CONFIDENCE INTERVALS FOR MEAN OF TWO-PARAMETER EXPONENTIAL DISTRIBUTION

BY

CHANIDA KRAISADAB

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CHANIDA KRAISADAB

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on June 22, 2022

Chairman	Wararit Panichkitkosolkul (Associate Professor Wararit Panichkitkosolkul, Ph.D.)
Member and Advisor	Patarawan Sangnawakij (Assistant Professor Patarawan Sangnawakij, Ph.D.)
Member and Co-advisor	(Professor Andrei Igorevich Volodin, Ph.D.)
Member	(Professor Sa-Aat Niwitpong, Ph.D.)
Member	Rattana Lerdsuwansri (Assistant Professor Rattana Lerdsuwansri, Ph.D.)
Dean	(Associate Professor Nuttanont Hongwarittrorn, Ph.D.)

Thesis Title	CONFIDENCE INTERVALS FOR MEAN OF TWO-PARAMETER EXPONENTIAL DISTRIBUTION
Author	Chanida Kraisadab
Degree	Master of Science (Applied Statistics)
Department/Faculty/University	Department of Mathematics and Statistics Faculty of Science and Technology Thammasat University
Thesis Advisor	Assistant Professor Patarawan Sangnawakij, Ph.D.
Thesis Co-Advisor	Professor Andrei Igorevich Volodin, Ph.D.
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ABSTRACT

The two-parameter exponential distribution is widely used in applications, such as lifetime, survival, medical sciences, and reliability. Since the data related to this distribution are continuous outcomes, the characteristic of population in mean is of interest. The objective of this thesis is to construct the confidence intervals for the population mean of a two-parameter exponential distribution, using the method of variance of estimates recovery (MOVER) and Wald-type method. Unbiased estimators for the parameters given in this probability model are also applied in interval estimation. The new estimated variance formulas for the mean estimator are derived in this current work. The performance of these confidence intervals is investigated in terms of coverage probability and expected length via Monte Carlo simulations. The results indicate that the MOVER confidence interval using pivotal method performs well in all situations in the study. This is because it provides coverage probability greater than or close to the target probability level with an acceptable expected length. The confidence interval based on the Wald-type method can be used as an alternative statistical tool for large sample sizes. We also illustrate our confidence intervals using a real-world example in the area of environmental pollution.

Keywords: Simulation, Unbiased estimator, Method of variance of estimates recovery, Wald-type method, Interval estimation.

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CHAPTER 1
INTRODUCTION

1.1 Statement of the problems and importance of the research

A direct investigation of population characteristics is quite difficult to do, it takes both a long time and many resources. As a result, to draw a conclusion, it is necessary to depend on the characteristics of the samples that will refer to or characterize the population. So that the sample data must be able to properly characterize the characteristics of population. The previous statistical procedures are referred to as statistical inference. The mainly statistical inference has two main branches: parameter estimation and testing statistical hypothesis. This thesis is classified as part of parameter estimation because we are interested in point estimator and constructing a confidence interval. Generally, the main objective of the estimation method is to estimate an unknown parameter value. The theory of statistical hypothesis testing, on the other hand, is used to accept or reject the hypothesis concerning the unknown parameter or is concerned with the question of whether a sample is compatible with a specified hypothesis or not. In other words, it may be used to see if the sample data supports or refuse the investigator's hypothesis regarding the parameter's real value. However, in fact, there have been researches done confidence intervals for testing the hypotheses, see Barr (1969), Gardner and Altman (1986), Payton et al. (2000), and Greenlan et al. (2016). This means confidence interval is more useful approaches in statistical inference.

One of the most important parts of statistical inference is estimation of unknown parameter. This is useful in forming conclusions about the population of interest. There are two types of parameter estimations that are commonly used. The first type is known as a point estimation, while the second is known as interval estimation. Point estimators are a function of a random samples that is used to estimate the value of a true parameter. This gives a statistic or single estimate used to refer to a parameter of interest. However, in fact, there has no probability guarantee or confidence level that a given statistic will capture a parameter of interest. Moreover, sometimes a point estimator may provide more than one value, for example mode. Interval estimation, on the other hand, is an interval defined by two integers generated from computations on the observed values of the random variable and is assumed to contain the true value of the parameter on its inside. As a result, range of estimate is more comprehensive of interest than point estimation and can discover possible inaccuracies (Casella & Berger, 2002, p. 419). The parameter is defined assuming that it falls inside a $(1-\alpha)100\%$, say 95%, or greater probability interval, also referred to as the confidence level that can define form, which is $1-\alpha$, where α is a significant level. The confidence interval is a representation of how reliable an estimate is. The standard form of the estimate range is denoted as (L,U) or $L \leq \theta \leq U$, where θ is an interesting generic parameter, L is the lower limit, and U is the upper limit. The parameters that is often interesting in practical applications are including mean, variance, coefficient of variation, and proportion of a population. However, in this work, the parameter of interest is the mean. The reason is that the mean or expected value of a random variable is not only an average value that is weighted according to the probability distribution but is also considered as the parameter of interest to represent a measure of the center of a population. We hope to obtain a number that summarizes a typical or expected value of an observation of the data.

The three major theories in statistical inference are as follows, First, the classical theory, which is a frequentist concept. The inference is based on the sample distribution and assumes that the parameter of interest is a constant. Second, likelihood inference is the Fisher's concept, which differs from Frequentist in that statistical inference, is the process of evaluating the statistical evidence for parameter values by likelihood function. Third, the Bayesian statistic is the process of connecting the observed data and the inference (statements about the parameters). The parameter of interest is a random variable that has a density function or is called prior distribution. It has a probability distribution referred to as the prior distribution and then calculates the posterior distribution that is used for inference (Rohde, 2014, p. 13-14). In this thesis, we focus on the first approach, as this method is widely used, simple, and always applied in several applications.

In probability and statistics, the two-parameter exponential distribution is a probability model related to the data on the time to failure of the unit being observed. It has extensive applications in the field of reliability, queuing theory, lifetime analysis, survival, medicine, environmental pollution, economics, life insurance, and reliability analysis. For example, Freireich et al. (1963) used the data on the time (weeks) to relapse of patients after being treated by a drug 6-mercaptopurine (6-MP) and a placebo. Dunsmore (1983) described a rock crushing machine operating in such a way that it must be reset if the size of the rock being crushed is larger than any other that has been crushed before. Rahman and Pearson (2001) presented the time between successive failures that the data on the 25 system failures have occurred in 100 days period. AL-Ani et al. (2020) applied the actual data taken from Babel Tires Factory, where the working time (hours) between failures were deducted by the time recorded in the internal statements of the factory for six months. Li et al. (2022), originally given in Bain & Engel, 1973; Gui, 2018, presented a dataset of floods collected from two stations on Fox River, Wisconsin, and a dataset of a high-voltage current in a P-type high-voltage metal oxide semiconductor (MOS) transistor (HPM) data from a wafer acceptable testing. The data used in the previous literature reviews fitted the two-parameter exponential distribution. This shows that the distribution can be found in areas.

In probability theory, we suppose that X is a random variable. If X is a two-parameter exponential distribution, denoted as $Exp(\lambda,\theta)$, the probability density function of X is given by

$$f_X(x;\lambda,\theta)=\frac{1}{\lambda}exp\left\{-\frac{(x-\theta)}{\lambda}\right\}, \tag{1-1}$$

where $x>\theta$, $\lambda>0$, and $\theta\in\mathbb{R}$. The mean or expected value of X is given by $\lambda+\theta$. Here, λ and θ are the scale and location parameters, respectively. We describe and give more details for this model in Chapter 2. Since this distribution is applied in many fields, accurate parameter estimation is then important, especially interval estimation because the confidence interval more accurately represents and provides information on the parameter of interest than the point estimator, as we pointed out in the beginning of this section. There have been several works related to confidence intervals for parameters. For example, Roy and Mathew (2005) constructed an exact lower confidence limit for the reliability function of two-parameter exponential distribution using the generalized confidence interval (GCI) approach. Fernandez (2007) developed the confidence interval from Roy and Mathew (2005), and presented a modified pivotal quantity related to reliability for constructing the confidence intervals. Jiang and Wong (2012) constructed confidence intervals for both scale and location parameters in two-parameter exponential distribution using the approximate Studentization method. Jianhong and Hongmei (2013) introduced the methods of interval estimation for the difference and the ratio of means based on the concept of generalized variable and proposed the generalized pivotal quantity for the mean. Li et al. (2015) proposed the simultaneous confidence intervals of differences for mean in two- parameter exponential distribution based on the parametric bootstrap (PB) approach. Sangnawakij and Niwitpong (2017) considered confidence intervals for the single and the difference of coefficients of variation using the method of variance estimates recovery (MOVER), the generalized confidence interval (GCI), and the asymptotic confidence interval (ACI), where the maximum likelihood (ML) estimators for scale and location parameters were used. Thangjai and Niwitpong (2018) created the simultaneous confidence intervals (SCIs) based on PB approach and the two new SCIs using GCI and MOVER. Khooriphan and Niwitpong (2020) provide confidence intervals for the mean of a delta two-parameter exponential distribution based on PB method, standard bootstrapping (SB), GCI, and MOVER. The latter works calculated the two-single parameters based on the ML estimators. However, the estimation of the parameter to obtain a good estimator is also significant, because these estimators may produce more effective ranges when used to build confidence intervals. There have been studies introduced the estimation of parameters in the two-parameter exponential distribution. For instance, Cohen and Helm (1973) proposed the point estimator in the two-parameter exponential distribution using a variation of the ordinary method of moments to obtain the best linear unbiased estimators (BLUE). They compared this estimator with the ML estimator, moment estimator, and the modified moment estimator. Rahman and Larry (2001) presented point estimation for scale and location parameters in two-parameter exponential distribution and compare parameter estimators which are calculated from the ML estimator, unbiased estimator, the method of product spacing, and the method of quantile estimated. Rashid and Akhter (2011) introduced the estimation of parameters using the least squares method (LSM), relative least squares method, ridge regression method, moment method, modified moment method, and ML method. Although many papers have studied estimation of parameters and examined confidence intervals for parameters of the two-parameter exponential distribution, those used the ML estimation. There has no research applied the unbiased estimators for scale and location parameters to construct the confidence interval for the population mean. Therefore, this is an interesting method we would like to address in this thesis.

The rest of this thesis is organized as follows. Chapter 2 introduces the definitions of the interested distribution, the methods used to construct the confidence interval, and literature reviews according to point and interval estimation for the two-parameter exponential distribution. The proposed confidence intervals related to the unbiased estimator for the two-parameter exponential distribution are derived in Chapter 3. We used two main approaches in construction: the Wald-type method and the MOVER approach. Chapter 4 presents the performance of the proposed interval estimators. It is evaluated in terms of coverage probability and expected length through simulation studies. In addition, all proposed confidence intervals are demonstrated using a real-world example in the field of environmental pollution in Thailand. The thesis ends with Chapter 5, containing some concluding remarks.

1.2 Research objectives

The objectives of this research are as follows.

1. To propose the confidence intervals for the population mean of two-parameter exponential distribution.
2. To compare the performance of the parameter estimators obtained from the maximum likelihood estimator and unbiased estimator for parameters of the two-parameter exponential distribution.
3. To study the performance of confidence intervals for the population mean of two-parameter exponential distribution.

1.3 Research scope

The scope of this research is presented as follows.

1. Let $X=(X_1,X_2,...,X_n)$ be an independent random sample of size n from the two- parameter exponential distribution, denoted as $X_i \sim Exp(\lambda,\theta)$. The probability density function of X given by

$$f_{X_i}(x_i;\lambda,\theta)=\frac{1}{\lambda}exp\left\{-\frac{(x_i-\theta)}{\lambda}\right\}, \tag{1-2}$$

   where $x=(x_1,x_2,...,x_n)$ is the observed value of $X=(X_1,X_2,...,X_n)$, $x_i>\theta$, $\lambda>0$, and $\theta\in\mathbb{R}$, for $i=1,2,...,n$. λ is scale parameter and θ is location parameter. The mean and variance of X_i are denoted as $E(X_i)=\lambda+\theta$ and $Var(X_i)=\lambda^2$, respectively. In this work, we focus on estimating the mean parameter, $\lambda+\theta$.
2. The maximum likelihood estimators and the unbiased estimator for parameters λ and θ are studied. The properties of these estimators are conducted using simulations. The details will be given in Chapter 3. We investigate the performance of point estimator in this step- in order to make it clear that we choose the best method for constructing interval for the population mean.
3. The confidence intervals for the single parameter λ and θ are derived using the asymptotic method, profile likelihood method, classical method, and pivotal method.
4. The confidence intervals for the parameter of interest or $\lambda+\theta$ are constructed based on the method of variance estimates recovery (MOVER) and the Wald-type method. Moreover, these confidence intervals are conducted the performance via simulations, again the details are given in Chapter 3.
5. Absolute bias and mean squared error are used as the measures to evaluate the performance of the point estimator are given in step 2.

6. The performance of the confidence interval is investigated in terms of coverage probability and expected length.
7. The RStudio programing language (https://www.rstudio.com/) is used in the simulation. All simulation settings given in Chapter 3.

1.4 Research advantage

This research would provide the following advantages.

1. The confidence intervals proposed in this thesis can be used as an alternative method to estimate the mean of real-world data that have a two-parameter exponential distribution, such as environmental pollution and economics.
2. In this thesis, we provide the statistical knowledge of the MOVER method and the Wald-type method for the population mean of two-parameter exponential distribution. These methods can be applied to construct the confidence interval for other functions of parameters and distribution.
3. Example of R-code used in applications is provided with this thesis. This will be useful in the data analysis.

1.5 Abbreviations

Symbols	**Terms**
AD	: Anderson-Darling statistics
ACI	: The asymptotic confidence interval
$Cov(.)$	: Covariance
CP	: Coverage probability
$E(.)$	: Expectation
EL	: Expected length
$Exp(.)$	: The exponential distribution
MLE	: Maximum likelihood estimator
MME	: Method of moment estimator
MOVER	: The method of variance of estimates recovery

n	: Sample size
$Var(.)$	: Variance
$\chi^2_{df=a}$	: The chi-square distribution with a degrees of freedom
$Z_{1-\frac{\alpha}{2}}$	: The value corresponding to a cumulative area of $1-\dfrac{\alpha}{2}$ from the standardized normal distribution

1.6 Notations

Symbols	**Terms**
λ	: Scale parameter
θ	: Location parameter
μ	: The mean in two-exponential distribution
σ^2	: The variance in two-exponential distribution
$\hat{\lambda}$	: Maximum likelihood estimator for parameter λ
$\hat{\theta}$	: Maximum likelihood estimator for parameter θ
$\hat{\lambda}_{unbias}$	: Unbiased estimator for parameter λ
$\hat{\theta}_{unbias}$	: Unbiased estimator for parameter θ
$\hat{\lambda}_{MME}$	: Method of moment estimator for parameter λ
$\hat{\theta}_{MME}$	: Method of moment estimator for parameter θ
$MSE\left(\hat{\lambda}\right)$	: Mean squared error of an estimator $\hat{\lambda}$
$MSE\left(\hat{\theta}\right)$	: Mean squared error of an estimator $\hat{\theta}$
$MSE\left(\hat{\lambda}_{unbias}\right)$	: Mean squared error of an estimator $\hat{\lambda}_{unbias}$
$MSE\left(\hat{\theta}_{unbias}\right)$	: Mean squared error of an estimator $\hat{\theta}_{unbias}$
$ABS\left(\hat{\lambda}\right)$	: Absolute bias of an estimator $\hat{\lambda}$

$ABS(\hat{\theta})$: Absolute bias of an estimator $\hat{\theta}$

$ABS(\hat{\lambda}_{unbias})$: Absolute bias of an estimator $\hat{\lambda}_{unbias}$

$ABS(\hat{\theta}_{unbias})$: Absolute bias of an estimator $\hat{\theta}_{unbias}$

CI_{A1} : The asymptotic confidence interval for λ

CI_{A2} : The asymptotic confidence interval for θ

CI_{m1} : The asymptotic confidence interval based on the MOVER for $\lambda + \theta$

CI_{w} : The profile likelihood confidence interval for λ

CI_{m2} : The profile likelihood confidence interval based on the MOVER for $\lambda + \theta$

CI_{c} : The classical confidence interval for λ

CI_{m3} : The classical confidence interval based on the MOVER for $\lambda + \theta$

CI_{p} : The pivotal confidence interval for λ

CI_{m4} : The pivotal confidence interval based on the MOVER for $\lambda + \theta$

CI_{m5} : The asymptotic confidence interval based on the Wald-type method for $\lambda + \theta$

CHAPTER 2
REVIEW OF LITERATURE

In this part, theoretical backgrounds of distributions, methods used to construct the confidence interval, and literature reviews related to this thesis are presented. The important statistical topics related are provided as follows.

2.1 Two-parameter exponential distribution

Let $X = \left(X_1, X_2, ..., X_n\right)$ be a random variable of size n drawn from a two-parameter exponential distribution with parameters λ and θ. This is denoted by $X \sim Exp\left(\lambda, \theta\right)$. The probability density function of X is represented in equation (1-1). The scale parameter, or λ, is the mean lifetime, while the location parameter, or θ, is the minimum or guaranteed time before, which no failure event arises. Thus, the two-parameter exponential probability model is used to represent the time to failure, where failure will never occur prior time θ. This distribution can be used to model the data as the service times of agents in a system (Queuing theory), the time it takes before the next telephone call, the time until a radioactive particle decays, the distance between mutations on a DNA strand, and the extreme values of annual snowfall or rainfall. However, if $\theta=0$, the model given in (1-1) becomes the one-parameter exponential distribution. The latter is a special case of the two-parameter exponential distribution.

The two-parameter exponential distribution is a right-skewed distribution. Various densities of this distribution are shown in Figures 2-1 and 2-2. In Figure 2-1, the values of λ are varied as 1, 2, 3, and 7, and θ is fixed at 3. It can be seen that the probability distribution has longer tail if λ is increased. Meanwhile, in Figure 2-2, λ is fixed at 3 but θ are set as 1, 3, 5, and 7. We can see that the probability density function will be more skewed to the right when θ is decreased.

The properties of the two-parameter exponential distribution are as follows. Let $X \sim Exp(\lambda, \theta)$. The moment generating function of X is $M_X(t) = \frac{exp\{\theta t\}}{1-\lambda t}$, where $t < 1/\lambda$.

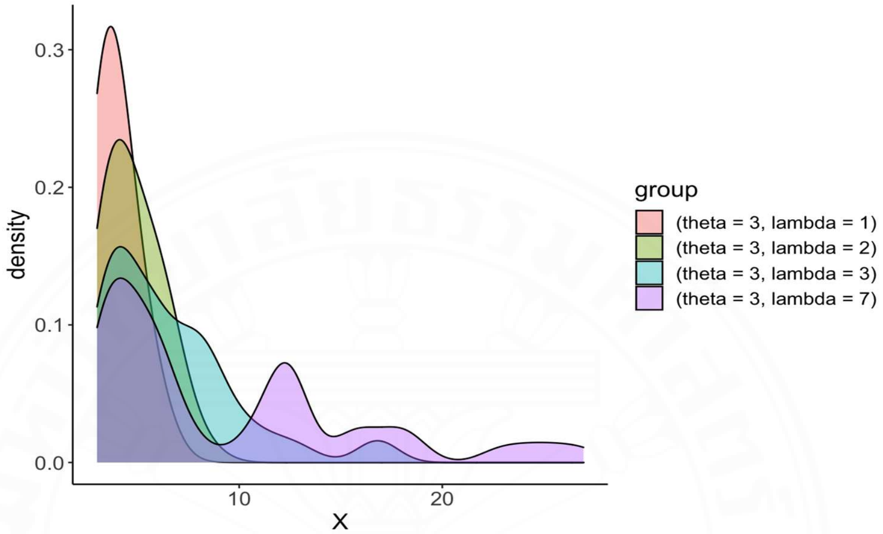

FIGURE 2-1 Two-parameter exponential density curve with various values of λ

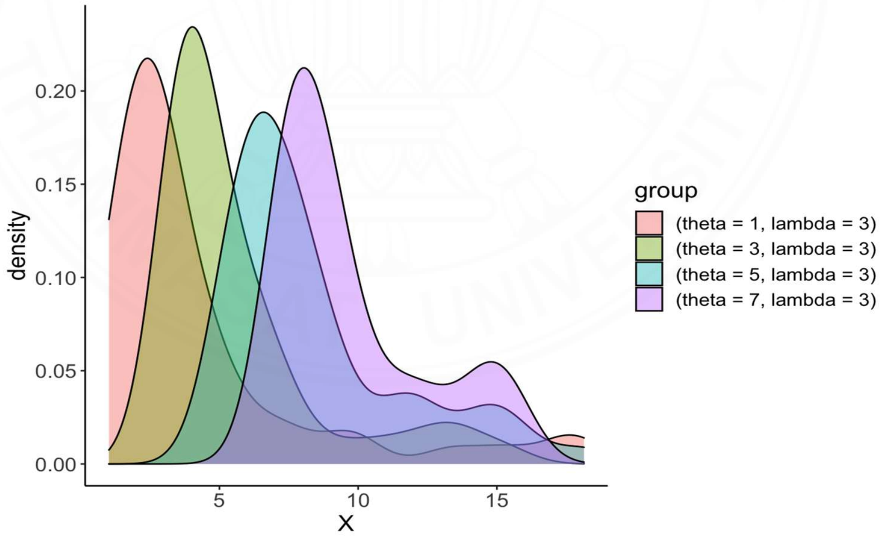

FIGURE 2-2 Two-parameter exponential density curve with various values of θ

Since λ and θ are unknown parameters, we consider the point estimator for the parameters shown in equation (1-1). In general, the maximum likelihood estimation is often used. This method based on defining a likelihood for calculating the conditional probability of observed data given probability distribution. For $X \sim Exp(\lambda,\theta)$, the likelihood function of λ and θ is given by

$$L(\lambda,\theta|x_1,x_2,\ldots,x_n) = \frac{1}{\lambda_n} exp\left\{\frac{-1}{\lambda_n}\sum_{i=1}^{n}(X_i-\theta)\right\} \qquad (2\text{-}1)$$

It is more convenient to work with the logarithm of the likelihood function as follows

$$l(\lambda,\theta) = \ln L(\lambda,\theta) = -\frac{1}{\lambda}\sum_{i=1}^{n} X_i + \frac{n\theta}{\lambda} - n\ln\lambda\ . \qquad (2\text{-}2)$$

Note that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are order statistics of $X_1, X_2, \ldots, X_n$. Since the first-order statistics $X_{(1)}$ maximizes in $\ln L(\lambda,\theta)$, the likelihood function is maximized with respect to θ by taking $\hat{\theta} = X_{(1)} = min(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$, the first-order statistic. Furthermore, the cumulative density function of $X_{(1)}$ is given by

$$F_X(X_{(1)}) = 1 - exp\left\{\frac{-n}{\lambda}(X_{(1)}-\theta)\right\},$$

with the probability density function of $X_{(1)}$

$$f_{X_{(1)}}(X_{(1)}) = F_{(1)}^{'}(X) = \frac{n}{\lambda} exp\left\{\frac{-n}{\lambda}(X_{(1)}-\theta)\right\},$$

where $X_{(1)} \geq \theta$. Thus, $X_{(1)}$ follows the distribution of $Exp\left(\lambda/n, \theta\right)$. We take the first-order partial derivatives of $l(\lambda,\theta)$ into equation (2-2) with respect to λ and solve the equation. The expression can be written as

$$\frac{\partial}{\partial\lambda} l(\lambda,\theta) = \frac{1}{\lambda^2}\sum_{i=1}^{n} X_i - \frac{n\theta}{\lambda^2} - \frac{n}{\lambda} = 0\ .$$

The maximum likelihood estimator for λ is therefore given by

$$\hat{\lambda} = \bar{X} - \hat{\theta} = \bar{X} - X_{(1)}\ .$$

In this thesis, the parameter of interest is $\lambda + \theta$. Based on the invariant property of the maximum likelihood estimator (Casella & Berger, 2002, p. 320), $\hat{\lambda}$ and $\hat{\theta}$ are substituted into λ and θ , respectively. The maximum likelihood estimator of $\lambda+\theta$ is therefore given as $\overline{X}$.

2.2 Normal distribution

The normal distribution, also called the Gaussian distribution, is one of the most widely used in statistics. There are three main reasons for this. First, the normal distribution and distributions associated with it are very tractable analytically. Second, the normal distribution has the familiar bell shape, whose symmetry makes it appealing choice for many population models. Although there are many other distributions that also bell-shaped, most do not possess the analytic tractability of the normal. Finally, there is the Central limit theorem, which shows that, under mild conditions, the normal distribution can be used to approximate a large variety of distributions in large samples (Casella & Berger, 2002, p. 102). The normal distribution is described by two parameters: the mean parameter (μ) which describes the location and the variance parameter, σ^2, which presents the spread of the distribution. If $\mu = 0$ and $\sigma=1$, we refer to distribution as the standard normal distribution. In many connections, it is sufficient to use this simpler form, because μ and σ simply may be regarded as a location and scale parameter, respectively. In Figure 2-3, we show plots of probability density of normal distributions.

In probability theory, let X be a random sample. If X is a normal distribution, denoted as $X \sim N\left(\mu,\sigma^2\right)$, the probability density function of X is written as

$$f_X\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, and $\sigma > 0$. The cumulative distribution function of the normal distribution is given by

$$F\left(x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]dt .$$

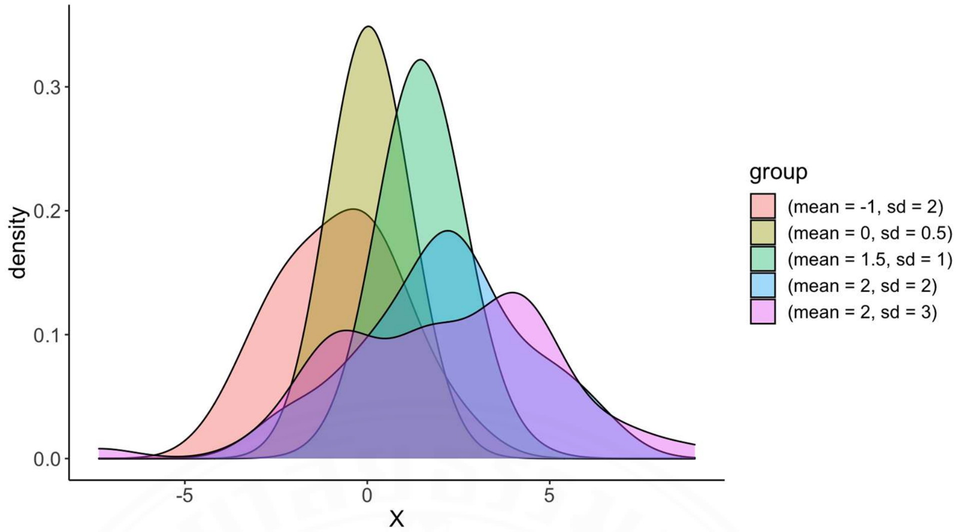

FIGURE 2-3 Probability density plot of the normal distributions

The moment generating function of X is $exp\left\{\mu t+\frac{1}{2}\sigma^2t^2\right\}$, it has developed into a standard of reference for many probability problems, including regression analysis, hypothesis testing, and confidence interval.

One of the most important results in probability theory is the Central Limit Theorem. This basically states that the z-transform of the sample mean is asymptotically standard normal. The amazing thing about the Central Limit Theorem is that no matter what the shape of the original distribution is, the (sampling) distribution of the mean approaches a normal probability distribution.

Definition 2.1 (Central Limit Theorem). If $X_1, X_2, \ldots, X_n$ is a random sample from an infinite population with mean μ, variance σ^2, and the moment-generating function $M_X(t)$, then the limiting distribution of

$$Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

as $n \to \infty$ is the standard normal probability distribution. That is,

$$\lim_{n\to\infty} P(Z_n \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}dt$$

(Ramachandran & Tsokos, 2009, p. 168).

2.3 Method of moments

The moment method is a parameter estimation that assumes the sample moment equal to the population moment. Then it uses a simple method for solving the equation to obtain the estimator for the parameter.

Definition 2.1 (Order k^{th} moment). The k^{th} moment of a random variable $X=(X_1,X_2,\ldots,X_n)$ taken about the origin is defined to be $E\left(X^k\right)$, for $k=1,2,\ldots$, and is denoted by μ_k'.

Definition 2.2 (Sample order k^{th} moment). The sample k^{th} moment of random variable $X=(X_1,X_2,\ldots,X_n)$ taken about the origin is defined as $m_k'=\frac{1}{n}\sum_{i=1}^{n}X_i^k$, for $k=1,2,\ldots$ (Wackerly et al., 2008, p. 138).

Let $X=(X_1,X_2,\ldots,X_n)$ be a sample of size n drawn from a population with the probability function $f(x\,|\,\theta_1,\ldots,\theta_k)$ and $\theta_1,\ldots,\theta_k$ are parameters. Method of moments estimators are found by equating the first k sample moments to the corresponding k population moments and solving the resulting system of simultaneous equations. More precisely, we define

$$m_1=\frac{1}{n}\sum_{i=1}^{n}X_i^1, \qquad \mu_1'=E\left(X^1\right),$$

$$m_2=\frac{1}{n}\sum_{i=1}^{n}X_i^2, \qquad \mu_2'=E\left(X^2\right),$$

$$\vdots$$

$$m_k=\frac{1}{n}\sum_{i=1}^{n}X_i^k, \qquad \mu_k'=E\left(X^k\right).$$

The population moment $\mu_i^{'}$ is a function of $\theta_1,\ldots,\theta_k$, say $\mu_i^{'}(\theta_1,\ldots,\theta_k)$. The method of moment estimators $(\tilde{\theta}_1,\ldots,\tilde{\theta}_k)$ of $(\theta_1,\ldots,\theta_k)$ are obtained by solving the following system of equation for $(\theta_1,\ldots,\theta_k)$ in terms of $(m_1,\ldots,m_k)$:

$$m_1^{'} = \mu_1(\theta_1,\ldots,\theta_k),$$

$$m_2^{'} = \mu_2(\theta_1,\ldots,\theta_k),$$

$$\vdots$$

$$m_k^{'} = \mu_k(\theta_1,\ldots,\theta_k)$$

(Casella & Berger, 2002, p. 312-313). In this thesis, the moment method will be used to find the estimator for the mean parameter of the two- parameter exponential distribution. Again, we present this in Chapter 3.

2.4 Definition of confidence interval

Interval estimation is a statistical technique determined by two numbers obtained from computation on the observed values of a random variable. The two statistics, lower and upper limits, are expected to contain the true value of the parameter in its interior with a given probability. Let $x = (x_1, x_2, \ldots, x_n)$ be a data set of observations taken from a population $X = (X_1, X_2, \ldots, X_n)$ with a distribution with unknown generic parameter θ. The set of all possible values of θ is called a parametric space and is denoted as Θ. Inference on θ for a set problem is the statement that $\theta \in C(x)$, where $C(x) \subset \Theta$ and $C(x)$ is determined by the value of the data x. If θ is a real-valued, then we usually prefer the set estimate C to be an interval (Casella & Berger, 2002, p. 417-418).

Definition 2.3 (Interval estimation). The estimation of interval for a true value parameter θ is depended on a sample value $x = (x_1, x_2, \ldots, x_n)$ from the sample a random variable $X = (X_1, X_2, \ldots, X_n)$ is a pair of functions $L(x)$ and $U(x)$, where

$L(x) \le U(x)$ for all $x \in X$. For the observed data x, the inference $L(x) \le \theta \le U(x)$ is made. The random interval $\left[L(x), U(x)\right]$ is called an interval estimator, and denoted as $\left[L(x), U(x)\right]$ a $(1-\alpha)100\%$ confidence interval for θ if

$$P\left(L(x) \le \theta \le U(x)\right) \ge 1-\alpha \text{, for any } \theta \in \Theta .$$

Here, a confidence level $1-\alpha$ is a probability of coverage of θ and $0 < \alpha < 1$. Note that $L(x)$ and $U(x)$ are the lower bound and upper bound of the confidence interval, respectively.

In most situations, we usually construct the two-sided confidence interval, that is $\left[L(x), U(x)\right]$. However, there is sometime interest in the one-sided confidence interval. That means, we define there is not mention of a lower bound or upper bound, which the interval is denoted as $\left(-\infty, U(x)\right]$ or $\left[L(x), \infty\right)$. There have been several methods interested in constructing the confidence interval for parameter, for example, asymptotic or large-sample method, profile likelihood method, pivotal method, Wald-type method, and method of variance of estimated recovery. However, in this thesis we focus on a two-sided confidence interval and the method of variance of estimated recovery is highlighted. This will be detailed in the following section.

2.5 Method of variance of estimates recovery

General details of the method of variance of estimates recovery are given in this section. Donner and Zou (2010) introduced the method to construct the confidence interval for the functions of generic parameter, including $\tau_1 + \tau_2$, $\tau_1 - \tau_2$, and τ_1 / τ_2. This method is named the method of variance of estimate recovery (MOVER) or so called the closed form method of variance estimation. The concept of this method is to find the separate confidence intervals for two-single parameters, τ_1 and τ_2, recover variance estimates from these confidence intervals, and then form the confidence interval for the function of parameters of interest. In this thesis, it is of interest to construct a $(1-\alpha)100\%$ two-sided confidence interval (L,U) for $\tau_1+\tau_2$, as it is similar to the form of our parameter of interest $\lambda+\theta$. By the central limit theorem and under the assumption of independence between the point estimate $\hat{\tau}_1$ and $\hat{\tau}_2$. The general form of a $(1-\alpha)100\%$ two-sided confidence interval for $\tau_1+\tau_2$ is given by

$$(L,U)=(\hat{\tau}_1+\hat{\tau}_2)\pm Z_{1-\frac{\alpha}{2}}\sqrt{Var(\hat{\tau}_1)+Var(\hat{\tau}_2)}, \qquad (2\text{-}3)$$

where $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution, and $Var(\hat{\tau}_i)$ is unknown variance of $\hat{\tau}_i$, for $i=1,2$.

Donner and Zou (2010) supposed that the given separate confidence limits for τ_i are given as (l_i',u_i'), for i = 1, 2. Therefore, (l_1',u_1') contains the possible parameter value for τ_1, and (l_2',u_2') contains the possible parameter value for τ_2. And then noted that $l_1'+l_2'$ is similar to the lower limit L, and $u_1'+u_2'$ is similar to the upper limit U because both L and U must be closer to $l_1'+l_2'$ and $u_1'+u_2'$ are than $\hat{\tau}_1+\hat{\tau}_2$, respectively. By (L,U) is the target confidence interval for $\tau_1+\tau_2$ in equation (2-3).

Since the variance of $\hat{\tau}_i$ is unknown, the estimation of $Var(\hat{\tau}_i)$ for obtaining L is computed based on the condition that $\tau_i=l_i'$. According to the central limit theorem,

$$Z_{1-\frac{\alpha}{2}}=\frac{\hat{\tau}_i-l_i'}{\sqrt{Var(\hat{\tau}_i)}} \text{ and } Z^2_{1-\frac{\alpha}{2}}=\frac{(\hat{\tau}_i-l_i')^2}{Var(\hat{\tau}_i)},$$

and the estimated variance for the lower limit L is computed by

$$Var_L(\hat{\tau}_i)=\frac{(\hat{\tau}_i-l_i')^2}{Z^2_{1-\frac{\alpha}{2}}}.$$

Similarly, under the condition that $\tau_i=u_i'$, the estimated variance for the upper limit U can be computed by

$$Z_{1-\frac{\alpha}{2}} = \frac{u_i^{'} - \hat{\tau}_i}{\sqrt{Var\left(\hat{\tau}_i\right)}} \text{ and } Z_{1-\frac{\alpha}{2}}^2 = \frac{(u_i^{'} - \hat{\tau}_i)^2}{Var\left(\hat{\tau}_i\right)}.$$

The estimated variance for the upper limit U is derived by

$$Var_U\left(\hat{\tau}_i\right) = \frac{\left(u_i^{'} - \hat{\tau}_i\right)^2}{Z_{1-\frac{\alpha}{2}}^2}.$$

Substituting the variances, $Var_L\left(\hat{\tau}_i\right)$ and $Var_U\left(\hat{\tau}_i\right)$ for $i = 1, 2$, into equation (2-3), the general form of the confidence interval for $\hat{\tau}_1 + \hat{\tau}_2$ based on the MOVER method is given by

$$\left(L,U\right) = \left((\hat{\tau}_1 + \hat{\tau}_2) - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{(\hat{\tau}_1 - l_1^{'})^2}{Z_{1-\frac{\alpha}{2}}^2} + \frac{(\hat{\tau}_2 - l_2^{'})^2}{Z_{1-\frac{\alpha}{2}}^2}}, (\hat{\tau}_1 + \hat{\tau}_2) + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{(u_1^{'} - \hat{\tau}_1)^2}{Z_{1-\frac{\alpha}{2}}^2} + \frac{(u_2^{'} - \hat{\tau}_2)^2}{Z_{1-\frac{\alpha}{2}}^2}} \right)$$

$$= \left(\left(\hat{\tau}_1 + \hat{\tau}_2\right) - \sqrt{\left(\hat{\tau}_1 - l_1^{'}\right)^2 + \left(\hat{\tau}_2 - l_2^{'}\right)^2}, \left(\hat{\tau}_1 + \hat{\tau}_2\right) + \sqrt{\left(u_1^{'} - \hat{\tau}_1\right)^2 + \left(u_2^{'} - \hat{\tau}_2\right)^2} \right). \quad (2\text{-}4)$$

Advantage of the MOVER is that i) it can be used to construct the confidence interval for a complex parameter function, ii) the basic statistical method is used however it provides a good performance for confidence interval, as noted in the review of literature of this chapter, and iii) it does not require computer-based method in computation as it has close-form solution. The confidence intervals for λ and θ, and the MOVER confidence intervals for $\lambda+\theta$ of the two-parameter exponential distribution introduced in this thesis will be discussed again in Chapter 3.

Besides the MOVER, the confidence interval for a parameter function can also be constructed in another method. When considering the general form and the estimated variance is unknown, we can construct the confidence interval for the function of parameter. The next section discusses one more approach as an alternative method to find a confidence interval.

2.6 Profile likelihood method

Visualization of a multidimensional likelihood function of parameters is in general difficult, so instead, one might want to look at one-dimensional likelihood function of the component. For a probability distribution that has two unknown parameters θ and η, the likelihood function will contain all available parameters that are θ and η and is denoted $L(\theta,\eta)$. In the case that θ is interested and needed to estimate only, but not for η although it involves in the model. The latter parameter is then called the nuisance parameter. In statistical inference, we can use the Fisherian concept to eliminate the nuisance parameter η by estimating it using the maximum likelihood estimator and obtaining the profile likelihood. The definition of the profile likelihood is as follows.

Definition 2.4 (Profile likelihood). Let $L(\theta,\eta)$ be the joint likelihood function of the parameter of interest θ and the nuisance parameter η. Let $\hat{\eta}_{ML}(\theta)$ be the maximum likelihood estimator with respect to $L(\theta,\eta)$ for fixed θ. Then,

$$L_p(\theta)=\max_{\eta} L(\theta,\eta)=L\{\theta,\hat{\eta}_{ML}(\theta)\}$$

is called the profile likelihood of θ. The value of the profile likelihood at a particular parameter value θ is obtained through maximizing the joint likelihood $L(\theta,\eta)$ with respect to the nuisance parameter η (Held & Bové, 2020, p. 130). This can be used in statistical inference, such as constructing score statistic and confidence interval.

2.7 Pivotal method

The pivotal method is a general method of constructing a confidence interval using a pivotal quantity function. General pivotal quantity is based on a random sample with has a probability distribution.

Definition 2.5 (Pivot). A pivotal quantity or pivot $t(X,\theta)$ is a function of a random sample $X=(X_1,X_2,..,X_n)$ and the true parameter θ in which its distribution does not depend on θ. The distribution of $t(X,\theta)$ is called pivotal distribution. Furthermore, an approximate pivot is a pivot, where its distribution does not asymptotically depend on the true parameter θ. This relies on the knowledge of sampling distribution. In conclusion, a pivotal quantity must be satisfied the following two characteristics.

i) It is a function of the random sample (a statistic or an estimator $\hat{\theta}$) and the unknown parameter θ, where θ is the only unknown quantity.

ii) It has a probability distribution that does not depend on the parameter θ and any nuisance parameter.

From i) and ii), it is important to note that the pivotal quantity depends on the parameter, but its distribution is independent of the parameter. The process of converting the probability statement about the pivot to a statement about the parameter is called the inversion (Ramachandran & Tsokos, 2009, p. 293).

In this thesis, the pivotal method is used to construct confidence intervals for single parameter and function of parameters in the two-parameter exponential distribution. The pivotal function will be given in the next chapter.

2.8 Definition of unbiased estimator

An estimator should be "close" in some sense to the true value of the unknown parameter. Formally, we say that $\hat{\theta}$ is an unbiased estimator of θ if the expected value of $\hat{\theta}$ is equal to θ. This is equivalent to say that the mean of the probability distribution of $\hat{\theta}$ (or the mean of the sampling distribution of $\hat{\theta}$) is equal to θ.

Definition 2.6 (Unbiased estimator). A point estimator $\hat{\theta}$ is called an unbiased estimator of parameter θ if $E(\hat{\theta})=\theta$ for all possible values of θ. When $\hat{\theta}$ is the unbiased estimator, its bias is equal to zero, that is $B(\hat{\theta})=E(\hat{\theta})-\theta=0$.

Definition 2.7 (Biased estimator). If $E\left(\hat{\theta}\right)\neq\theta$, $\hat{\theta}$ is said to be the biased estimator of parameter θ. Furthermore, the bias valued is given by $B\left(\hat{\theta}\right)=E\left(\hat{\theta}\right)-\theta\neq 0$.

There are many criteria for choosing a desired point estimator. The estimator satisfies the consistency property if the sample estimator has a high probability of being close to the population value θ for a large sample size. The concept of efficiency is based on comparing variances of the different unbiased estimators. If there are two unbiased estimators, it is desirable to have the one with the smaller variance. The estimator has the sufficiency property if it fully uses all the sample information. Minimal sufficient statistics are those that are sufficient for the parameter and are functions of every other set of sufficient statistics for those same parameters (Ramachandran & Tsokos, 2009, p. 227).

2.9 Definition of mean squared error

In theatrical and applied research, it is necessary to use the unbiased estimator in analysis. However, sometimes, unbiased estimators are found, mean squared error is then an important measure to determine the better point estimator. Let $\hat{\theta}$ is an estimator of parameter θ. The mean squared error of $\hat{\theta}$ is defined as the expected squared difference between $\hat{\theta}$ and θ.

Definition 2.8 (Mean squared error, MSE). The mean squared error of an estimator $\hat{\theta}$ of parameter θ is defined as

$$MSE\left(\hat{\theta}\right)=E\left(\hat{\theta}-\theta\right)^2.$$

This can be rewritten as

$$MSE\left(\hat{\theta}\right)=E\left(\hat{\theta}-E\left(\hat{\theta}\right)\right)^2+\left(\theta-E\left(\hat{\theta}\right)\right)^2=Var\left(\hat{\theta}\right)+B\left(\hat{\theta}\right)^2,$$

where $B\left(\hat{\theta}\right)$ is bias of an estimator (Montgomery & Runger, 2018, p. 161). If $\hat{\theta}$ is an unbiased estimator for θ, $B\left(\hat{\theta}\right)=0$ and $MSE\left(\hat{\theta}\right)=Var\left(\hat{\theta}\right)$.

2.10 Definition of coverage probability

Definition 2.9 (Coverage probability). A coverage probability for the confidence interval $\left[L(X),U(X)\right]$ is $P_{\theta}\left(\theta \in \left[L(X),U(X)\right]\right)$. It is the probability that the random interval $\left[L(X),U(X)\right]$ cover the true value of the parameter θ.

In comparison, we choose a confidence interval which has a coverage probability greater than or close to the nominal coverage level. This makes we have the confidence interval that more accurately estimates the true parameter. If a Monte Carlo simulation is studied, the estimated coverage probability is computed by

$$CP=\frac{c\left(L \leq \theta \leq U\right)}{M},$$

where $c\left(L \leq \theta \leq U\right)$ is the number of simulations runs for a generic parameter θ that lies within the confidence interval, L and U are the lower and upper confidence limits of parameter, respectively, and M is the total number of simulations runs or run simulations until the measurement of interest, here CP becomes stable.

2.11 Definition of expected length

The length of a confidence interval $\left[L(X),U(X)\right]$ is defined as the difference between the lower limit $L(X)$ and upper limit $U(X)$. By definition, the interval length is given as

$$l(X)=U(X)-L(X).$$

The expected length of a confidence interval is the expected value of $l(X)$, which is denoted as

$$E\left[l(X)\right]=E\left[U(X)-L(X)\right],$$

(Sangnawakij, 2016). In numerical simulation, the estimated expected length is computed by

$$EL = \frac{\sum_{h=1}^{M}\left(U_h - L_h\right)}{M},$$

where L_h and U_h are the lower bound and upper bound of the confidence interval in the h-th simulation run, respectively, and and M is the total number of simulations runs. Generally, we choose a confidence interval which has the short width interval. As a result, we obtain a confidence interval that more accurately estimates the true value.

2.12 Literature review

In this part, we would like to present a literature review on the authors and researchers used in this thesis. The two-parameter exponential distribution also known as a right-skewed probability model is frequently used in many studies. Most research is interested in estimating parameters first. After that have constructed a confidence interval for parameters and functions of parameters in two-parameter exponential distribution, including scale parameter, mean, and reliability functions, using a range of methods.

In 1973, Cohen and Helm introduced the point estimator in the two-parameter exponential distribution using a variation of the ordinary method of moments to obtain the best linear unbiased estimators (BLUE) and studied the estimator proposed by Mann (1969) and them (1970) that are estimated by maximum likelihood estimation, method of moment, and the modified method of moments. They compared the biased estimator and unbiased estimator of the scale parameter, location parameter, and mean in two-parameter exponential distribution.

Next, Rahman and Pearson (2001) presented the point estimation method for parameters of two-parameter exponential distribution. They compared the maximum likelihood method (MLE), the unbiased estimates which are linear function of maximum likelihood method (UMLE), the method of product spacing (MPS), and the method of quantile estimates (QE) using bias, root mean squared error (RMSE), average absolute (D_{abs}), and average of the maximum absolute (D_{max}). In this paper, UMLE is recommended for estimating both parameters because this method has the lowest bias, (D_{abs}), and (D_{max}).

Donner and Zou (2010) proposed the confidence intervals for functions of mean and standard deviation that are ratio (coefficient of variation) and difference ratio based on the method of variance estimates recovery (MOVER). The confidence limits for the ratio of MOVER are calculated by the normal percentile, $K_p = \mu + Z_p\sigma$, and the standardized mean μ/σ. They compared these confidence intervals from the MOVER with exact method. Moreover, they described constructing confidence intervals by replacing variance.

Next, in 2011, Akhter and Rashid compared the mean squared error (MSE) and total deviation (TD) of point estimation of parameters in two-parameter exponential distribution. The parameters are calculated from the following nine methods: the least squares method (LSM), relative least squares method (RELS), ridge regression method (RR), moment estimators (ME), modified moment estimators (MME), maximum likelihood estimators (MLE), and modified maximum likelihood estimation (MMLE). The results show that the estimates of parameters from the LSM are too close to the true values and the values of MSE and TD are very small.

In 2013, Zheng considered the estimation problem of two unknown parameters in two-parameter exponential distribution. He proposed new estimates for both parameters are unbiased and uniformly minimum variance unbiased estimators (UMVUE), using penalized maximum likelihood estimation.

In 2016, Sangnawakij et al. introduced new confidence intervals for the ratio of coefficients of variation in two-parameter exponential distribution based on the method of variance of estimates recovery (MOVER) and the generalized confidence interval (GCI). They calculated coverage probability (CP) and expected length (EL) of confidence intervals and compared the performances of CP and EL. It concluded that the CP of confidence intervals based on GCI has CP close to the nominal coverage level 0.95 in all cases. Meanwhile, the CP of the MOVER is much greater than 0.95 in many cases. In addition, the expected lengths of these confidence intervals decrease when sample size increase. As a result, the GCI and MOVER methods are recommended for constructing confidence intervals for the ratio of coefficient of variation in two-parameter exponential distribution.

In 2017, Sangnawakij and Niwitpong investigated the performance of new confidence intervals for the single coefficient of variation and the difference of coefficients of variation in the two-parameter exponential distributions. The confidence intervals are constructed using the method of variance of estimates recovery (MOVER), the generalized confidence interval (GCI), and the asymptotic confidence interval (ACI). They compared these confidence intervals in term of coverage probability and average length are evaluated by a Monte Carlo simulation. The results show that the coverage probabilities of the GCI maintain the nominal level in general cases. The MOVER performs well in terms of coverage probability when data only consist of positive values, but it has a wider expected length. The coverage probabilities of the ACI satisfy the target for large sample sizes. They also illustrate our confidence intervals using a real-world example in the area of medical science.

Next, Thangjai and Niwitpong (2017) proposed new confidence intervals for the weighted coefficients of variation (CV) of two- parameter exponential distributions based on the adjusted method of variance estimates recovery method (adjusted MOVER), then compare with the generalized confidence interval method (GCI) and the large sample method. The adjusted MOVER method is motivated based on concepts of the large sample method and MOVER method. The performance of four confidence intervals in terms of coverage probability and average length are compared via a Monte Carlo simulation. The results found that the generalized confidence interval CI_{GCI} performs the best confidence interval compared with the other confidence intervals when θ is a negative value. The adjusted MOVER confidence interval CI_{AM2} and CI_{GCI} performs well for large sample sizes and θ is a positive value. Moreover, CI_{AM2} performs as well as the large sample confidence interval CI_{LS}. The adjusted MOVER confidence interval CI_{AM1} is a conservative confidence interval. Hence, the CI_{GCI} should be chosen to estimate the weighted coefficients of variation of two-parameter exponential distributions.

Next, Somsamai and Srisuradetchai (2017) constructed the confidence intervals for shape parameters in Weibull distribution, using modified and non-modified profile likelihood functions. They compared the coverage probability and length of these confidence intervals under different sample sizes. The results indicated that modified profile likelihood has coverage probability close to 0.95 in all sample sizes while coverage probability of profile likelihood is close to 0.95 when the sample size is about 35.

In 2018, Thangjai et al. applied some of the results from Li et al (2015) to develop simultaneous confidence intervals (SCIs) using the parametric bootstrap (PB) for differences of means of several two-parameter exponential distributions and propose new SCIs based on the generalized confidence interval (GCI) method and method of variance estimates recovery (MOVER). Then, these approaches are compared with PB approach. The performance of these three approaches is evaluated through the coverage probabilities, the average lengths, and the standard errors of the confidence intervals. The GCI method and MOVER method perform much better than the PB approach in terms of coverage probability for all sample sizes, and by comparing the average lengths, it is seen that the MOVER method performs shorter than GCI method. In addition, all performance methods do not depend on the value of β_i and λ_i. As a result, the MOVER method is recommended for this paper.

In 2018, Saothayanun and Thangjai examined confidence intervals for the signal to noise ratio (SNR) of two-parameter exponential distribution, using the method of variance estimates recovery (MOVER), the generalized confidence interval (GCI), and large sample (LS). After that compare the coverage probability and average length of the confidence intervals. The results indicate that the GCI method performs well in terms of coverage probability for all cases. Furthermore, as a result of analyzing real-world data, the length of the generalized confidence interval is shorter than those of the other confidence intervals. Consequently, the GCI method is proposed as an alternative for estimating the confidence interval for the SNR of a two-parameter exponential distribution.

Next, Thangjai and Niwitpong (2018) extended the paper of Saothayanun and Thangjai (2018) from one population to two populations and constructed new confidence intervals for difference of signal to noise ratios (SNRs) based on parametric bootstrap (PB). They compared this method with the generalized confidence interval (GCI) approach, large sample approach, method of variance estimates recovery (MOVER) approach. The results show that the PB method provides better in terms of coverage probability, but the average lengths of the GCI approach are shorter than the other approaches. In addition, the result of analyzing data from the medical science, the interval length of the GCI is smaller than the interval length of other confidence intervals. It is clear that the results correspond simulation study in term of interval length. However, the performance of PB method is satisfactory as compared with other methods. As a result, the PB method is recommended to establish the confidence intervals for the difference of SNRs of two-parameter exponential distribution.

In 2022, Khooriphan et al. proposed confidence intervals for the mean of a delta two-parameter exponential distribution, using parametric bootstrapping (PB), standard bootstrapping (SB), the generalized confidence interval (GCI), and the method of variance estimates recovery (MOVER) and compared the performance these methods. The results show that GCI performs well in terms coverage probability for small-to-moderate sample sizes and small delta whereas the PB and the SB methods provide the best for large sample and large delta. Moreover, GCI is the best method for constructing confidence intervals for the mean of the sulfur dioxide data because it provided coverage probabilities close to 0.95 and shorter average lengths than PB and SB. Therefore, this paper introduces the GCI method for constructing confidence intervals for the mean of delta in two-parameter exponential distribution.

From the review, it can be seen that several papers have done on confidence intervals for parameters in the two-parameter exponential distribution. The estimators used in construction are based on maximum likelihood estimation. This is because it is a popular method. However, in fact, maximum likelihood estimator may lack of a good property of point estimation, especially unbiasedness. This thesis will show the behaviour of the maximum likelihood estimators for λ and θ in the two-parameter exponential distribution in terms of bias and compare to that of unbiased estimators. As it can be seen that there has not been research studied the confidence intervals for mean of the two- parameter exponential distribution using unbiased estimator. This is addressed in the next section.

CHAPTER 3
RESEARCH METHODOLOGY

In this chapter, theoretical results and proofs are focused. The maximum likelihood estimator, method of moments estimator, and unbiased estimator are used in point estimation. Moreover, we construct the confidence intervals for single parameters λ and θ in the two-parameter exponential distribution using the asymptotic method, profile likelihood method, classical method, and pivotal method. Then, these confidence intervals are applied to establish the newly proposed confidence intervals for the mean in the two-parameter exponential distribution using method of variance of estimates recovery. The Wald-type method is noted in this work, as a traditional method in estimation.

3.1 Method of moments and maximum likelihood estimation

Let $X = (X_1, X_2, ..., X_n)$ be a random sample of size n from a two-parameter exponential distribution with parameters λ and θ. The probability model of X is given in (1-1). The mean and variance of X are given as $E(X) = \lambda + \theta$ and $Var(X) = \lambda^2$, respectively. To estimate λ and θ, maximum likelihood estimation is firstly noted in this section. The proofs of maximum likelihood estimators for λ and θ are given in Chapter 2. They are given as

$$\hat{\lambda} = \overline{X} - X_{(1)} \text{ and } \hat{\theta} = X_{(1)}, \quad (3\text{-}1)$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $X_{(1)}$ is the first order statistic. Approximating $\hat{\lambda}$ and $\hat{\theta}$ in λ and θ, the maximum likelihood estimator for $\lambda + \theta$ or the population mean of X is therefore given as $\overline{X}$.

Not only maximum likelihood estimation is useful in parameter estimation, but the method of moments is so. Next, we find the estimators for the two parameters λ and θ using this approach. Based on the moment method, it is assumed that the estimators of population mean and variance are found by equating the first two sample moments about the origin ($M_X^{(k)} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$, for $k = 1, 2$) to the corresponding two population moments ($\mu_k^{'} = E(X^k)$, for $k = 1, 2$), and then solving the resulting system of simultaneous equations. From this concept, it can be given as follows.

Consider the first-order population and sample moments about the origin, if they are assumed to be equal, we have

$$\mu_1^{'} = M_X^{(1)}$$

$$E\left(X\right) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\lambda + \theta = \bar{X} . \tag{3-2}$$

We next evaluate the second-order moments,

$$\mu_2^{'} = M_X^{(2)}$$

$$E\left(X^2\right) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$Var\left(X\right) + E\left(X\right)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\lambda^2 + \left(\lambda + \theta\right)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\lambda^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right). \tag{3-3}$$

From (3-2) and (3-3), hence $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)$ are the method of moments estimators of $\lambda + \theta$ and λ^2, respectively.

It is important to note that the estimated mean estimators obtained from the moment and maximum likelihood methods are *identical*. Therefore, the mean or expected value of $\overline{X}$ is given by

$$E\left(\overline{X}\right) = \frac{1}{n}\sum_{i=1}^{n} E\left(X_i\right) = \lambda + \theta . \tag{3-4}$$

In conclusion, $\overline{X}$ is the unbiased estimator for $\lambda + \theta$.

3.2 Unbiased estimator for λ and θ

From $X \sim Exp(\lambda, \theta)$, we know that $E(X) = \lambda + \theta$. It is estimated by $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. According to section 2.1 in Chapter 2, $X_{(1)}$ has a two-parameter exponential distribution, denoted as $X_{(1)} \sim Exp\left(\lambda/n, \theta\right)$. So that the mean of $X_{(1)}$ is given as $E(X_{(1)}) = \frac{\lambda}{n} + \theta$. To find the unbiased estimators of λ and θ, Cohen and Helm (1973) proved that $\hat{\lambda}_{unbias} = \frac{n\left(\overline{X} - X_{(1)}\right)}{n-1} = \frac{n\hat{\lambda}}{n-1}$ and $\hat{\theta}_{unbias} = \frac{nX_{(1)} - \overline{X}}{n-1}$. These are obtained from the basic idea as follows.

Cohen and Helm (1973) supposed that $E(X) = \overline{X}$ and $E(X_{(1)}) = X_{(1)}$, where $\overline{X}$ and $X_{(1)}$ are the mean and the first-order statistic in a random sample of size n, obtained from the method of modified moments proposed in Helm et al. (1970). From the above information, hence it is easy to see that

$$E(X) = \lambda + \theta = \overline{X}, \quad \theta = \overline{X} - \lambda \quad \text{or} \quad \lambda = \overline{X} - \theta$$

and

$$E\left(X_{(1)}\right) = \frac{\lambda}{n} + \theta = X_{(1)}. \tag{3-5}$$

Substituting $\overline{X} - \lambda$ into θ of equation (3-5). It can be expressed as

$$\frac{\lambda}{n} + \left(\overline{X} - \lambda\right) = X_{(1)}$$

$$\frac{\lambda}{n}(n-1) = \overline{X} - X_{(1)}$$

$$\hat{\lambda}_{unbias} = \frac{n\left(\overline{X} - X_{(1)}\right)}{n-1} = \frac{n\hat{\lambda}}{n-1}. \tag{3-6}$$

We next substitute $\overline{X} - \theta$ into λ of equation (3-5), the desired estimator is given by

$$\frac{\overline{X} - \theta}{n} + \theta = X_{(1)}$$

$$-\theta + n\theta = nX_{(1)} - \overline{X}$$

$$\hat{\theta}_{unbias} = \frac{nX_{(1)} - \overline{X}}{n-1}. \qquad (3\text{-}7)$$

The estimators given in (3-6) and (3-7) are conducted in terms of unbiasedness. The expected values for $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ are

$$E\left(\hat{\lambda}_{unbias}\right) = E\left[\frac{n\left(\overline{X} - X_{(1)}\right)}{n-1}\right] = \frac{n}{n-1}\left[\left(\lambda + \theta - \left(\frac{\lambda}{n} + \theta\right)\right)\right] = \frac{n}{n-1}\left(\frac{\lambda(n-1)}{n}\right) = \lambda$$

and

$$E\left(\hat{\theta}_{unbias}\right) = E\left[\frac{nX_{(1)} - \overline{X}}{n-1}\right] = \frac{1}{n-1}\left[n\left(\frac{\lambda}{n} + \theta\right) - (\lambda + \theta)\right] = \frac{\theta(n-1)}{n-1} = \theta.$$

Hence, $\hat{\lambda}_{unbias} = \frac{n\hat{\lambda}}{n-1}$ and $\hat{\theta}_{unbias} = \frac{nX_{(1)} - \overline{X}}{n-1}$ are unbiased estimators of λ and θ, respectively. Furthermore, we can find the variance of $\hat{\lambda}_{unbias}$. It is given as

$$Var\left(\hat{\lambda}_{unbias}\right) = Var\left(\frac{n\left(\overline{X} - X_{(1)}\right)}{n-1}\right) = \frac{n^2}{(n-1)^2}\left(Var\left(\overline{X}\right) + Var\left(X_{(1)}\right) - 2Cov\left(\overline{X}, X_{(1)}\right)\right)$$

$$= \frac{n^2}{(n-1)^2}\left(Var\left(\overline{X}\right) + Var\left(X_{(1)}\right) - 2Var\left(X_{(1)}\right)\right)$$

$$= \frac{n^2}{(n-1)^2}\left(\frac{\lambda^2}{n} + \frac{\lambda^2}{n} - 2\left(\frac{\lambda^2}{n^2}\right)\right) = \frac{\lambda^2}{n-1}.$$

The variance of $\hat{\theta}_{unbias}$ is

$$Var\left(\hat{\theta}_{unbias}\right) = Var\left(\frac{nX_{(1)} - \overline{X}}{n-1}\right) = \frac{1}{(n-1)^2}\left(n^2 Var\left(X_{(1)}\right) + Var\left(\overline{X}\right) - 2Cov\left(nX_{(1)}, \overline{X}\right)\right)$$

$$= \frac{1}{(n-1)^2}\left(n^2 Var\left(X_{(1)}\right) + Var\left(\overline{X}\right) - 2nVar\left(X_{(1)}\right)\right)$$

$$= \frac{1}{(n-1)^2}\left(n^2\left(\frac{\lambda^2}{n^2}\right) + \frac{\lambda^2}{n} - 2n\left(\frac{\lambda^2}{n^2}\right)\right) = \frac{\lambda^2}{n(n-1)}.$$

Note that the covariance of $X_{(1)}$ and $\overline{X}$ is shown in Sangnawakij and Niwitpong (2017). If λ and θ are estimated, as well as $Var\left(\hat{\lambda}_{unbias}\right)$ and $Var\left(\hat{\theta}_{unbias}\right)$, the estimated variances can be used in inferential statistic, including confidence interval estimation.

In this paper, the performance of $\hat{\lambda}_{unbias} = \dfrac{n\hat{\lambda}}{n-1}$ and $\hat{\theta}_{unbias} = \dfrac{nX_{(1)} - \overline{X}}{n-1}$ are investigated in terms of the absolute bias (ABS) and mean squared error (MSE), and compare to $\hat{\lambda} = \overline{X} - X_{(1)}$ and $\hat{\theta} = X_{(1)}$, before applying in interval estimation. All details are given in Chapter 4. It is heighted here that $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ perform better than those two estimators in both criteria. Hence, $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ are suitable to use in the next method for constructing the confidence intervals for mean in $Exp\left(\lambda, \theta\right)$.

3.3 Proposed confidence interval for the population mean based on MOVER

To construct the confidence interval by MOVER interval, the separate confidence intervals for all single parameters should be firstly considered. Then we will combine them into the confidence interval for parameter of interest. Here, $\lambda + \theta$ is of interest, the confidence intervals for $\lambda + \theta$ based on MOVER are presented as follows.

3.3.1 MOVER confidence interval using asymptotic method

Let $X \sim Exp\left(\lambda, \theta\right)$. As shown in the previous section, we know that $\hat{\lambda}_{unbias} = \dfrac{n\hat{\lambda}}{n-1}$, $\hat{\theta}_{unbias} = \dfrac{nX_{(1)} - \overline{X}}{n-1}$, $E\left(\hat{\lambda}_{unbias}\right) = \lambda$, $E\left(\hat{\theta}_{unbias}\right) = \theta$, $Var\left(\hat{\lambda}_{unbias}\right) = \dfrac{\lambda^2}{n-1}$, and $Var\left(\hat{\theta}_{unbias}\right) = \dfrac{\lambda^2}{n\left(n-1\right)}$. Hence the general pivotal statistic for $\hat{\lambda}_{unbias}$ is obtained by

$$Z_{\lambda} = \frac{\hat{\lambda}_{unbias} - E\left(\hat{\lambda}_{unbias}\right)}{\sqrt{\widehat{Var}\left(\hat{\lambda}_{unbias}\right)}} = \frac{\dfrac{n\hat{\lambda}}{n-1} - \lambda}{\sqrt{\dfrac{\hat{\lambda}^2}{n-1}}},$$

where Z_λ converges to the standard normal distribution $N(0,1)$. By the central limit theorem, we establish the confidence limits for λ based on the probability statement

$$P\left(-Z_{1-\frac{\alpha}{2}} \leq Z \leq Z_{1-\frac{\alpha}{2}}\right) = 1-\alpha \,, \tag{3-8}$$

where Z is a generic pivotal quantity and $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution. For instance, if $\alpha = 0.05$, we obtain the probability statement

$$P\left(-Z_{1-\frac{0.05}{2}} < Z < Z_{1-\frac{0.05}{2}}\right) = 1-0.05$$

$$P\left(-Z_{0.975} < Z < Z_{0.975}\right) = 0.95$$

$$P\left(-1.96 < Z < 1.96\right) = 0.95$$

(Berenson et al., 2011, p. 283). Using Z_λ, we have

$$P\left(-Z_{1-\frac{\alpha}{2}} \leq \frac{\frac{n\hat{\lambda}}{n-1}-\lambda}{\sqrt{\frac{\hat{\lambda}^2}{n-1}}} \leq Z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n-1}} + \frac{n\hat{\lambda}}{n-1} \leq \lambda \leq Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n-1}} + \frac{n\hat{\lambda}}{n-1}\right) = 1-\alpha$$

$$P\left(\hat{\lambda}\left(\frac{n-Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right) \leq \lambda \leq \hat{\lambda}\left(\frac{n+Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right)\right) = 1-\alpha \,.$$

Thus the $(1-\alpha)100\%$ confidence interval for λ is

$$CI_{A1} = \left(l_{A1}, u_{A1}\right) = \left(\hat{\lambda}\left(\frac{n-Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right), \hat{\lambda}\left(\frac{n+Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right)\right), \tag{3-9}$$

where $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

Similarly, the general pivotal statistic for $\hat{\theta}_{unbias}$ can be derived as

$$Z_\theta = \frac{\hat{\theta}_{unbias} - E\left(\hat{\theta}_{unbias}\right)}{\sqrt{\widehat{Var}\left(\hat{\theta}_{unbias}\right)}} = \frac{\frac{nX_{(1)} - \overline{X}}{n-1} - \theta}{\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}},$$

where Z_θ converges to the $N(0,1)$. Again, using the probability statement shown in (3-8), we obtain

$$P\left(-Z_{1-\frac{\alpha}{2}} \le \frac{\frac{nX_{(1)} - \overline{X}}{n-1} - \theta}{\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}} \le Z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} - \frac{nX_{(1)} - \overline{X}}{n-1} \le -\theta \le Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} - \frac{nX_{(1)} - \overline{X}}{n-1}\right) = 1-\alpha$$

$$P\left(\frac{nX_{(1)} - \overline{X}}{n-1} - Z_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}}{\sqrt{n(n-1)}} \le \theta \le \frac{nX_{(1)} - \overline{X}}{n-1} + Z_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}}{\sqrt{n(n-1)}}\right) = 1-\alpha.$$

Therefore, the $(1-\alpha)100\%$ confidence interval for θ is

$$CI_{A2} = (l_{A2}, u_{A2}) = \left(\frac{nX_{(1)} - \overline{X}}{n-1} - Z_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}}{\sqrt{n(n-1)}}, \frac{nX_{(1)} - \overline{X}}{n-1} + Z_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}}{\sqrt{n(n-1)}}\right), \quad (3\text{-}10)$$

where $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

To establish the confidence interval for $\lambda+\theta$, MOVER with the above confidence limits is applied. The confidence interval from equations (3-9) and (3-10) is applied to construct the lower and upper limits in equation (2-4) of Chapter 2. We set the estimators $\hat{\tau}_1 = \hat{\lambda}$, $\hat{\tau}_2 = \hat{\theta}$, $l_1^{'} = l_{A1}$, $l_2^{'} = l_{A2}$, $u_1^{'} = u_{A1}$, and $u_2^{'} = u_{A2}$. Thus, the proposed MOVER confidence interval using asymptotic method for $\lambda+\theta$ is given by

$$CI_{m1} = \left(L_1, U_1\right)$$

$$= \left(\left(\hat{\lambda}+\hat{\theta}\right) - \sqrt{\left(\hat{\lambda} - \hat{\lambda}\left(\frac{n - Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right)\right)^2 + \left(\hat{\theta} - \left(\hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}\right)\right)^2}, \right.$$

$$\left. \left(\hat{\lambda}+\hat{\theta}\right) + \sqrt{\left(\hat{\lambda}\left(\frac{n + Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right) - \hat{\lambda}\right)^2 + \left(\left(\hat{\theta}_{unbias} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}\right) - \hat{\theta}\right)^2} \right)$$

$$= \left(\overline{X} - \sqrt{\hat{\lambda}^2\left(1 + \frac{n - Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1}\right)^2 + \left(\hat{\theta} - \hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}\right)^2}, \right.$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2\left(\frac{n + Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1} - 1\right)^2 + \left(\left(\hat{\theta}_{unbias} - \hat{\theta} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}\right)\right)^2} \right), \qquad (3\text{-}11)$$

where $\hat{\theta}_{unbias} = \dfrac{nX_{(1)} - \overline{X}}{n-1}$ and $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\dfrac{\alpha}{2}$ from the standardized normal distribution.

3.3.2 MOVER confidence interval using profile likelihood method

In this section, the confidence interval is constructed based on the profile likelihood method. Suppose that X is distributed as $Exp(\lambda,\theta)$, we consider the situation where λ is the parameter of interest and θ is the nuisance parameter. Here, we first consider the likelihood and the log-likelihood functions of λ and θ that are shown in (2-1) and (2-2), respectively. As a result, the maximum likelihood estimators for λ and θ are

$$\hat{\lambda} = \overline{X} - X_{(1)} \text{ and } \hat{\theta} = X_{(1)}. \qquad (3\text{-}12)$$

Thus the profile likelihood of λ is obtained by substituting $\hat{\theta}=X_{(1)}$ into θ of equation (2-2). It can be derived as

$$l_p(\lambda)=\max_{\theta} L(\lambda,\theta,x)=L(\lambda,\hat{\theta}(\lambda))$$

$$=-n\ln\lambda-\frac{1}{\lambda}\sum_{i=1}^{n}X_i+\frac{1}{\lambda}nX_i$$

$$=-n\ln\lambda-\frac{1}{\lambda}\sum_{i=1}^{n}(X_i-X_{(1)}), \tag{3-13}$$

where (3-13) depends on only λ. The score function is obtained by taking the first-order partial derivatives of $l_p(\lambda)$ with respect λ as follows

$$S_p(\lambda)=\frac{\partial l_p(\lambda)}{\partial\lambda}$$

$$=\frac{\partial}{\partial\lambda}\left(-n\ln\lambda-\frac{1}{n}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)\right)$$

$$=\frac{-n}{\lambda}+\frac{1}{\lambda^2}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right). \tag{3-14}$$

The solution of the score equation $S_p(\lambda)=0$ is the maximum likelihood estimator for λ, i.e. $\hat{\lambda}=\overline{X}-X_{(1)}$.

We next find the Fisher information from (3-14) by taking partial derivatives of $S_p(\lambda)$ with respect λ. For this we calculate

$$\frac{\partial S_p(\lambda)}{\partial\lambda}=\frac{\partial}{\partial\lambda}\left(\frac{-n}{\lambda}+\frac{1}{\lambda^2}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)\right)$$

$$=-n(-1)\lambda^{-2}+\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)(-2)\lambda^{-3}$$

$$=\frac{n}{\lambda^2}-\frac{2}{\lambda^3}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right).$$

Hence, the Fisher information is

$$I_p(\lambda)=\frac{-\partial S_p(\lambda)}{\partial\lambda}=-\left(\frac{n}{\lambda^2}-\frac{2}{\lambda^3}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)\right) \tag{3-15}$$

with the expectation of the Fisher information

$$J_p(\lambda)=I_p(\lambda)=E\left(-\frac{n}{\lambda^2}+\frac{2}{\lambda^3}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)\right)$$

$$=E\left(-\frac{n}{\lambda^2}\right)+E\left(\frac{2}{\lambda^3}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)\right)$$

$$=-\frac{n}{\lambda^2}+\frac{2}{\lambda^3}\sum_{i=1}^{n}X_i-nX_{(1)}$$

$$=-\frac{n}{\lambda^2}+\frac{2}{\lambda^3}\left(n(\lambda+\theta)-n\left(\frac{\lambda}{n}+\theta\right)\right)$$

$$=-\frac{n}{\lambda^2}+\frac{2}{\lambda^3}\lambda(n-1)=\frac{n-2}{\lambda^2}. \tag{3-16}$$

We calculate the inverse of $J_p(\lambda)$ by

$$J_p(\lambda)^{-1}=\frac{1}{J_p(\lambda)}=\frac{\lambda^2}{n-2}. \tag{3-17}$$

As a result, the variance estimates of $\hat{\lambda}$ is $\frac{\lambda^2}{n-2}$. Based on the large- sample approximation, the $(1-\alpha)100\%$ confidence interval for λ is therefore given by

$$CI_w=(l_w,u_w)=\left(\hat{\lambda}-Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n-2}},\hat{\lambda}+Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n-2}}\right), \tag{3-18}$$

where $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

According to the MOVER, we set the estimators $\hat{\tau}_1=\hat{\lambda}$, $\hat{\tau}_2=\hat{\theta}$, $l_1^{'}=l_w$, $l_2^{'}=l_{A2}$, $u_1^{'}=u_w$, and $u_2^{'}=u_{A2}$, and replace them in equation (2-4), where (l_w,u_w) and (l_{A2},u_{A2}) are the confidence intervals for λ and θ, given in equations (3-18) and (3-10), respectively. Hence, the $(1-\alpha)100\%$ MOVER confidence interval using profile likelihood method for $\lambda+\theta$ is given by

$$CI_{m2} = \left(L_2, U_2\right)$$

$$= \left(\left(\hat{\lambda} + \hat{\theta}\right) - \sqrt{\left(\hat{\lambda} - \left(\hat{\lambda} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n-2}}\right)\right)^2 + \left(\hat{\theta} - \left(\hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n\left(n-1\right)}}\right)\right)^2}, \right.$$

$$\left. \left(\hat{\lambda} + \hat{\theta}\right) + \sqrt{\left(\left(\hat{\lambda} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n-2}}\right) - \hat{\lambda}\right)^2 + \left(\left(\hat{\theta}_{unbias} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n\left(n-1\right)}}\right) - \hat{\theta}\right)^2} \right)$$

$$= \left(\overline{X} - \sqrt{Z^2_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}^2}{n-2} + \left(\hat{\theta} - \left(\hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}}{\sqrt{n\left(n-1\right)}}\right)\right)^2}, \right.$$

$$\left. \overline{X} + \sqrt{\frac{\hat{\lambda}^2}{n-2} + \left(\hat{\theta}_{unbias} - \hat{\theta} + Z_{1-\frac{\alpha}{2}}\frac{\hat{\lambda}}{\sqrt{n\left(n-1\right)}}\right)^2} \right), \qquad (3\text{-}19)$$

where $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

3.3.3 MOVER confidence interval using classical method

The confidence interval based on the basic method described in this section is the most common and easiest method for calculating the confidence interval. However, there have been researches applied in areas. This is called the pivotal method for large-sample confidence interval.

Since we know that the unbiased estimator of λ is $\frac{n\hat{\lambda}}{n-1}$. By the normal approximation we take the z-transform of $\frac{n\hat{\lambda}}{n-1}$. The pivot quantity is

$$Z = \frac{\frac{n\hat{\lambda}}{n-1} - E\left(\frac{n\hat{\lambda}}{n-1}\right)}{\sqrt{Var\left(\frac{n\hat{\lambda}}{n-1}\right)}} = \frac{n\hat{\lambda}}{\lambda\sqrt{n-1}} - \sqrt{n-1},$$

where Z has a limiting standard normal distribution for large sample size. Using the statement in (3-8) and Z given in the above equation, so we have

$$P\left(-Z_{1-\frac{\alpha}{2}} \le \frac{n\hat{\lambda}}{\lambda\sqrt{n-1}} - \sqrt{n-1} \le Z_{1-\frac{\alpha}{2}}\right) = 1-\alpha .$$

Suppose that

$$L'\left(\hat{\lambda}\right) = \left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)\frac{\lambda\sqrt{n-1}}{n}$$

and inverse function is $\hat{\lambda} = L(\lambda)$. The inequality above is rearranged as follows

$$\hat{\lambda} = \left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)\frac{\lambda\sqrt{n-1}}{n}$$

$$\frac{1}{\lambda} = \left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)\frac{\lambda\sqrt{n-1}}{n\hat{\lambda}}$$

$$\lambda = \frac{n\hat{\lambda}}{\sqrt{n-1}\left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)} . \qquad (3\text{-}20)$$

The statistic given on the right hand side of (3-20) is the upper confidence limit of λ. Similarly, assuming that

$$U'\left(\hat{\lambda}\right) = \left(Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)\frac{\lambda\sqrt{n-1}}{n}$$

and the inverse function is $\hat{\lambda} = U(\lambda)$. So, the lower inequality above be rearranged as

$$\hat{\lambda} = \left(Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)\frac{\lambda\sqrt{n-1}}{n}$$

$$\frac{1}{\lambda} = \left(Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)\frac{\lambda\sqrt{n-1}}{n\hat{\lambda}}$$

$$\lambda = \frac{n\hat{\lambda}}{\sqrt{n-1}\left(Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)} . \qquad (3\text{-}21)$$

The lower limit for λ is on the right hand side of (3-21). In conclusion, (3-20) and (3-21) can be written in the form

$$CI_c = (l_c, u_c) = \left(\frac{n\hat{\lambda}}{\sqrt{n-1}\left(Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)} \; , \; \frac{n\hat{\lambda}}{\sqrt{n-1}\left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1}\right)} \right), \qquad (3\text{-}22)$$

which is the confidence interval for λ.

To construct the $(1-\alpha)100\%$ confidence interval for $\lambda+\theta$ based on MOVER in equation (2-4). We use (l_c, u_c) and (l_{A2}, u_{A2}) given in (3-22) and (3-10), respectively, and define the estimators $\hat{\tau}_1 = \hat{\lambda}$ and $\hat{\tau}_2 = \hat{\theta}$. As a result, the $(1-\alpha)100\%$ MOVER confidence interval using classical method for $\lambda+\theta$ is given by

$$CI_{m3} = (L_3, U_3)$$

$$= \left((\hat{\lambda}+\hat{\theta}) - \sqrt{\left(\hat{\lambda} - \frac{n\hat{\lambda}}{\sqrt{n+1}\left(Z_{1-\frac{\alpha}{2}} + \sqrt{n+1}\right)} \right)^2 + \left(\hat{\theta} - \left(\hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} \right) \right)^2} \; , \right.$$

$$\left. (\hat{\lambda}+\hat{\theta}) + \sqrt{\left(\frac{n\hat{\lambda}}{\sqrt{n+1}\left(Z_{1-\frac{\alpha}{2}} + \sqrt{n+1}\right)} - \hat{\lambda} \right)^2 + \left(\left(\hat{\theta}_{unbias} + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} \right) - \hat{\theta} \right)^2} \right)$$

$$= \left(\overline{X} - \sqrt{\left(\frac{c\hat{\lambda} - n\hat{\lambda}}{c} \right)^2 + \left(\frac{\overline{X} - X_{(1)}}{n-1} - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} \right)^2} \; , \right.$$

$$\left. \overline{X} + \sqrt{\left(\frac{n\hat{\lambda} - k\hat{\lambda}}{k} \right)^2 + \left(\frac{X_{(1)} - \overline{X}}{n-1} + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} \right)^2} \right)$$

(Continuous)

$$= \left(\overline{X} - \sqrt{\hat{\lambda}^2 \left(\frac{c-n}{c} \right)^2 + \left(\frac{\hat{\lambda}}{n-1} - \frac{\hat{\lambda} Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right. ,$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2 \left(\frac{n-k}{k} \right)^2 + \left(\frac{-\hat{\lambda}}{n-1} + \frac{\hat{\lambda} Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right), \tag{3-23}$$

where $c = \sqrt{n-1}\left(Z_{1-\frac{\alpha}{2}} + \sqrt{n+1} \right)$ and $k = \sqrt{n-1}\left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1} \right)$ and $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

3.3.4 MOVER confidence interval using pivotal method

Pivotal quantity is similar to the general pivot, but the former provides an *exact distribution* for a random function and can be used to construct the confidence interval. This confidence interval is then valid for any sample size. Here, suppose that X be distributed as $Exp(\lambda, \theta)$. The pivot of λ can be given by

$$W = \frac{2n\hat{\lambda}}{\lambda} \tag{3-24}$$

(Lawless, 2003, p. 192). Since $S = \sum_{i=1}^{n} \left(X_i - X_{(1)} \right)$ follows a gamma distribution with parameter $n-1$ and λ, denoted as $Gamma(n-1, \lambda)$, and it has moment generating function $M_S(t) = \frac{1}{(1-\lambda t)^{n-1}}$ (Rohatgi and Saleh, 2015). It follows that $M_w(t)$ is

$$M_w(t) = E\left(exp\left\{ 2St/\lambda \right\} \right) = M_S\left(\frac{2t}{\lambda} \right)$$

$$= \frac{1}{\left(1 - \lambda \frac{2t}{\lambda} \right)^{n-1}} = \frac{1}{(1-2t)^{n-1}} = \frac{1}{(1-2t)^{\frac{2(n-1)}{2}}},$$

where the distribution of W does not depend on λ, it is called the pivot. Using W, it can be derived by

$$P\left(\chi^2_{\frac{\alpha}{2},2n-2} \le \frac{2n\hat{\lambda}}{\lambda} \le \chi^2_{1-\frac{\alpha}{2},2n-2}\right) = 1-\alpha$$

$$P\left(\frac{1}{2n\hat{\lambda}}\chi^2_{\frac{\alpha}{2},2n-2} \le \frac{1}{\lambda} \le \frac{1}{2n\hat{\lambda}}\chi^2_{1-\frac{\alpha}{2},2n-2}\right) = 1-\alpha$$

$$P\left(\frac{2n\hat{\lambda}}{\chi^2_{1-\frac{\alpha}{2},2n-2}} \le \lambda \le \frac{2n\hat{\lambda}}{\chi^2_{\frac{\alpha}{2},2n-2}}\right) = 1-\alpha .$$

Therefore, the $(1-\alpha)100\%$ confidence interval for λ is

$$CI_p = (l_p, u_p) = \left(\frac{2n\hat{\lambda}}{\chi^2_{1-\frac{\alpha}{2},2n-2}}, \frac{2n\hat{\lambda}}{\chi^2_{\frac{\alpha}{2},2n-2}}\right), \tag{3-25}$$

where $\chi^2_{\frac{\alpha}{2},2n-2}$ and $\chi^2_{1-\frac{\alpha}{2},2n-2}$ are the lower and upper $\left(\frac{\alpha}{2}\right)100th$ percentile of the chi-square distribution with $2n-2$ degrees of freedom $\left(\chi^2_{2n-2}\right)$.

The confidence intervals shown in (3-10) and (3-25) are applied to construct the lower and upper limits using MOVER. We set the estimators $\hat{\tau}_1 = \hat{\lambda}$, $\hat{\tau}_2 = \hat{\theta}$, $l_1^{'} = l_p$, $l_2^{'} = l_{A2}$, $u_1^{'} = u_p$, and $u_2^{'} = u_{A2}$. Hence, the MOVER confidence interval using pivotal method for $\lambda + \theta$ is given by

$$CI_{m4} = (L_4, U_4)$$

$$= \left((\hat{\lambda}+\hat{\theta}) - \sqrt{\left(\hat{\lambda} - \frac{2n\hat{\lambda}}{\chi^2_{1-\frac{\alpha}{2},2n-2}}\right)^2 + \left(\hat{\theta} - \left(\hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}\right)\right)^2}\ ,\right.$$

$$\left. (\hat{\lambda}+\hat{\theta}) + \sqrt{\left(\frac{2n\hat{\lambda}}{\chi^2_{\frac{\alpha}{2},2n-2}} - \hat{\lambda}\right)^2 + \left(\left(\hat{\theta}_{unbias} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n(n-1)}}\right) - \hat{\theta}\right)^2} \right)$$

(Continuous)

$$= \left(\overline{X} - \sqrt{\hat{\lambda}^2 \left(1 - \frac{2n}{\chi^2_{1-\frac{\alpha}{2},2n-2}} \right)^2 + \left(\frac{\hat{\lambda}}{n-1} - \frac{\hat{\lambda} Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right.,$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2 \left(\frac{2n}{\chi^2_{\frac{\alpha}{2},2n-2}} - 1 \right)^2 + \left(\frac{-\hat{\lambda}}{n-1} + \frac{\hat{\lambda} Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right)$$

$$= \left(\overline{X} - \sqrt{\hat{\lambda}^2 \left(1 - \frac{2n}{\chi^2_{1-\frac{\alpha}{2},2n-2}} \right)^2 + \hat{\lambda}^2 \left(\frac{1}{n-1} - \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right.,$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2 \left(\frac{2n}{\chi^2_{\frac{\alpha}{2},2n-2}} - 1 \right)^2 + \hat{\lambda}^2 \left(\frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} - \frac{1}{n-1} \right)^2} \right). \qquad (3\text{-}26)$$

where $\chi^2_{\frac{\alpha}{2},2n-2}$ and $\chi^2_{1-\frac{\alpha}{2},2n-2}$ are the lower and upper $\left(\frac{\alpha}{2}\right)100th$ percentile of the chi-square distributions with $2n-2$ degrees of freedom $\left(\chi^2_{2n-2}\right)$ and $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative area of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

3.4 Wald-type confidence interval

In this section, we construct the confidence interval for $\lambda+\theta$ by using the Wald-type method. The idea of this approach is based on variance approximation. Here, we use the moment estimator for $\lambda+\theta$ shown in (3-1). By the normal approximation, the mean and variance of $\overline{X}$ are given by $E\left(\overline{X}\right) = E\left(X\right) = \lambda + \theta$ and $Var\left(\overline{X}\right) = \frac{\lambda^2}{n}$, respectively. Using the central limit theorem, we can be derived the general pivot for $\overline{X}$ as

$$Z = \frac{\overline{X} - E(X)}{\sqrt{\widehat{Var}(\overline{X})}} = \frac{\overline{X} - E(X)}{\sqrt{\frac{\hat{\lambda}^2}{n}}}, \tag{3-27}$$

where Z has the standard normal distribution for large sample size. Using the probability statement in equation (3-8), we now obtain

$$P\left(-Z_{1-\frac{\alpha}{2}} \le \frac{\overline{X} - E(X)}{\sqrt{\frac{\hat{\lambda}^2}{n}}} \le Z_{1-\frac{\alpha}{2}} \right) = 1 - \alpha$$

$$P\left(\overline{X} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n}} \le \lambda + \theta \le \overline{X} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n}} \right) = 1 - \alpha .$$

Therefore, the $(1-\alpha)100\%$ Wald-type confidence interval for $\lambda+\theta$ is given by

$$CI_{m5} = \left(\overline{X} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n}}, \overline{X} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\lambda}^2}{n}} \right), \tag{3-28}$$

where $Z_{1-\frac{\alpha}{2}}$ is the value corresponding to a cumulative of $1-\frac{\alpha}{2}$ from the standardized normal distribution.

A question arises at the end of this section that which interval method has a good performance in estimating the mean in two-parameter exponential distribution. In this thesis, the behaviour of confidence intervals for $\lambda+\theta$ presentes in equations (3-11), (3-19), (3-23), (3-26), and (3-28) will be investigated via simulations. We provide the step and simulation settings in the next section.

3.5 Simulation study

The main objective of the computational part is to compare the coverage probabilities and average interval lengths of confidence intervals for the population mean in two-parameter exponential distribution. In order to perform simulations, we first need to fix the values of parameters and sample size. Then, we perform the following steps to get the desired results for this work. Each simulation is repeated M times using the following steps.

1. We generate a random variable $X=(X_1,X_2,...,X_n)$ from the two-parameter exponential distribution with parameters λ and θ. Since there has no function or package in R to generate data in two-parameter exponential distribution, we use the following steps based on the inverse transform method

   Step 1: Generate $X \sim U(0,1)$

   Step 2: Set $X = F^{-1}(U)$

   Let $U = F(X)$

   $$U = 1 - exp\left\{-\frac{X-\theta}{\lambda}\right\}$$

   $$-\frac{X-\theta}{\lambda} = \ln(1-U)$$

   $$X = \theta - \lambda\ln(1-U)$$

   Step 3: Compute $X = \theta - \lambda\ln(1-U)$, which follows a two-parameter exponential distribution.

2. The population mean $\mu = \lambda + \theta$ is fixed at 1.5, 2, 3.5, 4, 6, and 8, reflecting small to large values and varying distributions as shown in Figures (2-1) and (2-2). If λ is fixed at 1 and 3, θ is computed by $\theta = \mu - \lambda$. Therefore, the values of (λ,θ) are (1,0.5), (1,1), (1,3), (1,5), (3,0.5), (3,1), (3,3), (3,5).
3. The sample size (n) is given as 10, 30, 50, 100, 200, and 500.
4. The confidence level $(1-\alpha)$ is 0.95.
5. We calculate the point estimators, both biased and unbiased estimators, of λ and θ. They are given by $\hat{\lambda} = \overline{X} - X_{(1)}$, $\hat{\theta} = X_{(1)}$, $\hat{\lambda}_{unbias} = \frac{n\hat{\lambda}}{n-1}$, and $\hat{\theta}_{unbias} = \frac{nX_{(1)} - \overline{X}}{n-1}$.
6. We compute the $(1-\alpha)100\%$ confidence intervals for the population mean by the following formulas:

$$CI_{m1} = \left(\overline{X} - \sqrt{\hat{\lambda}^2 \left(1 + \frac{n - Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1} \right)^2 + \left(\hat{\theta} - \hat{\theta}_{unbias} + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} \right)^2} \right.,$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2 \left(\frac{n + Z_{1-\frac{\alpha}{2}}\sqrt{n-1}}{n-1} - 1 \right)^2 + \left(\left(\hat{\theta}_{unbias} - \hat{\theta} + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n(n-1)}} \right) \right)^2} \right),$$

$$CI_{m2} = \left(\overline{X} - \sqrt{Z^2_{1-\frac{\alpha}{2}} \frac{\hat{\lambda}^2}{n-2} + \left(\hat{\theta} - \left(\hat{\theta}_{unbias} - Z_{1-\frac{\alpha}{2}} \frac{\hat{\lambda}}{\sqrt{n(n-1)}} \right) \right)^2} \right.,$$

$$\left. \overline{X} + \sqrt{\frac{\hat{\lambda}^2}{n-2} + \left(\hat{\theta}_{unbias} - \hat{\theta} + Z_{1-\frac{\alpha}{2}} \frac{\hat{\lambda}}{\sqrt{n(n-1)}} \right)^2} \right),$$

$$CI_{m3} = \left(\overline{X} - \sqrt{\hat{\lambda}^2 \left(\frac{c-n}{c} \right)^2 + \left(\frac{\hat{\lambda}}{n-1} - \frac{\hat{\lambda} Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right.,$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2 \left(\frac{n-k}{k} \right)^2 + \left(\frac{-\hat{\lambda}}{n-1} + \frac{\hat{\lambda} Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right)$$

where $c = \sqrt{n-1}\left(Z_{1-\frac{\alpha}{2}} + \sqrt{n+1} \right)$ and $k = \sqrt{n-1}\left(-Z_{1-\frac{\alpha}{2}} + \sqrt{n-1} \right)$,

$$CI_{m4} = \left(\overline{X} - \sqrt{\hat{\lambda}^2 \left(1 - \frac{2n}{\chi^2_{1-\frac{\alpha}{2},2n-2}} \right)^2 + \hat{\lambda}^2 \left(\frac{1}{n-1} - \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} \right)^2} \right.,$$

$$\left. \overline{X} + \sqrt{\hat{\lambda}^2 \left(\frac{2n}{\chi^2_{\frac{\alpha}{2},2n-2}} - 1 \right)^2 + \hat{\lambda}^2 \left(\frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{n(n-1)}} - \frac{1}{n-1} \right)^2} \right),$$

and

$$CI_{m5} = \left(\overline{X} - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n}}, \overline{X} + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}^2}{n}} \right).$$

7. Steps 1 to 6 are repeated for $M = 5,000$ times to obtain the 5,000 estimators for each method.
8. The mean squared error (MSE) and absolute bias (ABS) are computed by

$$MSE(T) = \frac{1}{5,000} \sum_{i=1}^{5,000} (T_i - \omega)^2$$

and

$$ABS(T) = \left| \sum_{i=1}^{5,000} \frac{T_i}{5,000} - \omega \right|,$$

where T is an estimator for its parameter ω. Here, T will be $\hat{\lambda}$, $\hat{\theta}$, $\hat{\lambda}_{unbias}$, or $\hat{\theta}_{unbias}$.

9. We compute the estimated coverage probability and expected length from

$$CP = \frac{c(L \le \tau \le U)}{5,000}$$

and

$$EL = \frac{\sum_{h=1}^{5,000} (U_h - L_h)}{5,000},$$

where $c(L \le \tau \le U)$ is the number of simulations runs for τ that lie within the confidence interval, L_h and U_h are lower bound and upper bound of the confidence interval in the h-th simulation run, respectively.

Monte Carlo simulation used in this work is carried out using the R statistical package to investigate the coverage probability and expected length of the confidence intervals. The simulation process is shown by chart given in Figure 3-1. On decision, a confidence interval which has a coverage probability greater than or close to the nominal coverage probability level $(1-\alpha)$ and a short length interval is preferred to be the suitable method in a situation.

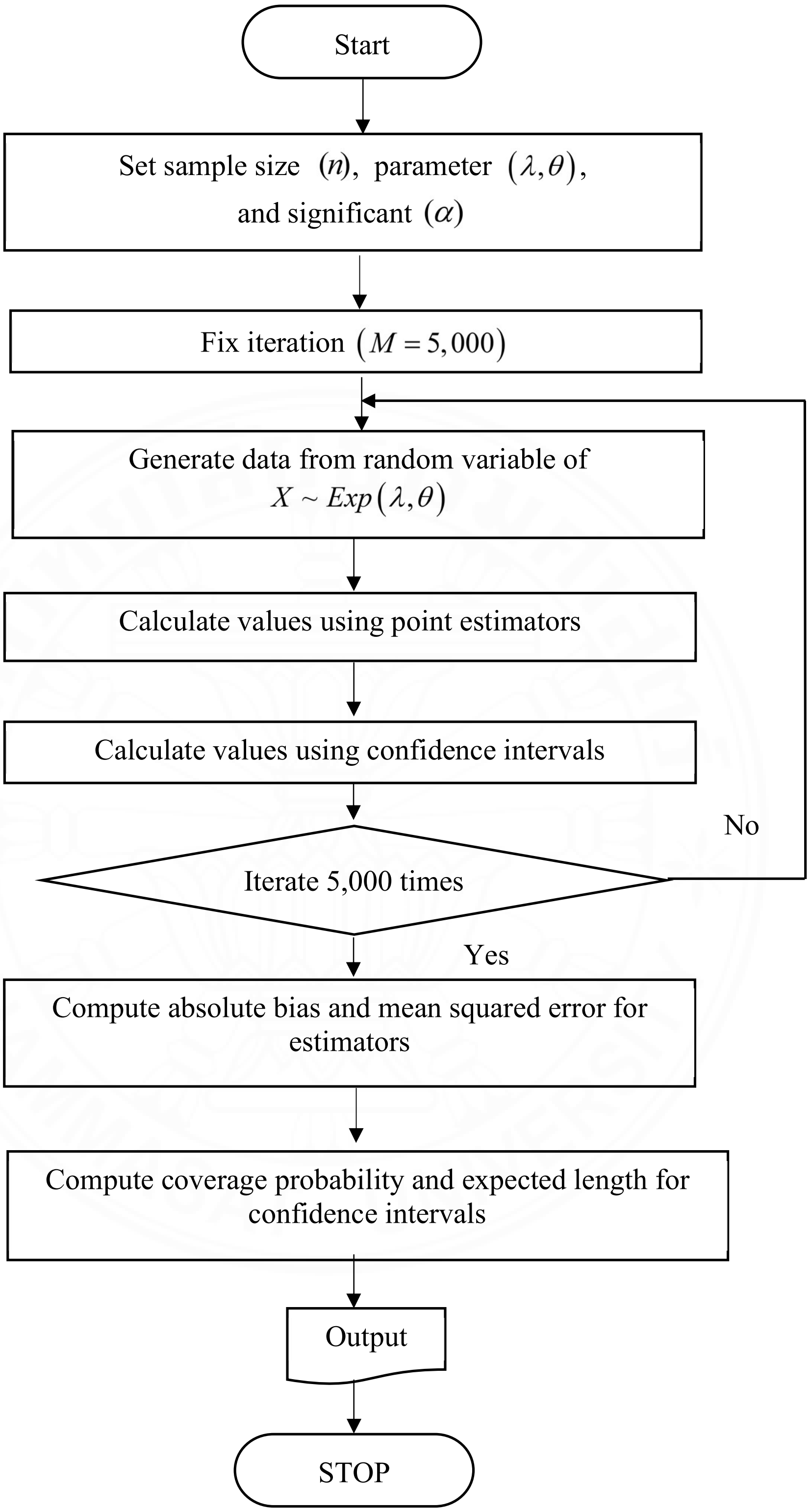

FIGURE 3-1 Flowchart of simulation study

CHAPTER 4
RESULTS AND DISCUSSION

In this chapter, we investigate the performance of estimators given in Chapter 3. The criteria for comparing the maximum likelihood and unbiased estimators are the absolute bias and mean squared error. Thes are based on simulations in various situations, for example sample sizes and parameter values. Moreover, we compare the coverage probabilities and expected lengths of the confidence intervals for the mean. In order to demonstrate the practical application of all confidence intervals for the population mean, the last section of this chapter these confidence intervals are applied to a real-world example in the field of environmental pollution.

A Monte Carlo simulation is carried out using the RStudio program to compute the absolute bias (ABS) and mean squared error (MSE) for point estimators, and estimate the coverage probability (CP) and expected length (EL) for interval estimator. In the simulation, the data are generated from $Exp(\lambda,\theta)$, where the parameters (λ,θ) are $(1,0.5)$, $(1,1)$, $(1,3)$, $(1,5)$, $(3,0.5)$, $(3,1)$, $(3,3)$, and $(3,5)$. The sample sizes are 10, 30, 50, 100, 200, and 500. The nominal coverage probability is 0.95. The symbols used in the Figures and Tables of this section are defined as follows.

Symbols	Terms
CI_{m1}	The MOVER confidence interval using asymptotic method
CI_{m2}	The MOVER confidence interval using profile likelihood method
CI_{m3}	The MOVER confidence interval using classical method
CI_{m4}	The MOVER confidence interval using pivotal method
CI_{m5}	The Wald-type method

4.1 Comparative analysis of point estimators

We start with an analysis of absolute bias and mean squared error for the point estimators for λ and θ. The results are summarized numerically in Table 4-1. The results show that the absolute biases of the unbiased estimator of λ, or $\hat{\lambda}_{unbias}$, are close to zero than those of the maximum likelihood estimator of λ, or $\hat{\lambda}$, in all simulations in the study. Similarly, the unbiased estimator of θ, or $\hat{\theta}_{unbias}$ has the absolute bias closer to zero than the maximum likelihood estimator of θ, or $\hat{\theta}$. This means that $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ provide the estimated values closer to its parameter values than $\hat{\lambda}$ and $\hat{\theta}$, respectively. For the mean squared errors of $\hat{\theta}_{unbias}$, they are smaller than the mean squared errors of $\hat{\theta}$, whereas the mean squared errors of $\hat{\lambda}_{unbias}$ are slightly greater than those of $\hat{\lambda}$. But the mean squared errors of $\hat{\lambda}_{unbias}$ and $\hat{\lambda}$ are approximately if the sample sizes are large. These results confirm that $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ are more precisely estimators than $\hat{\lambda}$ and $\hat{\theta}$. Moreover, the absolute biases and mean squared errors of $\hat{\lambda}$, $\hat{\theta}$, $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ do not depend on parameter values of λ and θ, also the population mean. However, they depend on sample sizes. That means the absolute biases and mean squared errors of the estimators decrease when sample sizes are increased.

Furthermore, the absolute biases and mean squared errors of these four estimators decrease when the sample sizes are increased. This corresponds to the central limit theorem. In clonclusion, $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ outperform $\hat{\lambda}$ and $\hat{\theta}$ in terms of their efficiency and accuracy estimation. Therefore, the unbiased estimators $\hat{\lambda}_{unbias}$ and $\hat{\theta}_{unbias}$ for the parameters $\hat{\lambda}$ and $\hat{\theta}$ are applied to use in confidence interval estimation.

TABLE 4-1 The Absolute Bias (ABS) and Mean Squared Error (MSE) of estimators in the two-parameter distribution

n	True Values		ABS				MSE			
	(λ,θ)	$\lambda+\theta$	$\hat{\lambda}$	$\hat{\lambda}_{unbias}$	$\hat{\theta}$	$\hat{\theta}_{unbias}$	$\hat{\lambda}$	$\hat{\lambda}_{unbias}$	$\hat{\theta}$	$\hat{\theta}_{unbias}$
10	(1,0.5)	1.5	0.1004	**0.0004**	0.1005	**0.0005**	**0.0995**	0.1104	0.0201	**0.0112**
	(1,1)	2	0.0981	**0.0021**	0.0999	**0.0003**	**0.0959**	0.1065	0.0201	**0.0112**
	(1,3)	4	0.0941	**0.0066**	0.0992	**0.0014**	**0.1024**	0.1156	0.0193	**0.0107**
	(1,5)	6	0.0964	**0.0040**	0.1001	**0.0003**	**0.1019**	0.1144	0.0206	**0.0116**
	(3,0.5)	3.5	0.3113	**0.0126**	0.2997	**0.0009**	**0.9072**	1.0005	0.1771	**0.0981**
	(3,1)	4	0.3044	**0.0049**	0.3040	**0.0045**	**0.9168**	1.0175	0.1829	**0.0993**
	(3,3)	6	0.2785	**0.0239**	0.2996	**0.0027**	**0.8844**	0.9967	0.1828	**0.1015**
	(3,5)	8	0.3175	**0.0195**	0.3048	**0.0068**	**0.8822**	0.9650	0.1880	**0.1056**
30	(1,0.5)	1.5	0.0349	**0.0017**	0.0331	**0.0002**	**0.0342**	0.0353	0.0022	**0.0011**
	(1,1)	2	0.0365	**0.0032**	0.0329	**0.0004**	**0.0337**	0.0346	0.0021	**0.0011**
	(1,3)	4	0.0337	**0.0004**	0.0337	**0.0004**	**0.0337**	0.0348	0.0023	**0.0012**
	(1,5)	6	0.0364	**0.0032**	0.0338	**0.0006**	**0.0327**	0.0336	0.0023	**0.0012**
	(3,0.5)	3.5	0.1022	**0.0023**	0.1009	**0.0009**	**0.3068**	0.3171	0.0207	**0.0109**
	(3,1)	4	0.1084	**0.0087**	0.1006	**0.0009**	**0.2954**	0.3036	0.0201	**0.0103**
	(3,3)	6	0.1069	**0.0071**	0.0991	**0.0007**	**0.2930**	0.3014	0.0191	**0.0097**
	(3,5)	8	0.0932	**0.0071**	0.0997	**0.0006**	**0.2955**	0.3070	0.0199	**0.0105**
50	(1,0.5)	1.5	0.0179	**0.0021**	0.0196	**0.0004**	**0.0197**	0.0202	0.0008	**0.0004**
	(1,1)	2	0.0162	**0.0039**	0.0199	**0.0001**	**0.0195**	0.0201	0.0008	**0.0004**
	(1,3)	4	0.0196	**0.0004**	0.0201	**0.0000**	**0.0205**	0.0210	0.0008	**0.0004**
	(1,5)	6	0.0186	**0.0014**	0.0200	**0.0000**	**0.0195**	0.0200	0.0008	**0.0004**
	(3,0.5)	3.5	0.0564	**0.0037**	0.0593	**0.0008**	**0.1795**	0.1836	0.0070	**0.0035**
	(3,1)	4	0.0582	**0.0018**	0.0605	**0.0005**	**0.1803**	0.1842	0.0072	**0.0036**
	(3,3)	6	0.0638	**0.0038**	0.0607	**0.0008**	**0.1774**	0.1804	0.0073	**0.0037**
	(3,5)	8	0.0510	**0.0092**	0.0601	**0.0001**	**0.1853**	0.1903	0.0074	**0.0039**

Note: Bold text reports that the estimator performs well in terms of absolute bias and mean squared error for the situation.

TABLE 4-1 (CONTINUED)

n	True Values		ABS				MSE			
	(λ,θ)	$\lambda+\theta$	$\hat{\lambda}$	$\hat{\lambda}_{unbias}$	$\hat{\theta}$	$\hat{\theta}_{unbias}$	$\hat{\lambda}$	$\hat{\lambda}_{unbias}$	$\hat{\theta}$	$\hat{\theta}_{unbias}$
100	(1,0.5)	1.5	0.0102	**0.0002**	0.0099	**0.0001**	**0.0099**	0.0100	0.0002	**0.0001**
	(1,1)	2	0.0116	**0.0016**	0.0100	**0.0000**	**0.0103**	0.0104	0.0002	**0.0001**
	(1,3)	4	0.0122	**0.0022**	0.0100	**0.0000**	**0.0100**	0.0101	0.0002	**0.0001**
	(1,5)	6	0.0108	**0.0008**	0.0102	**0.0002**	**0.0101**	0.0102	0.0002	**0.0001**
	(3,0.5)	3.5	0.0299	**0.0001**	0.0300	**0.0000**	**0.0945**	0.0955	0.0018	**0.0009**
	(3,1)	4	0.0303	**0.0003**	0.0304	**0.0004**	**0.0888**	0.0896	0.0018	**0.0009**
	(3,3)	6	0.0280	**0.0021**	0.0297	**0.0003**	**0.0913**	0.0924	0.0018	**0.0009**
	(3,5)	8	0.0275	**0.0026**	0.0302	**0.0002**	**0.0947**	0.0959	0.0018	**0.0009**
200	(1,0.5)	1.5	0.0042	**0.0008**	0.0051	**0.0001**	**0.0049**	0.0049	0.0001	**0.0000**
	(1,1)	2	0.0052	**0.0002**	0.0050	**0.0000**	**0.0050**	0.0050	0.0000	**0.0000**
	(1,3)	4	0.0058	**0.0008**	0.0051	**0.0001**	**0.0051**	0.0051	0.0001	**0.0000**
	(1,5)	6	0.0053	**0.0003**	0.0050	**0.0000**	**0.0049**	0.0050	0.0000	**0.0000**
	(3,0.5)	3.5	0.0205	**0.0056**	0.0147	**0.0003**	**0.0441**	0.0442	0.0004	**0.0002**
	(3,1)	4	0.0149	**0.0001**	0.0153	**0.0003**	**0.0445**	0.0448	0.0005	**0.0002**
	(3,3)	6	0.0172	**0.0022**	0.0153	**0.0003**	**0.0452**	0.0454	0.0005	**0.0002**
	(3,5)	8	0.0147	**0.0003**	0.0153	**0.0003**	**0.0469**	0.0471	0.0005	**0.0002**
500	(1,0.5)	1.5	0.0020	**0.0000**	0.0020	**0.0000**	**0.0020**	0.0021	0.0000	**0.0000**
	(1,1)	2	0.0020	**0.0000**	0.0020	**0.0000**	**0.0019**	0.0019	0.0000	**0.0000**
	(1,3)	4	0.0017	**0.0003**	0.0020	**0.0000**	**0.0020**	0.0020	0.0000	**0.0000**
	(1,5)	6	0.0018	**0.0002**	0.0020	**0.0000**	**0.0020**	0.0020	0.0000	**0.0000**
	(3,0.5)	3.5	0.0113	**0.0053**	0.0061	**0.0001**	**0.0178**	0.0177	0.0001	**0.0000**
	(3,1)	4	0.0053	**0.0007**	0.0059	**0.0001**	**0.0175**	0.0175	0.0001	**0.0000**
	(3,3)	6	0.0045	**0.0015**	0.0060	**0.0000**	**0.0176**	0.0176	0.0001	**0.0000**
	(3,5)	8	0.0071	**0.0011**	0.0060	**0.0000**	**0.0179**	0.0179	0.0001	**0.0000**

Note: Bold text reports that the estimator performs well in terms of absolute bias and mean squared error for the situation.

4.2 Comparative analysis of confidence intervals

The performance of confidence intervals for the mean in two-parameter exponential distribution is shown in Table 4-2. For small sample sizes, where $n=10$ and $n=30$, the coverage probabilities of CI_{m3} and CI_{m4} are greater than the nominal confidence level at 0.95. Meanwhile, CI_{m1} , CI_{m2} , and CI_{m5} have the coverage probabilities lower than 0.95. Moreover, CI_{m3} and CI_{m4} provide the coverage probabilities close to 0.95 when the sample sizes are increased. CI_{m1} and CI_{m2} give the coverage probabilities lower than 0. 95 when $n \le 100$. Conversely, when $n > 100$, the coverage probabilities of CI_{m1} and CI_{m2} are increased and are also close to the 0.95. Similarly, when $n \ge 200$, CI_{m5} performs well in term coverage probability. These results are also shown by graphs given in Figure 4-1 to Figure 4-12. It can be seen that CI_{m4} has the highest efficiency of the coverage probability, followed by CI_{m3}, CI_{m1}, CI_{m2}, and CI_{m5}, respectively in all cases. Furthermore, the coverage probabilities of all methods do not depend on the parameter mean, but they depend on the sample sizes. As we can seen in Table 4-2 that the coverage probabilities of CI_{m1}, CI_{m2}, and CI_{m5} increase and are close to 0.95, and CI_{m3} and CI_{m4} decrease and go to 0.95 if the sample sizes are increased.

The expected lengths of the confidence interval for the mean in two-parameter exponential distribution are given in Table 4-3. It can be seen that the expected lengths of the confidence interval bases on Wald-type method, or CI_{m5} are the shortest. However, the expected lengths of this method are slightly shorter than that of CI_{m1}. The results also show that the expected lengths of all methods decrease if the sample sizes are increased.

TABLE 4-2 The coverage probability (CP) of the 95% confidence intervals for the mean in the two-parameter distribution

n	True Values			CP				
	λ	θ	$\lambda+\theta$	CI_{m1}	CI_{m2}	CI_{m3}	CI_{m4}	CI_{m5}
10	1	0.5	1.5	0.9188	0.9056	**0.9682**	**0.9624**	0.8822
		1	2	0.9232	0.9098	**0.9712**	**0.9690**	0.8830
		3	4	0.9170	0.9008	**0.9658**	**0.9628**	0.8776
		5	6	0.9162	0.9058	**0.9626**	**0.9616**	0.8784
	3	0.5	3.5	0.9160	0.9030	**0.9700**	**0.9682**	0.8772
		1	4	0.9168	0.9036	**0.9642**	**0.9594**	0.8774
		3	6	0.9182	0.9060	**0.9652**	**0.9610**	0.8796
		5	8	0.9168	0.9046	**0.9720**	**0.9626**	0.8808
30	1	0.5	1.5	0.9402	0.9326	**0.9488**	**0.9488**	0.9240
		1	2	0.9388	0.9340	**0.9534**	**0.9516**	0.9258
		3	4	0.9404	0.9340	**0.9480**	**0.9514**	0.9248
		5	6	0.9440	0.9380	**0.9516**	**0.9554**	0.9262
	3	0.5	3.5	0.9374	0.9324	**0.9506**	**0.9512**	0.9218
		1	4	0.9460	0.9356	**0.9548**	**0.9574**	0.9262
		3	6	0.9446	0.9362	**0.9516**	**0.9542**	0.9284
		5	8	0.9440	0.9404	**0.9552**	**0.9534**	0.9314
50	1	0.5	1.5	0.9480	0.9436	**0.9488**	**0.9502**	0.9376
		1	2	0.9486	0.9456	**0.9498**	**0.9514**	0.9390
		3	4	0.9440	0.9406	**0.9474**	**0.9490**	0.9352
		5	6	0.9464	0.9418	**0.9492**	**0.9506**	0.9364
	3	0.5	3.5	0.9424	0.9396	**0.9496**	**0.9478**	0.9336
		1	4	0.9436	0.9414	**0.9492**	**0.9478**	0.9342
		3	6	0.9446	0.9412	**0.9498**	**0.9488**	0.9358
		5	8	0.9446	0.9440	**0.9394**	**0.9430**	0.9372

Note: Bold text reports that the confidence interval performs well in terms of coverage probability for the situation.

TABLE 4-2 (CONTINUED)

n	True Values			CP				
	λ	θ	$\lambda+\theta$	CI_{m1}	CI_{m2}	CI_{m3}	CI_{m4}	CI_{m5}
100	1	0.5	1.5	0.9496	0.9476	**0.9460**	**0.9480**	0.9450
		1	2	0.9470	0.9418	**0.9484**	**0.9498**	0.9392
		3	4	0.9460	0.9452	**0.9488**	**0.9500**	0.9410
		5	6	0.9432	0.9428	**0.9468**	**0.9498**	0.9408
	3	0.5	3.5	0.9406	0.9402	**0.9448**	**0.9424**	0.9374
		1	4	0.9486	0.9486	**0.9484**	**0.9504**	0.9456
		3	6	0.9438	0.9436	**0.9452**	**0.9454**	0.9404
		5	8	0.9418	0.9380	**0.9406**	**0.9400**	0.9354
200	1	0.5	1.5	**0.9538**	**0.9520**	**0.9488**	**0.9512**	0.9504
		1	2	**0.9446**	**0.9456**	**0.9466**	**0.9482**	0.9436
		3	4	**0.9474**	**0.9450**	**0.9488**	**0.9490**	0.9434
		5	6	**0.9486**	**0.9484**	**0.9480**	**0.9488**	0.9476
	3	0.5	3.5	**0.9526**	**0.9528**	**0.9528**	**0.9544**	0.9508
		1	4	**0.9542**	**0.9530**	**0.9540**	**0.9546**	0.9512
		3	6	**0.9500**	**0.9478**	**0.9520**	**0.9512**	0.9462
		5	8	**0.9420**	**0.9418**	**0.9432**	**0.9438**	0.9396
500	1	0.5	1.5	**0.9500**	**0.9486**	**0.9426**	**0.9448**	0.9484
		1	2	**0.9510**	**0.9516**	**0.9524**	**0.9518**	0.9506
		3	4	**0.9502**	**0.9488**	**0.9496**	**0.9498**	0.9484
		5	6	**0.9466**	**0.9444**	**0.9436**	**0.9448**	0.9440
	3	0.5	3.5	**0.9524**	**0.9516**	**0.9510**	**0.9520**	0.9508
		1	4	**0.9484**	**0.9498**	**0.9484**	**0.9478**	0.9490
		3	6	**0.9496**	**0.9504**	**0.9506**	**0.9502**	0.9488
		5	8	**0.9478**	**0.9466**	**0.9518**	**0.9508**	0.9458

Note: Bold text reports that the confidence interval performs well in terms of coverage probability for the situation.

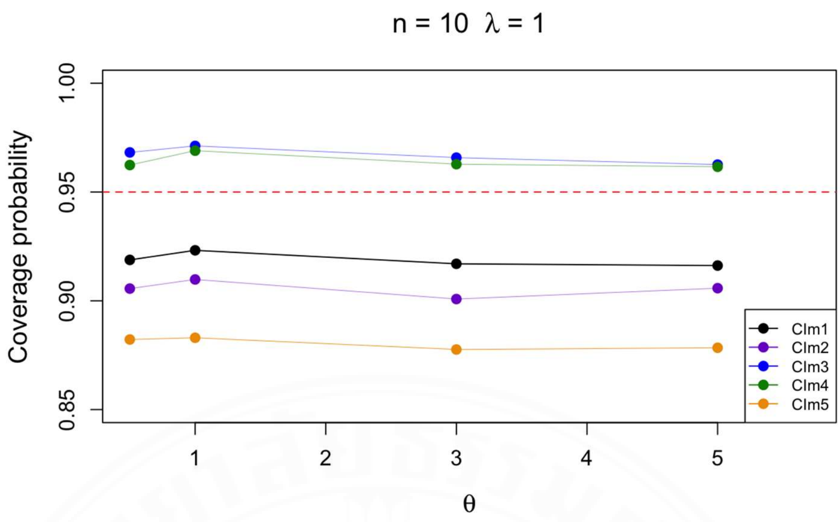

FIGURE 4-1 Coverage probability of confidence intervals when $n = 10$, $\lambda = 1$

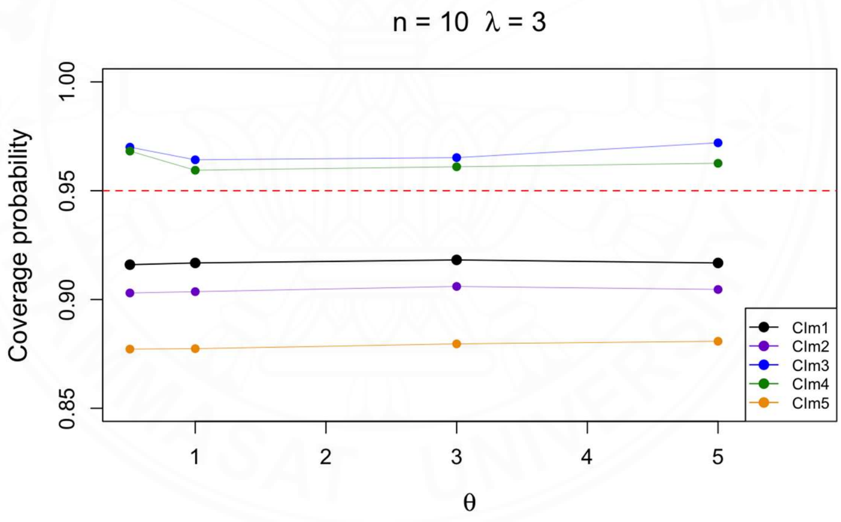

FIGURE 4-2 Coverage probability of confidence intervals when $n = 10$, $\lambda = 3$

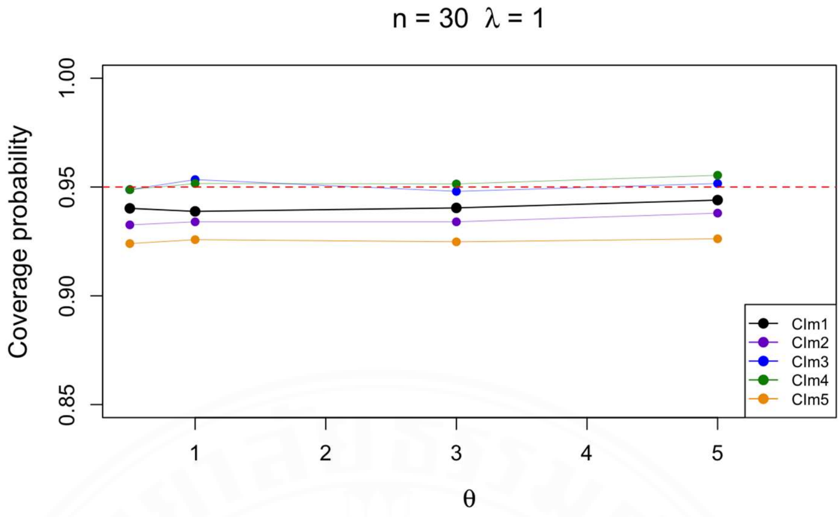

FIGURE 4-3 Coverage probability of confidence intervals when $n = 30$, $\lambda = 1$

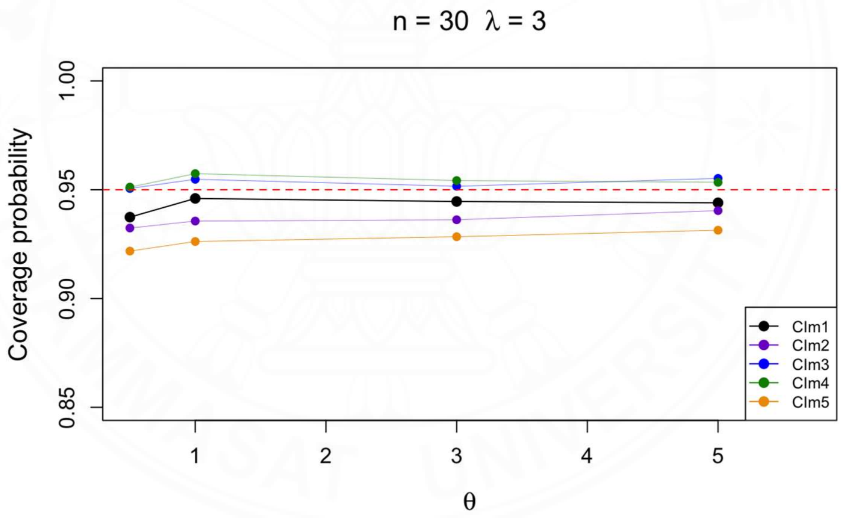

FIGURE 4-4 Coverage probability of confidence intervals when $n = 30$, $\lambda = 3$

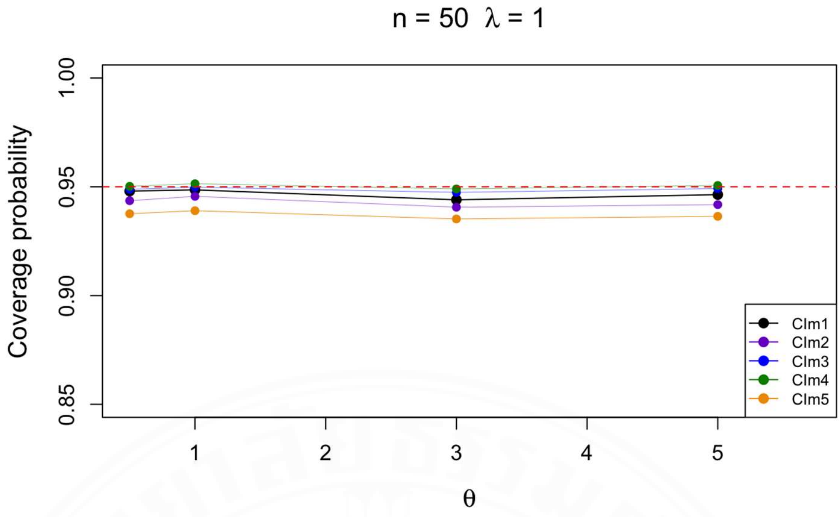

FIGURE 4-5 Coverage probability of confidence intervals when $n = 50$, $\lambda = 1$

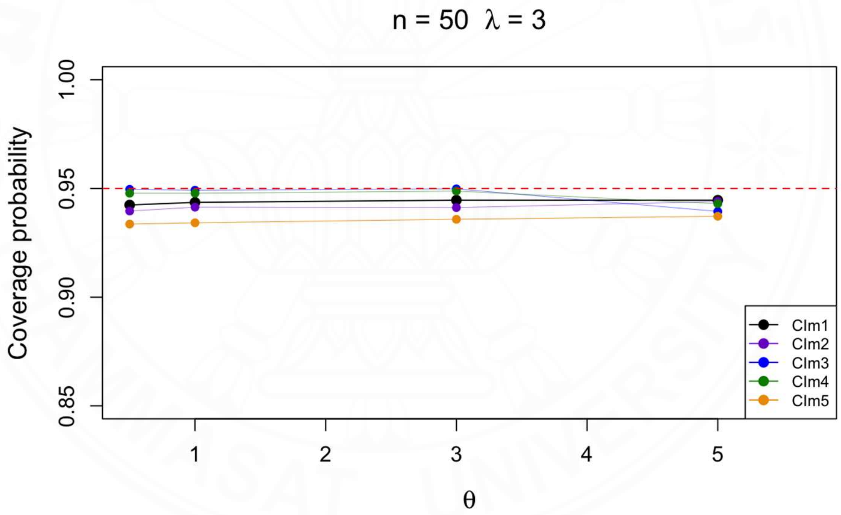

FIGURE 4-6 Coverage probability of confidence intervals when $n = 50$, $\lambda = 3$

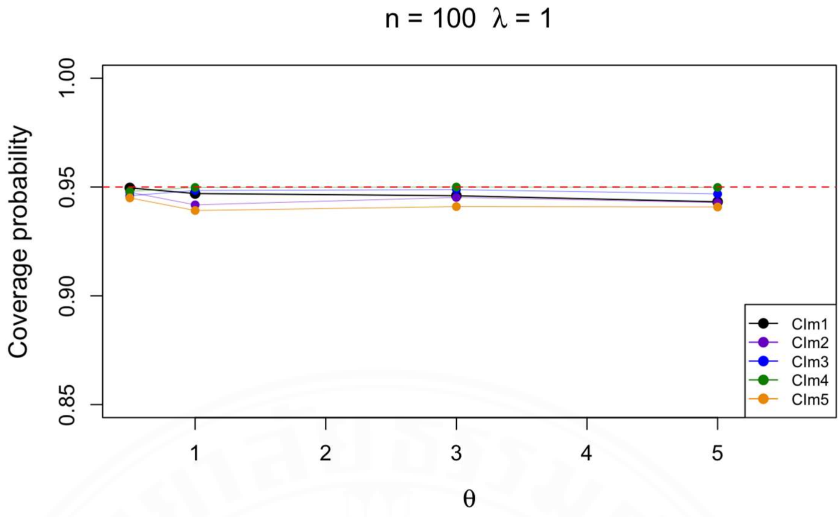

FIGURE 4-7 Coverage probability of confidence intervals when $n = 100$, $\lambda = 1$

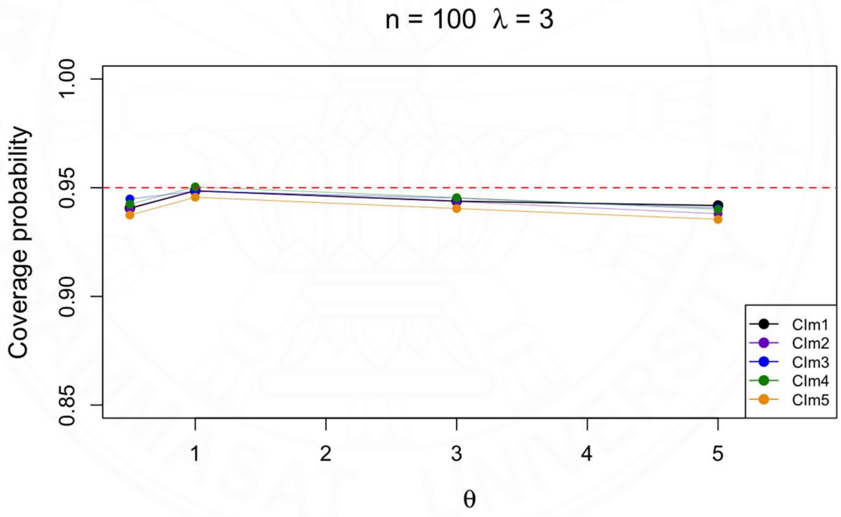

FIGURE 4-8 Coverage probability of confidence intervals when $n = 100$, $\lambda = 3$

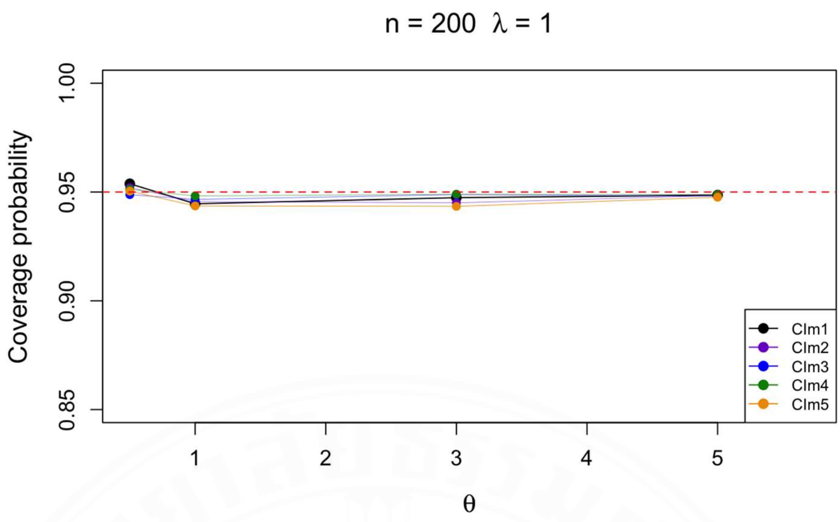

FIGURE 4-9 Coverage probability of confidence intervals when $n = 200$, $\lambda = 1$

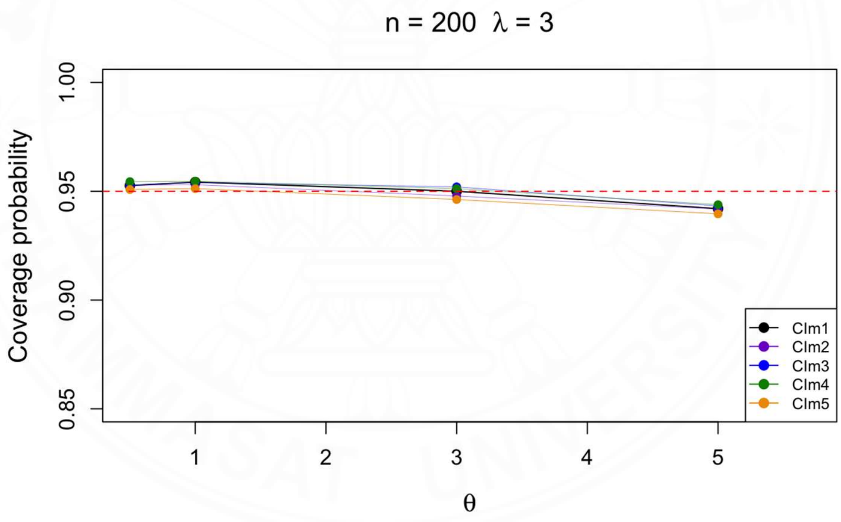

FIGURE 4-10 Coverage probability of confidence intervals when $n = 200$, $\lambda = 3$

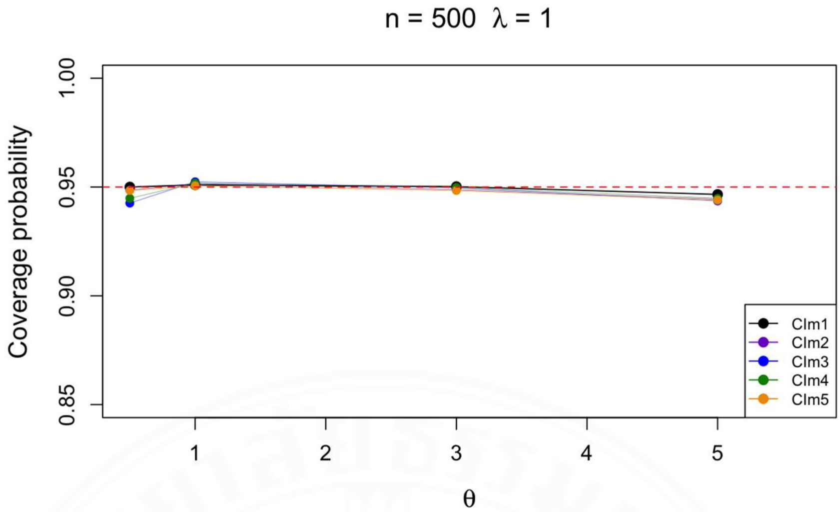

FIGURE 4-11 Coverage probability of confidence intervals when $n = 500$, $\lambda = 1$

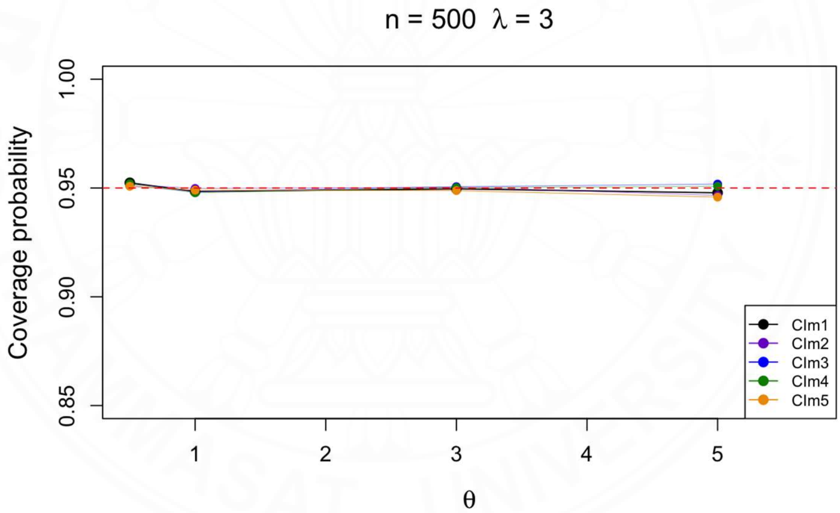

FIGURE 4-12 Coverage probability of confidence intervals when $n = 500$, $\lambda = 3$

TABLE 4-3 The Expected Length (EL) of the 95% confidence intervals for the mean in the two-parameter distribution

n	True Values			EL				
	λ	θ	$\lambda+\theta$	CI_{m1}	CI_{m2}	CI_{m3}	CI_{m4}	CI_{m5}
10	1	0.5	1.5	**1.2584**	1.3150	2.3963	1.7249	**1.1151**
		1	2	**1.2615**	1.3183	2.4023	1.7293	**1.1179**
		3	4	**1.2672**	1.3243	2.4131	1.7370	**1.1230**
		5	6	**1.2640**	1.3209	2.4069	1.7326	**1.1201**
	3	0.5	3.5	**3.7610**	3.9304	7.1619	5.1555	**3.3329**
		1	4	**3.7707**	3.9405	7.1803	5.1688	**3.3415**
		3	6	**3.8069**	3.9784	7.2493	5.2184	**3.3736**
		5	8	**3.7522**	3.9212	7.1452	5.1435	**3.3251**
30	1	0.5	1.5	**0.7183**	0.7293	0.8580	0.7945	**0.6907**
		1	2	**0.7172**	0.7281	0.8566	0.7932	**0.6896**
		3	4	**0.7192**	0.7302	0.8591	0.7955	**0.6915**
		5	6	**0.7172**	0.7282	0.8567	0.7933	**0.6896**
	3	0.5	3.5	**2.1568**	2.1898	2.5763	2.3856	**2.0739**
		1	4	**2.1523**	2.1851	2.5708	2.3806	**2.0695**
		3	6	**2.1534**	2.1863	2.5721	2.3818	**2.0706**
		5	8	**2.1636**	2.1966	2.5843	2.3931	**2.0803**
50	1	0.5	1.5	**0.5573**	0.5625	0.6179	0.5916	**0.5444**
		1	2	**0.5583**	0.5635	0.6190	0.5926	**0.5454**
		3	4	**0.5563**	0.5615	0.6168	0.5906	**0.5435**
		5	6	**0.5569**	0.5621	0.6174	0.5911	**0.5440**
	3	0.5	3.5	**1.6704**	1.6859	1.8520	1.7731	**1.6318**
		1	4	**1.6693**	1.6849	1.8509	1.7720	**1.6308**
		3	6	**1.6662**	1.6817	1.8474	1.7687	**1.6277**
		5	8	**1.6734**	1.6890	1.8554	1.7764	**1.6348**

Note: Bold text reports that the confidence interval performs well in terms of expected length for the situation.

TABLE 4-3 (CONTINUED)

n	True Values			EL				
	λ	θ	$\lambda+\theta$	CI_{m1}	CI_{m2}	CI_{m3}	CI_{m4}	CI_{m5}
100	1	0.5	1.5	**0.3925**	0.3944	0.4128	0.4043	**0.3880**
		1	2	**0.3919**	0.3938	0.4122	0.4037	**0.3874**
		3	4	**0.3917**	0.3936	0.4119	0.4034	**0.3872**
		5	6	**0.3923**	0.3941	0.4125	0.4040	**0.3878**
	3	0.5	3.5	**1.1778**	1.1834	1.2386	1.2130	**1.1643**
		1	4	**1.1776**	1.1832	1.2384	1.2128	**1.1641**
		3	6	**1.1786**	1.1842	1.2394	1.2138	**1.1650**
		5	8	**1.1788**	1.1844	1.2396	1.2140	**1.1652**
200	1	0.5	1.5	**0.2776**	0.2783	0.2846	0.2817	**0.2760**
		1	2	**0.2773**	0.2780	0.2843	0.2814	**0.2757**
		3	4	**0.2772**	0.2778	0.2841	0.2812	**0.2756**
		5	6	**0.2773**	0.2780	0.2843	0.2814	**0.2757**
	3	0.5	3.5	**0.8306**	0.8326	0.8515	0.8428	**0.8259**
		1	4	**0.8322**	0.8342	0.8531	0.8444	**0.8274**
		3	6	**0.8315**	0.8335	0.8524	0.8438	**0.8268**
		5	8	**0.8322**	0.8343	0.8531	0.8445	**0.8275**
500	1	0.5	1.5	**0.1754**	0.1755	0.1771	0.1764	**0.1750**
		1	2	**0.1754**	0.1755	0.1771	0.1764	**0.1750**
		3	4	**0.1754**	0.1756	0.1771	0.1764	**0.1750**
		5	6	**0.1754**	0.1756	0.1771	0.1764	**0.1750**
	3	0.5	3.5	**0.5251**	0.5257	0.5303	0.5282	**0.5239**
		1	4	**0.5262**	0.5267	0.5314	0.5292	**0.5250**
		3	6	**0.5263**	0.5268	0.5315	0.5294	**0.5251**
		5	8	**0.5259**	0.5264	0.5311	0.5289	**0.5247**

Note: Bold text reports that the confidence interval performs well in terms of expected length for the situation.

4.3 Application

To illustrate the computation of the confidence intervals proposed in this thesis, the data of particulate matter mass concentration denoted as PM 2.5 $(\mu g / m^3)$ in Bang Na district of Bangkok are used. The PM 2.5 data are obtained from 1 January 2019 to 31 December 2021, 36 monthly observations. These are reported by the Division of Air Quality and Noise Management Bureau, Pollution Control Department, Thailand (http://air4thai.pcd.go.th/webV2/history/). We show the data of PM 2.5 by graph that are given in Figure 4-13. The result shows that the data have the right-skewed distribution. It seems these data are similar to a two-parameter exponential distribution. To clarify we will use the test of significance and compare the minimum Akaike information criterion (AIC) and Bayesian information criterion (BIC) of five distributions.

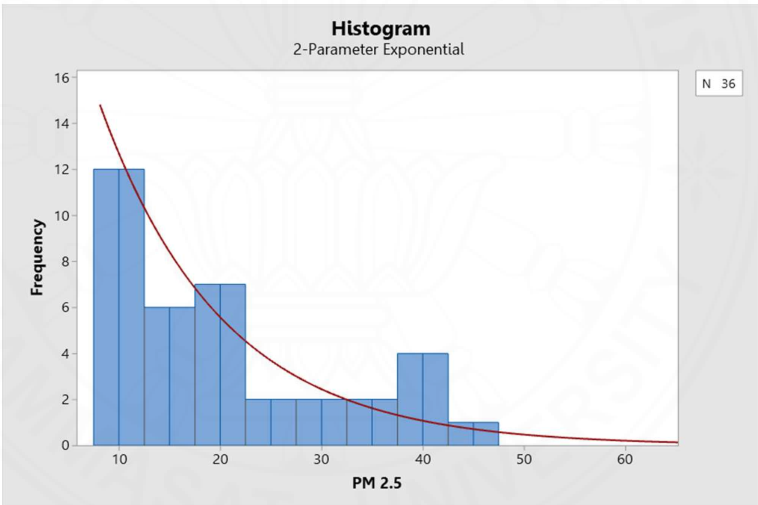

FIGURE 4-13 Histogram of PM 2.5

Now, we consider the distribution of these data by testing. The related hypotheses are

H_0: the data are a two-parameter exponential distribution

H_1: the data are not a two-parameter exponential distribution.

Using the Anderson-Darling test, these data follow the two-parameter exponential distribution with a p-value of 0.25. Furthermore, the results in Table 4-4 show that the lowest AIC = 253.7837 and BIC = 264.1178 for two-parameter exponential model, which is thus the most suitable distribution. It seems the PM 2.5 dataset is reasonable to use for the study in this chapter. This result is matched the P-P plot given in Figure 4-14.

TABLE 4-4 AIC and BIC results of PM 2.5 data

Models	AIC	BIC
Exponential	290.6398	292.2233
Gamma	266.6492	269.8162
Weibull	270.0079	273.1749
Cauchy	284.2087	287.3757
Two-parameter exponential	253.7837	264.1178

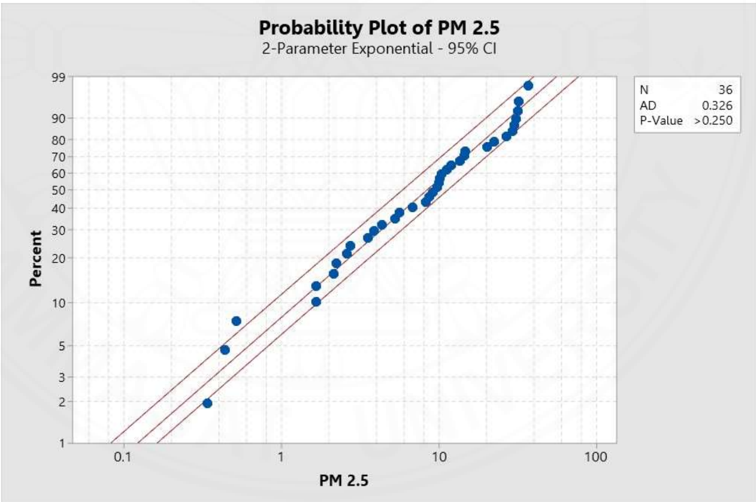

FIGURE 4-14 Probability plot of PM 2.5

The summary basic statistics for the PM 2.5 data are $\bar{X} = 20.2648$, $n = 36$ with the maximum likelihood estimator and unbiased estimators for λ being $\hat{\lambda} = 11.8132$ and $\hat{\lambda}_{unbias} = 12.1507$, for θ being $\hat{\theta} = 8.4516$ and $\hat{\theta}_{unbias} = 8.1141$, respectively.

TABLE 4-5 The 95% two-sided confidence intervals for the mean of PM 2.5 data in Thailand

Methods	Confidence intervals for $\lambda+\theta$ $(\mu g/m^3)$		Length of intervals
	Lower	Upper	
CI_{m1}	16.5542	24.5276	7.9734
CI_{m2}	16.1725	24.2481	8.0755
CI_{m3}	17.4021	26.6299	9.2279
CI_{m4}	17.2363	25.9049	8.6686
CI_{m5}	16.4059	24.1237	7.7178

We then compute the 95% two-sided confidence interval for $\lambda+\theta$ using the MOVER approach and Wald-type method. The estimated means are reported in Table 4-5. The Wald-type method, or CI_{m5} has the shortest length. However, the expected length of this method slightly differs from that of other methods. According to the results of simulation, in case of $n=30$, the coverage probabilities of CI_{m3} and CI_{m4} are greater than the nominal confidence level of 0.95 but the expected lengths of CI_{m4} are lower than these of CI_{m3}. We then use CI_{m4}. Although CI_{m5} gives the shortest length, it cannot work well in estimating the mean espeacially for $n<100$. As a result, the numerical results from a real data set confirm the simulation studies from the previous section.

Finally, in the period of study the mean of PM 2.5 in Bang Na district should is between 17.2363 and 25.9049 $\mu g/m^3$. It is reported that the air quality is very good.

CHAPTER 5
CONCLUSIONS AND RECOMMENDATIONS

The objective of this thesis is to estimate the parameter and construct the confidence interval for the population mean of a two- parameter exponential distribution. The first section is summarized in two main parts: theoretical and computational parts. The final section ends with the future work. We give more details as in the following.

5.1 Conclusions

In theoretical part, we estimate the parameters of the two- parameter exponential distribution. The maximum likelihood (bias estimator) and unbiased estimator of λ and θ are given by $\hat{\lambda} = \bar{X} - X_{(1)}$, $\hat{\theta} = X_{(1)}$, $\hat{\lambda}_{unbias} = \dfrac{n\left(\bar{X} - X_{(1)}\right)}{n-1} = \dfrac{n\hat{\lambda}}{n-1}$, and $\hat{\theta}_{unbias} = \dfrac{nX_{(1)} - \bar{X}}{n-1}$. After that, we construct the confidence intervals for the population mean of the two-parameter exponential distribution. The method of variance of estimates recovery (MOVER) using the asymptotic theory, the MOVER using profiled likelihood method, the MOVER using classical method, the MOVER using pivotal method, and the Wald-type method are applied to construct the interval estimators. These confidence intervals have the close-form solutions. Thus, it is easy to use in computation without software package in applications.

In the computational part, the two-parameter exponential random number are generated by using the inverse transform method in RStudio. For each situation, the experiment is repeated 5,000 times to obtain the absolute bias, mean squared error, coverage probability, and expected length. We first consider the performance of the point estimators for λ and θ in terms of absolute bias and mean squared error. We prefer the estimator which has a small absolute bias and low mean squared error. Furthermore, the criteria for deciding about the performance of interval estimators is the coverage probability and expected length. We choose a confidence interval that has a coverage probability greater than or close to the nominal coverage level and has a short length interval. The simulation results of the point and interval estimators are given in Chapter 4. For point estimation, the unbiased estimators perform better than the maximum likelihood estimator. This is because the absolute bias of unbiased estimators are lower than the comparators in all situations. Whereas the mean squared error of unbiased estimators are slightly differs from that of maximum likelihood estimators. In general theory, the maximum likelihood estimators can be biased or unbiased estimator; in this case, the estimators from this method are biases, especially for small sample sizes. According to the formula in Chapter 3, the unbiased estimators are derived from maximum likelihood estimators. The performance of this estimator is adjusted which has more efficient. In summary, it is noticed that when sample sizes are small, the absolute bias and mean squared error of the estimators are greater than the large sample size. Most significantly, as sample sizes increase, the magnitude of the absolute bias and mean square error for estimators decreases and approaches zero as $n \to \infty$, which corresponds to the central limit theorem.

For interval estimation, when sample sizes are large, the coverage probabilities of all methods are close to the nominal confidence level at 0.95. This corresponds to the central limit theorem. Although the Wald-type method gives the shortest expected lengths when compared to other methods. However, its coverage probability is less than 0.95 in all situations. This confidence interval should not be used to estimate the population mean of the two-parameter exponential distribution. The coverage probabilities of the MOVER using pivotal method are satisfied the nominal level in all cases of the study. Moreover, the coverage probability and expected length of the MOVER using pivotal method are nearly the same as the MOVER using classical method and are different only at the third or the fourth decimal place. It can be seen that although these methods are based on central limit theorem, they perform well in terms of coverage probability when the sample sizes are small. However, the expected length of the MOVER using pivotal method is shorter than the expected length of the MOVER using classical method in all cases.

Therefore, the MOVER using pivotal method is recommended to use for estimating the population mean in the two-parameter exponential distribution. Each estimator has advantages in different situations, and we propose a method for each case in the table below.

TABLE 5-1 The most appropriate confidence interval for each situation

Situation	Suggested method
$n = 10$	The MOVER using pivotal method and MOVER using classical method
$n = 30$	The MOVER using pivotal method and MOVER using classical method
$n = 50$	The MOVER using pivotal method and MOVER using classical method
$n = 100$	The MOVER using pivotal method and MOVER using classical method
$n = 200$	The MOVER using pivotal method, MOVER using classical method, MOVER using asymptotic method, and MOVER using profile likelihood method
$n = 500$	The MOVER using pivotal method, MOVER using classical method, MOVER using asymptotic method, and MOVER using profile likelihood method

Noted that the first method is the best and the other is the alternative method.

5.2 Recommendations for future work

The following ideas are interesting for further works.

1. In this thesis, we studied the confidence interval for the mean in one population. Our methods can be extended to construct confidence intervals in the two populations.
2. There are many exceptionally interesting methods for constructing confidence intervals. The generalized pivot method and bootstrap approach may be used in estimation.
3. According to very wide applications of the two-parameter exponential distribution, we can try to apply it in other fields such as economy and medical sciences.

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APPENDICES

APPENDIX A

TESTING THE DISTRIBUTION OF RANDOM VARIABLE

Since there has no package in the R language directly to find the random number in the two-parameter exponential distribution, we used the inverse transformation method to derive the solution to obtain the random number. A question is then arised whether this method is correct. We used the simulation given in Chapter 3 with the settings $n = 20$, $\lambda = 3$, and $\theta = 5$ to generate the data based on the inverse transform method. Simulated data were given by

10.5322,	8.4385,	8.5231,	9.8699,	18.7857,
6.5371,	8.3126,	9.8337,	7.8737,	5.1364,
7.6128,	5.7611,	5.9770,	6.3036,	7.2338,
5.7186,	6.4431,	18.1292,	6.8815,	5.4951

We consider the distribution of these simulated data using the Anderson-Darling test. It was found that these data follow the two-parameter exponential distribution with a probability value of 0.25 (see also Figure A1). As a result, the inverse transform method we used to generate the random variable provide the data followed a two-parameter exponential distribution.

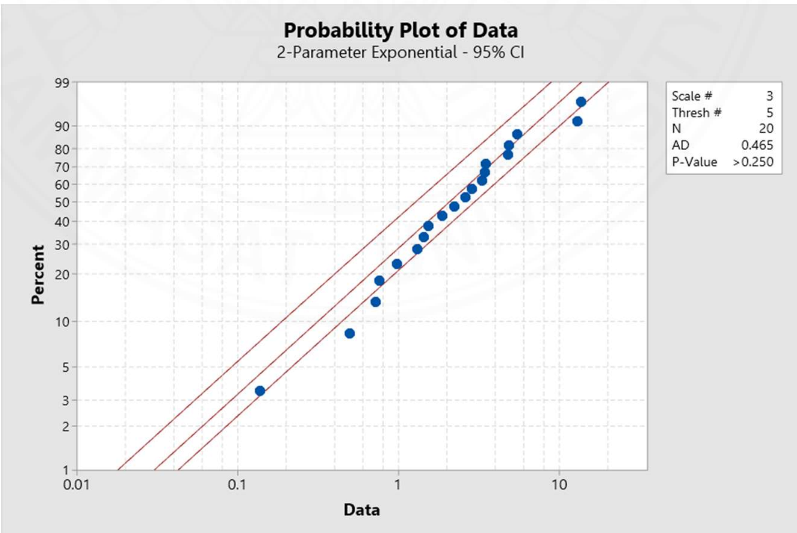

FIGURE A1 The probability plot of data when $n = 20$, $\lambda = 3$ and $\theta = 5$

For the case, $n=100$, $\lambda=3$ and $\theta=5$, the simulated data are shown in Figure A2. Again, these data obtained from the inverse transform method based on simulation using the R programming followed the two-parameter exponential distribution with a probability value of 0.25 (see Figure A3).

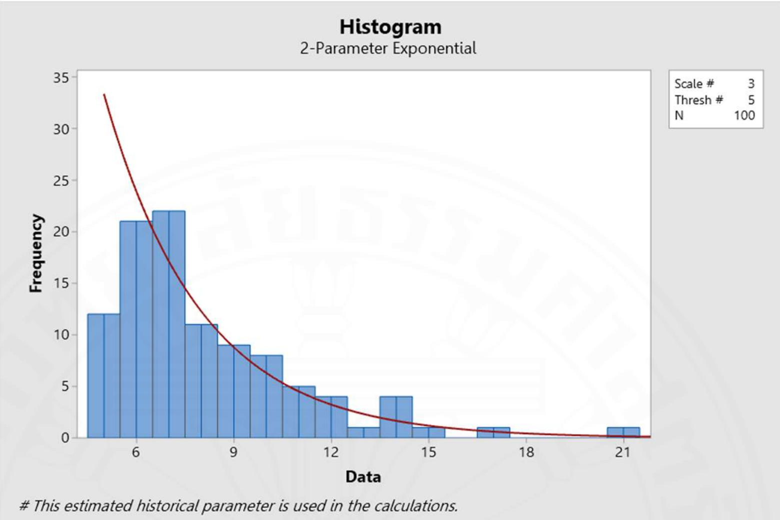

FIGURE A2 Histogram of data when $n=100$, $\lambda=3$ and $\theta=5$

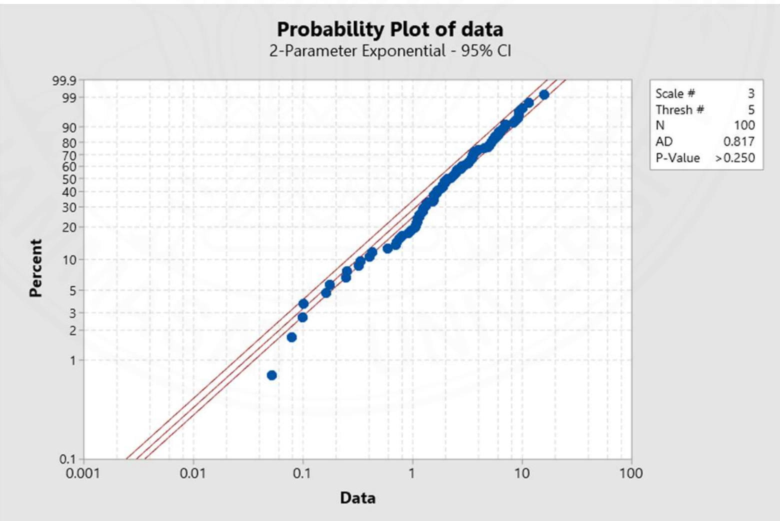

FIGURE A3 The probability plot of data when $n=100$, $\lambda=3$ and $\theta=5$

APPENDIX B

THE REAL DATASET

The real data of PM 2.5 mass concentration in Bang Na area (36 observations) used in Chapter 4 are given in the following.

45.1613,	16.8214,	18.5484,	13.7000,	18.1667,
12.0000,	14.9677,	9.7742,	19.3000,	22.7742,
30.6667,	34.9032,	9.7742,	19.3000,	22.7742,
30.6667,	34.9032,	40.3871,	38.3448,	18.2258,
17.9000,	12.4839,	8.6333,	9.7742,	10.3548,
10.2667,	16.3871,	22.7000,	28.3226,	39.7419,
39.0714,	21.8065,	17.2333,	11.6774,	10.7000,
8.5484,	8.4516,	10.8667,	13.4194,	20.0333,
37.4194				

APPENDIX C

THE PROGRAM FOR CALCULATING CONFIDENCE INETERVAL FOR MEAN IN TWO-PARAMETER EXPONENTIAL

```
#FUNTION 2EXP
EXP <- function(lambda, theta, n, alpha, M){
 Var1 <- rep(0, M)
 Var2 <- rep(0, M)
 Var3 <- rep(0, M)
 Var4 <- rep(0, M)

 mse1 <- rep(0, M)
 mse2 <- rep(0, M)
 mse3 <- rep(0, M)
 mse4 <- rep(0, M)

 L1 <- rep(0, M) ; U1 <- rep(0, M) ; CP1 <- rep(0, M) ; len1 <- rep(0, M)
 L2 <- rep(0, M) ; U2 <- rep(0, M) ; CP2 <- rep(0, M) ; len2 <- rep(0, M)
 L3 <- rep(0, M) ; U3 <- rep(0, M) ; CP3 <- rep(0, M) ; len3 <- rep(0, M)
 L4 <- rep(0, M) ; U4 <- rep(0, M) ; CP4 <- rep(0, M) ; len4 <- rep(0, M)
 L5 <- rep(0, M) ; U5 <- rep(0, M) ; CP5 <- rep(0, M) ; len5 <- rep(0, M)

  mean.para <- lambda + theta     #parameter used to find CP EL

  ##start for loop
  for(i in 1:M) {

## Generate data from 2-exp distribution
   # random sample
   u = runif(n, 0, 1)
   X = ifelse(u < 0, 0, theta - lambda*log(1 - u))
   lambda.MLE <- mean(X) - min(X)
   Var1[i] <- lambda.MLE
```

```
    lambda.un <- n* lambda.MLE /(n - 1)
    Var2[i] <- lambda.un

    theta.MLE <- min(X)
    Var3[i] <- theta.MLE

    theta.un <- (n*min(X) - mean(X)) / (n - 1)
    Var4[i] <- theta.un

# Confidence Interval for mean
    z <- qnorm(1 - alpha/2)
    CIL.theta.un <- theta.un - z*lambda.MLE*sqrt(1/(n^2-n))
    CIU.theta.un <- theta.un + z*lambda.MLE*sqrt(1/(n^2-n))

    ##### 1 Asymptotic -> both unbiased

    mean.un = lambda.un + theta.un
    #lambda.MLE + theta.MLE
    k2 <- (z*sqrt(n - 1) + n)/(n - 1)
    c2 <- (-z*sqrt(n - 1) + n)/(n - 1)

    U1[i] <- mean(X) + sqrt((lambda.MLE*k2 - lambda.MLE)^2 + (CIU.theta.un -
theta.MLE)^2)
    L1[i] <- mean(X) - sqrt((lambda.MLE - lambda.MLE*c2)^2 + (theta.MLE -
CIL.theta.un)^2)
    #cbind(L1,U1)

    #basic method
    #L1.new <- (lambda.MLE + theta.MLE) - sqrt((lambda.MLE - (lambda.un -
z*sqrt(lambda.MLE^2/(n-1))))^2 + (theta.MLE - CIL.theta.un)^2)
```

```
##### 2 Profile likelihood

U2[i] <- (lambda.MLE + theta.MLE) + sqrt((z^2*lambda.MLE^2)/(n-
2)+(CIU.theta.un - theta.MLE)^2)
L2[i] <- (lambda.MLE + theta.MLE) - sqrt((z^2*lambda.MLE^2)/(n-
2)+(theta.MLE - CIL.theta.un)^2)
#cbind(L2,U2)

##### 3 Classical

# Pivot-lambda, PN-theta
k1 <- sqrt(n - 1)*(-z + sqrt(n - 1))   #upper
c1 <- sqrt(n - 1)*(z + sqrt(n - 1))    #lower
U3[i] <- lambda.MLE + theta.MLE + sqrt((n*lambda.MLE/k1 - lambda.MLE)^2 +
(CIU.theta.un - theta.MLE)^2)
L3[i] <- lambda.MLE + theta.MLE - sqrt((lambda.MLE - n*lambda.MLE/c1)^2 +
(theta.MLE - CIL.theta.un)^2)
#cbind(L3,U3)

##### 4 Pivot function

chiU <- qchisq(alpha/2, 2*n - 2)
chiL <- qchisq(1 - alpha/2, 2*n - 2)
U4[i] <- lambda.MLE + theta.MLE + sqrt((((2*n*lambda.MLE)/chiU -
lambda.MLE)^2 + (CIU.theta.un - theta.MLE)^2)
L4[i] <- lambda.MLE + theta.MLE - sqrt((lambda.MLE -
(2*n*lambda.MLE)/chiL)^2 + (theta.MLE - CIL.theta.un)^2)
#cbind(L4,U4)

##### 5 Delta-method
```

```
  # X-bar
  U5[i] <- mean(X) + qnorm(1 - alpha/2) * sqrt(lambda.MLE^2 / n)
  L5[i] <- mean(X) - qnorm(1 - alpha/2) * sqrt(lambda.MLE^2 / n)
  #cbind(L5,U5)

## Find MSE
  mse1[i] <- (lambda - lambda.MLE)^2
  mse2[i] <- (lambda - lambda.un)^2
  mse3[i] <- (theta - theta.MLE)^2
  mse4[i] <- (theta - theta.un)^2

## Find CP and EL  ##
  ### CP
  if(L1[i] <= mean.para & mean.para <= U1[i]) { CP1[i] <- 1
  } else { CP1[i] <- 0 }
  CP2[i] <- ifelse(L2[i] <= mean.para & mean.para <= U2[i], 1, 0)
  CP3[i] <- ifelse(L3[i] <= mean.para & mean.para <= U3[i], 1, 0)
  CP4[i] <- ifelse(L4[i] <= mean.para & mean.para <= U4[i], 1, 0)
  CP5[i] <- ifelse(L5[i] <= mean.para & mean.para <= U5[i], 1, 0)

  ### EX length
  len1[i] <- U1[i] - L1[i]
  len2[i] <- U2[i] - L2[i]
  len3[i] <- U3[i] - L3[i]
  len4[i] <- U4[i] - L4[i]
  len5[i] <- U5[i] - L5[i]

 } # end loop

## Standrad erorr
  V1 <- sqrt(var(Var1))/M      #lambda.MLE
  V2 <- sqrt(var(Var2))/M      #lambda.UN
```

```
  V3 <- sqrt(var(Var3))/M      #theta.MLE
  V4 <- sqrt(var(Var4))/M      #theta.UN

## MSE
  M1 <- mean(mse1)
  M2 <- mean(mse2)
  M3 <- mean(mse3)
  M4 <- mean(mse4)

## Abs bias
  B1 <- abs(mean(Var1) - lambda)    #lambda.MLE
  B2 <- abs(mean(Var2) - lambda)    #lambda.UN
  B3 <- abs(mean(Var3) - theta)     #theta.MLE
  B4 <- abs(mean(Var4) - theta)     #theta.UN

## Average values from M simulation runs
 CP.1 <- mean(CP1)
 CP.2 <- mean(CP2)
 CP.3 <- mean(CP3)
 CP.4 <- mean(CP4)
 CP.5 <- mean(CP5)

 len.1 <- mean(len1)
 len.2 <- mean(len2)
 len.3 <- mean(len3)
 len.4 <- mean(len4)
 len.5 <- mean(len5)


 Var <- cbind(V1, V2, V3, V4, NA)
 Bias <- cbind(B1, B2, B3, B4, NA)
 MSE <- cbind(M1, M2, M3, M4, NA)
```

```
  cp <- cbind(CP.1, CP.2, CP.3, CP.4, CP.5)
  len <- cbind(len.1, len.2, len.3, len.4, len.5)

  output <- data.frame(cp, len, Var, Bias, MSE)

  return(output)

} # end EXP program

library("writexl")

M = 5000
alpha = 0.05

# n = 10
case1.0 <- EXP(lambda = 1, theta = 0.5, n = 10, alpha, M)
case2.0 <- EXP(lambda = 1, theta = 1, n = 10, alpha, M)
case3.0 <- EXP(lambda = 1, theta = 3, n = 10, alpha, M)
case4.0 <- EXP(lambda = 1, theta = 5, n = 10, alpha, M)
case5.0 <- EXP(lambda = 3, theta = 0.5, n = 10, alpha, M)
case6.0 <- EXP(lambda = 3, theta = 1, n = 10, alpha, M)
case7.0 <- EXP(lambda = 3, theta = 3, n = 10, alpha, M)
case8.0 <- EXP(lambda = 3, theta = 5, n = 10, alpha, M)

C0 <- rbind(case1.0, case2.0, case3.0, case4.0, case5.0, case6.0, case7.0, case8.0)
#write_xlsx(C0,"Downloads/CASE0.xls")

# n = 30
case1.1 <- EXP(lambda = 1, theta = 0.5, n = 30, alpha, M)
case2.1 <- EXP(lambda = 1, theta = 1, n = 30, alpha, M)
case3.1 <- EXP(lambda = 1, theta = 3, n = 30, alpha, M)
```

```
case4.1 <- EXP(lambda = 1, theta = 5, n = 30, alpha, M)
case5.1 <- EXP(lambda = 3, theta = 0.5, n = 30, alpha, M)
case6.1 <- EXP(lambda = 3, theta = 1, n = 30, alpha, M)
case7.1 <- EXP(lambda = 3, theta = 3, n = 30, alpha, M)
case8.1 <- EXP(lambda = 3, theta = 5, n = 30, alpha, M)

C1 <- rbind(case1.1, case2.1, case3.1, case4.1, case5.1, case6.1, case7.1, case8.1)
#write_xlsx(C1,"Downloads/CASE1.xls")

# n = 50
case1.2 <- EXP(lambda = 1, theta = 0.5, n = 50, alpha, M)
case2.2 <- EXP(lambda = 1, theta = 1, n = 50, alpha, M)
case3.2 <- EXP(lambda = 1, theta = 3, n = 50, alpha, M)
case4.2 <- EXP(lambda = 1, theta = 5, n = 50, alpha, M)
case5.2 <- EXP(lambda = 3, theta = 0.5, n = 50, alpha, M)
case6.2 <- EXP(lambda = 3, theta = 1, n = 50, alpha, M)
case7.2 <- EXP(lambda = 3, theta = 3, n = 50, alpha, M)
case8.2 <- EXP(lambda = 3, theta = 5, n = 50, alpha, M)

C2 <- rbind(case1.2, case2.2, case3.2, case4.2, case5.2, case6.2, case7.2, case8.2)
#write_xlsx(C2,"Downloads/CASE2.xls")

# n = 100
case1.3 <- EXP(lambda = 1, theta = 0.5, n = 100, alpha, M)
case2.3 <- EXP(lambda = 1, theta = 1, n = 100, alpha, M)
case3.3 <- EXP(lambda = 1, theta = 3, n = 100, alpha, M)
case4.3 <- EXP(lambda = 1, theta = 5, n = 100, alpha, M)
case5.3 <- EXP(lambda = 3, theta = 0.5, n = 100, alpha, M)
case6.3 <- EXP(lambda = 3, theta = 1, n = 100, alpha, M)
case7.3 <- EXP(lambda = 3, theta = 3, n = 100, alpha, M)
case8.3 <- EXP(lambda = 3, theta = 5, n = 100, alpha, M)
```

```r
C3 <- rbind(case1.3, case2.3, case3.3, case4.3, case5.3, case6.3, case7.3, case8.3)
#write_xlsx(C3,"Downloads/CASE3.xls")

# n = 200
case1.4 <- EXP(lambda = 1, theta = 0.5, n = 200, alpha, M)
case2.4 <- EXP(lambda = 1, theta = 1, n = 200, alpha, M)
case3.4 <- EXP(lambda = 1, theta = 3, n = 200, alpha, M)
case4.4 <- EXP(lambda = 1, theta = 5, n = 200, alpha, M)
case5.4 <- EXP(lambda = 3, theta = 0.5, n = 200, alpha, M)
case6.4 <- EXP(lambda = 3, theta = 1, n = 200, alpha, M)
case7.4 <- EXP(lambda = 3, theta = 3, n = 200, alpha, M)
case8.4 <- EXP(lambda = 3, theta = 5, n = 200, alpha, M)

C4 <- rbind(case1.4, case2.4, case3.4, case4.4, case5.4, case6.4, case7.4, case8.4)
#write_xlsx(C4,"Downloads/CASE4.xls")

# n = 500
case1.5 <- EXP(lambda = 1, theta = 0.5, n = 500, alpha, M)
case2.5 <- EXP(lambda = 1, theta = 1, n = 500, alpha, M)
case3.5 <- EXP(lambda = 1, theta = 3, n = 500, alpha, M)
case4.5 <- EXP(lambda = 1, theta = 5, n = 500, alpha, M)
case5.5 <- EXP(lambda = 3, theta = 0.5, n = 500, alpha, M)
case6.5 <- EXP(lambda = 3, theta = 1, n = 500, alpha, M)
case7.5 <- EXP(lambda = 3, theta = 3, n = 500, alpha, M)
case8.5 <- EXP(lambda = 3, theta = 5, n = 500, alpha, M)

C5 <- rbind(case1.5, case2.5, case3.5, case4.5, case5.5, case6.5, case7.5, case8.5)
#write_xlsx(C5,"Downloads/CASE5.xls")

row <- rbind(C0,C1,C2,C3,C4,C5)
write_xlsx(row,"Downloads/row.xls")
```

```
######## Comparing confidence interval ########
library(readxl)
Results <- read_excel("Desktop/Thesis/works/Results.xlsx", sheet = "Plot")
#View(Results)
#attach(Results)
#names(Results)
x <- c(0.5, 1, 3, 5)  #parameter lambda

par(mfrow = c(1,2))

## n=10, lambda=1
plot(x, CP1[1:4], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,6))
lines(x, CP2[1:4], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[1:4], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16, add =
T)
lines(x, CP4[1:4], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[1:4], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
     col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
     cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 10  ", lambda, " = 1")), cex.main = 1.2, col.lab =
"black")


## n=10, lambda=3
plot(x, cp1[5:8], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
```

```
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,6))
lines(x, CP2[5:8], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[5:8], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16, add =
T)
lines(x, CP4[5:8], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[5:8], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 10  ", lambda, " = 3")), cex.main = 1.2, col.lab =
"black")

## n=30, lambda=1
plot(x, cp1[9:12], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[9:12], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[9:12], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16, add
= T)
lines(x, CP4[9:12], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[9:12], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
```

```
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 30  ", lambda, " = 1")), cex.main = 1.2, col.lab =
"black")

## n=30, lambda=3
plot(x, cp1[13:16], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[13:16], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[13:16], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[13:16], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[13:16], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 30  ", lambda, " = 3")), cex.main = 1.2, col.lab =
"black")

## n=50, lambda=1
plot(x, cp1[17:20], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[17:20], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[17:20], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[17:20], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
```

```
lines(x, CP5[17:20], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 50  ", lambda, " = 1")), cex.main = 1.2, col.lab =
"black")

## n=50, lambda=3
plot(x, cp1[21:24], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[21:24], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[21:24], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[21:24], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[21:24], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 50  ", lambda, " = 3")), cex.main = 1.2, col.lab =
"black")

## n=100, lambda=1
plot(x, cp1[25:28], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
```

```
lines(x, CP2[25:28], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[25:28], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[25:28], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[25:28], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
    col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
    cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 100  ", lambda, " = 1")), cex.main = 1.2, col.lab =
"black")

## n=100, lambda=3
plot(x, cp1[29:32], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
  xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[29:32], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[29:32], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[29:32], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[29:32], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
    col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
    cex = 0.6, pt.cex = 1)
```

```
title(main = expression(paste("n = 100  ", lambda, " = 3")), cex.main = 1.2, col.lab =
"black")

## n=200, lambda=1
plot(x, cp1[33:36], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[33:36], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[33:36], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[33:36], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[33:36], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 200  ", lambda, " = 1")), cex.main = 1.2, col.lab =
"black")

## n=200, lambda=3
plot(x, cp1[37:40], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
    xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[37:40], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[37:40], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[37:40], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
```

```
lines(x, CP5[37:40], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 200  ", lambda, " = 3")), cex.main = 1.2, col.lab =
"black")

## n=500, lambda=1
plot(x, cp1[41:44], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
     xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
lines(x, CP2[41:44], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[41:44], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[41:44], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[41:44], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
      col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
      cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 500  ", lambda, " = 1")), cex.main = 1.2, col.lab =
"black")

## n=500, lambda=3
plot(x, cp1[45:48], ylim = c(0.85,1), lty = 1, cex = 1, type = "o", pch = 16,
     xlab = expression(theta), ylab = "Coverage probability", xlim = c(0.5,5.5))
```

```
lines(x, CP2[45:48], col = "purple3", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP3[45:48], col = "blue", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP4[45:48], col = "green4", lwd = 0.3, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
lines(x, CP5[45:48], col = "orange2", lwd = 0.5, lty = 1, cex = 1, type = "o", pch = 16,
add = T)
abline(h = 0.95, lty = 2, col = "red")
legend("bottomright", legend = paste(c("CI1", "CI2", "CI3", "CI4", "CI5")), lwd = 1,
    col = c("black", "purple3", "blue","green4","orange2"), pch = c(16, 16, 16, 16,
16),
    cex = 0.6, pt.cex = 1)
title(main = expression(paste("n = 500  ", lambda, " = 3")), cex.main = 1.2, col.lab =
"black")

##### Application #####
library(readxl)
Results <- read_excel("Desktop/Thesis/works/Results.xlsx",sheet = "DATA")
View(PM)
X <- Results$PM
X
n <- length(Results$PM)
n
mean(X)
lambda.MLE <- mean(X) - min(X)
lambda.MLE # 11.8132
lambda.un <- n*lambda.MLE / (n - 1)
lambda.un  # 12.15072
theta.MLE <- min(X)
theta.MLE  # 8.451613
theta.un <- (n*min(X) - mean(X)) / (n - 1)
```

```
theta.un   # 8.114093

## CI of theta unbiased estimator
alpha <- 0.05
z <- qnorm(1 - alpha/2)
CIL.theta.un <- theta.un - z*lambda.MLE*sqrt(1/(n^2-n))
CIU.theta.un <- theta.un + z*lambda.MLE*sqrt(1/(n^2-n))

##### 1 Asymptotic -> both unbiased
k2 <- (z*sqrt(n - 1) + n)/(n - 1)
c2 <- (-z*sqrt(n - 1) + n)/(n - 1)
CIL.theta.un <- theta.un - z*lambda.MLE*sqrt(1/(n^2-n))
CIU.theta.un <- theta.un + z*lambda.MLE*sqrt(1/(n^2-n))

U1 <- mean(X) + sqrt((lambda.MLE*k2 - lambda.MLE)^2 + (CIU.theta.un -
theta.MLE)^2)
L1 <- mean(X) - sqrt((lambda.MLE - lambda.MLE*c2)^2 + (theta.MLE -
CIL.theta.un)^2)
#cbind(L1,U1)
el1 <- U1 - L1

#basic method
#L1.new <- (lambda.MLE + theta.MLE) - sqrt((lambda.MLE - (lambda.un -
z*sqrt(lambda.MLE^2/(n-1))))^2 + (theta.MLE - CIL.theta.un)^2)

##### 2 Profile likelihood
z <- qnorm(1 - alpha/2)
CIL.theta.un <- theta.un - z*lambda.MLE*sqrt(1/(n^2 - n))
CIU.theta.un <- theta.un + z*lambda.MLE*sqrt(1/(n^2 - n))
U2 <- (lambda.MLE + theta.MLE) + sqrt((z^2*lambda.MLE^2)/(n-2)+(CIU.theta.un
- theta.MLE)^2)
```

```
L2 <- (lambda.MLE + theta.MLE) - sqrt((z^2*lambda.MLE^2)/(n-2)+(theta.MLE -
CIL.theta.un)^2)
#cbind(L2,U2)
el2 <- U2 - L2

##### 3 Classical
# Pivot-lambda, PN-theta
k1 <- sqrt(n - 1)*(-z + sqrt(n - 1))   #upper
c1 <- sqrt(n - 1)*(z + sqrt(n - 1))    #lower
U3 <- lambda.MLE + theta.MLE + sqrt((n*lambda.MLE/k1-lambda.MLE)^2 +
(CIU.theta.un - theta.MLE)^2)
L3 <- lambda.MLE + theta.MLE - sqrt((lambda.MLE - n*lambda.MLE/c1)^2 +
(theta.MLE - CIL.theta.un)^2)
#cbind(L3,U3)
el3 <- U3 - L3

##### 4 Pivot function
z <- qnorm(1 - alpha/2)
chiU <- qchisq(alpha/2, 2*n - 2)
chiL <- qchisq(1 - alpha/2, 2*n - 2)
U4 <- lambda.MLE + theta.MLE + sqrt((((2*n*lambda.MLE)/chiU - lambda.MLE)^2
+ (CIU.theta.un - theta.MLE)^2)
L4 <- lambda.MLE + theta.MLE - sqrt((lambda.MLE - (2*n*lambda.MLE)/chiL)^2 +
(theta.MLE - CIL.theta.un)^2)
cbind(L4,U4)
el4 <- U4 - L4

##### 5 Delta-method
# X-bar
U5 <- mean(X) + qnorm(1 - alpha/2) * sqrt(lambda.MLE^2 / n)
L5 <- mean(X) - qnorm(1 - alpha/2) * sqrt(lambda.MLE^2 / n)
#cbind(L5,U5)
```

```
el5 <- U5-L5

L <- c(L1, L2, L3, L4, L5)
U <- c(U1, U2, U3, U4, U5)
EL <- c(el1, el2, el3, el4, el5)
output <- data.frame(L, U, EL)
output

##### Fitting distributions #####
#install 'fitdistrplus' package if not already installed
#load package
install.packages('fitdistrplus')
install.packages('MASS')
library(fitdistrplus)
library(MASS)

#fit our dataset to a gamma distribution using mle

###### 2-parameter expo
k = 2
l2expo <- (1/(lambda.MLE^36)) * exp(-(1/(lambda.MLE))*(sum(X) - n*theta.MLE))
l2expo
aic2expo <- -2*log(l2expo ) + 2*(2)
aic2expo

bic2expo <- -2*log(l2expo ) + 2*(2)*log(36)
bic2expo

###### gamma
fitgam = fitdist(X, "gamma")
gofstat(fitgam)
summary(fitgam)
```

```
aicgam <- -2*(-131.3246) + 2*(2)
aicgam

bicgam <- -2*(-131.3246) + 2*(2)*log(36)
bicgam

######  weibull
fitwei = fitdist(X, "weibull")
gofstat(fitwei)
summary(fitwei)

######  cauchy
fitcau = fitdist(X, "cauchy")
gofstat(fitcau)
summary(fitcau)

#######  Expo
fitexpo = fitdist(X, "exp")
gofstat(fitexpo)
summary(fitexpo)
lexpo <- (1/(mean(X)^36)) * exp(-(1/(mean(X)))*sum(X))
lexpo
log(lexpo)
aicexpo <- -2*log(lexpo) + 2*(1)
aicexpo

bicexpo <- -2*log(lexpo) + (2*log(36))
bicexpo
```

BIOGRAPHY

Name	Chanida Kraisadab
Education Attainment	2019: B.Sc. in Statistics, Thammasat University, Thailand.
Scholarship	2020-2021: Scholarship for talent student to study graduate program in Faculty of Science and Technology, Thammasat University.
Experiences	October 2021- March 2022, Visiting Graduate Research Student at University of Regina, Canada funded by Canada ASEAN Scholarships and Educational Exchanges for Development (SEED)

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