EXISTENCE OF EXTREME POINTS OF COMPACT CONVEX SETS IN ASYMMETRIC CONE NORMED SPACES

BY

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
ACADEMIC YEAR 2021
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THESIS

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was approved as partial fulfillment of the requirements for the degree of Master of Science (Mathematics)

on June , 2022

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Thesis Title	EXISTENCE OF EXTREME POINTS OF COMPACT CONVEX SETS IN ASYMMETRIC CONE NORMED SPACES
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Thesis Academic Year	2021

ABSTRACT

One of the most popular applications in real-world problems from the past into the present is optimization. It is well-known that the concept of extreme points plays a vital role in optimization, and the existence of extreme points of compact convex subsets of a locally convex (Hausdorff) vector space can be obtained from the Krein-Milman theorem. As the usefulness of this application, many researchers follow this idea to investigate the existence of extreme points in compact convex subsets of various abstract spaces. In 2016, the research in this direction in asymmetric normed spaces was proved in Jonard-Pérez and Sánchez-Pérez (2016). Another important concept parallel to the concept of extreme points in optimization is the concept of cones. Most recently, the idea of cones is used to extend asymmetric normed spaces to asymmetric cone normed spaces, and its topological properties are studied. Surprisingly, nobody considered the existence of extreme points of compact convex subsets of asymmetric cone normed spaces. Our goal in this research is to fulfill this direction. Hence, the sufficient condition for the existence of extreme points of nonempty compact convex subsets of asymmetric cone normed spaces is invented. An example to illustrate the main result presented herein is given.

Keywords: Asymmetric cone normed space, Extreme point, Krein-Milman theorem

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ACKNOWLEDGEMENTS

First, for the results in this thesis, I would like to thank Associate Professor Dr. Wutiphol Sintunavarat for interesting research topics and ideas in the development of the application of asymmetric cone normed space and including hints on the use of language.

Second, I would also like to thank Thammasat University for giving me the Thammasat University Scholarship. Sincere thanks are also extended to all staff at the Department of Mathematics and Statistics, Thammasat University, for their support, to all my Classmates and seniors who have participated in giving opinions and exchanging knowledge as well, and I would like to thank my family for their inspiration and support all along.

Finally, the researcher hopes that this thesis will be useful for those interested in studying.

Wasin Supakul

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CHAPTER 1
INTRODUCTION

In 1931, Wilson (Wilson (1931)) initiated the concept of an asymmetric metric (or a quasi-metric) by deleting the symmetric property from the properties of metrics. In addition, some properties of asymmetric metric spaces are proved, and relations between asymmetric metric spaces, metric spaces, and topological spaces are investigated. From the appearance of this space, several results related to asymmetric metric spaces were further developed by many authors. For instance, in 1993, following the idea of asymmetric metrics, Ferrer et al. (Ferrer et al. (1993)) introduced an asymmetric norm (or a quasi-norm) and investigated several properties in asymmetric normed spaces, which are similar to properties in normed spaces. Nowadays, there is a lot of research on asymmetric normed spaces in many fields.

Although the spaces mentioned earlier have many applications, a codomain of an asymmetric metric or an asymmetric norm is the set of real numbers. In practice, some real-world problems require more structure of the codomain of the measuring tools as an asymmetric metric or an asymmetric norm. Going back to the history of this direction, it seems that Kurepa (Kurepa (1934)) was the first researcher who replaced a codomain of a metric from the set of real numbers by with arbitrary partially ordered set. Many researchers used this approach in the mid-20th century, and hence it has various names such as abstract metric spaces, K-metric spaces, etc. The early appearance of this space did not receive much attention and prevalence until Long-Guang and Xian (Long-Guang and Xian (2007)) re-introduced the mentioned type of spaces under the name of cone metric spaces. In this study, interior points of the cone are used to define the partial order of the space. In (Long-Guang and Xian (2007)), the assumption of the normality of the cone is assumed, but later most of the results were obtained for non-normal cones. However, the results of non-normal cones have more complicated proofs.

It is well-known that cone metric spaces are metrizable. It causes some results in cone metric spaces to be obtained from the standard metric spaces. However, this does not necessarily mean that all developments in cone metric spaces reduce to their standard metric spaces. For instance, in the fixed point theory, at least some of

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them still depend on the behavior of cones which is used. Therefore, the research in cone metric spaces is still exciting and has continued until now.

Similar to cone metric spaces, cone normed spaces are presented as a generalization of normed spaces. The codomain of a norm is replaced by a partially ordered real Banach space. The reader can see more details of fundamental results in cone normed spaces in (Gordji et al. (2012)). It is fascinating that any asymmetric norm induces a non-symmetric topology on its domain, named X, generated by the set of all asymmetric open balls. Moreover, this topology is a T_0 topology in X such that the vector sum on X is continuous. Nonetheless, this topology is not even Hausdorff, and the scalar multiplication is not continuous in general cases. It leads to asymmetric norm spaces failing to be a topological vector space. The classic example of a non-Hausdorff asymmetric normed space is an asymmetric normed lattice. This particular asymmetric normed space is critical and exciting, mainly because of its applications in theoretical computer science and complexity theory.

Nowadays, the compactness of asymmetric norm spaces has been widely studied, and the general structure of compact sets in asymmetric normed spaces is presented in some exciting results (see (Alegre et al. (2008); Garcia-Raffi (2005); Conradie and Mabula (2013))). One of the interesting problems related to the compactness in asymmetric normed spaces is convexity. There are some results showing that convexity plays a vital role while working with compact sets in asymmetric normed spaces (see (Jonard-Pérez and Sánchez-Pérez (2016))). Concerning convexity, a significant effort has been made to improve the fundamental results of functional analysis in the non-asymmetric case (see (Garcia-Raffi et al. (2002))).

One of these classic results is Krein-Milman theorem, which is stated that every compact convex subset of a locally convex (Hausdorff) vector space, i.e., a locally convex vector space, whose topology is Hausdorff, is the closure of the convex hull of its extreme points. In particular, each compact convex subset of a locally convex (Hausdorff) vector space has at least an extreme point. However, in general, the Krein-Milman theorem is no longer valid in asymmetric normed spaces, not even in finite-dimensional asymmetric normed spaces. In 2022, Garcia-Raffi et al. (Garcia-Raffi et al. (2002)), it was proved that every Hausdorff asymmetric normed space satisfies Krein-Milman Theorem.

In another view of the Krein-Milman theorem, a compact convex subset of a locally convex (Hausdorff) vector space can be characterized by a set of extreme points. In practice, there are certain cases that are the set of extreme points, and the set of extreme rays of a convex set identifies its structure. Indeed, in 1957, Klee (Klee (1957)) was interested in studying the Krein-Milman theorem for locally compact closed convex subsets of a locally convex (Hausdorff) vector space such that these subsets contain no lines.

Based on Klee's theorem, in 2016, Jonard-Pérez and Sánchez-Pérez (Jonard-Pérez and Sánchez-Pérez (2016)) showed that for each compact convex set K in an asymmetric normed space (X, q), if $\varphi(\theta_X)$ is a kernel of q and $K + \varphi(\theta_X)$ is locally compact in the topology determined by q^s, where $q^s(x) := \max\{q(x), q(-x)\}$ for all $x \in X$, then $K + \varphi(\theta_X)$ is the closed convex hull of set of all its extreme points and extreme rays. Moreover, the existence of extreme points in nonempty compact convex subsets of asymmetric normed spaces is presented in such research. The appearance of this research opens an avenue to the interesting question: Is it possible to establish the existence result of extreme points of nonempty compact convex subsets of spaces having a general structure more than asymmetric normed spaces?

Inspired by the question in the previous paragraph, this thesis aims to present the sufficient condition for the existence of extreme points of a nonempty compact convex subset of an asymmetric cone normed space, which was most recently introduced by İlkhan (İlkhan (2020)). Finally, An illustrative example is given to demonstrate the validity of the hypotheses and the degree of utility of the main result in this thesis.

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CHAPTER 2
PRELIMINARIES

Throughout this thesis, unless otherwise specified, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}_+$ and $\mathbb{R}$ denote the set of integers, the set of integers modulo n where n is a natural number, the set of natural numbers, the set of rational number, the set of positive real numbers, the set of nonnegative real numbers and the set of real numbers, respectively. In this chapter, we give a basic definitions, examples, and properties needed in this thesis.

2.1 Relations

In mathematics, the *Cartesian product* of sets A and B, denoted by $A \times B$, is the set of all ordered pairs (x, y) such that x belongs to A and y belongs to B, that is,

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}.$$

Example 2.1.1. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then

- $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$,
- $A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$,
- $B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$,
- $B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

Definition 2.1.2. Let A and B be sets. The set R is called a *relation* from A to B, if R is a subset of $A \times B$. Moreover, if R is a relation from A to A, we call R a relation on A. For simplicity, we use the notion xRy whenever $(x, y) \in R$.

Example 2.1.3. From Example 2.1.1, if we define R_1, R_2 and R_3 by

- $R_1 = \{(a, 1), (b, 2), (b, 3)\}$,
- $R_2 = \{(a, 1), (a, 2)\}$,
- $R_3 = \{(a, a), (a, b)\}$,

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then $R_1, R_2 \subseteq A \times B$ and $R_3 \subseteq A \times A$. Therefore, R_1 and R_2 are relations from A to B, and R_3 is a relation on A.

There are several definitions related to the property of relations. The definitions involved in this thesis are as follows.

Definition 2.1.4. Let R be a relation on the set X.

- R is said to be *reflexive* if $(x, x) \in R$ for all $x \in X$.
- R is said to be *antisymmetric* if $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$.
- R is said to be *transitive* if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.

Definition 2.1.5. The relation R on a set X is called a *partial order relation* if R is reflexive, antisymmetric and transitive.

Example 2.1.6. Let $X = \mathbb{R}$ and the relation R on X be defined by

$$R = \{(x, y) \in X \times X | y - x \in \mathbb{R}_+\}.$$

Then R is a partial order relation on X, and R is denoted by "$\leq$".

Definition 2.1.7. A relation f from the set X to the set Y is called a *function* from X into Y if for all $x \in X$ and $y_1, y_2 \in Y$, $(x, y_1), (x, y_2) \in f \implies y_1 = y_2$. We use the notion $f : X \longrightarrow Y$ whenever f is a function from X into Y.

Definition 2.1.8. Let X be a set. A function $\sim: X \times X \longrightarrow X$ is called a *binary operation* on X. For $(x, y) \in X \times X$, the value of $\sim (x, y)$ can be written as $x \sim y$.

2.2 Fields

Definition 2.2.1. Let $\mathbb{F}$ be a set with binary operations "$\oplus$" (addition) and "$\odot$" (multiplication). The set $\mathbb{F}$ is called a *field* if for every $a, b, c \in \mathbb{F}$, these operations satisfy the following properties:

1. (Associativity of addition)

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c;$$

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2. (Commutativity of addition)

$$a \oplus b = b \oplus a;$$

3. (Additive identity) there exists $\theta_\oplus \in \mathbb{F}$ such that

$$a \oplus \theta_\oplus = a = \theta_\oplus \oplus a;$$

4. (Additive inverses) for each $a \in \mathbb{F}$, there exists $a_\oplus$ such that

$$a \oplus a_\oplus = \theta_\oplus = a_\oplus \oplus a;$$

5. (Associativity of multiplication)

$$a \odot (b \odot c) = (a \odot b) \odot c;$$

6. (Commutativity of multiplication)

$$a \odot b = b \odot a;$$

7. (Multiplicative identity) there exists $\theta_\odot \in \mathbb{F} \setminus \{\theta_\oplus\}$ such that

$$a \odot \theta_\odot = a = \theta_\odot \odot a;$$

8. (Multiplicative inverses) for each $a \in \mathbb{F} \setminus \{\theta_\oplus\}$, there exists $a_\odot$ such that

$$a \odot a_\odot = \theta_\odot = a_\odot \odot a;$$

9. (Distributivity of multiplication over addition)

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c).$$

Example 2.2.2. $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$ and $(\mathbb{C},+,\cdot)$ are fields.

Example 2.2.3. For a prime number p, we obtain $\mathbb{Z}_p$ be a finite field.

2.3 Vector spaces

Definition 2.3.1. Let V be a set and $\mathbb{F}$ be a field. Suppose that $\oplus : V \times V \longrightarrow V$ and $\odot : \mathbb{F} \times V \longrightarrow V$ are mappings, which are called a *vector addition* and a *scalar multiplication*, respectively. For simplicity, we use the notion $x \oplus y$ instead of $\oplus(x, y)$ and $\alpha \odot x$ (or αx) instead of $\odot(\alpha, x)$. The set V is called a *vector space* over a field $\mathbb{F}$, if for every $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$, the following properties hold:

1. (Associativity of vector addition)

$$u \oplus (v \oplus w) = (u \oplus v) \oplus w;$$

2. (Commutativity of vector addition)

$$u \oplus v = v \oplus u;$$

3. (Vector additive identity) there exists $\theta_V \in V$, which is called a *zero vactor*, such that

$$v \oplus \theta_V = v = \theta_V \oplus v;$$

4. (Vector additive inverses) for each $v \in V$, there exists $v_\oplus$ such that

$$v \oplus v_\oplus = \theta_V = v_\oplus \oplus v;$$

5. (Compatibility of scalar multiplication)

$$\alpha \odot (\beta \odot v) = (\alpha \odot \beta) \odot v;$$

6. (Identity element of scalar multiplication)

$$1_{\mathbb{F}} \odot v = v = v \odot 1_{\mathbb{F}},$$

where $1_{\mathbb{F}}$ is a multiplicative identity element in field $\mathbb{F}$;

7. (Distributivity of a scalar multiplication with respect to a vector addition)

$$\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v);$$

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8. (Distributivity of scalar multiplication with respect to field addition)

$$(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v).$$

We can also call $(V, \oplus, \odot)$ a vector space. Each element in V is called a vector.

Remark 2.3.2. *The vector space over a field $\mathbb{R}$ $(\mathbb{C})$ is called a real (complex) vector space.*

Example 2.3.3. Let $n \in \mathbb{N}$. Define mappings $+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$(x_1, x_2, x_3, \ldots, x_n) + (y_1, y_2, y_3, \ldots, y_n) = ((x_1 + y_1), (x_2 + y_2), (x_3 + y_3), \ldots, (x_n + y_n))$$

and

$$\alpha \cdot (x_1, x_2, x_3, \ldots, x_n) = (\alpha x_1, \alpha x_2, \alpha x_3, \ldots, \alpha x_n)$$

for every $(x_1, x_2, x_3, \ldots, x_n), (y_1, y_2, y_3, \ldots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then $(\mathbb{R}^n, +, \cdot)$ is a real vector space.

Definition 2.3.4. Let X be a real vector space. A subset A of X is called an *absorbing set* if for every $a \in X$, there exists $r > 0$ such that $ra \in A$ (see an example in Figure 2.1).

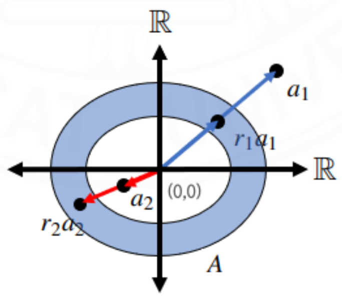

Figure 2.1 An example of an absorbing set A in $\mathbb{R}^2$

Example 2.3.5. Let $(\mathbb{R}, +, \cdot)$ be a real vector space defined in Example 2.3.3.

- Let $A_1 = [-1, 1]$ and $a \in \mathbb{R}$. If $a = 0$, there is $r = 1$ such that $ra = 1 \cdot 0 = 0 \in A_1$. If $a \neq 0$, there exists $r = \frac{1}{|a|} > 0$ such that $ra = \frac{a}{|a|} \in A_1$. Therefore, A_1 is an absorbing set.

- Let $A_2 = \{-y, 0, x\}$ for some $x, y \in \mathbb{R}^+$. Assume that $a \in \mathbb{R}$. If $a = 0$, there is $r = 1$ such that $ra = 1 \cdot 0 = 0 \in A_2$. If $a > 0$, there exists $r = \frac{x}{a} > 0$ such that $ra = \frac{x}{a} \cdot a = x \in A_2$. If $a < 0$, there exists $r = -\frac{y}{a} > 0$ such that $ra = -\frac{y}{a} \cdot a = -y \in A_2$. Therefore, A_2 is an absorbing set.

Definition 2.3.6. Let $(X, +_X, \cdot_X)$ and $(Y, +_Y, \cdot_Y)$ be vector spaces over the same field $\mathbb{F}$. A mapping $T : X \longrightarrow Y$ is called a *linear mapping* if

$$T(\alpha \cdot_X x +_X y) = \alpha \cdot_Y Tx +_Y Ty$$

for all $x, y \in X$ and $\alpha \in \mathbb{F}$.

2.4 Convexity

Definition 2.4.1. Let $(X, +, \cdot)$ be a real vector space and $x, y \in X$.

1. A set $R := \{x + \alpha y | \alpha \geq 0\}$ is called a *ray*.
2. A set $L := \{x + \alpha y | \alpha \in \mathbb{R}\}$ is called a *line*.
3. A *line passing through x and y* is defined to be the following set

$$\{(1-\alpha)x + \alpha y | \alpha \in \mathbb{R}\}.$$

4. A *closed line segment joining x and y*, denoted by $[x, y]$, is defined by

$$[x, y] = \{(1-\alpha)x + \alpha y | \alpha \in [0, 1]\}.$$

A point $(1-\alpha)x + \alpha y$, where $\alpha \in [0, 1]$, is called a *convex combination* of x and y.

5. An *open line segment joining x and y*, denoted by (x, y), is defined by

$$(x, y) = \{(1-\alpha)x + \alpha y | \alpha \in (0, 1)\}.$$

Definition 2.4.2. Let $(X, +, \cdot)$ be a real vector space. A subset A of X is called *convex* if for every distinct vectors $x, y \in A$, the convex combination of x and y contains in A, that is,

$$(1-\alpha)x + \alpha y \in A$$

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for every $\alpha \in [0,1]$. This definition indicates, in geometry, that for any two points $x, y \in A$, the closed line segment joining x and y is entirely contained in A (see in Figure 2.2).

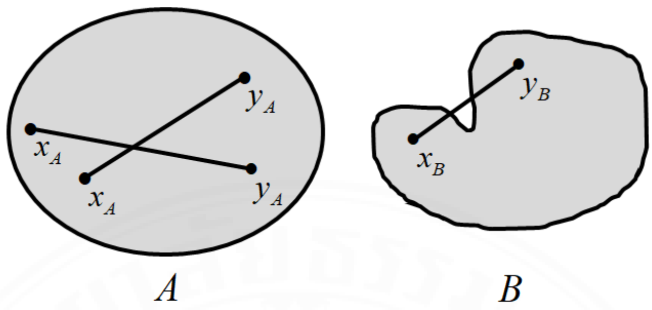

Figure 2.2 An example of a convex set and a non-convex set in $\mathbb{R}^2$

Definition 2.4.3. Let $(X,+,\cdot)$ be a real vector space and A be a subset of X. The set

$$\operatorname{conv}(A) := \bigcap \{B \mid A \subseteq B \text{ and } B \text{ is convex set}\}$$

is called a *convex hull* of A (see an example in Figure 2.3).

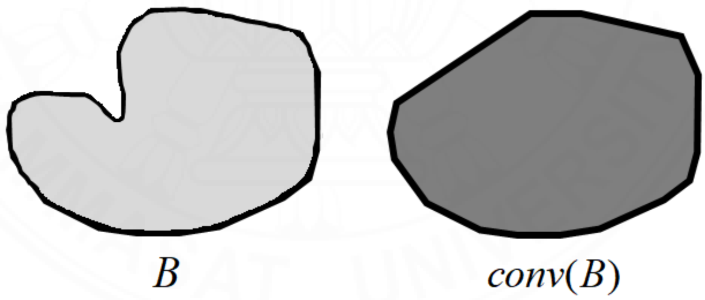

Figure 2.3 An example of a convex hull of a non-convex set B in $\mathbb{R}^2$

Example 2.4.4. Let $X = \mathbb{R}$ be a real vector space with operators $+, \cdot$ on $\mathbb{R}$ and $A = [0,1]$. Suppose that $x, y \in A$ and $\alpha \in [0,1]$. Since $x, y \in A$ and $\alpha, (1-\alpha) \geq 0$, we get $(1-\alpha)x + \alpha y \geq 0$. Without loss of generality, we may assume that $x \leq y$, we obtain

$$(1-\alpha)x + \alpha y \leq (1-\alpha)y + \alpha y = y \leq 1.$$

Therefore, A is a convex set.

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Example 2.4.5. Let $X = \mathbb{R}$ be a real vector space. By using the same technique in Example 2.4.4, it is easy to show that for all $x, y \in \mathbb{R}$, the open interval (x, y) and closed interval $[x, y]$ in $\mathbb{R}$ are convex sets. Moreover, $\operatorname{conv}((x, y)) = (x, y)$ and $\operatorname{conv}([x, y]) = [x, y]$.

Example 2.4.6. Let $(\mathbb{R}^2, +, \cdot)$ be a real vector space defined in Example 2.3.3. For each $a_1, a_2, b_1, b_2 \in \mathbb{R}$, the following sets are convex sets:

- $A_1 = [a_1, b_1] \times [a_2, b_2]$;
- $A_2 = (a_1, b_1) \times (a_2, b_2)$;
- $A_3 = [a_1, b_1] \times (a_2, b_2)$;
- $A_4 = \{(x, y) | x + y < 1 \text{ and } x, y \geq 0\}$.

Moreover, we get $\operatorname{conv}(A_i) = A_i$ for all $i \in \{1, 2, 3, 4\}$.

Definition 2.4.7. Let X be a real vector space and A a convex subset of X. An element $x \in A$ is called an *extreme point* of A, if the following condition holds: if $x = (1 - \alpha)y + \alpha z$ for some $\alpha \in [0, 1]$ and $y, z \in A$, then $x = y = z$ (see examples of extreme points in Figures 2.4 and 2.5). The set of all extreme points of A is denoted by $\operatorname{Ext}(A)$.

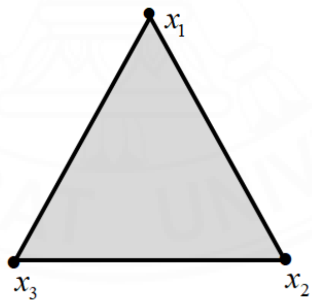

Figure 2.4 $x_1, x_2, x_3 \in \mathbb{R}^2$ are extreme points of a convex set in $\mathbb{R}^2$

Example 2.4.8. From Example 2.4.4, we get $\operatorname{Ext}(A) = \{0, 1\}$. Form Example 2.4.6, we get $\operatorname{Ext}(A_1) = \{(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)\}$, $\operatorname{Ext}(A_2) = \operatorname{Ext}(A_3) = \emptyset$ and $\operatorname{Ext}(A_4) = \{(0, 0)\}$. Moreover, if $A^* = [x, y]$ for some $x, y \in \mathbb{R}$, we get $\operatorname{Ext}(A^*) = \{x, y\}$.

Definition 2.4.9. Let X be a real vector space and A be a convex subset of X. A non zero vector $d \in X$ is called a *direction* of A if for each $x \in A$ and $\alpha \geq 0$, $x + \alpha d \in A$. Two directions d_1 and d_2 of A are called *distinct* if $d_1 \neq \alpha d_2$ for some $\alpha > 0$.

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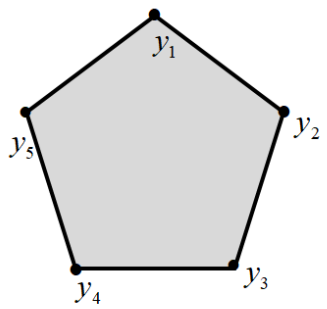

Figure 2.5 $y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}^2$ are extreme points of a convex set in $\mathbb{R}^2$

Definition 2.4.10. Let X be a real vector space and A be a convex subset of X. A direction of A is called an *extreme direction* of A if d cannot be expressed as a positive combination of two distinct directions. If d is an extreme direction of A, then $R = \{x+\alpha d | \alpha \geq 0\}$, where $x \in A$ is fixed, is called an *extreme ray* of A. The set of all points on all extreme rays of A is denoted by $\mathrm{Extr}(A)$.

Example 2.4.11. Consider a real vector space $(\mathbb{R}^2, +, \cdot)$ defined in Example 2.3.3, and a convex subset $A = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$ of $\mathbb{R}^2$. Let $(a, b) \in A$ be fixed. It is easy to see that

- $R_1 = \{(a, b) + \alpha_1(0, 1) | \alpha_1 \geq 0\} \subseteq A$,
- $R_2 = \{(a, b) + \alpha_2(1, 1) | \alpha_2 \geq 0\} \subseteq A$,
- $R_3 = \{(a, b) + \alpha_3(1, 0) | \alpha_3 \geq 0\} \subseteq A$.

Thus, $(0, 1), (1, 1)$ and $(1, 0)$ are directions of A, and $(0, 1)$ and $(1, 0)$ are extreme directions. Moreover,

$$\mathrm{Extr}(A) = \{(0, y) | y \geq 0\} \cup \{(x, 0) | x \geq 0\}.$$

2.5 Normed spaces

Definition 2.5.1. Let X be a vector space over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A mapping $\|\cdot\| : X \longrightarrow \mathbb{R}$ is called a *norm* on X if it satisfies the following conditions for all $x, y \in X$ and $\alpha \in \mathbb{F}$:

1. $\|x\| = 0 \Longleftrightarrow x = \theta_X$;
2. $\|\alpha x\| = |\alpha| \, \|x\|$;

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3. $\|x + y\| \leq \|x\| + \|y\|$.

The ordered pair $(X, \|\cdot\|)$ is called a *normed space*.

Remark 2.5.2. *From Definition 2.5.1, it easy to see that* $\|x\| \geq 0$.

Example 2.5.3. Let $X = \mathbb{R}$ be a real vector space. Define a mapping $|\cdot| : X \longrightarrow \mathbb{R}$ by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Then $(\mathbb{R}, |\cdot|)$ is a normed space, and $|\cdot|$ is called an absolute-value norm.

Example 2.5.4. Let $X = \mathbb{R}^n$ be a real vector space. Define a mapping $\|\cdot\|_2 : X \longrightarrow \mathbb{R}$ by

$$\|(x_1, x_2, x_3, \ldots, x_n)\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2},$$

for all $(x_1, x_2, x_3, \ldots, x_n) \in X$. Then $(\mathbb{R}^n, \|\cdot\|_2)$ is a normed space, and $\|\cdot\|_2$ is called a Euclidean norm.

Example 2.5.5. Let $X = \mathbb{R}^n$ be a real vector space. Define a mapping $\|\cdot\|_\infty : X \longrightarrow \mathbb{R}$ by

$$\|(x_1, x_2, x_3, \ldots, x_n)\|_\infty = \sup\{|x_1|, |x_2|, |x_3|, \ldots, |x_n|\},$$

for all $(x_1, x_2, x_3, \ldots, x_n) \in X$. Then $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a normed space.

Example 2.5.6. Let $X = C_{\mathbb{R}}[0, 1]$ be a set of all continuous mappings $f : [0, 1] \longrightarrow \mathbb{R}$. Define a mapping $\|\cdot\|_\infty : X \longrightarrow \mathbb{R}$ by

$$\|f\|_\infty = \sup\{|f(x)||x \in [0, 1]\}$$

for all $f \in X$. Then $(C_{\mathbb{R}}[0, 1], \|\cdot\|_\infty)$ is a normed space (see in Figure 2.6).

Definition 2.5.7. Let $(X, \|\cdot\|)$ be a normed space and (x_n) a sequence in X. A point $x \in X$ is called a *limit* of the sequence (x_n), if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\|x_n - x\| < \epsilon.$$

Moreover, the sequence (x_n) is said to converge to x, which is denoted by $\lim_{n\to\infty} x_n = x$.

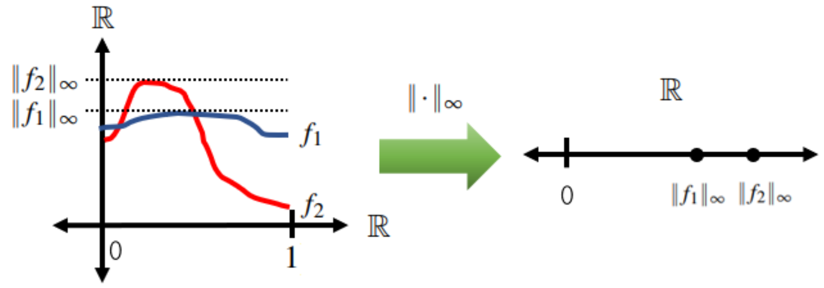

Figure 2.6 Examples of $\|f_1\|_\infty$ and $\|f_2\|_\infty$, where $f_1, f_2 \in X$ in Example 2.5.6

Definition 2.5.8. A sequence (x_n) in a normed space $(X, \|\cdot\|)$ is called a *Cauchy sequence* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n, m \geq N$,

$$\|x_n - x_m\| < \epsilon.$$

Definition 2.5.9. A normed space X is said to be *complete* if every Cauchy sequence in X converges to an element in X. A complete normed space is called a *Banach space*.

Definition 2.5.10. Let $(X, \|\cdot\|)$ be a normed space and $x \in X$. For each $\epsilon > 0$, the *open ball* of radius ϵ with centered at x, denoted by $B(x, \epsilon)$, is defined by

$$B(x, \epsilon) := \{y \in X \mid \|y - x\| < \epsilon\},$$

and the *closed ball* of radius ϵ with centered at x, denoted by $B[x, \epsilon]$, is defined by

$$B[x, \epsilon] := \{y \in X \mid \|y - x\| \leq \epsilon\}.$$

Definition 2.5.11. Let $(X, \|\cdot\|)$ be a normed space and A a subset of X. A point $x \in A$ is called an *interior point* of A if there exists $\epsilon > 0$ such that

$$B(x, \epsilon) \subseteq A.$$

The set of all interior points of A is denoted by Int$A$.

2.6 Asymmetric normed spaces

In the definition of normed spaces, if a field is $\mathbb{R}$, we can see that all scalars in the whole set $\mathbb{R}$ are considered. In the next definition, this situation is weeks to half set. It means that we will consider only the nonnegative scalars in the definition. This becomes the new definition as follows:

Definition 2.6.1. Let X be a real vector space. A mapping $q : X \to \mathbb{R}$ is called an *asymmetric norm* on X if it satisfies the following conditions for all $x, y \in X$ and $\alpha \geq 0$:

1. $q(x) \geq 0$;
2. $q(x) = q(-x) = 0 \Leftrightarrow x = 0$;
3. $q(\alpha x) = \alpha q(x)$;
4. $q(x + y) \leq q(x) + q(y)$.

The ordered pair (X, q) is called an *asymmetric normed space*.

Example 2.6.2. Let $X = \mathbb{R}$ be a real vector space. Define a mapping $q : X \longrightarrow \mathbb{R}$ by

$$q(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then $(\mathbb{R}, q)$ is an asymmetric normed space.

Proposition 2.6.3. *Let (X, q) be an asymmetric normed space and a mapping $q^s : X \longrightarrow \mathbb{R}$ be defined by*

$$q^s(x) = \max\{q(x), q(-x)\}$$

for all $x \in X$. We obtain (X, q^s) is a normed space.

Definition 2.6.4. Let (X, q) be an asymmetric normed space and $x \in X$. For every $\epsilon > 0$, the *q-open ball* of radius ϵ with centered at x is denote by $B_q(x, \epsilon)$, is defined by

$$B_q(x, \epsilon) := \{y \in X | q(y - x) < \epsilon\},$$

and the *q-closed ball* of radius ϵ with centered at x is denoted by $B_q[x, \epsilon]$, is defined by

$$B_q[x, \epsilon] := \{y \in X | q(y - x) \leq \epsilon\}.$$

For an asymmetric normed space (X, q), the family of open balls $B_q(x, \epsilon)$ for all $\epsilon > 0$ is a base of neighborhoods of the point x with respect to the topology τ_q on X generated by an asymmetric norm q.

Example 2.6.5. From Example 2.6.2, for $x \in \mathbb{R}$ and $\epsilon > 0$, we get $B_q(x, \epsilon) = \{y | y < x + \epsilon\} = (-\infty, x + \epsilon)$. In the same way, $B_q[x, \epsilon] = (-\infty, x + \epsilon]$.

Definition 2.6.6. Let A be a subset of an asymmetric normed space (X, q).

1. A set A is called a *q-open set* if for each $x \in A$, there exists $\epsilon > 0$ such that $B_q(x, \epsilon) \subseteq A$.

2. A set A is called a *q-closed set* if the complement of A is q-open.

Definition 2.6.7. Let A be a subset of an asymmetric normed space (X, q). The family $C \subseteq \tau_q$ is called an *open cover* of a set A if A is contained in the union of C.

Definition 2.6.8. Let (X, q) be an asymmetric normed space. A set $K \subseteq X$ is said to be *q-compact* if every open covers of K has a finite subcover.

2.7 Cones

Definition 2.7.1. Let $(E, \|\cdot\|_E)$ be a real Banach space. A set $P \subset E$ is called a *cone* if it satisfies the following conditions:

1. P is closed, $P \neq \{\theta_E\}$ and $P \neq \emptyset$;

2. $\alpha, \beta \in \mathbb{R}_+$ and $u, v \in P \Longrightarrow \alpha u + \beta v \in P$;

3. $P \cap (-P) = \{\theta_E\}$, that is, $u \in P$ and $-u \in P \Longrightarrow u = \theta_E$.

Example 2.7.2. From Example 2.5.4, let $(E, \|\cdot\|_E) = (\mathbb{R}^2, \|\cdot\|_2)$ be a real Banach space. We obtain $P = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$ is a cone in E.

For a given cone P on a real Banach space $(E, \|\cdot\|_E)$, we can define a partial order $\preceq_P$ with respect to P by $u \preceq_P v$ if $v - u \in P$. We also write $u \prec v$ to indicate that $u \preceq_P v$ but $u \neq v$ and $u \prec\prec_P v$ to indicate that for $v - u \in \text{Int}P$. In this study, the notations $\preceq$, $\prec$, $\prec\prec$ are used for the respective cone.

Next, we give two necessary lemmas which are helpful for the proof of main results in the next chapter.

Lemma 2.7.3 (Turkoglu and Abuloh (2010)). *Let P be a cone on a real Banach space $(E, \|\cdot\|_E)$. Then for each $c \in E$ with $\theta_E \prec\prec c$, there exists $\delta > 0$ such that $u \prec\prec c$ whenever $u \in E$ with $\|u\|_E < \delta$.*

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Lemma 2.7.4 (Turkoglu and Abuloh (2010))**.** *Let P be a cone on a real Banach space $(E, \|\cdot\|_E)$. Then for each $c_1, c_2 \in E$ with $\theta_E << c_1$ and $\theta_E << c_2$, there exists $c \in E$ with $\theta_E << c$ such that $c << c_1$ and $c << c_2$.*

Lemma 2.7.5. *Let P be a cone on a real Banach space $(E, \|\cdot\|_E)$. If $x \in IntP$, then $\frac{x}{n} \in IntP$ for every $n \in \mathbb{N}$.*

Proof. Suppose that $x \in \text{Int}P$. Then there exists $\epsilon > 0$ such that $B_{\|\cdot\|_E}(x, \epsilon) \subseteq P$. Let $n \in \mathbb{N}$. Consider $y \in B_{\|\cdot\|_E}(\frac{x}{n}, \frac{\epsilon}{n})$. Then $\|y - \frac{x}{n}\|_E < \frac{\epsilon}{n}$, that is,

$$\|ny - x\|_E = n\|y - \frac{x}{n}\|_E < n\left(\frac{\epsilon}{n}\right) = \epsilon.$$

Thus $ny \in B_{\|\cdot\|_E}(x, \epsilon) \subseteq P$ and so $y = \left(\frac{1}{n}\right) ny \in P$. This means that $B_{\|\cdot\|_E}(\frac{x}{n}, \frac{\epsilon}{n}) \subseteq P$. Therefore, $\frac{x}{n} \in \text{Int}P$ for every $n \in \mathbb{N}$. □

Lemma 2.7.6. *Let P be a cone on a real Banach space $(E, \|\cdot\|_E)$ and $y \in P$. If $x \in IntP$, then $x + y \in IntP$.*

Proof. Suppose that $x \in \text{Int}P$. Then there exists $\epsilon > 0$ such that $B_{\|\cdot\|_E}(x, \epsilon) \subseteq P$. Consider $z \in B_{\|\cdot\|_E}(x + y, \epsilon)$. Then $\|z - (x + y)\|_E < \epsilon$, that is, $\|(z - y) - x\| < \epsilon$. Thus, $(z - y) \in B_{\|\cdot\|_E}(x, \epsilon) \subseteq P$ and so $z = (z - y) + y \in P$. This means that $B_{\|\cdot\|_E}(x + y, \epsilon) \subseteq P$. Therefore, $x + y \in \text{Int}P$. □

Definition 2.7.7. Let P be a cone on a real Banach space $(E, \|\cdot\|_E)$.

1. The cone P is said to be *normal* if there exists a constant $k > 0$ such that $x, y \in E$ and $\theta_E \preceq x \preceq y$ implies that $\|x\|_E \leq k\|y\|_E$. The least positive number satisfying the last inequality is called the *normal constant* of P. Equivalently, the cone P is normal if the sandwich theorem holds, i.e., if (x_n), (y_n) and (z_n) are sequences in E such that

$$x_n \preceq y_n \preceq z_n \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x, \text{ then } \lim_{n\to\infty} y_n = x.$$

2. If the cone P is normal with the normal constant 1, i.e., $x, y \in E$ and $\theta_E \preceq x \preceq y$ implies that $\|x\|_E \leq \|y\|_E$, then P is said to be *monotone*.

3. The cone P is said to be *regular* if every increasing sequence which is bounded from above is convergent, that is, if (x_n) is a sequence in E such that

$$x_1 \preceq x_2 \preceq x_2 \preceq \cdots \preceq x_n \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} x_n = x$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent (see in (Rezapour and Hamlbarani (2008))).

4. The cone P is said to be *minihedral* if $\sup\{x, y\}$ exists (or equivalently $\inf\{x, y\}$ exists) for all $x, y \in E$.

5. The cone P is said to be *strongly minihedral* if every subset of E which is bounded above has a supremum.

6. The cone P is said to be *solid* if $\text{Int}P \neq \emptyset$.

Lemma 2.7.8 (Rezapour and Hamlbarani (2008); Karapınar (2010)). *1. A regular cone is a normal cone (see in (Rezapour and Hamlbarani (2008))).*

2. Every strongly minihedral normal (not necessarily closed) cone is regular (see in (Karapınar (2010))).

3. Every strongly minihedral closed cone is normal (see in (Karapınar (2010))).

From the past to the present, there are numerous examples of several types of cones. Moreover, a lot of spaces which are important in theory of functional analysis have cones which are normal and non-solid. The following example shows that there exists a cone which is non-normal and solid.

Example 2.7.9 (Vandergraft (1967)). Let $E = C^1_{\mathbb{R}}[0,1]$ be a set of all continuously differentiable functions $x : [0,1] \longrightarrow \mathbb{R}$. Define a norm $\|\cdot\|_E$ on E for each $x \in E$ by

$$\|x\|_E = \|x\|_\infty + \|x^{(1)}\|_\infty,$$

where $x^{(1)}$ is the first derivative of x. It is clear that $(E, \|\cdot\|_E)$ is a real Banach space. Define $P = \{x \in E | x(t) \geq 0 \text{ for all } t \geq 0\}$. Then P is a non-normal cone on E. Indeed, for each $n \in \mathbb{N}$, if $x_n(t) := \frac{t^n}{n}$ and $y_n(t) := \frac{1}{n}$ for all $t \in [0,1]$, then

$$\theta_E \preceq x_n \preceq y_n \text{ for all } n \in \mathbb{N}$$

and $\lim_{n\to\infty} y_n = \theta_E$ but $\lim_{n\to\infty} x_n \neq \theta_E$. This means that the sandwich theorem does not hold and so P is a non-normal cone on E. Furthermore, it is easy to see that P is a solid cone. Therefore, P is a solid non-normal cone.

2.8 Cone normed spaces

Based on the idea of cones on real Banach spaces, we can extend the concept of a normed space by replacing the codomain of its norm by a given real Banach space $(E, \|\cdot\|_E)$, and replacing the usual order in properties of norms by $\preceq_P$, where P is a cone on E.

Definition 2.8.1 (Gordji et al. (2012))**.** Let $\preceq$ be a partial ordering with respect to a solid cone P on a real Banach space $(E, \|\cdot\|_E)$ and X a vector space over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A mapping $\|\cdot\|_P : X \to E$ is called a *cone norm* on X if it satisfies the following conditions for all $x, y \in X$ and $\alpha \in F$:

1. $\theta_E \preceq \|x\|_P$;
2. $\|x\|_P = \theta_E \iff x = \theta_X$;
3. $\|\alpha x\|_P = |\alpha|\, \|x\|_P$;
4. $\|x + y\|_P \preceq \|x\|_P + \|y\|_P$.

The ordered pair $(X, \|\cdot\|_P)$ is called a *cone normed space*.

Example 2.8.2. Let $X = \mathbb{R}$ be a real vector space. Define a Banach space $(E, \|\cdot\|_E)$ and a cone P on E as in Example 2.7.9. Define a mapping $\|\cdot\|_P : X \to E$ by

$$\|x\|_P = f_x, \text{ where } f_x \text{ is define by } f_x(y) = |x| \text{ for all } y \in [0, 1].$$

Thus, $(\mathbb{R}, \|\cdot\|_P)$ is a cone normed space (see in Figure 2.7).

Example 2.8.3. Let $X = \mathbb{R}^2$ be a real vector space. Define a Banach space with the cone in Example 2.7.2. Define a mapping $\|\cdot\|_P : X \to E$ by

$$\|(x, y)\|_P = (|x|, |y|)$$

for all $(x, y) \in X$. Then $(X, \|\cdot\|_P)$ is a cone normed space (see in Figure 2.8).

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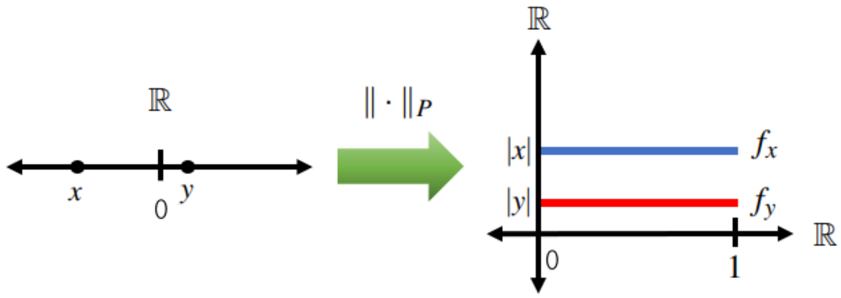

Figure 2.7 An example of $\|x\|_P$ and $\|y\|_P$, where $x, y \in X$ in Example 2.8.2

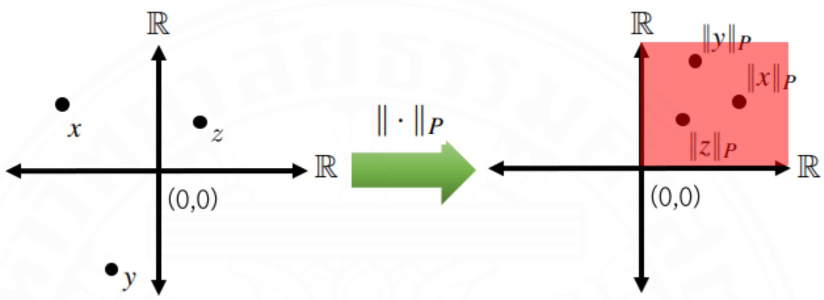

Figure 2.8 An example of $\|x\|_P, \|y\|_P$ and $\|z\|_P$, where $x, y, z \in X$ in Example 2.8.3

Definition 2.8.4. Let $(X, \|\cdot\|_P)$ be an cone normed space and $x \in X$. For every $c \in \text{Int}P$, the $\|\cdot\|_P$-*open ball* of radius c with centered at x, denoted by $B_{\|\cdot\|_P}(x,c)$, is defined by

$$B_{\|\cdot\|_P}(x,c) := \{y \in X | \parallel y - x \parallel_P << c\},$$

and the $\|\cdot\|_P$-*closed ball* of radius c with centered at x, denoted by $B_{\|\cdot\|_P}[x,c]$, is defined by

$$B_{\|\cdot\|_P}[x,c] := \{y \in X | \parallel y - x \parallel_P \preceq c\}.$$

For a cone normed space $(X, \|\cdot\|_P)$, the family of open balls $B_{\|\cdot\|_P}(x,c)$ for all $c \in \text{Int}P$ is a base of neighborhoods of the point x with respect to the topology $\tau_{\|\cdot\|_P}$ on X generated by an asymmetric norm $\|\cdot\|_P$.

Definition 2.8.5. Let A be a subset of a cone normed space $(X, \|\cdot\|_P)$.

1. The set A is called a $\|\cdot\|_P$-*open set* if for each $x \in A$, there exists $c \in \text{Int}P$ such that $B_{\|\cdot\|_P}(x,c) \subseteq A$.

2. The set A is called a $\|\cdot\|_P$-*closed set* if the complement of A is $\|\cdot\|_P$-open.

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Definition 2.8.6. Let A be a subset of a cone normed space $(X, \|\cdot\|_P)$. The set A is called a $\|\cdot\|_P$-*closure set*, if A is a smallest closed subset of A, and it is denoted by cl-$\tau_{\|\cdot\|_P}(A)$.

Definition 2.8.7. A sequence (x_n) in a cone normed space $(X, \|\cdot\|_P)$ is called $\|\cdot\|_P$-*convergent* to $x \in X$ if for every $c \in \text{Int}P$, there exists $n_c \in \mathbb{N}$ such that $\|x_n - x\|_P << c$ for all $n \geq n_c$, and it is denoted by $x_n \overset{\|\cdot\|_P}{\longrightarrow} x$.

Lemma 2.8.8. *Let $(X, \|\cdot\|_P)$ be a cone normed space, $x \in X$ and $c \in IntP$.*

$$y \in cl\text{-}\tau_{\|\cdot\|_P}(B_{\|\cdot\|_P}(x,c)) \iff \exists (z_n) \subseteq B_{\|\cdot\|_P}(x,c), z_n \overset{\|\cdot\|_P}{\longrightarrow} y.$$

Definition 2.8.9. Let $(X, \|\cdot\|_P)$ and $(Y, \|\cdot\|_P)$ be cone normed spaces. A mapping $T : X \longrightarrow Y$ is called a *continuous mapping* if for each $x \in X$ and $c \in \text{Int}P$, there is $t \in \text{Int}P$ such that $\|Tx - Ty\|_P << c$, whenever $y \in X$ and $\|y - x\|_P << t$.

2.9 Asymmetric cone normed spaces

Cone normed spaces play an essential role in various fields such as fixed point theory, optimization theory, control theory, computer science, and some other branches in functional analysis. In the definition of cone normed spaces, if a field is $\mathbb{R}$, we can see that all scalars in the whole set $\mathbb{R}$ are considered. In the next definition, this situation is weeks to half set. It means that we will consider only the nonnegative scalars in the definition. This becomes the new definition as follows:

Definition 2.9.1 (İlkhan (2020))**.** Let $\preceq$ be a partial ordering with respect to a solid cone P on a real Banach space $(E, \|\cdot\|_E)$ and X be a real vector space. A mapping $p_c : X \to E$ is called an *asymmetric cone norm* on X if it satisfies the following conditions for all $x, y \in X$ and $\alpha \geq 0$:

1. $\theta_E \preceq p_c(x)$;

2. $p_c(x) = p_c(-x) = \theta_E \Leftrightarrow x = \theta_X$;

3. $p_c(\alpha x) = \alpha p_c(x)$;

4. $p_c(x + y) \preceq p_c(x) + p_c(y)$.

The ordered pair (X, p_c) is also called an *asymmetric cone normed space*.

Example 2.9.2. Let $X = \mathbb{R}^2$ be a real vector space. Define a Banach space with the cone in Example 2.7.2. Define a mapping $p_c : X \to E$ by

$$p_c(x, y) = (\max\{x, 0\}, \max\{y, 0\})$$

for all $(x, y) \in X$. Then (X, p_c) is an asymmetric cone normed space (see in Figure 2.9).

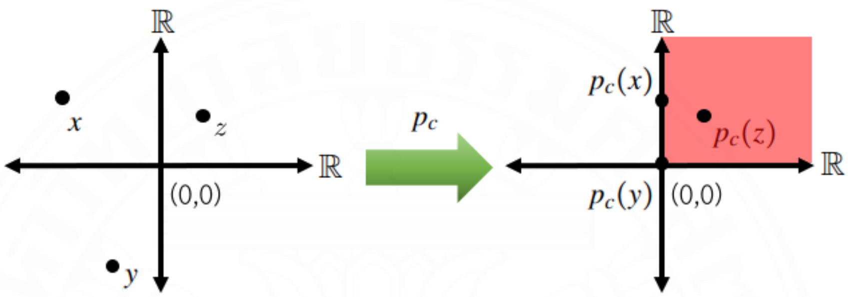

Figure 2.9 An example of $p_c(x), p_c(y)$ and $p_c(z)$, where $x, y, z \in X$ in Example 2.9.2

If (X, p_c) is an asymmetric cone normed space with respect to a minihedral cone P, we can define a cone norm p_c^s from an asymmetric cone norm p_c by

$$p_c^s(x) = \sup\{p_c(x), p_c(-x)\}$$

for every $x \in X$, and so (X, p_c^s) is a cone normed space.

Definition 2.9.3. Let (X, p_c) be an asymmetric cone normed space respect to a cone P and $x \in X$. For every $c \in \text{Int}P$, the *p_c-open ball* of radius c with centered at x, denoted by $B_{p_c}(x, c)$, is defined by

$$B_{p_c}(x, c) := \{y \in X | p_c(y - x) << c\},$$

and the *p_c-closed ball* of radius c with centered at x, denoted by $B_{p_c}(x, c)$, is defined by

$$B_{p_c}[x, c] := \{y \in X | p_c(y - x) \preceq c\}.$$

Example 2.9.4. From Example 2.9.2, the p_c-open ball of radius $(1, 1)$ with centered $(0, 0)$ is

$$B_{p_c}((0, 0), (1, 1)) = \{(x, y) \in \mathbb{R}^2 | x, y < 1\},$$

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and the p_c-closed ball of radius $(1,1)$ with centered $(0,0)$ is

$$B_{p_c}[(0,0),(1,1)] = \{(x,y) \in \mathbb{R}^2 | x,y \leq 1\}$$

(see in Figure 2.10).

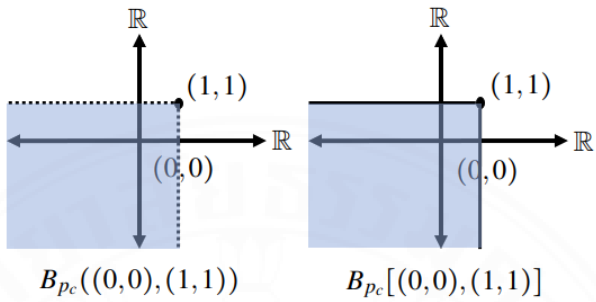

Figure 2.10 An p_c-open ball and a p_c-closed ball in Example 2.9.4

Definition 2.9.5. Let (X, p_c) be an asymmetric cone normed space respect to cone P and A is a subset of X. The smallest p_c-closed containing A is called a p_c-*closure* of A, and it is denoted by $\overline{A}$, that is,

$$\overline{A} = \bigcap \{B \subseteq X | B \text{ is } p_c\text{-closed and } A \subseteq B\}$$

(see in Figure 2.11).

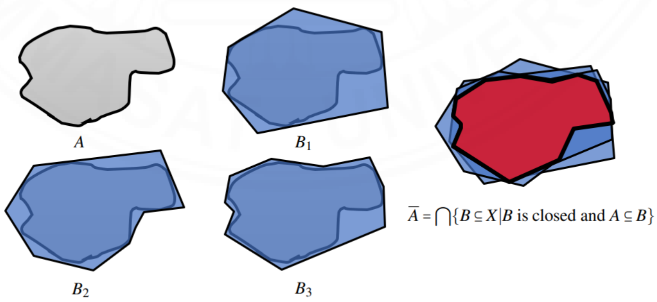

Figure 2.11 A closure of A

Definition 2.9.6. Let (X, p_c) be an asymmetric cone normed space respect to a cone P and A be a subset of X.

1. The family $\mathcal{U}$ of subsets of X is called an *open cover* of A if $\mathcal{U}$ satisfies the following conditions:

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- $U \in \mathcal{U}$ is p_c-open;
- $A \subseteq \bigcup_{U \in \mathcal{U}} U$

(see in Figure 2.12).

2. The set A is called *p_c-compact* if for every open cover of A, there exists a finite subcover $\mathcal{V}$ of $\mathcal{U}$ such that $A \subseteq \bigcup_{V \in \mathcal{V}} V$.

Figure 2.12 An open cover $\mathcal{U}$ of A

2.10 Klee's theorem

In this section, the important theorem that helps to prove the main results are discussed. Before giving the important theorem, the following definitions are needed.

Definition 2.10.1. Let X be a set. A collection τ of subsets of X is called a *topology* on X if it satisfies the following the following conditions:

1. $\emptyset, X \in \tau$;
2. the arbitrary union of members of τ belongs to τ;
3. the finite intersection of members of τ belongs to τ.

The ordered pair (X, τ) is also called a *topological space*. Moreover, the elements of τ are called an *open set* and a subset $C \subseteq X$ is called a *closed set* if the complement of C is open.

Definition 2.10.2. Let (X,τ) be a topology and $S \subseteq X$. A subset $V \subseteq X$ is called a *neighborhood* of S if V includes an open set U containing S, that is,

$$S \subseteq U \subseteq V.$$

Definition 2.10.3. A topology X is said to be *Hausdorff* if any two distinct points $x, y \in X$, there are disjoint neighborhood of x and y (see in Figure 2.13).

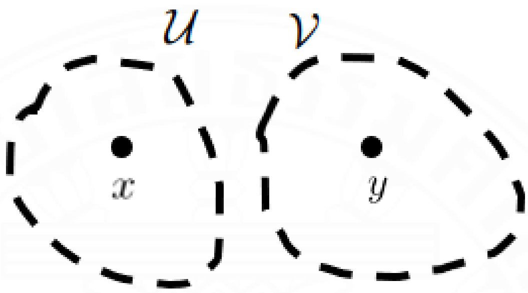

Figure 2.13 A disjoint neighborhood $\mathcal{U}$ of x and a neighborhood $\mathcal{V}$ of y

Definition 2.10.4. A topology X is said to be *regular* if for each closed set $A \subseteq X$ and a point $x \in X$ with $x \notin A$, there are a neighborhoods $\mathcal{U}$ of A and a neighborhood $\mathcal{V}$ of x such that $\mathcal{U} \cap \mathcal{V} = \emptyset$ (see in Figure 2.14).

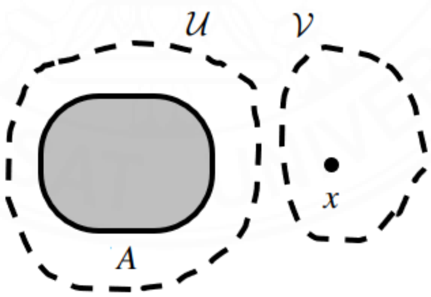

Figure 2.14 A disjoint neighborhood $\mathcal{U}$ of A and a neighborhood $\mathcal{V}$ of x

In 1940, Krein-Milman (Krein and Milman (1940)) presented the theory of extremes which is stated that “If a space X is regular, then any bounded convex closed set is the convex closed envelope of the set of its extreme points”.

Definition 2.10.5. Let X be a topological space. A set $A \subseteq X$ is called *locally compact* if for each $x \in A$ there is a compact neighborhood of x.

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Next, we give the statement of the Klee's theorem, which is an extension of the Krein-Milman theorem for locally compact closed convex subset of a locally convex vector space as follows:

Theorem 2.10.6 (Klee's theorem in (Klee (1957))). *Let C be a locally compact closed convex subset of a locally convex (Hausdorff) vector space. If C contains no line, then*

$$C = \overline{conv(Ext(C) \cup Extr(C))}.$$

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CHAPTER 3
THE SUFFICIENT CONDITION FOR THE EXISTENCE OF EXTREME POINTS

The aim of this chapter is to investigate the sufficient condition for the existence of extreme points of nonempty compact convex subsets of asymmetric cone normed spaces.

3.1 Main results in asymmetric cone normed spaces

Before showing this result, several propositions and lemmas are proved. Throughout this thesis, for each x in an asymmetric cone normed space (X, p_c), we use the notion $\varphi(x)$ as the set of all elements $y \in X$ such that $p_c(y - x) = \theta_E$, that is,

$$\varphi(x) = \{y \in X | p_c(y - x) = \theta_E\}.$$

Proposition 3.1.1. *Let (X, p_c) be an asymmetric cone normed space and $U, K \subseteq X$. The following statements are true:*

1. *If U is an p_c-open set, then $U = U + \varphi(\theta_X)$.*
2. *K is p_c-compact if and only if $K + \varphi(\theta_X)$ is p_c-compact.*

Proof. (1) Since $\theta_X \in \varphi(\theta_X)$, $U = U + \theta_X \subseteq U + \varphi(\theta_X)$. It suffices to prove that $U + \varphi(\theta_X) \subseteq U$. Let $x \in U + \varphi(\theta_X)$, that is, $x = u + t$, where $u \in U$ and $t \in \varphi(\theta_X)$. Since U is p_c-open, there exists an open ball $B_{p_c}(u, c)$ such that $B_{p_c}(u, c) \subseteq U$, where $c \in E$ with $\theta_E << c$. It follows form

$$p_c(x - u) = p_c(t) = \theta_E << c$$

that $x \in B_{p_c}(u, c) \subseteq U$ and so $U + \varphi(\theta_X) \subseteq U$. Therefore, $U = U + \varphi(\theta_X)$.

(2) For the first implication, suppose that K is a compact subset of (X, p_c) and $\mathcal{U}$ is an open cover of $K + \varphi(\theta_X)$. Since $K \subseteq K + \varphi(\theta_X)$, we obtain $\mathcal{U}$ is also an open

cover of K. Then there exists a finite subcover $\mathcal{V}$ of $\mathcal{U}$ with it covers K, that is, $K \subseteq \bigcup_{U\in\mathcal{V}} U$. By the assertion 1 of this proposition, we get

$$K + \varphi(\theta_X) \subseteq \bigcup_{U\in\mathcal{V}} U + \varphi(\theta_X) = \bigcup_{U\in\mathcal{V}} U.$$

Thus, $\mathcal{V}$ is a finite subcover of $\mathcal{U}$ with it covers $K+\varphi(\theta_X)$. Therefore, $K+\varphi(\theta_X)$ is a compact subset of (X, p_c). Next, we will show that the converge is true. Let $K + \varphi(\theta_X)$ be a compact subset of (X, p_c) and $\mathcal{U}$ be an open cover of K. Thus, $\Omega := \{U + \varphi(\theta_X) | U \in \mathcal{U}\}$ is an open cover of $K + \varphi(\theta_X)$. Then there exists a finite subcover $\mathcal{V}$ of Ω with it covers $K + \varphi(\theta_X)$, that is, $K + \varphi(\theta_X) \subseteq \bigcup_{V\in\mathcal{V}} V$, where $V = U_V + \varphi(\theta_X)$ for some $U_V \in \mathcal{U}$. By the assertion 1 of this proposition, we obtain

$$K \subseteq K + \varphi(\theta_X) \subseteq \bigcup_{V\in\mathcal{V}} V = \bigcup_{V\in\mathcal{V}} (U_V + \varphi(\theta_X)) = \bigcup_{V\in\mathcal{V}} U_V.$$

Thus $\{U_V | V \in \mathcal{V}\}$ is a finite subcover of $\mathcal{U}$ with it covers K. Therefore, K is a compact subset of (X, p_c).

□

Lemma 3.1.2. *If P is a solid cone on a real Banach space E and $x \in P\backslash(IntP \cup \{\theta_E\})$, then there exists $c \in IntP$ such that $c - x \notin IntP$.*

Proof. Suppose that $c - x \in \text{Int}P$ for all $c \in \text{Int}P$. Choosing $y \in \text{Int}P$, by Lemma 2.7.5, we get $\frac{y}{n} \in \text{Int}P$. By Lemma 2.7.6, we obtain $\frac{y}{n} + \frac{x}{2} \in \text{Int}P$ for all $n \in \mathbb{N}$. Hence,

$$z_n := \frac{y}{n} - \frac{x}{2} = \left(\frac{y}{n} + \frac{x}{2}\right) - x \in \text{Int}P \text{ for all } n \in \mathbb{N}.$$

Then $(z_n) \subseteq \text{Int}P$ convergs to $-\frac{x}{2}$. Since P is closed, we get $-\frac{x}{2} \in P$. This implies that $-x \in P$. Thus, $x = \theta_E$, which is a contradiction. □

Next we will show that for each an asymmetric cone normed space (X, p_c) and $c \in E$ with $\theta_E << c$, $B_{p_c}(\theta_X, c)$ is absorbing.

Lemma 3.1.3. *Let (X, p_c) be an asymmetric cone normed space and $c \in E$ with $\theta_E <<$ c. Then the open ball $B_{p_c}(\theta_X, c)$ is absorbing.*

Proof. Let $x \in X$. By Lemma 2.7.3, there exists $\delta > 0$ such that $u << c$ whenever $u \in E$ with $\|u\|_E < \delta$. Choosing $r = \frac{\delta}{2\|p_c(x)\|_E}$, we obtain

$$\|p_c(rx)\|_E = \|rp_c(x)\|_E = \left\| \left(\frac{\delta}{2\,\|p_c(x)\|_E} \right) p_c(x) \right\|_E = \frac{\delta\,\|p_c(x)\|_E}{2\,\|p_c(x)\|_E} = \frac{\delta}{2} < \delta.$$

Thus, $p_c(rx - \theta_X) = p_c(rx) << c$, that is, $rx \in B_{p_c}(\theta_X, c)$. Therefore, $B_{p_c}(\theta_X, c)$ is absorbing. □

Proposition 3.1.4. *Let (X, p_c) be an asymmetric cone normed space and $c \in E$ with $\theta_E << c$. Then $B_{p_c}(\theta_X, c)$ contains no line.*

Proof. Suppose that there are $x, y \in X$ with $x \neq \theta_X$ such that $L = \{y + tx | t \in \mathbb{R}\} \subseteq B_{p_c}(\theta_X, c)$. We will divide the proof into 2 cases.

Case 1. Let $y = \theta_X$. Then $L = \{tx | t \in \mathbb{R}\}$. In this case, there are 2 subcases.

Case 1.1: Assume that $p_c(l) \neq \theta_E$ for some $l \in L$. Let $x_n = \frac{c}{n} - p_c(l)$, where $n \in \mathbb{N}$. Since $nl \in B_{p_c}(\theta_X, c)$, we obtain $np_c(l) = p_c(nl) << c$, that is, $c - np_c(l) \in \mathrm{Int}P$. By Lemma 2.7.5, $\frac{c}{n} - p_c(l) \in \mathrm{Int}P$ and so (x_n) is a sequence in P. Let $\epsilon > 0$. For $n_c \in \mathbb{N}$ with $n_c > \frac{\|c\|_E}{\epsilon}$, we have

$$\|x_{n_c} - (-p_c(l))\|_E = \left\| \frac{c}{n_c} - p_c(l) + p_c(l) \right\|_E = \left\| \frac{c}{n_c} \right\|_E = \frac{\|c\|_E}{n_c} < \epsilon$$

Since P is closed, we obtain (x_n) converges in P. Thus $-p_c(l) \in P$, that is, $p_c(l) = \theta_E$, which is a contradiction.

Case 1.2: Assume that $p_c(l) = \theta_E$ for every $l \in L$. Then $p_c(l) = p_c(-l) = \theta_E$ for all $l \in L$. Since $x \in L$, we get $x = \theta_X$, which is a contradiction.

Case 2. Let $y \neq \theta_X$. By Lemma 3.1.3, we have $B_{p_c}(\theta_X, c)$ is absorbing and then there exists $r > 0$, such that $r(-y) \in B_{p_c}(\theta_X, c)$, that is, $(-r)y \in B_{p_c}(\theta_X, c)$. Since $B_{p_c}(\theta_X, c)$ is a convex set, we obtain

$$s((-r)y) + (1 - s)(y + tx) \in B_{p_c}(\theta_X, c) \text{ for every } s \in [0, 1] \text{ and } t \in \mathbb{R}.$$

Letting $s = \frac{1}{1+r}$ in the above relation, we obtain

$$\frac{(-r)}{1+r} y + \left(1 - \frac{1}{1+r} \right) (y + tx) = \left(1 - \frac{1}{1+r} \right) tx \in B_{p_c}(\theta_X, c) \text{ for every } t \in \mathbb{R}$$

which contradicts Case 1.

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From all cases, we obtain $B_{p_c}(\theta_X, c)$ contains no line. □

Corollary 3.1.5. *An p_c-open ball in an asymmetric cone normed space (X, p_c) contains no line.*

Proof. Let $x \in X$ and $c \in E$ with $\theta_E << c$. Suppose that $B_{p_c}(x, c)$ contains a line. We will show that $B_{p_c}(x, c) \subseteq B_{p_c}(\theta_X, c')$ for some $c' \in E$ with $\theta_E << c'$. For $y \in B_{p_c}(x, c)$, we obtain $p_c(y-x) << c$, that is, $c - p_c(y-x) \in \text{Int}P$. Since $p_c(y) \leq p_c(y-x) + p_c(x)$, we obtain $p_c(y-x) + p_c(x) - p_c(y) \in P$. For $c' := c + p_c(x)$, by Lemma 2.7.5, we get

$$c' - p_c(y - \theta_X) = (c + p_c(x)) - p_c(y) = (c - p_c(y-x)) + (p_c(y-x) + p_c(x) - p_c(y)) \in \text{Int}P.$$

Thus, $p_c(y - \theta_X) << c'$ and then $y \in B_{p_c}(\theta_X, c')$, that is, $B_{p_c}(x, c) \subseteq B_{p_c}(\theta_X, c')$. Hence, $B_{p_c}(\theta_X, c')$ contains a line, which contradicts Proposition 3.1.4. Therefore, a ball in an asymmetric cone normed space contains no line. □

Corollary 3.1.6. *If K is a p_c-compact subset of an asymmetric cone normed space (X, p_c) and P is a minihedral cone on X, then K contains no line.*

Proof. Assume that K is a compact subset of (X, p_c) and K contains a line. Let $c \in X$ with $\theta_E << c$. Then $\{B_{p_c}(x, c) | x \in K\}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{B_{p_c}(x_i, c) | x_i \in K \text{ and } i = 1, 2, 3, \ldots, n\}$ such that $K \subseteq \bigcup_{i=1}^{n} B_{p_c}(x_i, c)$. From the proof of Corollary 3.1.5, we obtain $B_{p_c}(x_i, c) \subseteq B_{p_c}(\theta_X, c + p_c(x_i))$ for every $x_i \in K$. Since P is minihedral cone, we obtain $c_{\sup} = \sup\{p_c(x_i) | i = 1, 2, 3, \ldots, n\}$ exists. Thus, $K \subseteq \bigcup_{i=1}^{n} B_{p_c}(x_i, c) \subseteq \bigcup_{i=1}^{n} B_{p_c}(\theta_X, c + p_c(x_i)) \subseteq B_{p_c}(\theta_X, c + c_{\sup})$, that is, $B_{p_c}(\theta_X, c + c_{\sup})$ contains a line, which contradicts Proposition 3.1.4. Therefore, K contains no line. □

Proposition 3.1.7. *Let $K \neq \emptyset$ be a p_c-compact set in an asymmetric cone normed space (X, p_c). Then the set $K + \varphi(\theta_X)$ is p_c^s-closed.*

Proof. Assume that (x_m) is a sequence in $K + \varphi(\theta_X)$ and it p_c^s-converges to x. We suppose that x is not an element in $K + \varphi(\theta_X)$. This implies that for every $a \in K$, we obtain $x - a \notin \varphi(\theta_X)$. Thus, $\theta_E \prec p_c(x-a)$, that is, $p_c(x-a) \in \text{Int}P$ or $p_c(x-a) \notin \text{Int}P$. From Lemma 3.1.2, if $p_c(x-a) \notin \text{Int}P$, then we can fine $c_{p_c(x-a)} \in \text{Int}P$ such that

$c_{p_c(x-a)} - p_c(x-a) \notin \text{Int}P$. Letting

$$\delta_a := \begin{cases} \dfrac{p_c(x-a)}{2}, & \text{if } p_c(x-a) \in \text{Int}P. \\ \dfrac{c_{p_c(x-a)}}{2}, & \text{otherwise.} \end{cases}$$

From this defining, we obtain $\delta_a \in \text{Int}P$. Thus, $\{B_{p_c}(a,\delta_a)\}_{a\in K}$ is an open cover of K. By Proposition 3.1.1, it is also an open cover of $K + \varphi(\theta_X)$. Since K is compact, there exists a finite subcover $\{B_{p_c}(a_i,\delta_{a_i})\}_{i=1}^n$ such that $K \subseteq \bigcup\limits_{i=1}^{n} B_{p_c}(a_i,\delta_{a_i})$ and so

$$(x_m) \in K + \varphi(\theta_X) \subseteq \bigcup_{i=1}^{n} B_{p_c}(a_i,\delta_{a_i}) + \varphi(\theta_X) = \bigcup_{i=1}^{n} B_{p_c}(a_i,\delta_{a_i}).$$

Then for each $m \in \mathbb{N}$ there is $i_m \in \{1,2,3,\ldots,n\}$ such that $x_m \in B_{p_c}(a_{i_m},\delta_{a_{i_m}})$, that is,

$$p_c(x_m - a_{i_m}) << \delta_{a_{i_m}}. \tag{3.1}$$

Since (x_m) p_c-converges to x, for each $i = \{1,2,3,\ldots,n\}$, there is $N_i \in \mathbb{N}$ with

$$p_c^s(x - x_m) << \delta_{a_i} \text{ for all } m \geq N_i. \tag{3.2}$$

Let $N := \max\{N_1,N_2,N_3,\ldots,N_n\}$. From (3.2), we obtain

$$p_c^s(x - x_N) << \delta_{a_i} \text{ for all } i = \{1,2,3,\ldots,n\}. \tag{3.3}$$

It follow from (3.1) that there is $i_N \in \{1,2,3,\ldots,n\}$ such that

$$p_c(x_N - a_{i_N}) << \delta_{a_{i_N}}. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\begin{aligned} p_c(x - a_{i_N}) &\leq p_c(x - x_N) + p_c(x_N - a_{i_N}) \\ &\leq p_c^s(x - x_N) + p_c(x_N - a_{i_N}) \\ &<< \delta_{a_{i_N}} + \delta_{a_{i_N}} \\ &= 2\delta_{a_{i_N}}. \end{aligned}$$

From the above relation, if $p_c(x - a_{i_N}) \in \text{Int}P$, we obtain

$$2\delta_{a_{i_N}} = p_c(x - a_{i_N}) << 2\delta_{a_{i_N}}$$

which is a contradiction. Also, if $p_c(x - a_{i_N}) \notin \text{Int}P$, we obtain

$$p_c(x - a_{i_N}) << 2\delta_{a_{i_N}} = 2\left(\frac{c_{p_c(x-a_{i_N})}}{2}\right) = c_{p_c(x-a_{i_N})}$$

which is a contradiction with $c_{p_c(x-a_{i_N})} - p_c(x - a_{i_N}) \notin \text{Int}P$. Therefore, $K + \varphi(\theta_X)$ is p_c^s-closed. □

Proposition 3.1.8. *Let K be a p_c-compact convex subset in an asymmetric cone normed space (X, p_c). Then the set of extreme points of $K + \varphi(\theta_X)$ contained in K.*

Proof. Suppose that $x \in K + \varphi(\theta_X)$ is an extreme point of $K + \varphi(\theta_X)$ with $x \notin K$. We will claim that for every $z \in K$, there exists $c_z \in \text{Int}P$ such that $x \notin B_{p_c}(z, c_z)$. Indeed, let $z \in K$. If $x \in z + \varphi(\theta_X)$ for some $z \in K$ and then there exists $y \in \varphi(\theta_X)\backslash\{\theta_X\}$ such that $x = z + y$. Since $y \in \varphi(\theta_X)$, we get $2y \in \varphi(\theta_X)$, that is $z + 2y \in K + \varphi(\theta_X)$. Now, we can write x as a convex combination of $z, z + 2y \in K + \varphi(\theta_X)$, that is,

$$x = z + y = \frac{1}{2}(z) + \frac{1}{2}(z + 2y),$$

which is a contradiction since x is an extreme point of $K + \varphi(\theta_X)$. Thus, $x \notin z + \varphi(\theta_X)$ for every $z \in K$, that is, $\theta_E \prec p_c(x - z)$. By using the same process with the proof of Proposition 3.1.7, we can define

$$c_z := \begin{cases} \dfrac{p_c(x-z)}{2}, & \text{if } p_c(x-z) \in \text{Int}P. \\ \dfrac{c_{p_c(x-z)}}{2}, & \text{otherwise,} \end{cases}$$

when $c_{p_c(x-z)} \in \text{Int}P$ such that $c_{p_c(x-z)} - p_c(x - z) \notin \text{Int}P$. Thus, $x \notin B_{p_c}(z, c_z)$. By Proposition 3.1.1, we obtain

$$K + \varphi(\theta_X) \subseteq \bigcup_{z\in K} B_{p_c}(z, c_z) + \varphi(\theta_X) = \bigcup_{z\in K} B_{p_c}(z, c_z) \subset X\backslash\{x\},$$

which is a contradiction with $x \in K + \varphi(\theta_X)$. Therefore, x is contained in K. □

Now, the existence of extreme points of compact convex sets in asymmetric cone normed spaces is described by the following core result, which is proved by helping of Propositions 3.1.1, 3.1.7, 3.1.8 and Corollary 3.1.6 together with Klee's theorem (Theorem 2.10.6).

Theorem 3.1.9. *Let $K \neq \emptyset$ be a p_c-compact convex subset in an asymmetric cone normed space (X, p_c) such that $K + \varphi(\theta_X)$ is p_c^s-locally compact. Then K has at least one extreme point.*

Proof. By Proposition 3.1.7, $K + \varphi(\theta_X)$ is p_c^s-closed. By Proposition 3.1.1, $K + \varphi(\theta_X)$ is p_c-compact. By Corollary 3.1.6, $K + \varphi(\theta_X)$ contains no line. Since K is convex, we obtain $K + \varphi(\theta_X)$ is convex. Then it follows from Theorem 2.10.6 that

$$K + \varphi(\theta_X) = \overline{\mathrm{conv}(\mathrm{Ext}(K + \varphi(\theta_X)) \cup \mathrm{Extr}(K + \varphi(\theta_X)))},$$

that is, $K + \varphi(\theta_X)$ is the smallest closed set of smallest convex set of its extreme points and extreme rays. By Proposition 3.1.8, we obtain the set of extreme points of $K + \varphi(\theta_X)$ contained in K. Therefore, K has at least one extreme point. □

3.2 Obtained results in asymmetric normed spaces

Same as the previous section, for each x in an asymmetric normed space (X, q), we use the notion $\varphi^*(x)$ for the set of all elements $y \in X$ such that $q(y - x) = 0$, that is,

$$\varphi^*(x) = \{y \in X | q(y - x) = 0\}.$$

Because any asymmetric normed space is also an asymmetric cone normed space, results in asymmetric normed spaces corresponding to all previous results can be obtained as follows:

Proposition 3.2.1 (Proposition 2.2 in (Jonard-Pérez and Sánchez-Pérez (2016))). *Let (X, q) be an asymmetric normed space.*

1. *For any q-open subset $U \subseteq X$, $U = U + \varphi^*(\theta_X)$.*

2. *A set $K \subseteq X$ is q-compact if and only if $K + \varphi^*(\theta_X)$ is q-compact.*

Corollary 3.2.2 (Corollary 3.5 in (Jonard-Pérez and Sánchez-Pérez (2016))). *If K is a q-compact subset of an asymmetric normed space (X, q), then K contains no line.*

Lemma 3.2.3 (Lemma 3.7 in (Jonard-Pérez and Sánchez-Pérez (2016))). *Let K be a q-compact set in an asymmetric normed space (X, q). Then the set $K + \varphi^*(\theta_X)$ is q^s-closed.*

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Proposition 3.2.4 (Proposition 4.1 in (Jonard-Pérez and Sánchez-Pérez (2016))). *Let K be a q-compact convex subset in an asymmetric normed space (X, q). Then the set of extreme points of $K + \varphi^*(\theta_X)$ contained in K.*

The following result is related to the existence of extreme points of compact convex sets in asymmetric normed spaces, which is derived from Theorem 3.1.9.

Proposition 3.2.5 (Proposition 4.2 in (Jonard-Pérez and Sánchez-Pérez (2016))). *Let $K \neq \emptyset$ be a q-compact convex subset in an asymmetric normed space (X, q) such that $K + \varphi(\theta_X)$ is q^s-locally compact. Then K has at least one extreme point.*

CHAPTER 4
AN ILLUSTRATIVE EXAMPLE AND AN OPEN PROBLEM

In this chapter, we furnish an illustrative example to demonstrate the validity of the hypotheses and degree of utility of Theorem 3.1.9. In addition, an open problem are given to the reader for further study.

4.1 An illustrative example

Example 4.1.1. Let $(E, \|\cdot\|_E)$ be a Euclidean Banach space $(\mathbb{R}^2, \|\cdot\|_E)$. Define a cone P on $(E, \|\cdot\|_E)$ by $P = \mathbb{R}^2_+$. Also, we let $X = \mathbb{R}^2$ and define a mapping $p_c : X \to E$ by

$$p_c((x, y)) = (x^+, y^+)$$

for all $(x, y) \in X$, where $a^+ := \max\{a, 0\}$ for any $a \in \mathbb{R}$. We obtain (X, p_c) is an asymmetric cone normed space with respect to a cone P and

$$\varphi((0,0)) = \{(x, y) \in X | x \le 0 \text{ and } y \le 0\}.$$

Let

$$K = \{(x, y) \in X | -1 \le x \le 1 \text{ and } -1 \le y \le 1\}.$$

It is easy to see that K is a nonempty convex set. Now, we will show that K is a p_c-compact subset in (X, p_c). Suppose that $\mathcal{U}$ is an p_c-open cover of K. Thus, $(1,1) \in U$ for some $U \in \mathcal{U}$. We know that for all $(x, y) \in K$, we have $(x, y) \preceq (1,1)$. This implies that $K \subseteq U$ and so K is p_c-compact.

Finally, we will claim that $K + \varphi((0,0)) = \{(x, y) \in X | x \le 1 \text{ and } y \le 1\}$ is locally p_c^s-compact. For each $(h, k) \in K + \varphi((0,0))$, there is $\epsilon > 0$ such that an open ball

$$B_{p_c^s}((h, k), (\epsilon, \epsilon)) := \{(x, y) \in X | |x - h| < \epsilon \text{ and } |y - k| < \epsilon\}$$

and a closed ball

$$B_{p_c^s}[(h, k), (\epsilon, \epsilon)] := \{(x, y) \in X | |x - h| < \epsilon \text{ and } |y - k| < \epsilon\}$$

satisfying

$$(h, k) \in B_{p_c^s}((h, k), (\epsilon, \epsilon)) \subseteq B_{p_c^s}[(h, k), (\epsilon, \epsilon)]$$

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and $B_{p_c^s}[(a,b),(r_1,r_2)]$ is p_c^s-compact. Therefore, $K + \varphi((0,0))$ is locally p_c^s-compact. By Theorem 3.1.9, K has at least one extreme point. In this case, we have

$$\mathrm{Ext}(K) = \{(1,1),(1,-1),(-1,-1),(-1,-1)\}.$$

4.2 An open problem

In this section, we would like to mention the interesting structure of an asymmetric cone locally convex space which was published in Proceedings of the 26th Annual Meeting in Mathematics. In this research, we define the definition of an asymmetric cone locally convex space by a family of asymmetric cone norms in the same Banach space and investigate the precompactness in this space. Based on the idea of an asymmetric cone locally convex space in such research, we will raise the following question to the reader for further study:

- Can use the idea in Chapter 3 to invent the novel results concerning the existence of extreme points of a given subset in asymmetric cone locally convex spaces?

REFERENCES

Articles

Alegre, C., Ferrando, I., Garcia-Raffi, L.M., & Sanchez-Perez, E.A. (2008). Compactness in asymmetric normed spaces. *Topol. Appl. 155*(6), 527-539.

Cobzas, S. (2012). Functional analysis in asymmetric normed spaces. *Springer Science & Business Media*.

Conradie, J.J., & Mabula, M.D. (2013). Completeness, precompactness and compactness in finite-dimensional asymmetrically normed lattices. *Topol. Appl. 160*, 2012-2024.

Deimling, K. (1985). Extremal Problems In Nonlinear Functional Analysis. *Springer, Berlin, Heidelberg*, 319-377.

Ferrer, J., Gregori, V., & Alegre, C. (1993). Quasi-uniform structures in linear lattices. *Rocky Mountain J. Math. 23*, 877-884.

Garcia-Raffi, L.M., Romaguera, S., & Sanchez-Perez, E.A. (2002). Sequence spaces and asymmetric norms in the theory of computational complexity. *Math. Comput. Model. 36*, 1-11.

Garcia-Raffi, L.M. (2005). Compactness and finite dimension in asymmetric normed linear spaces. *Topol. Appl. 153*, 844-853.

Gordji, M.E., Ramezani, M., Khodaei, H., & Baghani, H. (2012). Cone normed spaces. *Caspian J. Math. Sci. 1*(1), 7-12.

İlkhan, M., Zengin Alp, P., & Kara, E. E. (2018). On the spaces of linear operators acting between asymmetric cone normed spaces. *Mediterr. J. Math. 15*(136), 12 pages.

İlkhan, M. (2020). A study on nonsymmetric cone normed spaces. *J. BAUN Inst. Sci. Technol. 22*(1), 381-393.

Jonard-Pérez, N., & Sánchez-Pérez, E.A. (2016). Compact convex sets in 2-dimensional asymmetric normed lattices. *Quaestiones Mathematicae 39*(1), 73-82.

Jonard-Pérez, N., & Sánchez-Pérez, E.A. (2016). Extreme points and geometric aspects of compact convex sets in asymmetric normed spaces. *Topol. Appl. 203*, 12-21.

Kadelburg, Z., Paunović, L., Radenović, S., & Rad, S. G. (2016). Non-normal cone metric and cone b-metric spaces and fixed point results. *Scientific Publications*

Ref. code: 25646309031067IPM

of the State University of Novi Pazar Series A: Applied Mathematics, Informatics and mechanics, 8(2), 177-186.

Karapınar, E. (2010). Some nonunique fixed point theorems of Ćirić type on cone metric spaces. *Abstract and Applied Analysis Article ID 123094*, 14 pages

Kempner, Y., & Levit, V.E. (2019). Krein-Milman spaces. *Electronic Notes in Discrete Mathematics 80*, 203-213.

Klee Jr., V.L. (1957). Extremal structure of convex sets. *Arch. Math. 8*(3), 234-240.

Krein, M., & Milman, D. (1940). On extreme points of regular convex sets. *Studia Mathematica, 9*, 133-138.

Kurepa, D.R. (1934). Tableaux ramifies d'ensembles. *C. R.Acad. Sci. Paris 198*, 1563-1565.

Long-Guang, H., & Xian, Z. (2007). Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl. 332*, 1468-1476.

Rezapour, S., & Hamlbarani, R. (2008). Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings". *J. Math. Anal. Appl. 345*(2), 719-724.

Turkoglu, D., & Abuloh, M. (2010). Cone metric spaces and fixed point theorems in diametrically contractive mappings. *Acta Math. Sin. (Engl. Ser.) 26*(3), 489-496.

Vandergraft, J.S. (1967). Newton method for convex operators in partially ordered spaces. *SIAM J. Numer. Anal. 4*(3), 406-432.

Wilson, A.W. (1931). On quasi metric spaces. *Am. J. Math. 53*, 675-684.

Ref. code: 25646309031067IPM

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Publications

1. Supakul, W., Sintunavarat, W. *On the precompactness in asymmetric cone locally convex spaces*. Conference proceedings of the 1st International Annual Meeting in Mathematice (2022), 104-114.