid k_3 \}. \]
Let \( B_{3,p} = \{ k_3r : k_3 = 1, 2, 3, \ldots, pq - 1 \text{ and } p \mid k_3 \} \)
and \( B_{3,q} = \{ k_3r : k_3 = 1, 2, 3, \ldots, pq - 1 \text{ and } q \mid k_3 \} \).
Since \( p \) and \( q \) are distinct primes, \( B_{3,p} \cap B_{3,q} = \emptyset \), so \( |B_{3,p} \cap B_{3,q}| = 0 \). We get \( |B_{3,p}| = q - 1 \) and \( |B_{3,q}| = p - 1 \). Then
\[
|B_3| = (pq - 1) - (q - 1) - (p - 1) \\
= pq - q - p + 1 \\
= (p - 1)q - (p - 1) \\
= (p - 1)(q - 1).
\]
Thus \( |B_3| = (p - 1)(q - 1) \).

**Lemma 3.2.1.** Let \( i \in \{1, 2, \ldots, 6\} \), \( x_1, x_2 \in B_i \) and \( z \in V(\Gamma[\mathbb{Z}_{pqr}]) \). Then \( x_1z = 0 \) if and only if \( x_2z = 0 \).

**Proof.** Let \( i \in \{1, 2, \ldots, 6\} \), \( x_1, x_2 \in B_i \) and \( z \in V(\Gamma[\mathbb{Z}_{pqr}]) \).

**Case 1:** \( i \in \{1, 2, 3\} \). Without loss of generality, suppose that \( i = 1 \). Then \( x_1, x_2 \in B_1 \).
Since \( x_1 \in B_1 \), we have \( x_1 = k_1 p \) where \( q \nmid k_1 \) and \( r \nmid k_1 \). Thus \( p \mid x_1 \), \( q \nmid x_1 \) and \( r \nmid x_1 \). Similarly, because \( x_2 \in B_1 \), we have \( x_2 = k_2 p \) where \( q \nmid k_2 \) and \( r \nmid k_2 \). Therefore \( p \mid x_2 \), \( q \nmid x_2 \) and \( r \nmid x_2 \).

Suppose that \( x_1 z = 0 \). Then \( pqr \mid x_1 z \). Since \( q \nmid x_1 \) and \( r \nmid x_1 \), we get \( qr \mid z \).

Because \( p \mid x_2 \), we have \( pqr \mid x_2 z \). Thus \( x_2 z = 0 \).

Conversely, Assume that \( x_2 z = 0 \). Then \( pqr \mid x_2 z \). Since \( q \nmid x_2 \) and \( r \nmid x_2 \), we get \( qr \mid z \). Because \( p \mid x_1 \), we have \( pqr \mid x_1 z \). Thus \( x_1 z = 0 \).

**Case 2:** \( i \in \{4, 5, 6\} \). Without loss of generality, suppose that \( i = 4 \). Then \( x_1, x_2 \in B_4 \), so \( x_1 = k_1 pq \) and \( x_2 = k_2 pq \) where \( r \nmid k_1 \) and \( r \nmid k_2 \). Thus we have \( pq \mid x_1 \) and \( r \nmid x_1 \). We also get \( pq \mid x_2 \) and \( r \nmid x_2 \).

Suppose that \( x_1 z = 0 \), so \( pqr \mid x_1 z \). Since \( r \nmid x_1 \), we have \( r \mid z \). Because \( pq \mid x_2 \), we obtain \( pqr \mid x_2 z \), so \( x_2 z = 0 \).

Conversely, Assume that \( x_2 z = 0 \). Then \( pqr \mid x_2 z \). Since \( r \nmid x_2 \), we have \( r \mid z \). Because \( pq \mid x_1 \), we obtain \( pqr \mid x_1 z \), so \( x_1 z = 0 \). \( \square \)

From the definition of \( B_i \) for all \( i \in \{1, 2, \ldots, 6\} \), we have the following remark.

**Remark 3.2.2.** Let \( i \in \{1, 2, \ldots, 6\} \). Then

1. For any \( x_1, x_2 \in B_i \) such that \( x_1 \neq x_2 \), we have \( x_1 x_2 \neq 0 \).

2. For any \( x \in B_i \), there exist \( z \in B_j \) for some \( j \in \{1, 2, \ldots, 6\} \setminus \{i\} \) such that \( xz = 0 \).

**Lemma 3.2.3.** Let \( p, q \) and \( r \) be prime numbers such that \( p < q < r \) and \( \Gamma[Z_{pqr}] \) be the zero divisor graph of \( Z_{pqr} \). Let \( i, j \in \{1, 2, \ldots, 6\} \), \( x_1, x_2 \in B_i \) and \( y_1, y_2 \in B_j \) such that \( x_1 \neq y_1 \) and \( x_2 \neq y_2 \). Then \( d(x_1, y_1) = d(x_2, y_2) \).

**Proof.** Let \( p, q \) and \( r \) be prime numbers such that \( p < q < r \) and \( \Gamma[Z_{pqr}] \) be the zero divisor graph of \( Z_{pqr} \). Let \( i, j \in \{1, 2, \ldots, 6\} \) and \( x_1, x_2 \in B_i \) and \( y_1, y_2 \in B_j \) such that \( x_1 \neq y_1 \) and \( x_2 \neq y_2 \). Then \( d(x_1, y_1) \neq 0 \) and \( d(x_2, y_2) \neq 0 \). Let \( n = d(x_1, y_1) \in \mathbb{Z}^+ \). Then
there exist \(z_1, z_2, \ldots, z_{n-1} \in \mathbb{Z}_{pqr}\) such that
\[x_1, z_1, z_2, \ldots, z_{n-1}, y_1\]
is a path of length \(n\) from \(x_1\) to \(y_1\). By Lemma 3.2.1, we have
\[x_2, z_1, z_2, \ldots, z_{n-1}, y_2\]
is also a path of length \(n\) from \(x_2\) to \(y_2\).

To show that \(d(x_2, y_2) = n\), assume that \(d(x_2, y_2) = m < n\). Then there exist \(z_1', z_2', \ldots, z_{m-1}' \in \mathbb{Z}_{pqr}\) such that
\[x_2, z_1', z_2', \ldots, z_{m-1}', y_2\]
is a path of length \(m\) from \(x_2\) to \(y_2\). By Lemma 3.2.1,
\[x_1, z_1, z_2, \ldots, z_{m-1}, y_1\]
is a path of length \(m < n\) from \(x_1\) to \(y_1\), which contradicts \(d(x_1, y_1) = n\). Hence \(d(x_1, y_1) = n = d(x_2, y_2)\).  

We define the following definition by employing previous lemma.

**Definition 3.2.4.** Let \(i, j \in \{1, 2, \ldots, 6\}\). The distance between any two partition sets \(B_i\) and \(B_j\) is defined by
\[d(B_i, B_j) := d(x, y)\]
for any distinct \(x \in B_i\) and \(y \in B_j\).

**Remark 3.2.5.** Let \(i, j \in \{1, 2, \ldots, 6\}\). Let \(x \in B_i\) and \(y \in B_j\). If \(xy\) is a multiple of \(pqr\), then \(xy = 0\), which implies that \(x\) is adjacent to \(y\). From definition of \(B_i\) for all \(i \in \{1, 2, \ldots, 6\}\), we have the following notices.

(i) Every vertex in \(B_1\) is adjacent to every vertex in \(B_6\), but each vertex in \(B_1\) is not adjacent to vertex in \(B_1, B_2, B_3, B_4\) and \(B_5\).
(ii) Every vertex in $B_2$ is adjacent to every vertex in $B_5$, but each vertex in $B_2$ is not adjacent to vertex in $B_1, B_2, B_3, B_4$ and $B_6$.

(iii) Every vertex in $B_3$ is adjacent to every vertex in $B_4$, but each vertex in $B_3$ is not adjacent to vertex in $B_1, B_2, B_3, B_5$ and $B_6$.

(iv) Every vertex in $B_4$ is adjacent to every vertex in $B_3, B_5$ and $B_6$, but each vertex in $B_4$ is not adjacent to vertex in $B_1, B_2$ and $B_4$.

(v) Every vertex in $B_5$ is adjacent to every vertex in $B_2, B_4$ and $B_6$, but each vertex in $B_5$ is not adjacent to vertex in $B_1, B_3$ and $B_5$.

(vi) Every vertex in $B_6$ is adjacent to every vertex in $B_1, B_4$ and $B_5$, but each vertex in $B_6$ is not adjacent to vertex in $B_2, B_3$ and $B_6$.

In Figure 3.4, every vertex in $B_i$ is adjacent to every vertex in $B_j$ for any distinct $i, j \in \{1, 2, \ldots, 6\}$ if and only if there exists an edge joining $B_i$ and $B_j$.

Figure 3.4: Joining an edge between any two partition sets $B_i$ and $B_j$. 
**Theorem 3.2.6.** Let \( p, q \) and \( r \) be prime numbers such that \( p < q < r \) and \( \Gamma[Z_{pqr}] \) be the zero divisor graph of \( Z_{pqr} \). Let \( i \in \{1, 2, \ldots, 6\} \). The distance between any two partition sets \( B_i \) and \( B_j \) is equal to 2, that is

\[
d(B_i, B_j) = 2.
\]

**Proof.** Let \( p, q \) and \( r \) be prime numbers such that \( p < q < r \) and \( \Gamma[Z_{pqr}] \) be the zero divisor graph of \( Z_{pqr} \). Let \( i \in \{1, 2, \ldots, 6\} \). Suppose \( x, y \in B_i \) be distinct vertices. Then \( d(x, y) \neq 0 \). By Remark 3.2.2(1), we have \( xy \neq 0 \) which implies that \( d(x, y) \neq 1 \). By Remark 3.2.2(2), there exists \( z \in B_j \) such that \( xz = 0 \) where \( j \neq i \). By Lemma 3.2.1, \( yz = 0 \). Clearly, \( x, z, y \) is a path of length 2. Then \( d(x, y) = 2 \), so \( d(B_i, B_j) = 2 \). □

**Lemma 3.2.7.** Let \( i, j \in \{1, 2, \ldots, 6\} \). For any \( x \in B_i \) and \( y \in B_j \). Then

\[
d(B_i, B_j) = \begin{cases} 
0 & \text{if } x = y, \\
1 & \text{if } x \neq y \text{ and } (i, j) \in \{(1, 6), (2, 5), (3, 4), (4, 5), (4, 6), (5, 6)\}, \\
2 & \text{if } x \neq y \text{ and } (i, j) \in \{(1, 1), (1, 4), (1, 5), (2, 2), (2, 4), (2, 6), (3, 3), (3, 5), (3, 6), (4, 4), (5, 5), (6, 6)\}, \\
3 & \text{if } x \neq y \text{ and } (i, j) \in \{(1, 2), (1, 3), (2, 3)\}.
\end{cases}
\]

**Proof.** Let \( i, j \in \{1, 2, \ldots, 6\} \). Let \( x \in B_i \) and \( y \in B_j \). It is easy to see that if \( x = y \), then \( d(x, y) = 0 \). Suppose \( x \neq y \). We have the following three cases.

**Case 1:** \( d(B_i, B_j) = 1 \)

By Remark 3.2.5, we get \( d(B_i, B_j) = 1 \) for \( (i, j) \in \{(1, 6), (2, 5), (3, 4), (4, 5), (4, 6), (5, 6)\} \).

**Case 2:** \( d(B_i, B_j) = 2 \)

By Remark 3.2.5 and Theorem 3.2.6, we get \( d(B_i, B_j) = 2 \) for \( (i, j) \in \{(1, 1), (1, 4), (1, 5), (2, 2), (2, 4), (2, 6), (3, 3), (3, 5), (3, 6), (4, 4), (5, 5), (6, 6)\} \).

**Case 3:** \( d(B_i, B_j) = 3 \)

By Remark 3.2.5, we get \( d(B_i, B_j) = 3 \) for \( (i, j) \in \{(1, 2), (1, 3), (2, 3)\} \). □

By Theorem 3.2.6 and Lemma 3.2.7, we obtain the Wiener index of the zero divisor graph of \( Z_{pqr} \).
Theorem 3.2.8. Let \( p, q \) and \( r \) be the prime numbers such that \( p < q < r \). Then the Wiener index of \( \Gamma[Z_{pqr}] \) is

\[
W(\Gamma[Z_{pqr}]) = 3pqr^2 + 3pq^2r + 3p^2qr - 15pqr + p^2q^2 + p^2r^2 + q^2r^2
- 3pq^2 - 3pr^2 - 3qp^2 - 3qr^2 - 3rp^2 - 3rq^2 + 8pq + 8pr + 8qr
+ 2p^2 + 2q^2 + 2r^2 - 4p - 4q - 4r + 3.
\]

Proof. Let \( \Gamma[Z_{pqr}] \) be the zero divisor graph of \( Z_{pqr} \). Then

\[
Z[Z_{pqr}] = \bigcup_{i=1}^{6} B_i
\]

where \( B_i \cap B_j = \emptyset \) for any distinct \( i, j \in \{1, 2, \ldots, 6\} \). Let \( x \in B_i \) and \( y \in B_j \). It is clear that \( d(x,y) = 0 \) if \( x = y \). Thus we may assume that \( x \) and \( y \) are distinct. It is clear that \( d(B_i,B_j) = d(B_j,B_i) \). Without loss of generality, we consider \( i \leq j \), so we get

\[
W(\Gamma[Z_{pqr}]) = \sum_{i=j} d(B_i,B_j) + \sum_{i<j} d(B_i,B_j).
\]

We have the following two cases.

Case 1: \( i = j \)

By Theorem 3.2.6, we get \( d(B_i,B_i) = 2 \) for any \( i \in \{1, 2, \ldots, 6\} \). We obtain

\[
\sum_{i=j} d(B_i,B_i) = 2 \left[ \left( \frac{|B_1|}{2} \right) + \left( \frac{|B_2|}{2} \right) + \left( \frac{|B_3|}{2} \right) + \left( \frac{|B_4|}{2} \right) + \left( \frac{|B_5|}{2} \right) + \left( \frac{|B_6|}{2} \right) \right]
= 2 \left[ \frac{|B_1|(|B_1| - 1)}{2} + \frac{|B_2|(|B_2| - 1)}{2} + \frac{|B_3|(|B_3| - 1)}{2} + \frac{|B_4|(|B_4| - 1)}{2} 
+ \frac{|B_5|(|B_5| - 1)}{2} + \frac{|B_6|(|B_6| - 1)}{2} \right]
= |B_1|(|B_1| - 1) + |B_2|(|B_2| - 1) + |B_3|(|B_3| - 1) + |B_4|(|B_4| - 1)
+ |B_5|(|B_5| - 1) + |B_6|(|B_6| - 1)
= (q-1)(r-1) \left[ (q-1)(r-1) - 1 \right] + (p-1)(r-1) \left[ (p-1)(r-1) - 1 \right]
+ (p-1)(q-1) \left[ (p-1)(q-1) - 1 \right] + (r-1)(r-2) + (q-1)(q-2)
+ (p-1)(p-2)
= q^2r^2 - 2qr^2 - 2q^2r + 3qr + p^2r^2 - 2pr^2 - 2p^2r + 3pr + p^2q^2
- 2pq^2 - 2p^2q + 3pq + 3r^2 + 3q^2 + 3p^2 - 5r - 5q - 5p + 6. \quad (3.2.1)
\]
Case 2 : \( i < j \)

**Subcase 2.1 :** \( d(B_i, B_j) = 1 \)

By Lemma 3.2.7, we obtain \( d(B_1, B_6) = 1, d(B_2, B_5) = 1, d(B_3, B_4) = 1, \)
\( d(B_4, B_5) = 1, d(B_4, B_6) = 1 \) and \( d(B_5, B_6) = 1 \). Then

\[
\sum d(B_i, B_j) = \left[ B_1||B_6| + |B_2||B_5| + |B_3||B_4| + |B_4||B_5| + |B_4||B_6| + |B_5||B_6| \right]
\]
\[
= (q-1)(r-1)(p-1) + (p-1)(r-1)(q-1) + (p-1)(q-1)(r-1)
+ (r-1)(q-1) + (r-1)(p-1) + (q-1)(p-1)
\]
\[
= 3(p-1)(q-1)(r-1) + (q-1)(r-1) + (p-1)(r-1)
+ (p-1)(q-1)
\]
\[
= 3pqr - 2pr - 2qr - 2pq + r + p + q. \quad (3.2.2)
\]

**Subcase 2.2 :** \( d(B_i, B_j) = 2 \)

By Lemma 3.2.7, we get \( d(B_1, B_4) = 2, d(B_1, B_5) = 2, d(B_2, B_4) = 2, d(B_2, B_6) = 2, \)
\( d(B_3, B_5) = 2 \) and \( d(B_3, B_6) = 2 \). Then

\[
\sum d(B_i, B_j) = 2 \left[ B_1||B_4| + |B_1||B_5| + |B_2||B_4| + |B_2||B_6| + |B_3||B_5| + |B_3||B_6| \right]
\]
\[
= 2 \left[ (q-1)(r-1)^2 + (r-1)(q-1)^2 + (p-1)(r-1)^2
+ (r-1)(p-1)^2 + (p-1)(q-1)^2 + (q-1)(p-1)^2 \right]
\]
\[
= 2pr^2 + 2pq^2 + 2qp^2 - 8pr - 8pq - 8qr + 12p + 12q + 12r
+ 2qr^2 + 2rq^2 + 2rp^2 - 4r^2 - 4q^2 - 4p^2 - 12. \quad (3.2.3)
\]

**Subcase 2.3 :** \( d(B_i, B_j) = 3 \)

By Lemma 3.2.7, we have \( d(B_1, B_2) = 3, d(B_1, B_3) = 3 \) and \( d(B_2, B_3) = 3 \). Then

\[
\sum d(B_i, B_j) = 3 \left[ B_1||B_2| + |B_1||B_3| + |B_2||B_3| \right]
\]
\[
= 3 \left[ (p-1)(q-1)(r-1)^2 + (p-1)(r-1)(q-1)^2
+ (q-1)(r-1)(p-1)^2 \right]
\]
\[
= 3pq^2 + 3pr^2 + 3qr^2 - 18pqr + 3pq + 3pr + 3qr - 3p^2 - 3q^2
- 3p^2 + 6pr + 12pq - 12p - 12q - 12r - 3qr^2 - 3q^2 - 3pr^2
+ 12qr + 6pr + 3r^2 + 3q^2 + 3p^2 + 9. \quad (3.2.4)
\]
By combining (3.2.1) – (3.2.4), we get the following result

\[
W(\Gamma[\mathbb{Z}_{pqr}]) = 3pqr^2 + 3pq^2r + 3p^2qr - 15pqr + p^2q^2 + p^2r^2 + q^2r^2 \\
- 3pq^2 - 3pr^2 - 3qp^2 - 3qr^2 - 3rp^2 - 3rq^2 + 8pq + 8pr + 8qr \\
+ 2p^2 + 2q^2 + 2r^2 - 4p - 4q - 4r + 3.
\]

This completes the proof. \(\square\)

**Example 3.2.9.** Consider the ring \(\mathbb{Z}_{30}\). The zero divisor graph of \(\mathbb{Z}_{30}\) is shown below.

![Zero Divisor Graph of \(\mathbb{Z}_{30}\)](image)

Figure 3.5: The zero divisor graph of \(\mathbb{Z}_{30}\)
By Theorem 3.2.8 and \( p = 2, q = 3 \) and \( r = 5 \), so we get

\[
W(\Gamma[Z_{30}]) = 3(2)(3)(5)^2 + 3(2)(3)^2(5) + 3(2)^2(3)(5) - 15(2)(3)(5)
+ (2)^2(3)^2 + (2)^2(5)^2 + (3)^2(5)^2 - 3(2)(3)^2 - 3(2)(5)^2
- 3(3)(2)^2 - 3(3)(5)^2 - 3(5)(2)^2 - 3(5)(3)^2 + 8(2)(3)
+ 8(2)(5) + 8(3)(5) + 2(2)^2 + 2(3)^2 + 2(5)^2 - 4(2) - 4(3)
- 4(5) + 3
= 438.
\]
CHAPTER 4

LAPLACIAN MATRIX OF SOME ZERO DIVISOR GRAPHS

We next determine the Laplacian matrix of the zero divisor graph of $\mathbb{Z}_{p^n}$ where $p$ is a prime number and $n \in \{2, 3\}$. We then give the general form of the Laplacian matrix of the zero divisor graph of $\mathbb{Z}_{p^n}$ where $p$ is a prime number for any integer $n \geq 4$. Furthermore, we generalize the Laplacian matrix of the zero divisor graph of $\mathbb{Z}_{pqr}$ where $p, q$ and $r$ are prime numbers such that $p < q < r$.

Recall that, for any zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 2$, we partition the vertex set of $\Gamma[\mathbb{Z}_{p^n}]$ into $n - 1$ sets,

$$V(\Gamma[\mathbb{Z}_{p^n}]) = \bigcup_{i=1}^{n-1} A_i$$

where

$$A_i = \{k_ip^i : k_i = 1, 2, 3, \ldots, p^{n-i} - 1 \text{ and } p \nmid k_i\}$$

for all $i \in \{1, 2, \ldots, n - 1\}$. Then $|A_i| = p^{n-i-1}(p - 1)$ and $A_i \cap A_j = \emptyset$ for any $i, j \in \{1, 2, \ldots, n - 1\}$ such that $i \neq j$.

We also consider each $A_i$ as an ordered set sorting from the smallest number to the largest number for all $i \in \{1, 2, \ldots, n - 1\}$.

For any positive integers $m_1$ and $m_2$, we denoted $-I_{m_1 \times m_2}$ to be an $m_1 \times m_2$ matrix whose all entries are $-1$. Also, $O_{m_1 \times m_2}$ denotes an $m_1 \times m_2$ matrix whose all entries are 0. In case the dimension of matrix is known, we may write $1$ or $O$, respectively.

In the next section, we will discuss the Laplacian matrix of the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \in \{2, 3\}$. 
4.1 Laplacian matrix of the zero divisor graphs of $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^3}$

We begin this section by giving the result of Duane [5]. We then use this to determine the Laplacian matrix of the zero divisor graph of $\mathbb{Z}_{p^2}$.

**Theorem 4.1.1.** [5] The zero divisor graph of $\mathbb{Z}_{p^2}$ is $K_{p-1}$ where $p$ is a prime.

**Theorem 4.1.2.** Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^2}]$ be the zero divisor graph of $\mathbb{Z}_{p^2}$. Then the Laplacian matrix of $\Gamma[\mathbb{Z}_{p^2}]$ is a $(p-1) \times (p-1)$ matrix

$$L(\Gamma[\mathbb{Z}_{p^2}]) = \begin{bmatrix}
    p-2 & -1 & \cdots & -1 \\
    -1 & p-2 & \cdots & -1 \\
    \vdots & \vdots & \ddots & \vdots \\
    -1 & -1 & \cdots & p-2
\end{bmatrix} = -1_{(p-1)\times(p-1)} + (p-1)I_{p-1}.$$

**Proof.** Let $\Gamma[\mathbb{Z}_{p^2}]$ be the zero divisor graph of $\mathbb{Z}_{p^2}$. By Theorem 4.1.1, $\Gamma[\mathbb{Z}_{p^2}]$ is complete graph of order $p-1$. Then the degree of every vertex is $(p-1) - 1 = p-2$. Thus the elements on the main diagonal of $L(\Gamma[\mathbb{Z}_{p^2}])$ are $p-2$. Since each pair of distinct vertices is adjacent, all other entries are $-1$. So we get a $(p-1) \times (p-1)$ matrix

$$L(\Gamma[\mathbb{Z}_{p^2}]) = \begin{bmatrix}
    p-2 & -1 & \cdots & -1 \\
    -1 & p-2 & \cdots & -1 \\
    \vdots & \vdots & \ddots & \vdots \\
    -1 & -1 & \cdots & p-2
\end{bmatrix} = -1_{(p-1)\times(p-1)} + (p-1)I_{p-1}.$$
**Theorem 4.1.3.** Let $p$ be a prime number and $\Gamma[Z_p^3]$ be the zero divisor graph of $Z_p^3$.

Then the Laplacian matrix of $\Gamma[Z_p^3]$ is a $(p^2 - 1) \times (p^2 - 1)$ matrix

\[
\begin{bmatrix}
A_1 & A_2 \\
A_1^T & -I_{|A_1|} \\
A_2 & -1_{|A_2|} \\
\end{bmatrix}
\]

where \( J = \begin{bmatrix} p^2 - 2 & -1 & \cdots & -1 \\
-1 & p^2 - 2 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & p^2 - 2 \end{bmatrix} = -1_{|A_2| \times |A_2|} + (p^2 - 1)I_{|A_2|}. \)

**Proof.** Let $p$ be a prime number and $\Gamma[Z_p^3]$ be the zero divisor graph of $Z_p^3$. The row and the column of the Laplacian matrix of $\Gamma[Z_p^3]$ is labeled with elements of $A_1$ and $A_2$, respectively.

By Lemma 3.1.4, we obtain $d(A_1, A_1) = 2$, $d(A_1, A_2) = 1$ and $d(A_2, A_2) = 1$, which imply that every vertex in $A_1$ and $A_2$ is adjacent to every vertex in $A_2$, but each vertex in $A_1$ is not adjacent to vertex in $A_1$. Since zero divisor graph contains no loop, we get

\[
\text{deg}(x) = \begin{cases} |A_2| & \text{if } x \in A_1, \\
|A_1| + |A_2| - 1 & \text{if } x \in A_2. \end{cases}
\]

Then

\[
\text{deg}(x) = \begin{cases} p - 1 & \text{if } x \in A_1, \\
p^2 - 2 & \text{if } x \in A_2. \end{cases}
\]
So we get matrices \(-I_{|A_1| \times |A_2|}\) and \(-I_{|A_2| \times |A_1|}\) in \(L(\Gamma[Z_{p^3}])\) as follows:

\[
\begin{bmatrix}
  A_1 & A_2 \\
  A_1 & \begin{bmatrix} -I_{|A_1| \times |A_2|} \\
                    -I_{|A_2| \times |A_1|} \end{bmatrix}
\end{bmatrix}
\]

However, every vertex in \(A_1\) is not adjacent to vertex in \(A_1\). Thus we get diagonal matrix \((p - 1)I_{|A_1|}\) in \(L(\Gamma[Z_{p^3}])\), which yields

\[
\begin{bmatrix}
  A_1 & A_2 \\
  A_1 & \begin{bmatrix} (p - 1)I_{|A_1|} & -I_{|A_1| \times |A_2|} \\
                    -I_{|A_2| \times |A_1|} \end{bmatrix}
\end{bmatrix}
\]

Since every vertex in \(A_2\) is adjacent to every vertex in \(A_2\), we obtain matrix \(J\) which the main diagonal entries of \(J\) are \(\deg(x) = p^2 - 2\) for any \(x \in A_2\) and all other entries are \(-1\), which gives

\[
\begin{bmatrix}
  A_1 & A_2 \\
  A_1 & \begin{bmatrix} (p - 1)I_{|A_1|} & -I_{|A_1| \times |A_2|} \\
                    -I_{|A_2| \times |A_1|} & J \end{bmatrix}
\end{bmatrix}
\]

where \(J = \begin{bmatrix} p^2 - 2 & -1 & \cdots & -1 \\ -1 & p^2 - 2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & p^2 - 2 \end{bmatrix} = -I_{|A_2| \times |A_2|} + (p^2 - 1)I_{|A_2|}\)

Since the order of zero divisor graph of \(Z_{p^3}\) is \(p^2 - 1\), the Laplacian matrix of \(\Gamma[Z_{p^3}]\) is a \((p^2 - 1) \times (p^2 - 1)\) matrix. \(\square\)
4.2 Laplacian matrix of the zero divisor graphs of $\mathbb{Z}_{p^n}$ where $n \geq 4$

Here, we generalize the Laplacian matrix of $\Gamma[\mathbb{Z}_{p^n}]$. In the following lemma, we determine the degree of each vertex in $\Gamma[\mathbb{Z}_{p^n}]$. This will be used in Theorem 4.2.4.

**Lemma 4.2.1.** Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 4$ is even. Let $i \in \{1, 2, \ldots, n-1\}$ and $x \in A_i$. Then

$$\deg(x) = \begin{cases} \sum_{j \geq n-i} |A_j| & \text{if } i \leq \frac{n}{2} - 1, \\ \sum_{j \geq n-i} |A_j| - 1 & \text{if } i \geq \frac{n}{2}. \end{cases}$$

where $|A_j| = p^{n-j-1}(p-1)$.

**Proof.** Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 4$ is even. Let $i \in \{1, 2, \ldots, n-1\}$ and $x \in A_i$. Therefore we have the following two cases.

**Case 1 :** $i \leq \frac{n}{2} - 1$

If $i \leq \frac{n}{2} - 1$, then $i + i < n$. By Lemma 3.1.4, each vertex in $A_i$ is not adjacent to vertex in $A_i$. Also, if $j \geq n-i$, then $i + j \geq n$. Then every vertex in $A_j$ is adjacent to $x$, so we get

$$\deg(x) = \sum_{j \geq n-i} |A_j|$$

**Case 2 :** $i \geq \frac{n}{2}$

If $i \geq \frac{n}{2}$, then $i + i \geq n$. By Lemma 3.1.4, every vertex in $A_i$ is adjacent to every vertex in $A_i$. Also, if $j \geq n-i$, then $i + j \geq n$. Then every vertex in $A_j$ is adjacent to $x$, so we get

$$\deg(x) = \sum_{j \geq n-i} |A_j| - 1$$

This completes the proof. □
Lemma 4.2.2. Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 5$ is odd. Let $i \in \{1, 2, \ldots, n-1\}$ and $x \in A_i$. Then

$$\deg(x) = \begin{cases} 
\sum_{j \geq n-i}^n |A_j| & \text{if } i \leq \frac{n-1}{2}, \\
\sum_{j \geq n-i}^n |A_j| - 1 & \text{if } i \geq \frac{n+1}{2}.
\end{cases}$$

where $|A_j| = p^{n-j-1}(p-1)$.

**Proof.** Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 5$ is odd. Let $i \in \{1, 2, \ldots, n-1\}$ and $x \in A_i$. Therefore we have the following two cases.

**Case 1 :** $i \leq \frac{n-1}{2}$

If $i \leq \frac{n-1}{2}$, then $i + i = 2i \leq n - 1 < n$. By Lemma 3.1.4, each vertex in $A_i$ is not adjacent to vertex in $A_j$. Also, if $j \geq n - i$, then $i + j \geq n$. Then every vertex in $A_j$ is adjacent to $x$, so we get

$$\deg(x) = \sum_{j \geq n-i}^n |A_j|$$

**Case 2 :** $i \geq \frac{n+1}{2}$

If $i \geq \frac{n+1}{2}$, then $i + i = 2i \geq n + 1 > n$. By Lemma 3.1.4, every vertex in $A_i$ is adjacent to every vertex in $A_j$. Also, if $j \geq n - i$, then $i + j \geq n$. Then every vertex in $A_j$ is adjacent to $x$, so we get

$$\deg(x) = \sum_{j \geq n-i}^n |A_j| - 1$$

This completes the proof. \qed

**Corollary 4.2.3.** Let $p$ be a prime number and $\Gamma[\mathbb{Z}_{p^n}]$ be the zero divisor graph of $\mathbb{Z}_{p^n}$ where $n \geq 4$. Let $i \in \{1, 2, \ldots, n-1\}$ and $x \in A_i$. Then

$$\deg(x) = \begin{cases} 
\sum_{j \geq n-i}^{\lfloor \frac{n}{2} \rfloor} |A_j| & \text{if } i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
\sum_{j \geq n-i}^{\lfloor \frac{n}{2} \rfloor} |A_j| - 1 & \text{if } i \geq \left\lfloor \frac{n}{2} \right\rfloor,
\end{cases}$$

where $|A_j| = p^{n-j-1}(p-1)$. 

Ref. code: 25605909031071FGX
Proof. It follows from Lemma 4.2.1 and 4.2.2.

\[ \square \]

**Theorem 4.2.4.** Let \( n \geq 4 \) and \( i, j \in \{1, 2, \ldots, n-1\} \). Let \( p \) be a prime number and \( \Gamma[Z_{p^n}] \) be the zero divisor graph of \( Z_{p^n} \). Then the Laplacian matrix of \( \Gamma[Z_{p^n}] \) is a \( (n-1) \times (n-1) \) block matrix \( L(\Gamma[Z_{p^n}]) = [L_{i,j}] \), given by

\[
L_{i,j} = \begin{cases} 
\deg(x) J_{|A_i|} & \text{if } i = j \text{ and } i + j < n, \\
J_{|A_i|} & \text{if } i = j \text{ and } i + j \geq n, \\
O & \text{if } i \neq j \text{ and } i + j < n, \\
-1 & \text{if } i \neq j \text{ and } i + j \geq n
\end{cases}
\]

where \( J_{|A_i|} = \begin{bmatrix} \deg(x) & -1 & \cdots & -1 \\ -1 & \deg(x) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \deg(x) \end{bmatrix} \) and \( x \in A_i \).

Proof. Let \( n \geq 4 \) and \( i, j \in \{1, 2, \ldots, n-1\} \). Let \( p \) be a prime number and \( \Gamma[Z_{p^n}] \) be the zero divisor graph of \( Z_{p^n} \). We let \( L(\Gamma[Z_{p^n}]) = [L_{i,j}] \) where \( L_{i,j} \) is block matrix. The rows and the columns of the Laplacian matrix of \( \Gamma[Z_{p^n}] \) are labeled with elements of \( A_1, A_2, \ldots, A_{n-1} \), respectively.

\[
\begin{array}{c|cccc}
 & A_1 & A_2 & \cdots & A_{n-1} \\
\hline A_1 & & & & \\
A_2 & & & & \\
\vdots & & & & \\
A_{n-1} & & & & \\
\end{array}
\]

**Case 1 :** \( i = j \).

We have the following two subcases to consider.
Subcase 1.1 : \( i + j < n \).

By Lemma 3.1.4, every vertex in \( A_i \) is not adjacent to vertex in \( A_j \). Then we get diagonal matrix \( \deg(x)I_{|A_i|} \) where \( x \in A_i \).

Subcase 1.2 : \( i + j \geq n \).

By Lemma 3.1.4, every vertex in \( A_i \) is adjacent to every vertex in \( A_j \). Hence we get a matrix \( J \) where

\[
J_{|A_i|} = \begin{bmatrix}
\deg(x) & -1 & \cdots & -1 \\
-1 & \deg(x) & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & \deg(x)
\end{bmatrix}
\]

which its main diagonal are all \( \deg(x) \) and other entries are \(-1\) where \( x \in A_i \).

Case 2 : \( i \neq j \).

We have the following two subcases to consider.

Subcase 2.1 : \( i + j < n \).

By Lemma 3.1.4, every vertex in \( A_i \) is not adjacent to vertex in \( A_j \). We get a zero matrix \( O \).

Subcase 2.2 : \( i + j \geq n \).

By Lemma 3.1.4, every vertex in \( A_i \) is adjacent to every vertex in \( A_j \). Then we obtain a matrix \(-1\) which all element in matrix are all \(-1\).

This completes the proof. □
Example 4.2.5. Recall the zero divisors of \( \mathbb{Z}_{16} \) in Example 3.1.2. The set of all zero divisors of \( \mathbb{Z}_{16} \) is \( \{2, 4, 6, 8, 10, 12, 14\} \). We partition \( \mathbb{Z}[\mathbb{Z}_{16}] \) into

\[
\mathbb{Z}[\mathbb{Z}_{16}] = A_1 \cup A_2 \cup A_3
\]

where

\[
A_1 = \{2, 6, 10, 14\},
\]

\[
A_2 = \{4, 12\},
\]

\[
A_3 = \{8\},
\]

and

\[
\text{deg}(x) = 1 \text{ for all } x \in A_1,
\]

\[
\text{deg}(x) = 2 \text{ for all } x \in A_2,
\]

\[
\text{deg}(x) = 6 \text{ for all } x \in A_3.
\]

The Laplacian matrix of \( \Gamma[\mathbb{Z}_{16}] \) is given below

\[
\begin{bmatrix}
2 & 6 & 10 & 14 & 4 & 12 & 8 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
6 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
10 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
14 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
4 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\
12 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
8 & -1 & -1 & -1 & -1 & -1 & 6 & -1
\end{bmatrix}
\]
4.3 Laplacian matrix of the zero divisor graphs of $\mathbb{Z}_{pqr}$

In this section, we generalize the Laplacian matrix of $\Gamma[\mathbb{Z}_{pqr}]$. We first recall the definition of $B_i$ for all $i \in \{1,2,\ldots,6\}$.

Let $p$, $q$ and $r$ be prime numbers such that $p < q < r$. Let $\Gamma[\mathbb{Z}_{pqr}]$ be the zero divisor graph of $\mathbb{Z}_{pqr}$. We partition the vertex set of $\Gamma[\mathbb{Z}_{pqr}]$ into six sets,

$$V(\Gamma[\mathbb{Z}_{pqr}]) = \bigcup_{i=1}^{6} B_i$$

where

$B_1 = \{k_1 p : k_1 = 1,2,3,\ldots, qr - 1 \text{ where } q \nmid k_1 \text{ and } r \nmid k_1\},$

$B_2 = \{k_2 q : k_2 = 1,2,3,\ldots, pr - 1 \text{ where } p \nmid k_2 \text{ and } r \nmid k_2\},$

$B_3 = \{k_3 r : k_3 = 1,2,3,\ldots, pq - 1 \text{ where } p \nmid k_3 \text{ and } q \nmid k_3\},$

$B_4 = \{k_4 pq : k_4 = 1,2,3,\ldots, r - 1\},$

$B_5 = \{k_5 pr : k_5 = 1,2,3,\ldots, q - 1\},$

$B_6 = \{k_6 qr : k_6 = 1,2,3,\ldots, p - 1\}$

and

$|B_1| = (q - 1)(r - 1),$

$|B_2| = (p - 1)(r - 1),$

$|B_3| = (p - 1)(q - 1),$

$|B_4| = r - 1,$

$|B_5| = q - 1,$

$|B_6| = p - 1.$

From now on, we consider each $B_i$ as an ordered set sorting from smallest number to largest number for every $i \in \{1,2,\ldots,6\}$. The following lemma, we determine the degree of each vertex in $\Gamma[\mathbb{Z}_{pqr}]$ which will be used in Theorem 4.3.2.
Lemma 4.3.1. Let $p, q$ and $r$ be prime numbers such that $p < q < r$. Let $\Gamma[\mathbb{Z}_{pqr}]$ be the zero divisor graph of $\mathbb{Z}_{pqr}$. Then we get

$$\deg(x) = p - 1 \text{ for all } x \in B_1,$$
$$\deg(x) = q - 1 \text{ for all } x \in B_2,$$
$$\deg(x) = r - 1 \text{ for all } x \in B_3,$$
$$\deg(x) = pq - 1 \text{ for all } x \in B_4,$$
$$\deg(x) = pr - 1 \text{ for all } x \in B_5,$$
$$\deg(x) = qr - 1 \text{ for all } x \in B_6.$$

Proof. Let $\Gamma[\mathbb{Z}_{pqr}]$ be the zero divisor graph of $\mathbb{Z}_{pqr}$. We have the following six cases.

**Case 1 :** $x \in B_1$. By Remark 3.2.5, every vertex in $B_1$ is adjacent to every vertex in $B_6$, but each vertex in $B_1$ is not adjacent to vertex in $B_1, B_2, B_3, B_4$ and $B_5$. So

$$\deg(x) = |B_6| = p - 1.$$

**Case 2 :** $x \in B_2$. By Remark 3.2.5, every vertex in $B_2$ is adjacent to every vertex in $B_5$, but each vertex in $B_2$ is not adjacent to vertex in $B_1, B_2, B_3, B_4$ and $B_6$. Thus

$$\deg(x) = |B_5| = q - 1.$$

**Case 3 :** $x \in B_3$. By Remark 3.2.5, every vertex in $B_3$ is adjacent to every vertex in $B_4$, but each vertex in $B_3$ is not adjacent to vertex in $B_1, B_2, B_3, B_5$ and $B_6$. Then

$$\deg(x) = |B_4| = r - 1.$$

**Case 4 :** $x \in B_4$. By Remark 3.2.5, every vertex in $B_4$ is adjacent to every vertex in $B_3, B_5$ and $B_6$, but each vertex in $B_4$ is not adjacent to vertex in $B_1, B_2$ and $B_4$. Thus

$$\deg(x) = |B_3| + |B_5| + |B_6|$$
$$= (p - 1)(q - 1) + (q - 1) + (p - 1)$$
$$= pq - 1.$$
Case 5: \( x \in B_5 \). By Remark 3.2.5, every vertex in \( B_5 \) is adjacent to every vertex in \( B_2, B_4 \) and \( B_6 \), but each vertex in \( B_5 \) is not adjacent to vertex in \( B_1, B_3 \) and \( B_5 \). Thus

\[
\deg(x) = |B_2| + |B_4| + |B_6|
\]

\[
= (p - 1)(r - 1) + (r - 1) + (p - 1)
\]

\[
= pr - 1.
\]

Case 6: \( x \in B_6 \). By Remark 3.2.5, every vertex in \( B_6 \) is adjacent to every vertex in \( B_1, B_4 \) and \( B_5 \), but each vertex in \( B_6 \) is not adjacent to vertex in \( B_2, B_3 \) and \( B_6 \). Then

\[
\deg(x) = |B_1| + |B_4| + |B_5|
\]

\[
= (q - 1)(r - 1) + (r - 1) + (q - 1)
\]

\[
= qr - 1.
\]

This completes the proof.

Theorem 4.3.2. Let \( p, q \) and \( r \) be prime numbers such that \( p < q < r \). Then the Laplacian matrix of \( \Gamma[Z_{pqr}] \) is a \( 6 \times 6 \) block matrix

\[
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p - 1)I_{|B_1|} & O & O & O & O & -1 \\
O & (q - 1)I_{|B_2|} & O & O & -1 & O \\
O & O & (r - 1)I_{|B_3|} & -1 & O & O \\
O & O & -1 & I & (pq - 1)I_{|B_4|} & -1 \\
O & -1 & O & -1 & (pr - 1)I_{|B_5|} & -1 \\
-1 & O & O & -1 & -1 & (qr - 1)I_{|B_6|}
\end{bmatrix}
\]

Proof. Let \( \Gamma[Z_{pqr}] \) be the zero divisor graph of \( Z_{pqr} \). Recall that, the Laplacian matrix is a symmetric matrix. The rows and the columns of the Laplacian matrix are labeled with elements of \( B_1, B_2, \ldots, B_6 \), respectively. Then we have the following six cases.

Case 1: Consider \( B_1 \).

By Remark 3.2.5, every vertex in \( B_1 \) is adjacent to every vertex in \( B_6 \), but each vertex in \( B_1 \) is not adjacent to vertex in \( B_1, B_2, B_3, B_4 \) and \( B_5 \). By Lemma 4.3.1, we get \( \deg(x) = \)
\[ p - 1 \] for all \( x \in B_1 \). So we get

\[
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p - 1)I_{|B_1|} & O & O & O & O & -1 \\
B_2 & O & & & & \\
B_3 & O & & & & \\
B_4 & O & & & & \\
B_5 & O & & & & \\
B_6 & -1 & & & & \\
\end{bmatrix}
\]

**Case 2**: Consider \( B_2 \).

By Remark 3.2.5, every vertex in \( B_2 \) is adjacent to every vertex in \( B_5 \), but each vertex in \( B_2 \) is not adjacent to vertex in \( B_2, B_3, B_4 \) and \( B_6 \). By Lemma 4.3.1, we get \( \deg(x) = q - 1 \) for all \( x \in B_2 \). So we get

\[
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p - 1)I_{|B_1|} & O & O & O & O & -1 \\
B_2 & O & (q - 1)I_{|B_2|} & O & -1 & O \\
B_3 & O & O & & & \\
B_4 & O & O & & & \\
B_5 & O & O & & & \\
B_6 & -1 & O & & & \\
\end{bmatrix}
\]

**Case 3**: Consider \( B_3 \).

By Remark 3.2.5, every vertex in \( B_3 \) is adjacent to every vertex in \( B_4 \), but each vertex in \( B_3 \) is not adjacent to vertex in \( B_3, B_5 \) and \( B_6 \). By Lemma 4.3.1, we get \( \deg(x) = r - 1 \) for all \( x \in B_3 \). So we get

\[
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p - 1)I_{|B_1|} & O & O & O & O & -1 \\
B_2 & O & (q - 1)I_{|B_2|} & O & -1 & O \\
B_3 & O & O & (r - 1)I_{|B_3|} & -1 & O \\
B_4 & O & O & -1 & & \\
B_5 & O & O & & & \\
B_6 & -1 & O & & & \\
\end{bmatrix}
\]

**Case 4**: Consider \( B_4 \).

By Remark 3.2.5, every vertex in \( B_4 \) is adjacent to every vertex in \( B_5 \) and \( B_6 \), but each
vertex in $B_4$ is not adjacent to vertex in $B_4$. By Lemma 4.3.1, we get $\deg(x) = pq - 1$ for all $x \in B_4$. So we get

$$
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p-1)I_{B_1} & O & O & O & O & -1 \\
O & (q-1)I_{B_2} & O & O & -1 & O \\
O & O & (r-1)I_{B_3} & -1 & O & O \\
O & O & -1 & (pq-1)I_{B_4} & -1 & -1 \\
-1 & O & O & -1 & -1 & -1
\end{bmatrix}
$$

**Case 5** : Consider $B_5$.

By Remark 3.2.5, every vertex in $B_5$ is adjacent to every vertex in $B_6$, but each vertex in $B_5$ is not adjacent to vertex in $B_5$. By Lemma 4.3.1, we get $\deg(x) = pr - 1$ for all $x \in B_5$. So we get

$$
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p-1)I_{B_1} & O & O & O & O & -1 \\
O & (q-1)I_{B_2} & O & O & -1 & O \\
O & O & (r-1)I_{B_3} & -1 & O & O \\
O & O & -1 & (pq-1)I_{B_4} & -1 & -1 \\
-1 & O & O & -1 & (pr-1)I_{B_5} & -1 \\
-1 & O & O & -1 & -1 & -1
\end{bmatrix}
$$

**Case 6** : Consider $B_6$.

By Remark 3.2.5, each vertex in $B_6$ is not adjacent to vertex in $B_6$. By Lemma 4.3.1, we get $\deg(x) = qr - 1$ for all $x \in B_6$. So we get

$$
\begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
(p-1)I_{B_1} & O & O & O & O & -1 \\
O & (q-1)I_{B_2} & O & O & -1 & O \\
O & O & (r-1)I_{B_3} & -1 & O & O \\
O & O & -1 & (pq-1)I_{B_4} & -1 & -1 \\
O & -1 & O & -1 & (pr-1)I_{B_5} & -1 \\
-1 & O & O & -1 & (qr-1)I_{B_6} & -1
\end{bmatrix}
$$

This completes the proof.

□
Example 4.3.3. Recall the zero divisors of \( \mathbb{Z}_{30} \) in Example 3.2.9. The set of all zero divisors of \( \mathbb{Z}_{30} \) is \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}. We partition \( \mathbb{Z}[\mathbb{Z}_{30}] \) into

\[
\mathbb{Z}[\mathbb{Z}_{30}] = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6
\]

where

\[
B_1 = \{2, 4, 8, 14, 16, 22, 26, 28\},
\]

\[
B_2 = \{3, 9, 21, 27\},
\]

\[
B_3 = \{5, 25\},
\]

\[
B_4 = \{6, 12, 18, 24\},
\]

\[
B_5 = \{10, 20\},
\]

\[
B_6 = \{15\}.
\]

By Theorem 4.3.2, the Laplacian matrix of \( \Gamma[\mathbb{Z}_{30}] \) is

\[
\begin{bmatrix}
I_8 & O & O & O & O & -1 \\
O & 2I_4 & O & O & -1 & O \\
O & O & 4I_2 & -I & O & O \\
O & O & -I & 5I_4 & -1 & -1 \\
O & -I & O & -I & 9I_2 & -1 \\
-I & O & O & -I & -I & 14I_1
\end{bmatrix}
\]

That is,
5.1 Conclusions

In Chapter 3, we determine the Wiener index of zero divisor graphs of $\mathbb{Z}_{p^n}$ for $n \geq 4$ and $\mathbb{Z}_{pqr}$ where $p, q$ and $r$ are prime numbers such that $p < q < r$. We show the diameter of zero divisor graphs of $\mathbb{Z}_{p^n}$ for $n \geq 2$ is less than or equal to 2.

In Chapter 4, we determine general form of the Laplacian matrix of zero divisor graphs of $\mathbb{Z}_{p^n}$ for $n \in \{2, 3\}$. We then give general form of the Laplacian matrix of zero divisor graphs of $\mathbb{Z}_{p^n}$ for $n \geq 4$ and $\mathbb{Z}_{pqr}$ where $p, q$ and $r$ are prime numbers such that $p < q < r$.

5.2 Future work

This thesis focuses on the Wiener index and the Laplacian matrix of the zero divisor graphs of rings $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{pqr}$. More general analyses remain to be opened.
REFERENCES


BIOGRAPHY

Name Miss Benchamat Takhwan
Date of Birth December 20, 1993
Educational Attainment
Academic Year 2015: Bachelor of Science (Mathematics), Thammasat University, Thailand
Academic Year 2017: Master of Science (Mathematics), Thammasat University, Thailand
Scholarships
2016-2017 : Graduate Scholarship from the Faculty of Science and Technology (Thammasat University)
Publications