A NEW COMPOUNDING LIFE DISTRIBUTION: GAMMA ZERO-TRUNCATED POISSON DISTRIBUTION

BY

AUSAINA NIYOMDECHA

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (STATISTICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY
ACADEMIC YEAR 2023

THAMMASAT UNIVERSITY
FACULTY OF SCIENCE AND TECHNOLOGY

DISSERTATION

BY

AUSAINA NIYOMDECHA

ENTITLED

A NEW COMPOUNDING LIFE DISTRIBUTION: GAMMA ZERO-TRUNCATED POISSON DISTRIBUTION

was approved as partial fulfillment of the requirements for
the degree of Doctor of Philosophy (Statistics)
on December 18, 2023

Chairman ______________________
(Associate Professor Wikanda Phaphan, Ph.D.)

Member and Advisor Patchanok Srisuradetchai
(Associate Professor Patchanok Srisuradetchai, Ph.D.)

Member Wararit Panichkitkosolkul
(Associate Professor Wararit Panichkitkosolkul, Ph.D.)

Member Monthira Duangsaphon
(Assistant Professor Monthira Duangsaphon, Ph.D.)

Member Patarawan Sangnawakij
(Associate Professor Patarawan Sangnawakij, Ph.D.)

Dean ______________________
(Associate Professor Supet Jirakajohnkool, Ph.D.)

Dissertation Title	A NEW COMPOUNDING LIFE DISTRIBUTION: GAMMA ZERO-TRUNCATED POISSON DISTRIBUTION
Author	Ausaina Niyomdecha
Degree	Doctor of Philosophy (Statistics)
Department/Faculty/University	Mathematics and Statistics Faculty of Science and Technology Thammasat University
Dissertation Advisor	Associate Professor Patchanok Srisuradetchai
Academic Year	2023

ABSTRACT

Modeling of lifetimes is an important statistical work in many fields. This study introduces novel continuous three-parameter lifetime distributions known as the gamma zero-truncated Poisson (GZTP) and complementary gamma zero-truncated Poisson (CGZTP) distributions. These distributions compound minimum or maximum value from a set of independent, identical gamma-distributed random variables, with zero-truncated Poisson random variables. Properties of the proposed distributions, including the probability density, cumulative distribution, survival, and hazard functions as well as moments, are studied and mathematically proven. The estimation of unknown parameters is performed using the maximum likelihood method, and its asymptotic properties are analyzed. In addition, confidence intervals are calculated for GZTP and CGZTP parameters by Wald's method. Based on generated samples, Bayes estimates of unknown parameters and credible intervals are further computed. Simulation studies are performed to assess effectiveness of parameter estimates, while real-world cases illustrate useful applications of the proposed distributions.

Keywords: Compounding, Gamma distribution, Zero-truncated Poisson distributions

ACKNOWLEDGEMENTS

Firstly, I would like to express my sincere appreciation and gratitude from the bottom of my heart to my patient and supportive advisor, Associate Professor Dr. Patchanok Srisuradetchai, who has supported me throughout this dissertation.

I am grateful to all the committees, Associate Professor Dr. Wikanda Phaphan, Associate Professor Dr. Wararit Panichkitkosolkul, Assistant Professor Dr. Monthira Duangsaphon, and Associate Professor Dr. Patarawan Sangnawakij for their thoughtful comments and efforts towards improving my dissertation.

I express my sincere thankfulness to the entire staff of the Department of Mathematics and Statistics at Thammasat University for their invaluable support and substantial knowledge in the field of Statistics.

I would like to express my heartfelt gratitude to the Science Achievement Scholarship of Thailand (SAST Scholarship) for financial support. Last but not least, I would like to express my deep appreciation to my parents for supporting and encouraging me in my Ph.D. study. Without their helps, I could never have completed this dissertation.

Ausaina Niyomdecha

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

1.1 Overview

The modeling of lifetimes is an important statistical work in widely fields. The new lifetime distributions have been proposed in many literatures. A compounding of some lifetime model and some discrete distribution is one of the methods for generating lifetime distribution. The most common idea of compound model is that a lifetime of N (discrete random variable) components and the non-negative continuous random variable, X_i, can be denote by $Y = \min\{X_1, X_2, \ldots, X_N\}$ (the minimum of N positive continuous random variables) or $Z = \max\{X_1, X_2, \ldots, X_N\}$ (the maximum of N positive continuous random variables). The distribution of Y can be applied to model the time to a first failure of a system with N protected components. This means that the distribution of Y can be used to model the time to the first failure of a system with N protected parts or the time it takes for a person to get sick again after treatment. On some system, the system will fail if all subsystems fail, so the time to failure of this system with N protected components, can be modeled by the generated distribution of Z.

Several authors have proposed new distributions for the minimum of X_i. Adamidis and Loukas (1998) introduced the exponential-geometric (EG) distribution, which combines the geometric distribution and exponential distribution. This distribution is used to model the time it takes for devices to fail and the time intervals between explosions in coalmines, measured in days. Adamidis et al. (2005) investigated the extended exponential geometric (EEG) distribution. The different estimation procedures for the unknown parameters of the EEG distribution presented by Louzada et al. (2016). The Barreto-Souza et al. (2011) study introduced the Weibull-geometric (WG) distribution with the the minimum compounded function. This distribution can be applied to analyze the fatigue life data of 67 specimens of Alloy T7987. Zakerzadeh and Mahmoudi (2013) introduced a Lindley-geometric (LG) distribution, which is a strong competitor to other distributions used in fitting the data on the waiting times before service of 100 bank customers. Tahmasbi and Rezaei (2008) introduced an exponential-logarithmic (EL) distribution. Ciumara and Preda (2009) proposed a Weibull-logarithmic distribution that generalizes the EL distributions. Furthermore, various new compounds of Poisson distribution and a number of lifetime distributions have been presented in their closed forms. Kus (2007) introduced an Exponential-Poisson (EP) distribution and applied it to analyze the data on successive earthquakes. Barreto-Souza and Silva (2013) and Louzada et al. (2018), respectively, discuss a likelihood ratio test to discriminate between the EP and gamma distributions and various frequentist estimation techniques for the parameters of the EP distribution. In addition, Xu et al. (2016) examined Bayes estimations of the parameter of the EP distribution under some symmetrical and unsymmetrical loss functions. Hemmati et al. (2011) and Lu and Shi (2012) proposed a Weibull-Poisson (WP) and discussed various of its statistical properties along with its reliability features. Alkarni and Oraby (2012) defined the class of Poisson with some lifetime distributions, presented the density, survival, and hazard functions, and gave some of their properties. In their works, they also present some Rayleigh-Poisson and Pareto-Poisson distribution properties. Gui et al. (2014) developed the Lindley-Poisson (LP) distribution and used it to model the time between earthquakes and the length of time guinea pigs lived after being injected with varying amounts of tubercle bacilli.

A different approach that considers the maximum, instead of the minimum function, has been also considered, which is mostly referred to as the complementary distribution. A complementary version of the EG distribution, as suggested by Louzada et al. (2011), can be utilized for analyzing maximum lifetime data. Tojeiro et al. (2014) proposed a complementary Weibull-geometric (CWG) distribution and applied it to the recovering addict's data, the fatigue life for 67 specimen's data, and Serum-reversal time data. Gui et al. (2017) introduced a complementary Lindley-geometric distribution. Lastly, Ismail (2016) proposed a new distribution, the complementary Weibull-Poisson (CWP) distribution, for pricing catastrophic bonds for extreme earthquakes.

The gamma distribution is a frequently used model for lifetime data. However, the gamma distribution gives a monotone hazard function and does not provide a reasonable parametric fit for modeling phenomena with non-monotone hazard functions, such as bathtub or upside-down bathtub hazard functions. In many situations, the hazard function goes through three phases: it first goes up, then stays almost the same, and then goes down. This hazard function, which we shall refer to as upside-down bathtub-shaped, can be discovered through reliability and biological research. Consequently, life-cycle models that exhibit a hazard function with an upside-down bathtub shape are very helpful in survival analysis. This study combines the gamma and zero-truncated Poisson distributions to construct a novel lifetime distribution. This is achieved by using the minimum and maximum functions, resulting in a hazard function that exhibits an upside-down bathtub shape and a bathtub shape, respectively.

In this study, we propose new compound distributions named the gamma zero-truncated Poisson (GZTP) and complementary gamma zero-truncated Poisson (CGZTP) distributions. The rest of this work is organized as follows: In chapter 2, there is a review of some research methodology. In chapters 3 and 4, we define the density function of GZTP and CGZTP, respectively. The properties of distributions are introduced. Their moment-generating functions, quantiles, survival, and hazard rate functions are derived in these chapters. Estimation of the parameters by the maximum likelihood method and inference for a large sample are also discussed, and their estimation performance is evaluated by a simulation study. In Section 5, we introduced Bayesian parameter estimations and their intervals. The comparison of estimation methods is discussed in Chapter 6. An application to real datasets is provided in Chapter 7. Finally, there are some conclusions and recommendations.

1.2 Research objectives

1. To propose new compound distribution, the gamma zero-truncated Poisson (GZTP) distribution.

2. To propose new compound distribution, complementary gamma zero-truncated Poisson (CGZTP) distribution.

3. To evaluate the estimation performance under various parameter settings.

4. To compare the different estimation procedures for the gamma zero-truncated Poisson and complementary gamma zero-truncated Poisson distribution parameters.

1.3 Research advantages

1. To apply the proposed distributions to real lifetime data.
2. To provides an alternative to many existing lifetime distributions.

CHAPTER 2
REVIEW OF METHODOLOGY

2.1 Compound distribution

This approach focuses on the compounding of various discrete distributions, including the geometric, Poisson, logarithmic, binomial, negative-binomial (NB), Conway-MaxwellPoisson (COMP), and power-series with continuous lifetime distribution. Compound distribution are present on the idea that the lifetime of a system consisting of N components, each of which represents a positive continuous random variable, X_i (representing the lifetime of the ith component), can be represented by the non-negative random variable $Y = \min\{X_1, X_2, \ldots, X_N\}$ (representing the minimum of an unknown number of continuous random variables) or $Z = \max\{X_1, X_2, \ldots, X_N\}$ (representing the maximum of an unknown number of continuous random variables), depending on whether the components are arranged in a series or parallel system. In a parallel system, failure occurs only when all subsystems fail, whereas in a series system, the failure of any subsystem results in the failure of the entire system. The distribution of Y can be applied to the series system, such as modelling the time to a first failure of a system with N protected components or the shortest time that disease on the individual return after the treatment. For a parallel system, the time to failure can be modeled by the generated distribution of Z. Some useful references for the readers are Louzada et al. (2012), and Bidram and Alavi (2014).

2.2 Properties of distribution

2.2.1 The cumulative distribution function

The cumulative distribution function (cdf) of a real-valued random variable X is the function given by

$$F_X(x) = P(X \le x),$$

where the right-hand side represents the probability that the random variable X takes on a value less than or equal to x. The probability density function (pdf) of a continuous random variable can be determined from the cdf by differentiating using the fundamental theorem of calculus;

$$f_X(x) = \frac{d}{dx} F_X(x).$$

The cdf of a continuous random variable X can be expressed as the integral of its probability density function as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

A distribution function F of a distribution on $\mathbb{R}$ has the following properties:

1. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.
2. If $x_1 < x_2$ then $F(x_1) < F(x_2)$.
3. $\lim_{h\to 0^+} F(x+h) = F(x)$.
4. $\lim_{h\to 0^+} F(x-h) \equiv F(x^-) = F(x) - P(X = x) = P(X < x)$.

2.2.2 The distribution of the minimum and maximum

Suppose that $X_1, X_2, \ldots, X_N$ is a random sample from a continuous distribution with the probability density function (pdf) f and cumulative distribution function F. We will now derive the pdf for X_1, the minimum value of the sample. The cdf for the minimum is

$$\begin{aligned} F_{X_1}(x) &= P(X_1 \le x) = 1 - P(X_1 > x) \\ &= 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x) \\ &= 1 - P(X_1 > x) P(X_2 > x) \ldots P(X_n > x) \\ &= 1 - \left[P(X_1 > x)\right]^n \\ &= 1 - \left[1 - F(x)\right]^n. \end{aligned}$$

Therefore, the pdf for minimum is

$$f_{X_1}(x) = \frac{d}{dx} F_{X_1}(x) = n\left[1 - F(x)\right]^{n-1} f(x).$$

Consider random sample $X_1, X_2, \ldots, X_N$ from a continuous distribution with pdf f and cdf F. We will now derive the pdf for X_n, the maximum value of the sample. The cdf for the maximum is

$$\begin{aligned} F_{Xn}(x) &= P(X_n \le x) \\ &= P(X_1 \le x, X_2 \le x, \ldots, X_n \le x) \\ &= \left[P(X_1 \le x)\right]^n \\ &= \left[F(x)\right]^n. \end{aligned}$$

Therefore, the pdf for maximum is

$$f_{X_n}(x) = \frac{d}{dx} F_{X_n}(x) = n\left[F(x)\right]^{n-1} f(x).$$

2.2.3 Moment generating function, mean and variance

The moment generating function $M_X(t)$ of a random variable X is defined to be

$$M_X(t) = E\left(e^{tX}\right).$$

When X is a discrete random variable with mass function $p(x)$, we have

$$M_X(t) = \sum_x e^{tx} p(x).$$

When X is an absolutely continuous random variable with density $f(x)$, we have

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Suppose that $M_X(t)$ is finite on $-t_0 < t \le t_0$ for some $t_0 > 0$, $M_X(t) = \sum_{k=0}^{\infty} \frac{E\left[X^k\right]}{k!} t^k$, $t \in (-t_0, t_0)$. The Taylor series for $M_X(t)$ is $M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{d^k}{dt^k} M_X(t)\big|_{t=0}$. By comparing coefficients of t^k in the last two displays above, we get the k^{th} raw moment or the k^{th} moment about zero of a probability density function

$$E\left[X^n\right] = \frac{d^n}{dt^n} M_X(t)\big|_{t=0}.$$

In particular, the mean and variance of the distribution are given, respectively, by

$$E[X] = M'_X(0) \text{ and } Var(X) = M''_X(0) - (M'_X(0))^2.$$

2.2.4 The quantile

Suppose that X is a real-valued random variable with distribution function F. For $r \in (0,1)$, a value of x such that

$$F(x^-) = P(X < x) \le r \text{ and } F(x) = P(X \le x) \ge r$$

is called a quantile of order r for the distribution. The relationship between quantiles and cumulative distribution values is inverse. Nevertheless, the relationship between the distribution function and its ordinary inverse function is more complicated due to the fact that the distribution function, in general, is not one-to-one. A particular quantile selection for each order is useful for a variety of reasons. Defining a generalized inverse of the distribution function F, however, is required. The quantile function F^{-1} of X is defined by

$$F^{-1}(r) = min\{x \in \mathbb{R} : F(x) \ge r\}, r \in (0,1).$$

2.2.5 Survival function and Hazard function

We assume that T is a continuous random variable with probability density function $f(t)$ and cumulative distribution function $F(t)$, giving the probability that the event has occurred by duration t. The survival function of this population is defined as

$$S(t) = P(T \ge t) = 1 - F(t),$$

which gives the probability of being alive just before duration t, or more generally, the probability that the event of interest has not occurred by duration t. Here are some basic properties about $S(t)$

1. $S(0) = 1$ and $S(\infty) = 0$.
2. $S(t)$ is a non-increasing function.

An alternative characterization of the distribution of T is given by the hazard function, or failure rate, defined as

$$H(t)=\lim_{dt\to 0}\frac{P\left(t<T\leq t+dt\middle|T>t\right)}{dt}=\frac{f(t)}{S(t)}.$$

The hazard function describes the intensity of death at the time t given that the individual has already survived past time t. The hazard function or failure rate function can be decreasing or increasing on the parameters. Several parametric lifetime models, including the gamma, Weibull, and truncated normal distributions, exhibit a monotonically increasing or decreasing failure rate. If the $H(t)$ monotonically increases over time, the distribution is considered to have an increasing failure rate (IFR). If $H(t)$ monotonically decreases we obtain a decreasing failure rate (DFR). Nonmonotonic failure rates can be seen in a many of physical phenomena. A common model for human lifespan indicates three phases: an initial phase in which the failure rate decreases, a middle phase during which the failure rate is virtually constant, and a final phase in which the failure rate increases. Such failure rates are usually termed bathtub (BT) shaped. The three-phase condition, in which the failure rate initially increases, then becomes virtually constant, and finally decreases, is the conceptual opposite to the BT failure rate. We will call this failure rate the upside-down bathtub (UBT) shape. In order to figure out what failure rate's shape is, we can determine that by using Glaser (1980)'s theorem as follows:

(a) If $\eta'(t)>0$ for all $t>0$, then the failure rate function is increasing.

(b) If $\eta'(t)<0$ for all $t>0$, then the failure rate function is decreasing.

(c) Suppose there exists $t_0>0$ such that $\eta'(t)<0$ for all $t\in(0,t_0)$, $\eta'(t_0)=0$ and $\eta'(t)>0$ for all $t>t_0$.

(i) If there exist $y_0>0$ such that $g'(y_0)=0$, then the failure rate function is bathtub.

(ii) If there does not exist $y_0>0$ such that $g'(y_0)=0$, then the failure rate function is increasing.

(d) Suppose there exists $t_0 > 0$ such that $\eta'(t) > 0$ for all $t \in (0, t_0)$, $\eta'(t_0) = 0$ and $\eta'(t) < 0$ for all $t > t_0$.

(i) If there exist $y_0 > 0$ such that $g'(y_0) = 0$, then the failure rate function is upside-down bathtub.

(ii) If there does not exist $y_0 > 0$ such that $g'(y_0) = 0$, then the failure rate function is decreasing, where $\eta(t) = -\dfrac{f'(t)}{f(t)}$ and $g'(t) = \int_t^\infty \dfrac{f(y)}{f(t)}\left[\eta(t) - \eta(y)\right] dy$.

2.3 Rejection sampling

The rejection sampling algorithm is a way to simulate random samples from an unknown distribution (called the *target distribution*) by using random samples from a similar, more convenient distribution (called the *proposal distribution).* This proposal distribution $p(y)$ has to have an important property, namely, $p(y)$ has to envelope the target distribution $g(y)$. That means it has to be $cp(y) \geq g(y)$ for all y where $c > 1$ is an appropriate bound on $\dfrac{g(y)}{p(y)}$.

Algorithm

Input

target density , f

proposal density , p

a constant c, which $cp(y) \geq f(y)$

Algorithm

1. Obtain a sample y from density p
2. Obtain a sample u from *uniform(0,1)*
3. Check whether or not $cp(y)u \leq g(y)$
   - If this holds, accept y as a sample drawn from g
   - If not, reject the value of y and return to the sampling
4. Take M iterations to obtain a sample

2.4 Simulated annealing method

For solving optimization problems, the simulated annealing method (SA) algorithm is one of the most popular heuristic methods. Kirkpatrick et al. (1983)

introduced SA by inspiring the annealing procedure of the metal working. The annealing technique refers to the slow cooling of metals after they have been subjected to high heat and provides the optimal molecular arrangements of metal particles where the potential energy of the mass is minimized. In general, the SA algorithm develops in an iterative manner based on a changeable temperature parameter, imitating the annealing process of metals.

A simple optimization technique compares the outputs of the objective functions running with the current and neighboring points in the domain iteratively. If a neighboring point produces a better result than the current one, the neighboring point is saved as the base solution for the following iteration. Otherwise, the algorithm stops the procedure without looking for better results in a larger domain. As a result, the algorithm is sensitive to being trapped in local minima or maxima. Alternatively, the SA algorithm proposes an effective solution to this problem by merging two iterative loops, namely the annealing cooling procedure and the Metropolis criterion. Basic idea behind the Metropolis criterion is to be executed randomly to extra search the neighborhood of the candidate solution.

In the context of the minimization problem, let us establish an objective function $f(x)$ that is associated with the argument set of $x_i = \{x_1, x_2, \ldots, x_n\}$. Therefore, if $f(x_{i+1}) < f(x_i)$, then take x_{i+1} as a new candidate extreme point to check. Otherwise, define $w = \exp\left[-\left(f(x_{i+1}) - f(x_i)\right)/T_c\right]$ where T_c is the current temperature parameter and generate a random number s , such that $0 < s < 1$. Then, accept the x_{i+1} as a new candidate if the relationship of $w > s$ is true; otherwise, reject and return to the previous step, where another s is generated. As a result, the Metropolis criterion allows for the current step to move to some extent even when the objective function's path is convergent via the potential local minimum point. On the other hand, Metropolis algorithm proposes solution for a constant temperature. SA provides an iterative method that uses nested loops to change the temperature parameter and solution point (Erdinc, 2017). The summary of this procedure is illustrated in Figure 2.1

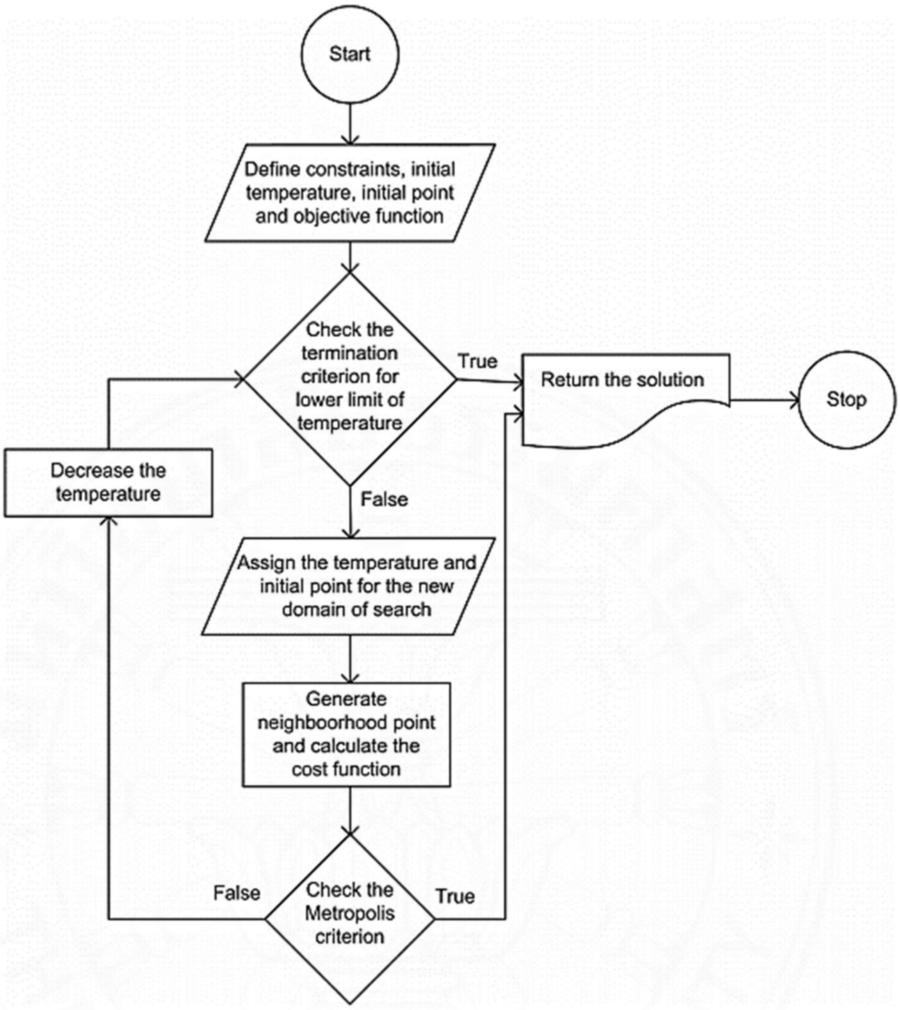

Figure 2.1 Flowchart of SA technique.

In the R program, we can employ the simulated-annealing strategy using either the maxLik or optim function. The candidate point is obtained from a Gaussian Markov kernel by default, where the scale is directly proportional to the current temperature. Simulated annealing, commonly known as "SANN," can be employed to solve stochastic optimization problems when provided with a function that generates a new candidate point. Temperatures are reduced according to

```
temp / log(((t-1) %/% tmax)*tmax + exp(1)),
```

where the current iteration step is denoted as `t`, and the values of `temp` and `tmax` can be specified via `control`. The variable `temp` represents the initial temperature for the cooling program. The default value is 10. `tmax` is the number of function evaluations performed at each temperature for the "SANN" approach. The default value is set to 10.

2.5 Meijer-G function

The Meijer-G function was introduced by Meijer (1936). The Meijer-G function is a very general function which reduces to simpler special functions in many common cases. It is defined by

$$G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,\dots a_p\\ b_1,\dots b_q\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-s\right)}z^s ds,$$

where Γ denote a gamma function. The definition holds under the following assumption:

(a) $0\le m\le q$ and $0\le n\le p$ where m,n,p and q are integer numbers.

(b) $a_k-b_j\ne 1,2,3,\dots$ for $k=1,2,\dots,n$ and $j=1,2,\dots,m$ which implies that no pole of any $\Gamma\left(b_j-s\right)$, $j=1,2,\dots,m$, coincide with any pole of any $\Gamma\left(1-a_k+s\right)$, $k=1,2,\dots,n$.

(c) $z\ne 0$.

Below is a list that demonstrates how the widely known elementary functions can be derived as special cases of the Meijer-G function:

$$e^x=G_{0,1}^{1,0}\left(-z\left|\begin{matrix}-\\ 0\end{matrix}\right.\right),\qquad \forall z$$

$$\cos\left(z\right)=\sqrt{\pi}\,G_{0,2}^{1,0}\left(\frac{z^2}{4}\left|\begin{matrix}-\\ 0,\frac{1}{2}\end{matrix}\right.\right),\qquad \forall z$$

$$\sin\left(z\right)=\sqrt{\pi}\,G_{0,2}^{1,0}\left(\frac{z^2}{4}\left|\begin{matrix}-\\ \frac{1}{2},0\end{matrix}\right.\right),\qquad \frac{-\pi}{2}<\arg z<\frac{\pi}{2}$$

$$\log(1+x)=G_{2,2}^{1,2}\left(z\left|\begin{matrix}1,1\\ 1,0\end{matrix}\right.\right),\qquad \forall z$$

$$\Gamma\left(a,z\right)=G_{0,2}^{2,0}\left(z\left|\begin{matrix}1\\ a,0\end{matrix}\right.\right),\qquad \forall z$$

For the upper incomplete gamma function $\Gamma(a,z)$, the derivative of $\Gamma(a,z) = G_{0,2}^{2,0}\left(z \middle| \begin{matrix} 1 \\ a,0 \end{matrix} \right)$

with respect to its first argument a is given by $\frac{\partial \Gamma(a,z)}{\partial a} = G_{2,3}^{3,0}\left(z \middle| \begin{matrix} 1,1 \\ 0,0,a \end{matrix} \right) + \log(z)\Gamma(a,z)$,

which will be used in the next chapter.

2.6 Likelihood background

In this section, we discuss maximum likelihood estimation, Fisher's information, and Wald statistics by looking at the appropriate likelihood function. Hogg et al. (2005) provide a nice overview of the likelihood function, which is summarized here.

In this section, we discuss the case where $\boldsymbol{\theta}$ is a vector of p parameters. Let $X_1, X_2, ..., X_n$ be iid with common pdf $f(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \Omega \subset \mathbb{R}^p$. The likelihood function and its log are given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i;\boldsymbol{\theta}),$$

$$l(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(x_i;\boldsymbol{\theta}).$$

The maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$, if it exists, is defined to be the value, $\hat{\boldsymbol{\theta}}$ such that

$$\hat{\boldsymbol{\theta}} = \arg\max L(\boldsymbol{\theta}).$$

$L(\boldsymbol{\theta})$ is maximized at the true value of $\boldsymbol{\theta}$. Hence, as an estimate of $\boldsymbol{\theta}$ we consider the value that maximizes $L(\boldsymbol{\theta})$ or equivalently solves the score equation $\frac{\partial}{\partial \boldsymbol{\theta}} l(\boldsymbol{\theta}) = 0$. The symmetric Fisher information matrix $I(\boldsymbol{\theta})$ is the $p \times p$ matrix with entries

$$I_{ij} = -\frac{\partial^2}{\partial \theta_i \partial \theta_j} l(\boldsymbol{\theta}) \quad ,1 \le i, j \le p,$$

and the observed Fisher information matrix is $I(\hat{\boldsymbol{\theta}})$.

The Wald statistic $\left(\hat{\theta}_{ML}-\theta\right)/se\left(\hat{\theta}_{ML}\right)\overset{a}{\sim}N(0,1)$, which is applicable to a single parameter, can be readily extended to multiparameter models by considering a parameter vector $\boldsymbol{\theta}=\left(\theta_1,\theta_2,\ldots,\theta_p\right)^T$. Here we have

$$\frac{\hat{\theta}_i-\theta_i}{se\left(\hat{\theta}_i\right)}\overset{a}{\sim}N(0,1),$$

where the square root of the ith diagonal entry of the inverse observed Fisher information matrix defines the standard error

$$se\left(\hat{\theta}_i\right)=\sqrt{\left[I\left(\hat{\boldsymbol{\theta}}_{ML}\right)^{-1}\right]_{ii}}.$$

This result can be used to calculate the limits of a $(1-\gamma)100\%$ Wald confidence interval for θ_i : $\hat{\theta}_i \pm z_{1-\frac{\gamma}{2}}\sqrt{se\left(\hat{\theta}_i\right)}$.

2.7 Bayesian inference

Let $X=x$ denote the observed realization of a random variable X with density function $f(x\,|\,\theta)$. Specifying a prior distribution with density function $f(\theta)$ allows us to compute the density function $f(\theta\,|\,x)$ of the posterior distribution using Bayes' theorem

$$f(\theta\,|\,x)=\frac{f(x\,|\,\theta)f(\theta)}{\int f(x\,|\,\theta)f(\theta)d\theta}. \tag{2.1}$$

When dealing with discrete parameters θ, the integral in the denominator needs to be replaced with a sum. The term $f(x\,|\,\theta)$ represents the likelihood function $L(\theta)$ previously denoted by $f(x;\theta)$. Given that θ is now random, we can explicitly consider a specific value of θ and express it as $L(\theta)=f(x\,|\,\theta)$. The denominator in (2.1) can be expressed as

$$\int f(x\,|\,\theta)f(\theta)d\theta=\int f(x,\theta)d\theta=f(x).$$

The posterior distribution's density is directly related to the likelihood and the prior distribution's density, with a constant of proportionality $1/f(x)$. It is typically represented as

$$f\left(x|\theta\right) \propto f\left(\theta|x\right) f\left(\theta\right) \text{ or } f\left(x|\theta\right) \propto L\left(\theta\right) f\left(\theta\right).$$

The posterior mean $E(\theta \mid x)$ is the expectation of the posterior distribution:

$$E(\theta \mid x) = \int \theta f(\theta \mid x) d\theta.$$

The posterior mode $Mod(\theta \mid x)$ is the mode of the posterior distribution:

$$Mod(\theta \mid x) = \arg\max_{\theta} f(\theta \mid x).$$

The posterior median $Med(\theta \mid x)$ is a value that can be found by calculating the median of the posterior distribution. It is represented by a number a that satisfies

$$\int_{-\infty}^{a} f(\theta \mid x) d\theta = 0.5 \text{ and } \int_{a}^{\infty} f(\theta \mid x) d\theta = 0.5.$$

Bayesian interval estimates are also calculated based on the posterior distribution. They are referred to as credible intervals to differentiate them from confidence intervals, as they have a distinct interpretation. For a given value of $\gamma \in (0,1)$, a $\gamma \cdot 100\%$ credible interval is determined by two real numbers t_l and t_u that satisfy

$$\int_{t_l}^{t_u} f(\theta \mid x) d\theta = \gamma.$$

The quantity γ is called the credible level of the credible interval $\left[t_l, t_u\right]$.

In multiparameter models with parameter vector $\boldsymbol{\theta}$, Bayesian inference works similarly to that of scalar parameters θ: the posterior density of $\boldsymbol{\theta}$ is obtained by multiplying a multivariate prior density $f(\boldsymbol{\theta})$ by the likelihood and then normalizing the result.

2.8 Metropolis–Hastings algorithm

Bayesian analysis considers all parameters as random and includes prior distributions to represent existing knowledge about parameter values. It then uses the posterior distribution, which is derived from the observed data, to make inferences. Due to the the complexity of the posterior distribution, statisticians have developed simulated methods like the Markov Chain Monte Carlo (MCMC) approach to generate samples from it. Because our posterior distributions do not have an analytic form, we must use the MCMC procedure to get posterior distributions and hence to extract characteristics of parameters such as Bayes estimators and credible intervals. The Metropolis–Hastings algorithm is one of a Markov chain Monte Carlo (MCMC) method for obtaining a sequence of random samples from a probability distribution from which direct sampling is difficult. The details of Metropolis–Hastings algorithm used are given below:

Algorithm

Input

target function , g

proposal distribution , $p: S \times S \to (0,\infty]$

$X_0 \in \{x \in S | g(x) > 0\}$

Define $\alpha : S \times S \to [0,1]$

$$\alpha(x,y) = \min\left(\frac{g(y)p(y,x)}{g(x)p(x,y)},1\right)$$

For all $x, y \in S$ and $g(x)p(x,y) > 0$

Algorithm

For $i = 1,2,3,\ldots$ do

4. draw Y_i from $p(X_{i-1},\cdot)$
5. draw U_i from uniform distribution $U(0,1)$
6. calculate ratio $\alpha(X_{i-1},Y_i)$
7. Check whether or not $U_i \le \alpha(X_{i-1},Y_i)$
   - If this holds, $X_i \leftarrow Y_i$
   - If not, $X_i \leftarrow X_{i-1}$

If the proposal are formed as $Y_i = X_{i-1} + \varepsilon_i$, where $\{\varepsilon_i\}$ is a sequence of independent draw from a known probability distribution. Then the algorithm is called random walk Metropolis. We have that

$$p(Y_i|X_{i-1}) = p(Y_i - X_{i-1}) = p(\varepsilon_i) \text{ and } p(X_{i-1}|Y_i) = p(X_{i-1} - Y_i) = p(-\varepsilon_i).$$

Therefore, the acceptance probability is

$$\begin{aligned}\alpha(x,y) &= \min\left(\frac{g(y_i)p(x_{i-1}|y_i)}{g(x_{i-1})p(y_i|x_{i-1})},1\right)\\ &= \min\left(\frac{g(x_{i-1}+\varepsilon_i)p(-\varepsilon_i)}{g(x_{i-1})p(\varepsilon_i)},1\right).\end{aligned}$$

In the case where the distribution of random walk increment ε_i is symmetric, $p(-\varepsilon_i)=p(\varepsilon_i)$, then the acceptance probability is $\alpha(x,y)=\min\left(\frac{g(x_{i-1}+\varepsilon_i)}{g(x_{i-1})},1\right)$ (Robert and Casella, 2013).

CHAPTER 3
GAMMA ZERO-TRUNCATED POISSON DISTRIBUTION

This chapter presents the derived of the density function of the gamma zero-truncated Poisson distribution and introduces the properties of the distribution. Its quantile, survival function, hazard function, and moment-generating functions are derived in this section. Furthermore, the discussion includes the generation of random numbers for the gamma zero-truncated Poisson distribution. Subsequently, estimation of parameters using maximum likelihood, inference for large samples, and simulation studies are presented.

3.1 Distribution function

Let $X_1, X_2, \ldots, X_N$ represent N random variables that are identically distributed and independent (iid) according to the gamma distribution using a subsequent probability density function (pdf):

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a rate parameter, and $\Gamma(\alpha)$ is a gamma function of α, and N is itself a random variable with a zero- truncated Poisson distribution and independence of X_i 's. The probability mass function of N is

$$P(N = n) = \frac{e^{-\lambda} \lambda^{n}}{n!\left(1 - e^{-\lambda}\right)}, \quad n = 1, 2, \ldots \text{ and } \lambda > 0.$$

Assuming that a random variables X and N are independent, we define $Y = \min\{X_1, X_2, \ldots, X_N\}$. Then $g(y|n) = n\left[1 - F(y)\right]^{n-1} f(y)$, where $f(y)$ is a pdf and $F(y)$ is the cumulative distribution function (cdf) of Y. The joint distribution between Y and N is obtained as follows:

$$g(y,n) = g(y\,|\,n)\,p(N=n)$$
$$= n\left[1-F(y)\right]^{n-1} f(y)\left(\frac{e^{-\lambda}\lambda^{n}}{n!\left(1-e^{-\lambda}\right)}\right)$$
$$= \frac{\lambda e^{-\lambda} f(y)}{\left(1-e^{-\lambda}\right)} \frac{\left[\lambda\left(1-F(y)\right)\right]^{n-1}}{(n-1)!}$$

and the marginal distribution for Y is

$$g(y;\lambda,\alpha,\beta) = \sum_{n=1}^{\infty} g(y,n)$$
$$= \sum_{n=1}^{\infty} \frac{\lambda e^{-\lambda} f(y)}{\left(1-e^{-\lambda}\right)} \frac{\left[\lambda\left(1-F(y)\right)\right]^{n-1}}{(n-1)!}$$
$$= \frac{\lambda e^{-\lambda} f(y)}{\left(1-e^{-\lambda}\right)} \sum_{n=1}^{\infty} \frac{\left[\lambda\left(1-F(y)\right)\right]^{n-1}}{(n-1)!}$$
$$= \frac{\lambda e^{-\lambda} f(y)}{\left(1-e^{-\lambda}\right)} e^{\lambda\left(1-F(y)\right)},$$

where $F(y) = 1 - \dfrac{\Gamma(\alpha, By)}{\Gamma(\alpha)}$ and $\Gamma(\alpha,\beta y) = \int_{\beta y}^{\infty} t^{\alpha-1} e^{-t} dt$ is the upper incomplete gamma function. The pdf of gamma zero-truncated Poisson distribution with shape parameter α and scale parameters λ and β is given by

$$g(y;\boldsymbol{\theta}) = \frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)} \left(\frac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}\right) e^{\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}, \quad y>0, \lambda>0, \alpha>0, \beta>0 \tag{3.1}$$

where $\boldsymbol{\theta} = (\lambda,\alpha,\beta)$.

In the following, the distribution of Y will be referred as the gamma zero-truncated Poisson distribution (GZTP) and the plots of its pdf are displayed in Figure 3.1 for selected parameter values. When $0<\alpha\leq 1$, the shape of density is strictly decreasing as shown in Figure 3.1(a), whereas when $\alpha>1$ the density becomes unimodal, and the curves show that the GZTP has a positively skewed distribution, as shown in Figure 3.1(b). For $\alpha=1$, the GZTP distribution simplifies to the density of the Exponential-Poisson distribution presented by Kus (2007).

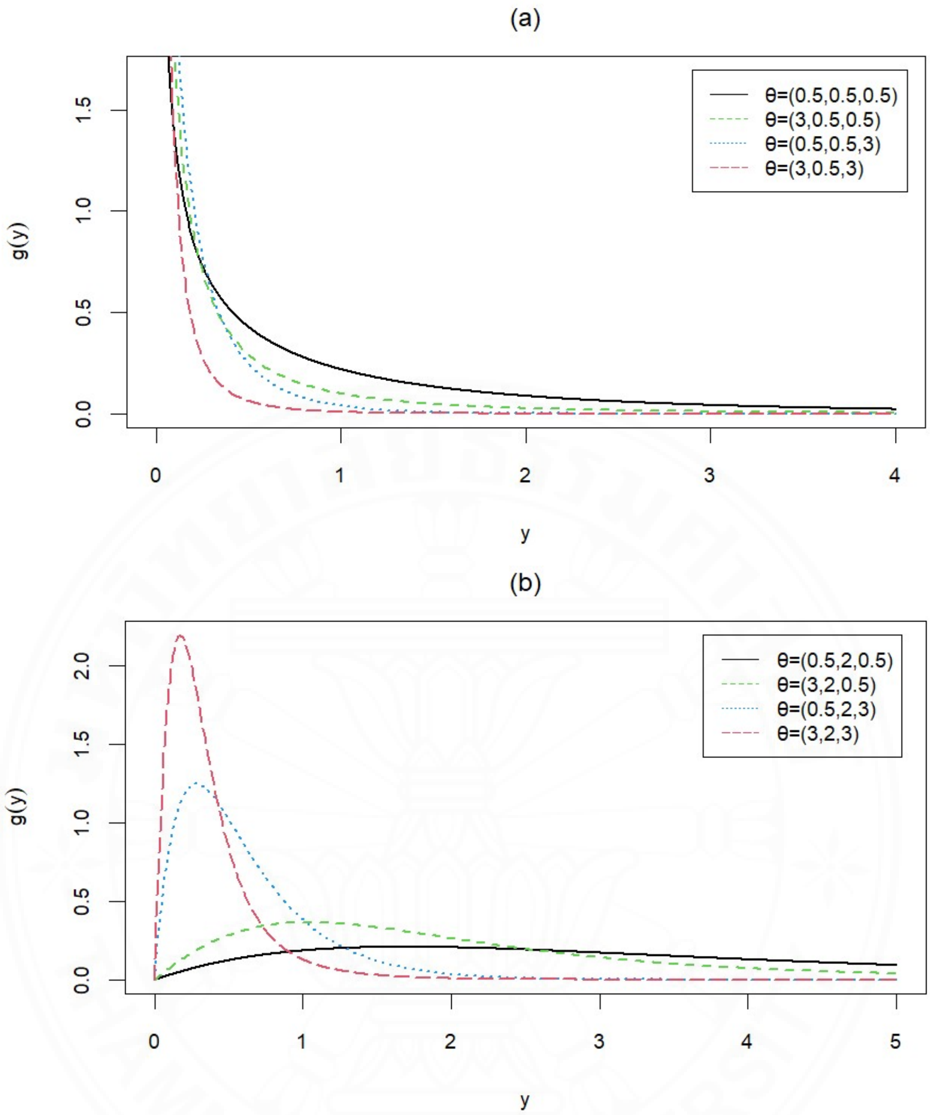

Figure 3.1. Probability density functions of the GZTP distribution with $\boldsymbol{\theta}=(\lambda,\alpha,\beta)$, (a) $\alpha=0.5$, (b) $\alpha=2$.

Theorem 3.1 Considering the GZTP distribution with the pdf of equation (3.1). The GZTP distribution reduces to two-parameter Gamma distribution as λ approaches 0.

Proof. If λ approaches to zero, then

$$\lim_{\lambda\to 0} g(y;\boldsymbol{\theta}) = \lim_{\lambda\to 0}\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}\right) e^{\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}$$

$$= \lim_{\lambda\to 0}\frac{\lambda\beta^{\alpha} y^{\alpha-1} e^{-\lambda-\beta y+\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{\lambda}{\left(1-e^{-\lambda}\right)} \right) \left(\lim_{\lambda \to 0} e^{-\lambda - \beta y + \lambda \left(\frac{\Gamma(\alpha, By)}{\Gamma(\alpha)} \right)} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{\lambda}{\left(1-e^{-\lambda}\right)} \right) \left(e^{-\beta y} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{\lambda}{e^{-\lambda}\left(e^{\lambda}-1\right)} \right) \left(e^{-\beta y} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{\lambda e^{\lambda}}{\left(e^{\lambda}-1\right)} \right) \left(e^{-\beta y} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{\lambda}{\left(e^{\lambda}-1\right)} \right) \left(\lim_{\lambda \to 0} e^{\lambda} \right) \left(e^{-\beta y} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{\frac{d\lambda}{d\lambda}}{\frac{d}{d\lambda}\left(e^{\lambda}-1\right)} \right) (1) \left(e^{-\beta y} \right) \qquad \text{, } \textit{L'hospital's rule}$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} \left(\lim_{\lambda \to 0} \frac{1}{e^{\lambda}} \right) (1) \left(e^{-\beta y} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1}}{\Gamma(\alpha)} (1)(1) \left(e^{-\beta y} \right)$$

$$= \frac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}$$

Then, the GZTP distribution reduces to two-parameter gamma distribution.

Theorem 3.2 The density function of GZTP distribution is strictly decreasing if $0 < \alpha \le 1$.

Proof. The first derivative of GZTP distribution is

$$g'(y;\boldsymbol{\theta}) = \frac{\lambda \beta^{\alpha} y^{\alpha-2} e^{\lambda \left(\frac{\Gamma(\alpha, By)}{\Gamma(\alpha)} \right) - \beta y - \lambda}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)} \left[\alpha - 1 - \beta y - \frac{\lambda (\beta y)^{\alpha} e^{-\beta y}}{\Gamma(\alpha)} \right].$$

The sign of $g'(y;\boldsymbol{\theta})$ depends on the sign of $\left[\alpha - 1 - \beta y - \frac{\lambda (\beta y)^{\alpha} e^{-\beta y}}{\Gamma(\alpha)} \right]$. Let $h = \beta y$,

we write $\left[\alpha - 1 - \beta y - \frac{\lambda (\beta y)^{\alpha} e^{-\beta y}}{\Gamma(\alpha)} \right] = \alpha - 1 - \left(h + \frac{\lambda h^{\alpha} e^{-h}}{\Gamma(\alpha)} \right)$.

If $0 < \alpha \le 1$, then $g'(y;\boldsymbol{\theta}) < 0$. Hence $g(y;\boldsymbol{\theta})$ is decreasing function.

The cumulative distribution function of the GZTP distribution is given by

$$G(y;\boldsymbol{\theta})=\int_{0}^{y} g(y;\boldsymbol{\theta})dy$$
$$=\int_{0}^{y}\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}\right)e^{\lambda\left(\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}\right)}dy$$
$$=\frac{\lambda e^{-\lambda}\beta^{\alpha}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\int_{0}^{y} y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy.$$

Now, find the value of $\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy$ by letting $u=\Gamma(\alpha,\beta y)$, $\frac{du}{dy}=-\beta^{\alpha}y^{\alpha-1}e^{-\beta y}$.

Therefore, $\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,y)}{\Gamma(\alpha)}-\beta y}dy=\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}\left(-\frac{y^{1-\alpha}e^{\beta y}}{\beta^{\alpha}}du\right)=-\frac{1}{\beta^{\alpha}}\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du$.

Consider $\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du$, let $v=\frac{\lambda u}{\Gamma(\alpha)}$, then $\frac{dv}{du}=\frac{\lambda}{\Gamma(\alpha)}$ and $du=\frac{\Gamma(\alpha)}{\lambda}dv$, and

$$\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du=\frac{\Gamma(\alpha)}{\lambda}\int e^{v}dv=\frac{\Gamma(\alpha)e^{v}}{\lambda}=\frac{\Gamma(\alpha)e^{\frac{\lambda u}{\Gamma(\alpha)}}}{\lambda}.$$

Therefore, $\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy=-\frac{1}{\beta^{\alpha}}\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du=-\frac{\Gamma(\alpha)e^{\frac{\lambda u}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}$, and then

$$G(y;\boldsymbol{\theta})=\left[\frac{\lambda e^{-\lambda}\beta^{\alpha}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\left(-\frac{\Gamma(\alpha)e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}\right)+C\right]_{0}^{y}=\left[-\frac{e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}}{\left(1-e^{-\lambda}\right)}+C\right]_{0}^{y}$$
$$=-\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}-e^{\frac{\lambda\Gamma(\alpha,0)}{\Gamma(\alpha)}-\lambda}\right]$$
$$=-\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}-e^{\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha)}-\lambda}\right]$$
$$=-\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}-e^{\lambda-\lambda}\right]=-\frac{\left(e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}-1\right)}{\left(1-e^{-\lambda}\right)}=\frac{\left(1-e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}\right)}{\left(1-e^{-\lambda}\right)}.$$

The cumulative distribution function of the GZTP distribution is

$$G(y;\boldsymbol{\theta})=\begin{cases}\dfrac{\left(1-e^{-\lambda+\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)} & , y>0 \\ 0 & , \text{Otherwise.}\end{cases} \tag{3.2}$$

The cdf mentioned above is non-decreasing function with $\lim_{y\to\infty} G(y;\boldsymbol{\theta})=1$ and $\lim_{y\to-\infty} G(y;\boldsymbol{\theta})=0$.

3.2 Some properties of distribution

3.2.1 The r^{th} quantile

The r^{th} quantile for this distribution is defined as the value y_r such thatn $\Gamma(\alpha,\beta y_r)=\dfrac{\Gamma(\alpha)}{\lambda}\ln\left(\dfrac{1-r\left(1-e^{-\lambda}\right)}{e^{-\lambda}}\right)$.

Proof. Since the r^{th} quantile denoted by y_r and $y_r=G^{-1}(r)$. This implies that $G(y_r)=r$. As $Y\sim GZTP$, then

$$G(y_r;\theta)=\frac{\left(1-e^{-\lambda+\frac{\lambda\Gamma(\alpha,\beta y_r)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)}=r$$

$$e^{-\lambda+\frac{\lambda\Gamma(\alpha,\beta y_r)}{\Gamma(\alpha)}}=1-r\left(1-e^{-\lambda}\right)$$

$$e^{\frac{\lambda\Gamma(\alpha,\beta y_r)}{\Gamma(\alpha)}}=\frac{1-r\left(1-e^{-\lambda}\right)}{e^{-\lambda}}$$

$$\frac{\lambda\Gamma(\alpha,\beta y_r)}{\Gamma(\alpha)}=\ln\left(\frac{1-r\left(1-e^{-\lambda}\right)}{e^{-\lambda}}\right)$$

$$\Gamma(\alpha,\beta y_r)=\frac{\Gamma(\alpha)}{\lambda}\ln\left(\frac{1-r\left(1-e^{-\lambda}\right)}{e^{-\lambda}}\right).$$

Hence, it can be solved analytically for y_r to obtain $\Gamma(\alpha,\beta y_r)=\dfrac{\Gamma(\alpha)}{\lambda}\ln\left(\dfrac{1-r\left(1-e^{-\lambda}\right)}{e^{-\lambda}}\right)$.

3.2.2 The moment generating function

The moment generating function is defined by

$$M_Y(t) = E\left(e^{tY}\right) = \int_0^\infty e^{ty} g(y;\boldsymbol{\theta})dy$$

$$= \int_0^\infty e^{ty} \frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)} \left(\frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}\right) e^{\lambda\left(\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}\right)} dy$$

$$= \frac{\lambda\beta^\alpha e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^\infty y^{\alpha-1} e^{ty-\beta y+\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} dy.$$

The numerical values of moment can be obtained by using numerical integration. The raw moments of Y are determined from equation (3.1) by direct integration. The k raw moments are given by

$$E(Y^k) = \frac{d^k}{dt^k} M_Z(t)\Big|_{t=0} = \frac{\lambda\beta^\alpha}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^\infty y^{\alpha-1+k} e^{-\beta y-\lambda+\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} dy \ , \ k \in \mathbb{N}.$$

The exact expression for the raw moments is not available, however the convergence of these moments can be proved by applying the comparison theorem to an improper integral. Suppose that $m(y) = y^{\alpha-1+k} e^{-\beta y-\lambda+\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}$ and $n(y) = y^{\alpha-1+k} e^{-\beta y}$ are continuous functions with $0 \le m(y) \le n(y)$ for $y \ge 0$. Since $\int_0^\infty n(y)dy = \beta^{-(\alpha+k)}\Gamma(\alpha+k)$,which means that this integral converges, $\int_0^\infty m(y)dy$ also converges. It follows that the raw moments of the distribution converge for all k.

3.2.3 The mean and variance

The mean and variance of the Y are given, respectively, by

$$E(Y) = \frac{\lambda\beta^\alpha e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^\infty y^\alpha e^{-\beta y+\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} dy, \text{ and}$$

$$Var(Y) = \frac{\lambda\beta^\alpha e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^\infty y^{\alpha+1} e^{-\beta y+\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} dy - \left[E(Y)\right]^2.$$

3.3 Survival function and Hazard function

From equation (3.1) and equation (3.2), survival function and hazard function of the GZTP distribution are given by

$$S(y;\boldsymbol{\theta}) = 1 - G(y;\boldsymbol{\theta}) = 1 - \frac{\left(1 - e^{-\lambda + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)} = \frac{\left(1-e^{-\lambda}\right) - \left(1 - e^{-\lambda + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)}$$

$$= \frac{1 - e^{-\lambda} - 1 + e^{-\lambda + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\left(1-e^{-\lambda}\right)} = \frac{e^{-\lambda + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} - e^{-\lambda}}{\left(1-e^{-\lambda}\right)} = -\frac{e^{-\lambda}\left(1 - e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)}, \text{ and}$$

$$H(y;\boldsymbol{\theta}) = \frac{g(y;\boldsymbol{\theta})}{s(y;\boldsymbol{\theta})}$$

$$= \frac{\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}\right) e^{\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}}{-\frac{e^{-\lambda}\left(1 - e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)}} = -\frac{\lambda\beta^{\alpha} y^{\alpha-1} e^{-\beta y + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\Gamma(\alpha)\left(1 - e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}.$$

Let $\eta(y) = -\frac{g'(y;\boldsymbol{\theta})}{g(y;\boldsymbol{\theta})}$, then $\eta(y) = \beta\left(1 + \frac{\lambda e^{-\beta y}(\beta y)^{\alpha-1}}{\Gamma(\alpha)}\right) - \frac{(\alpha-1)}{y}$ and

$$\eta'(y) = \frac{1}{\Gamma(\alpha)y^2}\left[(\alpha-1)\Gamma(\alpha) - \lambda(\beta y)^{\alpha}(\beta y - \alpha + 1)e^{-\beta y}\right].$$

For $0 < \alpha \le 1$, $\eta'(y) < 0$ for all y. Then the hazard function is decreasing that follows from Glaser (1980). Figure 3.2 illustrates some of the possible shapes of the hazard function for selected values of $\boldsymbol{\theta}$.

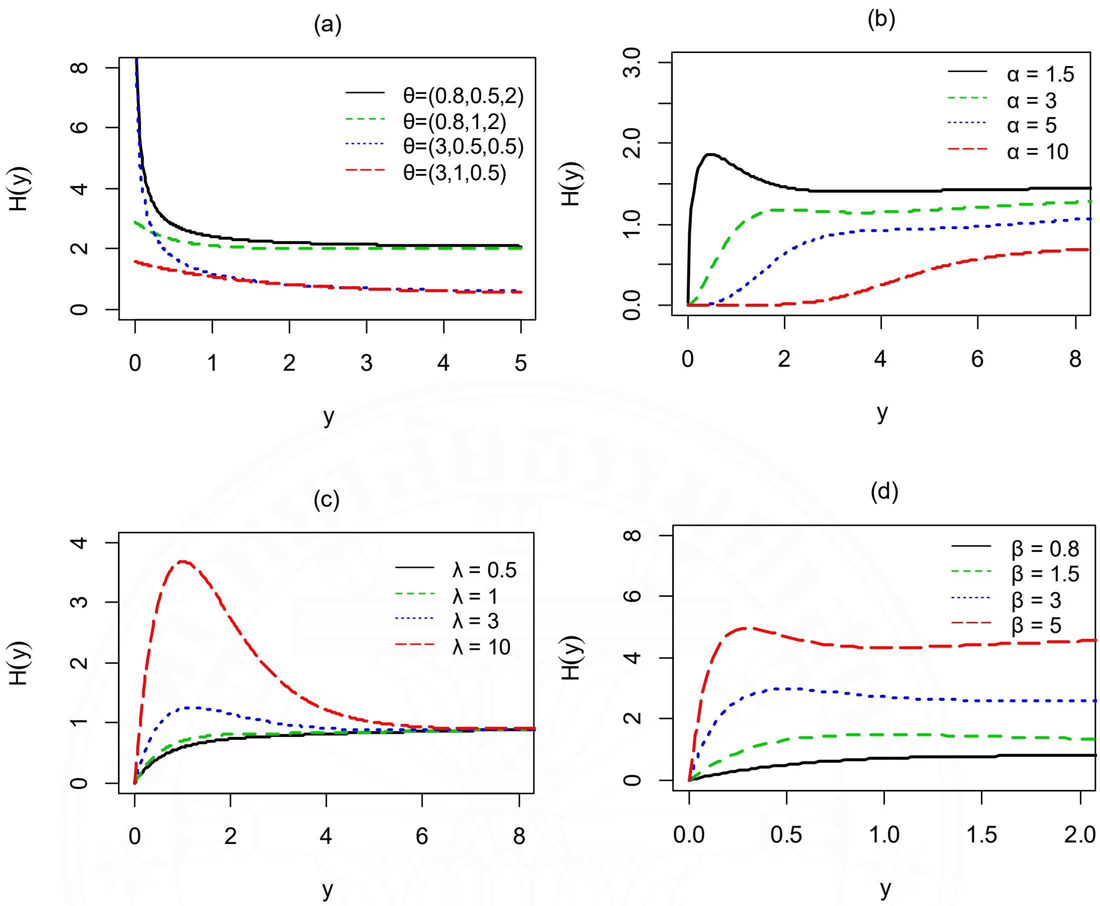

Figure 3.2 Hazard functions of the GZTP distribution. (a) $0<\alpha\le 1$, (b) $\lambda=2,\ \beta=1.5$, (c) $\alpha=2,\ \beta=1$, (d) $\lambda=2,\ \alpha=2$.

3.4 Random number generation

Samples from the GZTP distribution are generated using a rejection sampling algorithm in R programming. A continuous uniform distribution, $U(0,20)$, is used as a proposal distribution $p(y)$, and the smallest c that maximizes $\frac{g(y;\boldsymbol{\theta})}{p(y)}$ is selected. The algorithm is shown as follow:

Step 1: find a constant c such that $cp(y)\ge g(y;\boldsymbol{\theta})$;

Step 2: obtain a sample y from $U(0,20)$;

Step 3: obtain a sample u from $U(0,1)$;

Step 4: check whether $cp(y)u\le g(y;\boldsymbol{\theta})$. If this holds, accept y as a sample drawn from g. Otherwise, y will be rejected.

The Figure 3.3 presents the histograms of 10,000 samples, which were obtained through rejection sampling from selected cases of the GZTP distribution.

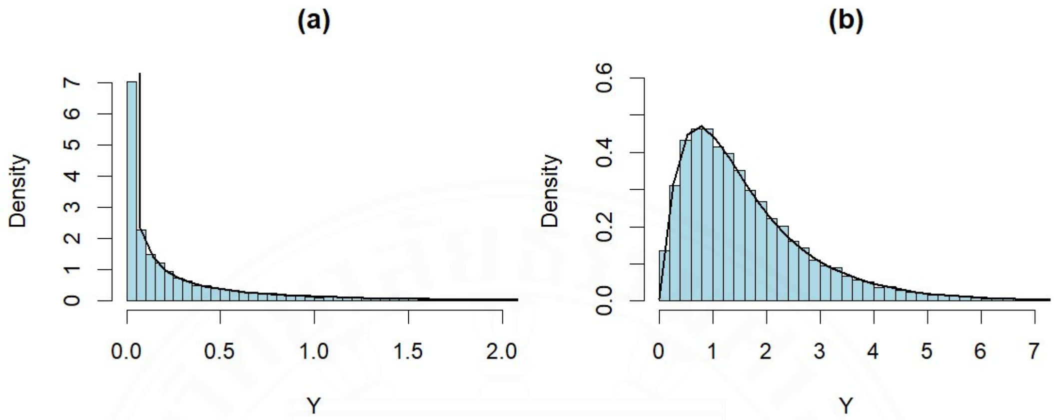

Figure 3.3 Histogram of 10,000 samples from GZTP distribution
(a) $\lambda = 1,\ \alpha = 0.5, \beta = 1$ and (b) $\lambda = 1,\ \alpha = 2, \beta = 1$.

3.5 Estimation of parameters

3.5.1 Estimation by maximum likelihood

Let $Y_1, Y_2, \ldots, Y_n$ be random samples with observed values $y_1, y_2, \ldots, y_n$ from a GZTP distributions with parameters $\boldsymbol{\theta}$. The likelihood function based on the observed random sample size of n, $w_{obs} = (y_1, y_2, \ldots, y_n)$ is given by

$$L(\boldsymbol{\theta}; w_{obs}) = \left(\frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\right) \left(\prod_{i=1}^{n} y_i\right)^{\alpha - 1} e^{-\beta\left(\sum_{i=1}^{n} y_i\right) + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{n} \Gamma(\alpha, \beta y_i)}.$$

The corresponding log-likelihood function is

$$l(\boldsymbol{\theta}; w_{obs}) = n\left(\log\lambda - \lambda - \log\left(1 - e^{-\lambda}\right)\right) + n\alpha\log\beta - n\log\Gamma(\alpha) + (\alpha - 1)\sum_{i=1}^{n}\log y_i - \beta\left(\sum_{i=1}^{n} y_i\right) + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha, \beta y_i).$$

The first derivatives of the log-likelihood function are the following:

$$\frac{\partial l(\boldsymbol{\theta};w_{obs})}{\partial\lambda}=n\left(\frac{1}{\lambda}-1-\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)+\frac{\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)}{\Gamma(\alpha)},$$

$$\frac{\partial l(\boldsymbol{\theta};w_{obs})}{\partial\alpha}=n\log\beta-n\psi_0(\alpha)+\sum_{i=1}^{n}\log y_i+\lambda\sum_{i=1}^{n}\left(\frac{\Gamma(\alpha)\frac{\partial\Gamma(\alpha,\beta y_i)}{\partial\alpha}-\Gamma(\alpha,\beta y_i)\frac{\partial\Gamma(\alpha)}{\partial\alpha}}{\left(\Gamma(\alpha)\right)^2}\right)$$

$$=n\log\beta-n\psi_0(\alpha)+\sum_{i=1}^{n}\log y_i$$
$$+\lambda\sum_{i=1}^{n}\left(\frac{\Gamma(\alpha)\left(G_{2,3}^{3,0}\left(\beta y_i\middle|\begin{matrix}1,1\\0,0,a\end{matrix}\right)+\Gamma(\alpha,\beta y_i)\log(\beta y_i)\right)-\Gamma(\alpha,\beta y_i)\Gamma(\alpha)\psi_0(\alpha)}{\left(\Gamma(\alpha)\right)^2}\right)$$

$$=n\log\beta-n\psi_0(\alpha)+\sum_{i=1}^{n}\log y_i+\lambda\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta y_i\middle|\begin{matrix}1,1\\0,0,a\end{matrix}\right)+\Gamma(\alpha,\beta y_i)\left(\log(\beta y_i)-\psi_0(\alpha)\right)}{\Gamma(\alpha)},$$

$$\frac{\partial l(\boldsymbol{\theta};w_{obs})}{\partial\beta}=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i+\frac{\lambda}{\Gamma(\alpha)}\frac{\partial\left(\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)\right)}{\partial\beta}$$
$$=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\frac{\partial\Gamma(\alpha,\beta y_i)}{\partial\beta}$$
$$=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(-y_i^{\alpha}\beta^{\alpha-1}e^{-\beta y_i}\right)$$
$$=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i-\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}y_i^{\alpha}e^{-\beta y_i}.$$

Then subsequently the associated gradients are found to be

$$\frac{\partial l(\boldsymbol{\theta};w_{obs})}{\partial\lambda}=n\left(\frac{1}{\lambda}-1-\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)+\frac{\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)}{\Gamma(\alpha)},\tag{3.3}$$

$$\frac{\partial l(\boldsymbol{\theta};w_{obs})}{\partial\alpha}=n\log\beta-n\psi_0(\alpha)+\sum_{i=1}^{n}\log y_i$$
$$+\lambda\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta y_i\middle|\begin{matrix}1,1\\0,0,\alpha\end{matrix}\right)+\Gamma(\alpha,\beta y_i)\left(\log(\beta y_i)-\psi_0(\alpha)\right)}{\Gamma(\alpha)},\tag{3.4}$$

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} y_i - \frac{\lambda \beta^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i}, \tag{3.5}$$

where $\psi_0(\alpha)$ is a digamma function that define as the 1^{st} derivative of the logarithm of gamma function and $G_{p,q}^{m,n}\left(\beta y_i \middle| \begin{matrix} a_1,\ldots a_p \\ b_1,\ldots b_q \end{matrix}\right)$ is Meijer G-function. The equation (3.5) could be solved exactly for λ as follows:

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial \beta} = 0$$

$$\frac{n\alpha}{\beta} + \sum_{i=1}^{n} y_i - \frac{\lambda \beta^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} = 0$$

$$\frac{\lambda \beta^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} = \frac{n\alpha}{\beta} + \sum_{i=1}^{n} y_i$$

$$\lambda = \frac{\Gamma(\alpha)}{\beta^{\alpha-1} \sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i}} \left(\frac{n\alpha}{\beta} + \sum_{i=1}^{n} y_i \right).$$

Therefore, the maximum likelihood estimator of λ is $\hat{\lambda} = \frac{\Gamma(\hat{\alpha})}{\hat{\beta}^{\hat{\alpha}-1} \sum_{i=1}^{n} y_i^{\hat{\alpha}} e^{-\hat{\beta} y_i}} \left(\frac{n\hat{\alpha}}{\hat{\beta}} + \sum_{i=1}^{n} y_i \right)$, conditional upon the value of $\hat{\alpha}$ and $\hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are maximum likelihood estimates for the parameter α and β, respectively. For α and β, there are no closed forms, but the estimates can be calculated by numerical methods such as the Newton–Raphson method or probabilistic methods such as simulated annealing.

In the following, Theorem 3.3 expresses what the conditions are there to obtain the existence of the MLEs.

Theorem 3.3

(a) Let $l_1(\lambda; \alpha, \beta, w_{obs}) = \frac{\partial l(\theta; w_{obs})}{\partial \lambda}$, If α, β are known, then $\hat{\lambda}$ is the uniquely exist root of $l_1(\lambda; \alpha, \beta, w_{obs}) = 0$ if $\frac{\sum_{i=1}^{n} \Gamma(\alpha, \beta y_i)}{\Gamma(\alpha)} > \frac{n}{2}$.

Proof. Since $l_1\left(\lambda;\alpha,\beta,w_{obs}\right)=n\left(\frac{1}{\lambda}-1-\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}$,

$$\lim_{\lambda\to 0} l_1\left(\lambda;\alpha,\beta,w_{obs}\right)=-\frac{n}{2}+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}, \text{ and}$$

$$\lim_{\lambda\to\infty} l_1\left(\lambda;\alpha,\beta,w_{obs}\right)=-n+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}.$$

It can be shown that $\lim_{\lambda\to 0} l_1\left(\lambda;\alpha,\beta,w_{obs}\right)=-\frac{n}{2}+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}>0$ as $\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}>\frac{n}{2}$.

Since $\frac{\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}<1$ for all y_i, then $\lim_{\lambda\to\infty} l_1\left(\lambda;\alpha,\beta,w_{obs}\right)=-n+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}<0$.

Therefore, there exist at least one solution of $l_1\left(\lambda;\alpha,\beta,w_{obs}\right)=0$.

For the proof of uniqueness of solution, it is needed to show function l_1 is strictly decreasing in λ. The first derivative of l_1 is considered and given by

$$l_1'\left(\lambda;\alpha,\beta,w_{obs}\right)=-\frac{n(1+e^{2\lambda}-e^{\lambda}(\lambda^2+2))}{\left(e^{\lambda}-1\right)^2\lambda^2}=-\frac{ne^{\lambda}(e^{-\lambda}+e^{\lambda}-(\lambda^2+2))}{\left(e^{\lambda}-1\right)^2\lambda^2}.$$

If $e^{-\lambda}+e^{\lambda}-(\lambda^2+2)>0$, then $l_1'\left(\lambda;\alpha,\beta,w_{obs}\right)<0$ and l_1 is strictly decreasing in λ.

Consider $e^{\lambda}=1+\lambda+\frac{1}{2}\lambda^2+\frac{1}{3!}\lambda^3+\ldots$ and $e^{-\lambda}=1-\lambda+\frac{1}{2}\lambda^2-\frac{1}{3!}\lambda^3+\ldots$, then

$e^{-\lambda}+e^{\lambda}=2+\lambda^2+\frac{2}{4!}\lambda^4+\ldots>\lambda^2+2$ or $e^{-\lambda}+e^{\lambda}-(\lambda^2+2)>0$.

Therefore, $l_1'\left(\lambda;\alpha,\beta,w_{obs}\right)<0$ for $\lambda>0$. This complete the proof.

(b) Let $l_3\left(\beta;\lambda,\alpha,w_{obs}\right)=\frac{\partial l\left(\theta;w_{obs}\right)}{\partial\beta}$, If λ and α are known, then there exist at least one solution of $l_3\left(\beta;\lambda,\alpha,w_{obs}\right)=0$.

Proof. Since $l_3\left(\beta;\lambda,\alpha,w_{obs}\right)=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i-\frac{\lambda\beta^{\alpha-1}}{\Gamma\left(\alpha\right)}\sum_{i=1}^{n}y_i^{\alpha}e^{-\beta y_i}$,

$$\lim_{\beta\to 0} l_3\left(\beta;\lambda,\alpha,w_{obs}\right) = \lim_{\beta\to 0}\frac{n\alpha}{\beta} - \lim_{\beta\to 0}\sum_{i=1}^{n} y_i - \lim_{\beta\to 0}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} = \infty \text{, and}$$

$$\begin{aligned}
\lim_{\beta\to\infty} l_3\left(\beta;\lambda,\alpha,w_{obs}\right) &= \lim_{\beta\to\infty}\frac{n\alpha}{\beta} - \lim_{\beta\to\infty}\sum_{i=1}^{n} y_i - \lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} \\
&= 0 - \sum_{i=1}^{n} y_i - \lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i}.
\end{aligned}$$

Consider

$$\begin{aligned}
\lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} &= \frac{\lambda}{\Gamma(\alpha)}\lim_{\beta\to\infty}\sum_{i=1}^{n}\beta^{\alpha-1} y_i^{\alpha} e^{-\beta y_i} \\
&= \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(\lim_{\beta\to\infty}\beta^{\alpha-1} y_i^{\alpha} e^{-\beta y_i}\right) \\
&= \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(y_i \lim_{\beta\to\infty}\frac{\left(\beta y_i\right)^{\alpha-1}}{e^{\beta y_i}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\beta\to\infty}\frac{\left(\beta y_i\right)^{\alpha-1}}{e^{\beta y_i}} &= \lim_{\beta\to\infty}\left(\frac{\beta y_i}{e^{\beta y_i/\alpha-1}}\right)^{\alpha-1} \\
&= \lim_{\beta\to\infty}\left(\alpha-1\right)^{\alpha-1}\left(\frac{c}{e^{c}}\right)^{\alpha-1}, c = \beta y_i/\alpha-1 \\
&= \left(\alpha-1\right)^{\alpha-1}\left(\lim_{\beta\to\infty}\frac{c}{e^{c}}\right)^{\alpha-1} \\
&= \left(\alpha-1\right)^{\alpha-1}\left(\lim_{\beta\to\infty}\frac{1}{e^{c}}\right)^{\alpha-1} \quad , L'Hospital's\ rule \\
&= 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{\beta\to\infty} l_3\left(\beta;\lambda,\alpha,w_{obs}\right) &= \lim_{\beta\to\infty}\frac{n\alpha}{\beta} - \lim_{\beta\to\infty}\sum_{i=1}^{n} y_i - \lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} \\
&= 0 - \sum_{i=1}^{n} y_i - 0 \\
&= -\sum_{i=1}^{n} y_i < 0.
\end{aligned}$$

Therefore, there exist at least one solution of $l_3\left(\beta;\lambda,\alpha,w_{obs}\right) = 0$.

3.5.2 Statistical inference and confidence intervals

The MLE of $\boldsymbol{\theta}$ is approximately multivariate normal with mean $\boldsymbol{\theta}$ and a variance-covariance matrix that is the inverse of expected information matrix $J(\boldsymbol{\theta}) = E\left[I(\boldsymbol{\theta})\right]$, where $I(\boldsymbol{\theta})$ is the observed Fisher information matrix with elements $I_{ij} = -\partial l^2 \big/ \partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j$, $i, j = 1,2,3$.

3.5.2.1 Case 1: All parameters are unknown

When λ, α and β are unknown, the parameter is a vector $\boldsymbol{\theta}$. Because the equations (3.3) - (3.5) are nonlinear, the MLEs of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}} = \left(\hat{\lambda},\ \hat{\alpha},\ \hat{\beta}\right)$ need to be determined numerically. By differentiating equations (3.3) - (3.5), the elements of the symmetric and second order observed information matrix, $I(\boldsymbol{\theta})$, are found as follows:

$$I_{11} = \frac{n(1+e^{2\lambda} - e^{\lambda}(\lambda^2+2))}{\left(e^{\lambda}-1\right)^2 \lambda^2},$$

$$I_{22} = n\psi^{(1)}(\alpha) - \lambda \sum_{i=1}^{n} \frac{\partial^2}{\partial \alpha^2}\left(\frac{\Gamma(\alpha, \beta y_i)}{\Gamma(\alpha)}\right)$$

$$= n\psi^{(1)}(\alpha) - \lambda \sum_{i=1}^{n} \left[\frac{1}{\Gamma(\alpha)} \left(\begin{aligned} &2G_{3,4}^{4,0}\left(\beta y_i \left| \begin{matrix} 1,1,1 \\ 0,0,0,\alpha \end{matrix} \right.\right) + 2\left(\log(\beta y_i) - \psi^{(0)}(\alpha)\right) G_{2,3}^{3,0}\left(\beta y_i \left| \begin{matrix} 1,1 \\ 0,0,\alpha \end{matrix} \right.\right) \\ &+\Gamma(\alpha, \beta y_i)\left(-2\psi^{(0)}(\alpha)\log(\beta y_i) + \psi^{(0)}(\alpha)^2 - \psi^{(1)}(\alpha) + \log^2(\beta y_i)\right) \end{aligned} \right)\right],$$

$$I_{33} = \frac{n\alpha}{\beta^2} + \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{n} y_i^{\alpha}\left((\alpha - 1 - \beta y_i)\beta^{\alpha-2} e^{-\beta y_i}\right),$$

$$I_{12} = I_{21} = -\sum_{i=1}^{n} \frac{G_{2,3}^{3,0}\left(\beta y_i \left| \begin{matrix} 1,1 \\ 0,0,\alpha \end{matrix} \right.\right) + \Gamma(\alpha, \beta y_i)\left(\log(\beta y_i) - \psi^{(0)}(\alpha)\right)}{\Gamma(\alpha)},$$

$$I_{13} = I_{31} = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i},$$

$$I_{23}=I_{32}=-\frac{\partial}{\partial\alpha}\left[\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i-\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}y_i^{\alpha}e^{-\beta y_i}\right]$$

$$=-\frac{n}{\beta}+\lambda\frac{\partial}{\partial\alpha}\left[\sum_{i=1}^{n}\frac{y_i^{\alpha}e^{-\beta y_i}\beta^{\alpha-1}}{\Gamma(\alpha)}\right]$$

$$=-\frac{n}{\beta}+\lambda\sum_{i=1}^{n}e^{-\beta y_i}\frac{\partial}{\partial\alpha}\left[\frac{y_i^{\alpha}\beta^{\alpha-1}}{\Gamma(\alpha)}\right]=-\frac{n}{\beta}+\lambda\sum_{i=1}^{n}e^{-\beta y_i}\left[\frac{y_i^{\alpha}\beta^{\alpha-1}\left(-\psi^{(0)}(\alpha)+\log(\beta)+\log(y_i)\right)}{\Gamma(\alpha)}\right].$$

The Fisher information matrix is given by

$$I\left(\hat{\boldsymbol{\theta}}\right)=\begin{bmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{bmatrix}.$$

The inverse observed Fisher information matrix is

$$I^{-1}\left(\hat{\boldsymbol{\theta}}\right)=\begin{bmatrix}I^{11} & I^{12} & I^{13}\\ I^{21} & I^{22} & I^{23}\\ I^{31} & I^{32} & I^{33}\end{bmatrix}.$$

Since $se\left(\hat{\boldsymbol{\theta}}_i\right)$ is defined as the square root of the ith diagonal entry of the inverse observed Fisher information matrix , Hence $(1-\gamma)100\%$ Wald confidence intervals for λ, α and β are $\hat{\lambda}\pm z_{1-\frac{\gamma}{2}}\sqrt{I^{11}}$, $\hat{\alpha}\pm z_{1-\frac{\gamma}{2}}\sqrt{I^{22}}$ and $\hat{\beta}\pm z_{1-\frac{\gamma}{2}}\sqrt{I^{33}}$, respectively.

3.5.2.2 Case 2: α and β are unknown

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample with observed values $y_1,y_2,\ldots,y_n$ from a GZTP distributions with known parameter λ and unknown parameters α and β . The log-likelihood function based on the observed random sample size of n, $w_{obs}=\left(y_1,y_2,\ldots,y_n\right)$ is given by

$$l\left(\alpha,\beta;w_{obs}\right)=n\left(\log\lambda-\lambda-\log\left(1-e^{-\lambda}\right)\right)+n\alpha\log\beta-n\log\Gamma\left(\alpha\right)+\left(\alpha-1\right)\sum_{i=1}^{n}\log y_i$$

$$-\beta\left(\sum_{i=1}^{n}y_i\right)+\frac{\lambda}{\Gamma\left(\alpha\right)}\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right),$$

where $\boldsymbol{\theta}=(\alpha,\ \beta)$, and the associated gradients are found to be

$$\frac{\partial l(\alpha,\beta;w_{obs})}{\partial\alpha}=n\log\beta-n\psi_0(\alpha)+\sum_{i=1}^{n}\log y_i + \lambda\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta y_i\left|\begin{matrix}1,1\\0,0,\alpha\end{matrix}\right.\right)+\Gamma(\alpha,\beta y_i)\left(\log(\beta y_i)-\psi_0(\alpha)\right)}{\Gamma(\alpha)}, \tag{3.6}$$

$$\frac{\partial l(\alpha,\beta;w_{obs})}{\partial\beta}=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}y_i-\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}y_i^{\alpha}e^{-\beta y_i}. \tag{3.7}$$

By differentiating equations (3.6) - (3.7), the elements of the symmetric and second order observed information matrix, $I(\boldsymbol{\theta})$, are found as follows:

$$I_{11}=n\psi^{(1)}(\alpha)-\lambda\sum_{i=1}^{n}\left[\frac{1}{\Gamma(\alpha)}\left(\begin{matrix}2G_{3,4}^{4,0}\left(\beta y_i\left|\begin{matrix}1,1,1\\0,0,0,\alpha\end{matrix}\right.\right)+2\left(\log(\beta y_i)-\psi^{(0)}(\alpha)\right)G_{2,3}^{3,0}\left(\beta y_i\left|\begin{matrix}1,1\\0,0,\alpha\end{matrix}\right.\right)\\+\Gamma(\alpha,\beta y_i)\left(-2\psi^{(0)}(\alpha)\log(\beta y_i)+\psi^{(0)}(\alpha)^2-\psi^{(1)}(\alpha)+\log^2(\beta y_i)\right)\end{matrix}\right)\right], \tag{3.8}$$

$$I_{22}=\frac{n\alpha}{\beta^2}+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}y_i^{\alpha}\left(\left(\alpha-1-\beta y_i\right)\beta^{\alpha-2}e^{-\beta y_i}\right), \tag{3.9}$$

$$I_{12}=I_{21}=-\frac{n}{\beta}+\lambda\sum_{i=1}^{n}e^{-\beta y_i}\left[\frac{y_i^{\alpha}\beta^{\alpha-1}\left(-\psi^{(0)}(\alpha)+\log(\beta)+\log(y_i)\right)}{\Gamma(\alpha)}\right]. \tag{3.10}$$

When α and β are unknown, and λ is known, the MLEs of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}=(\hat{\alpha},\ \hat{\beta})$ also need to be determined numerically. The observed Fisher information matrix is given by

$$I(\hat{\boldsymbol{\theta}})=\begin{bmatrix}I_{11} & I_{12}\\ I_{21} & I_{22}\end{bmatrix},$$

and the inverse observed Fisher information matrix is

$$I^{-1}\left(\hat{\boldsymbol{\theta}}\right)=\frac{1}{\det I\left(\hat{\boldsymbol{\theta}}\right)}\begin{bmatrix} I_{22} & -I_{21} \\ -I_{12} & I_{11} \end{bmatrix},$$

where $\det I\left(\hat{\boldsymbol{\theta}}\right)=I_{11}I_{22}-I_{12}I_{21}$. Hence $(1-\gamma)100\%$ Wald confidence intervals for α and β are, $\hat{\alpha}\pm z_{1-\frac{\gamma}{2}}\sqrt{I_{22}/\left(I_{11}I_{22}-I_{12}I_{21}\right)}$ and $\hat{\beta}\pm z_{1-\frac{\gamma}{2}}\sqrt{I_{11}/\left(I_{11}I_{22}-I_{12}I_{21}\right)}$, respectively.

3.5.2.3 Case 3: λ is unknown

Let $Y_1,Y_2,\ldots,Y_n$ be a random sample with observed values $y_1,y_2,\ldots,y_n$ from a GZTP distributions with unknown parameter λ and known parameters α and β. The log-likelihood function based on the observed random sample size of n, $w_{obs}=\left(y_1,y_2,\ldots,y_n\right)$ is given by

$$\begin{aligned} l\left(\lambda;w_{obs}\right)&=n\left(\log\lambda-\lambda-\log\left(1-e^{-\lambda}\right)\right)+n\alpha\log\beta-n\log\Gamma\left(\alpha\right)+\left(\alpha-1\right)\sum_{i=1}^{n}\log y_i \\ &-\beta\left(\sum_{i=1}^{n}y_i\right)+\frac{\lambda}{\Gamma\left(\alpha\right)}\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right), \end{aligned}$$

and the associated gradients is

$$\frac{\partial l\left(\lambda;w_{obs}\right)}{\partial\lambda}=n\left(\frac{1}{\lambda}-1-\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma\left(\alpha\right)}. \quad (3.8)$$

By differentiating equation (3.8), the observed Fisher information is found as follows:

$$I\left(\hat{\lambda}\right)=\frac{n(1+e^{2\hat{\lambda}}-e^{\hat{\lambda}}(\hat{\lambda}^2+2))}{\left(e^{\hat{\lambda}}-1\right)^2\hat{\lambda}^2}.$$

The MLE of λ has no closed form and also need to be determined numerically with $se\left(\hat{\lambda}\right)=\sqrt{1/I\left(\hat{\lambda}\right)}=\sqrt{\frac{\left(e^{\hat{\lambda}}-1\right)^2\hat{\lambda}^2}{n(1+e^{2\hat{\lambda}}-e^{\hat{\lambda}}(\hat{\lambda}^2+2))}}$. Therefore, a $(1-\gamma)100\%$ Wald confidence interval for λ is $\hat{\lambda} \pm z_{1-\frac{\gamma}{2}} \sqrt{\dfrac{\left(e^{\hat{\lambda}}-1\right)^2 \hat{\lambda}^2}{n(1+e^{2\hat{\lambda}}-e^{\hat{\lambda}}(\hat{\lambda}^2+2))}}$.

3.5.3 Simulation study

The samples were generated by using the rejection sampling method, where $\lambda = 0.5, 1, 3$, $\alpha = 0.5, 1, 2$, and $\beta = 0.5, 1, 3$. These values of parameters are selected such that all different shapes of distributions are represented. The MLEs of λ, α and β are numerically calculated by the simulated-annealing method via the function maxLik in R program. The maxLik package (Henningsen and Toomet, 2011) is used to calculate the MLEs, and the simulated annealing method is chosen because it gives stable solutions. The study was based on $m = 1{,}000$ simulated samples from the GZTPs with different sample sizes: $n = 50$, 100, and 1,000. The averages of the MLEs are calculated from

$$\hat{\boldsymbol{\theta}} = \frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\theta}}_i$$

, and mean-squared errors (MSEs) of estimates are obtained from

$$MSE(\hat{\boldsymbol{\theta}}) = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}\right)^2$$

, where $\boldsymbol{\theta} = (\lambda,\ \alpha,\ \beta)$ is parameter vector. Moreover, Monte Carlo simulations help estimate the coverage probability (CP) and average length (AL) of the confidence intervals (CIs).

3.5.3.1 Case 1: All parameters are unknown

Table 3.1 give the averages of the MLEs, $AV(\hat{\boldsymbol{\theta}})$, and the corresponding MSEs. As sample sizes increase, estimates tend to be close to their actual values, and MSE values decrease; however, it is noticed that, with the same sample size, MSE of $\hat{\lambda}$ is larger than MSE of $\hat{\alpha}$ and $\hat{\beta}$ in most situations. When $\alpha = 0.5$, $\hat{\alpha}$ exhibits the lowest MSE compared to the other two estimates. Conversely, when $\alpha = 1$ and $\alpha = 2$, $\hat{\beta}$ tends to have the smallest MSE. It is observed that the MSE of $\hat{\lambda}$ increases as λ increases given that α and β are fixed. For instance, comparing cases with $\lambda = 0.5$ and $\lambda = 3$ at $\alpha = 1$, $\beta = 1$ and $n = 50$, MSE of $\hat{\lambda}$ increases from 2.9216 to 5.5309, respectively. Also, when λ and β are fixed, MSE of $\hat{\alpha}$ increases as α increases. For example, for cases with $\alpha = 0.5$ and $\alpha = 2$ at $\lambda = 1$, $\beta = 1$ and $n = 100$, MSE of $\hat{\alpha}$ increases from 0.0062 to 0.0698. Likewise, when λ and α are fixed, the MSE of $\hat{\beta}$ increases as β increases.

Wald confidence intervals are calculated for every parameter related to GZTPs. The estimation of confidence intervals (CIs) average length (AL) and coverage probability (CP) can be simplified through using of 1,000 repetitions of Monte Carlo simulations. Table 3.2 displays all of the results. It is found that when $\alpha = 0.5$, the CPs of α are less than 0.95 in most situations and the CPs of λ are less than 0.95 when $n = 1{,}000$ and λ has a small value, i.e., $\lambda = 0.5$. Conversely, when $\alpha = 1$ and $\alpha = 2$, the CPs of all parameter will be closer to the nominal coverage probability, 0.95, and the ALs will decrease when the sample size (n) increases. When λ has a large value, i.e., $\lambda = 3$, the CPs of all parameter are less than 0.95 although the sample size is 1,000. Moreover, when $\alpha = 0.5$ and $\beta = 3$, the CPs of α are very low at $n = 1{,}000$.

3.5.3.2 Case 2: α and β are unknown

Tables 3.3–3.4 present the results obtained from simulated data sets where the values of α and β are unknown. As sample sizes increase, estimates become more accurate and the MSE values decrease. It is noticed that the MSE of $\hat{\alpha}$ increases as α increases given that β are fixed. Similarly, when α are fixed, the MSE of $\hat{\beta}$ increases as β increases. Wald confidence intervals using observed Fisher information are constructed for α and β of GZTPs. It is found that when the sample size (n) increases, the CPs will be closer to the nominal coverage probability, 0.95, and the ALs will decrease. In most cases, CPs are not less than 0.95, although the sample size is only 50.

3.5.3.3 Case 3: λ is unknown

Table 3.5 displays the results obtained from simulated data sets in which λ's value is unknown. As the size of the samples increases, the estimates get more accurate, and the MSE values decrease in most situations. The CPs approach the desired coverage probability of 0.95, and the average lengths decrease with increasing sample sizes. However, in cases when α is small and β is large, namely when $\alpha = 0.5$ and $\beta = 1$ or $\beta = 3$, the estimates of λ showed a bias and the CPs are below 0.95, although the sample size being 1,000.

Table 3.1 The averages of MLEs and mean-squared errors of λ, α, and β of three-parameter GZTP distribution.

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$			MSE		
		λ	α	β	λ	α	β
(0.5, 0.5, 0.5)	25	1.0587	0.5641	0.5739	2.0213	0.0891	0.0925
	50	1.0466	0.5528	0.5056	1.5515	0.0314	0.0393
	100	0.9276	0.5378	0.4894	1.1950	0.0194	0.0188
	1,000	0.7518	0.5320	0.4905	0.1666	0.0018	0.0022
(0.5, 0.5, 1)	25	1.0571	0.5768	1.1443	1.7945	0.0646	0.3750
	50	1.1495	0.5390	1.0127	2.5316	0.0857	0.1754
	100	1.0203	0.4884	0.9781	1.6493	0.1751	0.1071
	1,000	0.9620	0.5348	0.9566	1.4490	0.0206	0.0299
(0.5, 0.5, 3)	25	1.3051	0.5872	3.2644	4.9770	0.0678	2.5205
	50	1.1519	0.5843	3.1323	2.0726	0.0163	1.4857
	100	1.0415	0.5681	2.9943	1.2967	0.0098	0.6825
	1,000	0.9308	0.5601	2.8986	0.2754	0.0043	0.0784
(0.5, 1, 0.5)	25	1.1116	1.1287	0.5320	2.7562	0.1018	0.0475
	50	1.0916	1.0705	0.4925	2.5897	0.0463	0.0256
	100	0.8813	1.0359	0.4880	1.7116	0.0200	0.0139
	1,000	0.5312	1.0039	0.4992	0.1351	0.0032	0.0014
(0.5, 1, 1)	25	1.1853	1.1361	1.0551	3.1140	0.1045	0.1974
	50	1.1161	1.0707	0.9849	2.9216	0.0466	0.1108
	100	1.0162	1.0377	0.9564	2.3742	0.0213	0.0720
	1,000	0.5627	1.0039	0.9913	0.2854	0.0032	0.0077
(0.5, 1, 3)	25	1.4233	1.1396	3.1322	7.4235	0.1103	1.8164
	50	1.4255	1.0536	2.8168	7.6411	0.0406	0.9719
	100	1.0793	1.0420	2.8738	3.1477	0.0204	0.6380
	1,000	0.6164	1.0027	2.9758	1.0371	0.0028	0.1128
(0.5, 2, 0.5)	25	1.5234	2.2488	0.5184	6.3502	0.4247	0.0438
	50	1.3058	2.1102	0.4878	4.2820	0.1743	0.0246
	100	1.1188	2.0651	0.4797	2.9108	0.0819	0.0135
	1,000	0.5220	2.0045	0.5020	0.3159	0.0103	0.0017
(0.5, 2, 1)	25	1.3881	2.2218	1.0309	4.8437	0.4598	0.1822
	50	1.4515	2.1159	0.9540	5.6724	0.1737	0.0967
	100	1.3616	2.0517	0.9314	4.8830	0.0898	0.0755
	1,000	0.6906	2.0025	0.9793	1.0936	0.0103	0.0148
(0.5, 2, 3)	25	1.7127	2.2217	3.0046	8.6952	0.4007	1.5565
	50	1.5614	2.1195	2.8588	7.3809	0.1703	0.9110
	100	1.7219	2.0555	2.7099	8.5531	0.0834	0.8313
	1,000	0.5700	1.9997	2.9702	0.3472	0.0110	0.0597

Table 3.1 The averages of MLEs and mean-squared errors of λ, α, and β of three-parameter GZTP distribution. (Cont.)

$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	n	$AV(\hat{\boldsymbol{\theta}})$			MSE		
		λ	α	β	λ	α	β
(1, 0.5, 0.5)	25	1.2710	0.5551	0.6321	1.8475	0.0451	0.1474
	50	1.3914	0.5369	0.5367	1.7075	0.0353	0.0561
	100	1.3012	0.5305	0.5037	1.1588	0.0054	0.0263
	1,000	1.2525	0.5294	0.4859	0.1649	0.0014	0.0027
(1, 0.5, 1)	25	1.4258	0.5657	1.2178	2.2325	0.0334	0.4731
	50	1.3000	0.5376	1.0982	1.4911	0.0268	0.2163
	100	1.3952	0.5418	1.0123	1.3569	0.0062	0.1162
	1,000	1.3354	0.5384	0.9636	0.2494	0.0020	0.0136
(1, 0.5, 3)	25	1.5321	0.5705	3.5662	3.4772	0.0552	3.9184
	50	1.5002	0.5634	3.1910	2.0011	0.0123	1.8828
	100	1.4967	0.5580	3.0129	1.3034	0.0077	1.0662
	1,000	1.4598	0.5569	2.8518	0.3166	0.0037	0.1168
(1, 1, 0.5)	25	1.3000	1.0964	0.5776	2.7834	0.0894	0.0749
	50	1.3466	1.0334	0.5219	2.8932	0.0350	0.0371
	100	1.3719	1.0273	0.5045	3.0261	0.0212	0.0233
	1,000	1.0032	0.9969	0.4998	0.1713	0.0026	0.0023
(1, 1, 1)	25	1.3200	1.1005	1.1675	3.0216	0.1026	0.3108
	50	1.2637	1.0252	1.0322	2.6142	0.0355	0.1320
	100	1.3104	1.0166	0.9948	2.1904	0.0191	0.0829
	1,000	1.0219	0.9988	0.9986	0.1945	0.0027	0.0094
(1, 1, 3)	25	1.2924	1.0654	3.3424	3.6450	0.0829	2.3216
	50	1.3889	1.0396	3.0734	2.8758	0.0385	1.1814
	100	1.5086	1.0122	2.9152	4.2378	0.0169	0.8758
	1,000	0.9866	0.9965	3.0187	0.2256	0.0028	0.0884
(1, 2, 0.5)	25	1.5860	2.1543	0.5352	5.0030	0.3800	0.0513
	50	1.4729	2.0686	0.5097	3.5168	0.1576	0.0287
	100	1.3930	2.0188	0.4878	2.4579	0.0699	0.0155
	1,000	0.9585	1.9934	0.5048	0.2391	0.0091	0.0019
(1, 2, 1)	25	1.5066	2.1650	1.0866	4.8697	0.3719	0.2086
	50	1.5486	2.0625	0.9981	3.8682	0.1481	0.1109
	100	1.5543	2.0172	0.9654	3.7393	0.0698	0.0733
	1,000	1.0532	1.9910	0.9955	0.5855	0.0090	0.0124
(1, 2, 3)	25	1.9087	2.1546	3.1363	8.3165	0.3391	1.8556
	50	1.7698	2.0534	2.9155	6.4903	0.1436	1.0829
	100	1.8722	2.0026	2.7825	6.3461	0.0771	0.8426
	1,000	1.1084	1.9914	2.9564	0.6749	0.0088	0.1253

Table 3.1 The averages of MLEs and mean-squared errors of λ, α, and β of three-parameter GZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$			MSE		
		λ	α	β	λ	α	β
(3, 0.5, 0.5)	25	2.2581	0.5066	1.1557	3.1625	0.0798	1.4435
	50	2.5958	0.4979	0.8024	2.2398	0.0607	0.3609
	100	2.8465	0.5176	0.6779	1.5995	0.0029	0.1569
	1,000	3.3409	0.5252	0.4849	0.4609	0.0009	0.0124
(3, 0.5, 1)	25	2.3662	0.5290	2.0800	3.4972	0.0244	3.6110
	50	2.7170	0.5188	1.5965	2.0937	0.0358	1.4680
	100	2.9538	0.5286	1.3352	1.5106	0.0033	0.6079
	1,000	3.4454	0.5321	0.9513	0.6106	0.0013	0.0537
(3, 0.5, 3)	25	3.1659	0.5528	4.9514	8.7717	0.0135	13.6218
	50	3.2917	0.5500	3.9965	5.1780	0.0076	6.2965
	100	3.3447	0.5454	3.5835	2.5219	0.0042	3.7413
	1,000	3.5618	0.5501	2.8633	0.6532	0.0027	0.4091
(3, 1, 0.5)	25	1.8407	1.0242	0.8742	4.6063	0.0613	0.3655
	50	2.3703	0.9938	0.7247	4.7895	0.0286	0.1791
	100	2.6764	0.9866	0.6363	4.0772	0.0134	0.0966
	1,000	2.9950	0.9952	0.5212	0.9076	0.0012	0.0164
(3, 1, 1)	25	1.9583	1.0400	1.7700	5.3592	0.0723	1.7058
	50	2.4374	0.9844	1.3904	5.5309	0.0280	0.6318
	100	2.7028	0.9861	1.2371	3.9633	0.0124	0.3438
	1,000	3.0684	0.9936	1.0238	1.1616	0.0012	0.0680
(3, 1, 3)	25	2.4280	1.0067	4.6323	8.7838	0.0632	8.5958
	50	2.8545	0.9974	4.0063	9.0171	0.0261	5.0420
	100	2.7438	0.9906	3.7468	4.0199	0.0141	3.2714
	1,000	3.0290	0.9945	3.1027	1.1181	0.0011	0.6060
(3, 2, 0.5)	25	2.1893	2.1885	0.7528	6.4763	0.3891	0.2028
	50	2.3435	2.0725	0.6690	4.9520	0.1478	0.1043
	100	2.5475	2.0226	0.6047	3.5585	0.0715	0.0578
	1,000	3.1162	1.9880	0.5039	1.2259	0.0093	0.0122
(3, 2, 1)	25	2.0328	2.1913	1.5482	6.4470	0.3714	0.8331
	50	2.3123	2.0505	1.3129	5.1930	0.1424	0.3837
	100	2.5887	2.0112	1.1908	3.8080	0.0650	0.2196
	1,000	3.0010	1.9898	1.0287	1.1619	0.0089	0.0510
(3, 2, 3)	25	2.3679	2.1891	4.4930	9.1173	0.3852	6.7864
	50	2.6518	2.0368	3.8053	6.6572	0.1297	3.2989
	100	2.9839	2.0017	3.4082	6.1593	0.0699	1.8382
	1,000	2.9390	1.9959	3.1194	0.9928	0.0077	0.4305

Table 3.2 Coverage probabilities and average lengths of Wald CIs of λ, α, and β of three-parameter GZTP distribution.

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	CP			AL		
		λ	α	β	λ	α	β
(0.5, 0.5, 0.5)	25	0.9880	0.9540	0.9690	5.4513	0.7112	1.2275
	50	0.9920	0.9460	0.9660	4.2286	0.4947	0.8325
	100	0.9890	0.9310	0.9660	3.1824	0.3585	0.5914
	1,000	0.9260	0.8300	0.9530	1.2482	0.1165	0.1820
(0.5, 0.5, 1)	25	0.9900	0.9610	0.9730	5.3968	0.7132	2.4445
	50	0.9790	0.9340	0.9540	4.3558	0.4874	1.6881
	100	0.9570	0.8940	0.9440	3.1788	0.3022	1.1776
	1,000	0.8840	0.7200	0.9350	1.3870	0.1092	0.3614
(0.5, 0.5, 3)	25	0.9780	0.9470	0.9630	5.6556	0.7167	7.1720
	50	0.9920	0.9370	0.9780	4.4099	0.5113	5.2266
	100	0.9900	0.9120	0.9740	3.2785	0.3602	3.6956
	1,000	0.8220	0.4680	0.9510	1.3263	0.1155	1.1309
(0.5, 1, 0.5)	25	0.9930	0.9860	0.9720	6.0201	1.4056	1.0042
	50	0.9890	0.9650	0.9570	5.0474	1.0010	0.7359
	100	0.9850	0.9710	0.9660	3.7258	0.7406	0.5279
	1,000	0.9760	0.9560	0.9710	1.2920	0.2374	0.1571
(0.5, 1, 1)	25	0.9910	0.9850	0.9680	6.1005	1.3875	2.0094
	50	0.9860	0.9710	0.9530	4.9199	0.9727	1.4473
	100	0.9770	0.9700	0.9460	3.8920	0.7256	1.0490
	1,000	0.9700	0.9520	0.9660	1.3021	0.2350	0.3128
(0.5, 1, 3)	25	0.9720	0.9810	0.9550	6.1029	1.3698	5.9205
	50	0.9630	0.9660	0.9420	5.1336	0.9635	4.1594
	100	0.9700	0.9670	0.9510	3.8017	0.6919	3.0978
	1,000	0.9580	0.9630	0.9670	1.3047	0.2353	0.9318
(0.5, 2, 0.5)	25	0.9770	0.9800	0.9450	7.1705	2.7107	0.9107
	50	0.9730	0.9820	0.9430	5.5092	1.8794	0.6453
	100	0.9660	0.9670	0.9510	4.1372	1.3329	0.4721
	1,000	0.9650	0.9520	0.9580	1.3742	0.4363	0.1384
(0.5, 2, 1)	25	0.9850	0.9830	0.9450	6.9267	2.6711	1.7864
	50	0.9680	0.9750	0.9320	5.5183	1.8476	1.2752
	100	0.9440	0.9610	0.9110	4.3796	1.3200	0.9256
	1,000	0.9670	0.9520	0.9550	1.5377	0.4389	0.2893
(0.5, 2, 3)	25	0.9660	0.9830	0.9260	7.0185	2.6076	5.1895
	50	0.9550	0.9790	0.9290	5.6170	1.8638	3.8276
	100	0.9190	0.9600	0.8860	4.7021	1.2797	2.7239
	1,000	0.9690	0.9490	0.9660	1.4614	0.4397	0.8563

Table 3.2 Coverage probabilities and average lengths of Wald CIs of λ, α, and β of three-parameter GZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	CP			AL		
		λ	α	β	λ	α	β
(1, 0.5, 0.5)	25	0.9940	0.9790	0.9790	5.7482	0.6628	1.4028
	50	0.9900	0.9520	0.9650	4.8063	0.4541	0.9677
	100	0.9940	0.9530	0.9630	3.8393	0.3247	0.7014
	1,000	0.9550	0.8080	0.9630	1.4062	0.1021	0.2183
(1, 0.5, 1)	25	0.9920	0.9660	0.9670	5.7986	0.6470	2.7135
	50	0.9950	0.9500	0.9690	4.6244	0.4599	1.9530
	100	0.9880	0.9310	0.9610	3.7938	0.3209	1.3904
	1,000	0.9170	0.6830	0.9510	1.4298	0.1012	0.4434
(1, 0.5, 3)	25	0.9880	0.9540	0.9740	6.0610	0.6551	8.2274
	50	0.9910	0.9460	0.9690	4.8315	0.4574	5.7802
	100	0.9920	0.9090	0.9660	3.9617	0.3193	4.2610
	1,000	0.8450	0.3700	0.9430	1.4569	0.1004	1.3629
(1, 1, 0.5)	25	0.9870	0.9810	0.9760	6.3322	1.3129	1.1233
	50	0.9870	0.9750	0.9530	5.2818	0.9035	0.7987
	100	0.9640	0.9550	0.9350	4.2110	0.6491	0.5912
	1,000	0.9680	0.9490	0.9600	1.6316	0.2026	0.1917
(1, 1, 1)	25	0.9940	0.9780	0.9750	6.1752	1.3005	2.2503
	50	0.9910	0.9780	0.9640	5.2569	0.9311	1.6071
	100	0.9820	0.9690	0.9500	4.1571	0.6409	1.1810
	1,000	0.9620	0.9500	0.9670	1.6548	0.2030	0.3883
(1, 1, 3)	25	0.9880	0.9850	0.9640	6.1128	1.2790	6.4809
	50	0.9900	0.9730	0.9550	5.3091	0.9000	4.8128
	100	0.9630	0.9730	0.9170	4.2245	0.6223	3.4278
	1,000	0.9490	0.9520	0.9520	1.6065	0.2035	1.1412
(1, 2, 0.5)	25	0.9880	0.9700	0.9580	7.2002	2.5449	0.9647
	50	0.9750	0.9720	0.9540	5.6544	1.7725	0.7064
	100	0.9690	0.9740	0.9560	4.5368	1.2418	0.5177
	1,000	0.9650	0.9490	0.9530	1.7983	0.3781	0.1740
(1, 2, 1)	25	0.9790	0.9820	0.9440	6.6658	2.5445	1.8803
	50	0.9780	0.9760	0.9360	5.6802	1.7247	1.3758
	100	0.9500	0.9610	0.9250	4.5500	1.2048	1.0212
	1,000	0.9660	0.9480	0.9440	1.9536	0.3830	0.3609
(1, 2, 3)	25	0.9680	0.9710	0.9300	7.1747	2.4774	5.5280
	50	0.9560	0.9720	0.9250	5.6561	1.6977	3.9512
	100	0.9360	0.9520	0.8920	5.5527	1.3140	3.1918
	1,000	0.9520	0.9630	0.9380	1.9084	0.3690	1.0682

Table 3.2 Coverage probabilities and average lengths of Wald CIs of λ, α, and β of three-parameter GZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	CP			AL		
		λ	α	β	λ	α	β
(3, 0.5, 0.5)	25	0.9780	0.9760	0.9660	7.0184	0.5058	2.8384
	50	0.9270	0.9710	0.9390	6.6032	0.3243	1.7841
	100	0.9240	0.9720	0.9400	5.9674	0.2316	1.3447
	1,000	0.9620	0.7040	0.9520	2.9142	0.0683	0.5346
(3, 0.5, 1)	25	0.9810	0.9760	0.9800	7.3171	0.5165	5.2640
	50	0.9520	0.9770	0.9520	6.7792	0.3308	3.6215
	100	0.9500	0.9620	0.9530	6.1961	0.2311	2.7230
	1,000	0.9560	0.5400	0.9420	2.9932	0.0677	1.0679
(3, 0.5, 3)	25	0.9640	0.9770	0.9410	8.5732	0.5030	14.2200
	50	0.9550	0.9630	0.9440	7.6418	0.3377	9.9442
	100	0.9610	0.9310	0.9520	6.5839	0.2279	7.8336
	1,000	0.9530	0.1260	0.9640	3.4755	0.0690	3.5976
(3, 1, 0.5)	25	0.9810	0.9760	0.9650	6.9383	1.1356	1.8177
	50	0.9160	0.9630	0.9100	6.7072	0.7518	1.3075
	100	0.8550	0.9630	0.8600	5.8902	0.4944	0.9957
	1,000	0.8580	0.9570	0.8640	3.2721	0.1392	0.4575
(3, 1, 1)	25	0.9700	0.9700	0.9600	7.0218	1.1292	3.6442
	50	0.9180	0.9640	0.9010	6.6439	0.7411	2.5236
	100	0.8740	0.9710	0.8620	6.1201	0.4907	1.9509
	1,000	0.8550	0.9360	0.8460	3.3319	0.1380	0.9192
(3, 1, 3)	25	0.9570	0.9690	0.9610	7.4582	1.0586	10.2976
	50	0.9150	0.9660	0.9040	7.1191	0.7375	7.6598
	100	0.8850	0.9520	0.8750	6.1088	0.4911	5.9027
	1,000	0.8570	0.9600	0.8680	3.4053	0.1412	2.7556
(3, 2, 0.5)	25	0.9730	0.9630	0.9470	7.6358	2.4227	1.3939
	50	0.9230	0.9640	0.8860	6.7973	1.6406	1.0402
	100	0.8670	0.9560	0.8500	6.0633	1.1289	0.8032
	1,000	0.8430	0.9210	0.8430	3.5504	0.3665	0.3931
(3, 2, 1)	25	0.9700	0.9830	0.9560	7.6184	2.4848	2.8358
	50	0.9160	0.9680	0.8850	6.6524	1.6306	2.0046
	100	0.8860	0.9580	0.8570	6.0271	1.1183	1.5729
	1,000	0.8430	0.9250	0.8240	3.3930	0.3521	0.7507
(3, 2, 3)	25	0.9570	0.9670	0.9330	7.7846	2.4468	8.4552
	50	0.9260	0.9600	0.8920	7.1660	1.6039	6.0783
	100	0.8900	0.9380	0.8510	6.5345	1.0968	4.7316
	1,000	0.8410	0.9430	0.8430	3.3893	0.3592	2.2788

Table 3.3 The averages of MLEs and mean-squared errors of α and β of of two-parameter GZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
			α	β	α	β
0.5	(1, 0.5)	25	1.0863	0.5703	0.0837	0.0472
		50	1.0473	0.5390	0.0386	0.0202
		100	1.0237	0.5189	0.0154	0.0078
		1,000	1.0035	0.5028	0.0015	0.0007
	(1, 1)	25	1.1107	1.1634	0.1067	0.2090
		50	1.0546	1.0831	0.0410	0.0825
		100	1.0222	1.0335	0.0157	0.0347
		1,000	1.0025	1.0032	0.0014	0.0029
	(1, 3)	25	1.1102	3.4834	0.0996	1.8198
		50	1.0365	3.2002	0.0336	0.6986
		100	1.0252	3.1244	0.0154	0.3003
		1,000	1.0029	3.0180	0.0014	0.0247
	(2, 0.5)	25	2.2284	0.5770	0.4436	0.0453
		50	2.1062	0.5370	0.1668	0.0165
		100	2.0644	0.5218	0.0760	0.0072
		1,000	2.0144	0.5050	0.0066	0.0006
	(2, 1)	25	2.2438	1.1501	0.4768	0.1810
		50	2.1056	1.0662	0.1639	0.0646
		100	2.0601	1.0404	0.0846	0.0317
		1,000	2.0062	1.0030	0.0061	0.0022
	(2, 3)	25	2.2237	3.4450	0.4222	1.4955
		50	2.1325	3.2446	0.1753	0.6111
		100	2.0619	3.1110	0.0721	0.2462
		1,000	2.0022	3.0056	0.0064	0.0221
1	(1, 0.5)	25	1.0948	0.5834	0.0861	0.0576
		50	1.0522	0.5504	0.0341	0.0254
		100	1.0354	0.5305	0.0157	0.0102
		1,000	1.0010	0.5012	0.0012	0.0008
	(1, 1)	25	1.1025	1.1897	0.0946	0.2654
		50	1.0546	1.0831	0.0410	0.0825
		100	1.0222	1.0335	0.0157	0.0347
		1,000	1.0025	1.0032	0.0014	0.0029
	(1, 3)	25	1.1078	3.5130	0.0834	1.9694
		50	1.0466	3.2483	0.0344	0.8181
		100	1.0222	3.1240	0.0140	0.3268
		1,000	1.0027	3.0174	0.0012	0.0296

Table 3.3 The averages of MLEs and mean-squared errors of α and β of two-parameter GZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
			α	β	α	β
1	(2, 0.5)	25	2.2103	0.5706	0.4146	0.0447
		50	2.1201	0.5436	0.1681	0.0197
		100	2.0541	0.5173	0.0697	0.0072
		1,000	2.0101	0.5039	0.0067	0.0007
	(2, 1)	25	2.2186	1.1492	0.3595	0.1642
		50	2.1123	1.0762	0.1617	0.0732
		100	2.0594	1.0405	0.0684	0.0293
		1,000	2.0059	1.0055	0.0061	0.0027
	(2, 3)	25	2.2262	3.4247	0.4026	1.5185
		50	2.0966	3.1738	0.1537	0.5978
		100	2.0533	3.1091	0.0736	0.2937
		1,000	2.0035	3.0095	0.0060	0.0235
3	(1, 0.5)	25	1.0803	0.6087	0.0647	0.1006
		50	1.0364	0.5510	0.0275	0.0365
		100	1.0177	0.5257	0.0112	0.0135
		1,000	1.0030	0.5046	0.0011	0.0012
	(1, 1)	25	1.0845	1.2358	0.0695	0.4004
		50	1.0378	1.1056	0.0241	0.1329
		100	1.0138	1.0473	0.0114	0.0559
		1,000	1.0009	1.0005	0.0010	0.0050
	(1, 3)	25	1.0941	3.7408	0.0756	3.9448
		50	1.0391	3.2978	0.0259	1.2932
		100	1.0240	3.1851	0.0131	0.5590
		1,000	1.0023	3.0127	0.0010	0.0428
	(2, 0.5)	25	2.2175	0.6011	0.3851	0.0714
		50	2.1007	0.5474	0.1303	0.0238
		100	2.0522	0.5220	0.0649	0.0109
		1,000	2.0047	0.5029	0.0051	0.0009
	(2, 1)	25	2.2155	1.2030	0.3944	0.3164
		50	2.0773	1.0632	0.1312	0.0875
		100	2.0436	1.0356	0.0575	0.0387
		1,000	2.0036	1.0040	0.0054	0.0035
	(2, 3)	25	2.2060	3.5397	0.4129	2.7547
		50	2.0833	3.2492	0.1296	0.8799
		100	2.0459	3.1071	0.0613	0.3610
		1,000	2.0043	3.0095	0.0055	0.0328

Table 3.4 Coverage probabilities and average lengths of Wald CIs of α and β of two-parameter GZTP distribution.

λ	$\boldsymbol{\theta} = (\alpha, \beta)$	n	CP		AL	
			α	β	α	β
0.5	(1, 0.5)	25	0.9680	0.9450	1.0176	0.7388
		50	0.9360	0.9520	0.6901	0.4968
		100	0.9600	0.9580	0.4755	0.3394
		1,000	0.9420	0.9520	0.1470	0.1043
	(1, 1)	25	0.9640	0.9650	1.0428	1.5028
		50	0.9510	0.9550	0.6956	0.9973
		100	0.9440	0.9450	0.4748	0.6760
		1,000	0.9540	0.9530	0.1468	0.2082
	(1, 3)	25	0.9580	0.9650	1.0427	4.5019
		50	0.9550	0.9520	0.6823	2.9543
		100	0.9550	0.9580	0.4763	2.0433
		1,000	0.9560	0.9620	0.1469	0.6265
	(2, 0.5)	25	0.9570	0.9620	2.2254	0.6919
		50	0.9550	0.9560	1.4793	0.4570
		100	0.9550	0.9570	1.0235	0.3145
		1,000	0.9510	0.9480	0.3151	0.0965
	(2, 1)	25	0.9610	0.9610	2.2423	1.3787
		50	0.9590	0.9510	1.4789	0.9074
		100	0.9500	0.9530	1.0210	0.6271
		1,000	0.9560	0.9580	0.3137	0.1916
	(2, 3)	25	0.9700	0.9640	2.2171	4.1250
		50	0.9570	0.9530	1.5004	2.7610
		100	0.9570	0.9570	1.0215	1.8740
		1,000	0.9590	0.9530	0.3130	0.5741
1	(1, 0.5)	25	0.9570	0.9530	0.9835	0.7955
		50	0.9520	0.9510	0.6639	0.5344
		100	0.9520	0.9500	0.4608	0.3654
		1,000	0.9540	0.9560	0.1401	0.1099
	(1, 1)	25	0.9580	0.9630	0.9916	1.6189
		50	0.9510	0.9550	0.6956	0.9973
		100	0.9440	0.9450	0.4748	0.6760
		1,000	0.9540	0.9530	0.1468	0.2082
	(1, 3)	25	0.9630	0.9660	0.9970	4.7827
		50	0.9570	0.9560	0.6601	3.1576
		100	0.9540	0.9500	0.4541	2.1568
		1,000	0.9520	0.9500	0.1404	0.6611

Table 3.4 Coverage probabilities and average lengths of Wald CIs of α and β of two-parameter GZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
			α	β	α	β
1	(2, 0.5)	25	0.9560	0.9640	2.1421	0.7115
		50	0.9540	0.9580	1.4448	0.4807
		100	0.9530	0.9590	0.9860	0.3242
		1,000	0.9410	0.9330	0.3044	0.1001
	(2, 1)	25	0.9690	0.9710	2.1501	1.4321
		50	0.9560	0.9580	1.4393	0.9520
		100	0.9580	0.9560	0.9890	0.6521
		1,000	0.9480	0.9380	0.3037	0.1998
	(2, 3)	25	0.9730	0.9610	2.1652	4.2785
		50	0.9490	0.9570	1.4277	2.8102
		100	0.9490	0.9430	0.9859	1.9493
		1,000	0.9490	0.9460	0.3033	0.5981
3	(1, 0.5)	25	0.9660	0.9580	0.8848	1.0147
		50	0.9450	0.9550	0.5940	0.6585
		100	0.9570	0.9600	0.4103	0.4470
		1,000	0.9500	0.9540	0.1273	0.1362
	(1, 1)	25	0.9750	0.9630	0.8885	2.0569
		50	0.9620	0.9610	0.5950	1.3222
		100	0.9470	0.9500	0.4084	0.8911
		1,000	0.9500	0.9450	0.1270	0.2702
	(1, 3)	25	0.9580	0.9560	0.9114	6.3434
		50	0.9560	0.9570	0.5968	3.9462
		100	0.9350	0.9410	0.4134	2.7035
		1,000	0.9610	0.9570	0.1271	0.8134
	(2, 0.5)	25	0.9510	0.9550	2.0440	0.8671
		50	0.9630	0.9650	1.3543	0.5634
		100	0.9380	0.9520	0.9311	0.3810
		1,000	0.9480	0.9610	0.2862	0.1165
	(2, 1)	25	0.9460	0.9460	2.0474	1.7405
		50	0.9530	0.9510	1.3368	1.0953
		100	0.9460	0.9470	0.9266	0.7565
		1,000	0.9460	0.9510	0.2859	0.2324
	(2, 3)	25	0.9560	0.9470	2.0619	5.1945
		50	0.9570	0.9590	1.3420	3.3463
		100	0.9610	0.9550	0.9287	2.2719
		1,000	0.9460	0.9470	0.2861	0.6968

Table 3.5 The averages of MLEs, mean-squared errors, coverage probabilities, and average lengths of Wald CIs of λ of one-parameter GZTP distribution.

λ	(α, β)	n	$AV(\hat{\lambda})$	MSE	CP	AL
0.5	(0.5, 0.5)	25	0.7859	0.3581	0.9740	2.1347
		50	0.6336	0.1723	0.9730	1.5597
		100	0.5280	0.0896	0.9830	1.1537
		1,000	0.4848	0.0123	0.9560	0.4316
	(0.5, 1)	25	0.7765	0.3415	0.9800	2.1304
		50	0.6177	0.1678	0.9770	1.5466
		100	0.5323	0.0978	0.9780	1.1505
		1,000	0.4678	0.0127	0.9470	0.4315
	(0.5, 3)	25	0.7579	0.3391	0.9770	2.1077
		50	0.6095	0.1479	0.9860	1.5464
		100	0.5208	0.0854	0.9820	1.1507
		1,000	0.4490	0.0142	0.9360	0.4312
	(1, 0.5)	25	0.7904	0.3599	0.9780	2.1459
		50	0.6524	0.1954	0.9700	1.5697
		100	0.5558	0.1010	0.9690	1.1690
		1,000	0.5042	0.0127	0.9450	0.4323
	(1, 1)	25	0.7945	0.3921	0.9690	2.1429
		50	0.6746	0.2010	0.9690	1.5898
		100	0.5618	0.1035	0.9720	1.1722
		1,000	0.5049	0.0116	0.9540	0.4322
	(1, 3)	25	0.8242	0.4215	0.9670	2.1655
		50	0.6452	0.1725	0.9790	1.5722
		100	0.5539	0.0950	0.9720	1.1722
		1,000	0.5059	0.0122	0.9580	0.4321
	(2, 0.5)	25	0.8088	0.4188	0.9670	2.1522
		50	0.6605	0.1944	0.9710	1.5770
		100	0.5637	0.0961	0.9770	1.1773
		1,000	0.5002	0.0119	0.9560	0.4320
	(2, 1)	25	0.7741	0.3625	0.9750	2.1212
		50	0.6646	0.1974	0.9690	1.5807
		100	0.5580	0.0997	0.9730	1.1716
		1,000	0.4980	0.0116	0.9510	0.4322
	(2, 3)	25	0.8035	0.3868	0.9700	2.1507
		50	0.6700	0.1959	0.9730	1.5856
		100	0.5571	0.0989	0.9710	1.1717
		1,000	0.4925	0.0117	0.9590	0.4321

Table 3.5 The averages of MLEs, mean-squared errors, coverage probabilities, and average lengths of Wald CIs of λ of one-parameter GZTP distribution. (Cont.)

λ	(α, β)	n	$AV(\hat{\lambda})$	MSE	CP	AL
1	(0.5, 0.5)	25	1.0961	0.3932	0.9860	2.3947
		50	1.0359	0.2296	0.9740	1.8228
		100	0.9641	0.1253	0.9520	1.3527
		1,000	0.9775	0.0125	0.9520	0.4397
	(0.5, 1)	25	1.0975	0.4389	0.9760	2.3890
		50	1.0206	0.2251	0.9770	1.8161
		100	0.9693	0.1220	0.9590	1.3548
		1,000	0.9611	0.0134	0.9490	0.4394
	(0.5, 3)	25	1.0812	0.3897	0.9820	2.3834
		50	0.9867	0.2167	0.9840	1.7952
		100	0.9579	0.1234	0.9510	1.3499
		1,000	0.9301	0.0171	0.9040	0.4388
	(1, 0.5)	25	1.1555	0.4255	0.9790	2.4412
		50	1.0567	0.2354	0.9740	1.8300
		100	1.0188	0.1277	0.9480	1.3655
		1,000	1.0035	0.0119	0.9510	0.4404
	(1, 1)	25	1.1256	0.4531	0.9760	2.4057
		50	1.0375	0.2303	0.9790	1.8204
		100	1.0007	0.1301	0.9570	1.3605
		1,000	1.0016	0.0124	0.9610	0.4403
	(1, 3)	25	1.1389	0.4275	0.9760	2.4282
		50	1.0462	0.2320	0.9730	1.8282
		100	0.9921	0.1292	0.9510	1.3591
		1,000	1.0006	0.0120	0.9540	0.4402
	(2, 0.5)	25	1.1029	0.4031	0.9810	2.3998
		50	1.0485	0.2335	0.9790	1.8260
		100	1.0187	0.1331	0.9450	1.3644
		1,000	0.9980	0.0120	0.9540	0.4402
	(2, 1)	25	1.1546	0.4163	0.9780	2.4414
		50	1.0215	0.2225	0.9780	1.8143
		100	1.0090	0.1255	0.9580	1.3642
		1,000	1.0013	0.0129	0.9480	0.4403
	(2, 3)	25	1.1309	0.4340	0.9760	2.4138
		50	1.0163	0.2209	0.9790	1.8161
		100	1.0015	0.1276	0.9490	1.3583
		1,000	1.0018	0.0124	0.9500	0.4403

Table 3.5 The averages of MLEs, mean-squared errors, coverage probabilities, and average lengths of Wald CIs of λ of one-parameter GZTP distribution. (Cont.)

λ	(α, β)	n	$AV(\hat{\lambda})$	MSE	CP	AL
3	(0.5, 0.5)	25	3.0724	0.8139	0.9470	3.3821
		50	2.9447	0.3777	0.9460	2.3442
		100	2.9607	0.1779	0.9440	1.6544
		1,000	2.9563	0.0189	0.9470	0.5214
	(0.5, 1)	25	2.9995	0.7014	0.9670	3.3515
		50	2.9192	0.3447	0.9510	2.3349
		100	2.9565	0.1760	0.9530	1.6534
		1,000	2.9206	0.0221	0.9240	0.5193
	(0.5, 3)	25	2.9897	0.7357	0.9510	3.3456
		50	2.9023	0.3675	0.9420	2.3327
		100	2.8990	0.1765	0.9480	1.6423
		1,000	2.8772	0.0331	0.8360	0.5166
	(1, 0.5)	25	3.1068	0.8263	0.9540	3.3978
		50	3.0164	0.3689	0.9530	2.3628
		100	3.0065	0.1686	0.9620	1.6632
		1,000	2.9980	0.0174	0.9470	0.5241
	(1, 1)	25	3.0998	0.8007	0.9500	3.3940
		50	3.0205	0.3574	0.9610	2.3643
		100	3.0102	0.1916	0.9450	1.6645
		1,000	2.9961	0.0177	0.9490	0.5239
	(1, 3)	25	3.1218	0.8182	0.9570	3.4036
		50	3.0285	0.3987	0.9450	2.3673
		100	3.0366	0.1854	0.9470	1.6695
		1,000	2.9966	0.0179	0.9480	0.5241
	(2, 0.5)	25	3.1194	0.8179	0.9570	3.4036
		50	3.0409	0.3709	0.9490	2.3694
		100	3.0173	0.1820	0.9530	1.6663
		1,000	2.9960	0.0180	0.9590	0.5240
	(2, 1)	25	3.1299	0.7898	0.9610	3.4067
		50	3.0893	0.4009	0.9500	2.3834
		100	3.0124	0.1828	0.9500	1.6640
		1,000	3.0004	0.0177	0.9530	0.5242
	(2, 3)	25	3.0829	0.8012	0.9560	3.3870
		50	3.0579	0.3508	0.9570	2.3741
		100	3.0284	0.1743	0.9590	1.6680
		1,000	3.0033	0.0184	0.9500	0.5244

CHAPTER 4
COMPLEMENTARY GAMMA ZERO-TRUNCATED POISSON DISTRIBUTION

This chapter covers the properties of the CGZTP distribution and shows how its density function is derived. In this section, we derive the quantile, survival function, hazard function, and moment-generating functions. In addition, the discussion covers the generation of random numbers for the CGZTP distribution. Following this, further discussions are provided on parameter estimation via maximum likelihood, inference for large samples, and simulation studies.

4.1 Distribution function

Let $X_1, X_2, \ldots, X_N$ be N independent and identically distributed (iid) random variables from gamma distribution with following probability density function:

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a rate parameter, and $\Gamma(\alpha)$ is a gamma function of α, and N is itself a random variable with a zero-truncated Poisson distribution and independence of X_i's. The probability mass function of N is

$$P(N=n) = \frac{e^{-\lambda}\lambda^{n}}{n!\left(1-e^{-\lambda}\right)}, \quad n = 1, 2, \ldots \text{ and } \lambda > 0.$$

Assuming that a random variables X and N are independent, we define $Z = \max\{X_1, X_2, \ldots, X_N\}$. Then, $g(z|n) = n\left[F(z)\right]^{n-1} f(z)$, where $f(z)$ is pdf and $F(z)$ is the cdf of Z. The joint distribution between Z and N are obtained as follow:

$$
\begin{aligned}
g(z,n) &= g(z \mid n)\, p(N=n) \\
&= n\left[F(z)\right]^{n-1} f(z)\left(\frac{e^{-\lambda}\lambda^{n}}{n!\left(1-e^{-\lambda}\right)}\right) \\
&= \frac{\lambda e^{-\lambda} f(z)}{\left(1-e^{-\lambda}\right)} \frac{\left[\lambda F(z)\right]^{n-1}}{(n-1)!}
\end{aligned}
$$

and the marginal distribution for Z is

$$
\begin{aligned}
g(z;\alpha,\beta,\lambda) &= \sum_{n=1}^{\infty} g(z,n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda e^{-\lambda} f(z)}{\left(1-e^{-\lambda}\right)} \frac{\left[\lambda F(z)\right]^{n-1}}{(n-1)!} \\
&= \frac{\lambda e^{-\lambda} f(z)}{\left(1-e^{-\lambda}\right)} \sum_{n=1}^{\infty} \frac{\left[\lambda F(z)\right]^{n-1}}{(n-1)!} \\
&= \frac{\lambda e^{-\lambda} f(z)}{\left(1-e^{-\lambda}\right)} e^{\lambda F(z)},
\end{aligned}
$$

where $F(z) = 1 - \dfrac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}$ and $\Gamma(\alpha,\beta z) = \int_{\beta z}^{\infty} t^{\alpha-1} e^{-t} dt$ is the upper incomplete gamma function. The pdf of the gamma- Poisson distribution with maximum compound function is given by

$$
g(z;\boldsymbol{\theta}) = \frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)} \left(\frac{\beta^{\alpha} z^{\alpha-1} e^{-\beta z}}{\Gamma(\alpha)}\right) e^{\lambda\left(1-\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}\right)} \tag{4.1}
$$

where $\boldsymbol{\theta} = (\lambda,\alpha,\beta)$

The distribution of Z will be denoted as the Complementary gamma zero-truncated Poisson distribution (CGZTP), and the probability density function plots for specific parameter values are shown in Figure 4.1. When $\alpha = 1$, the CGZTP distribution simplifies to the density of the Complementary Exponential-Poisson distribution, which was presented by Cancho et al. (2011).

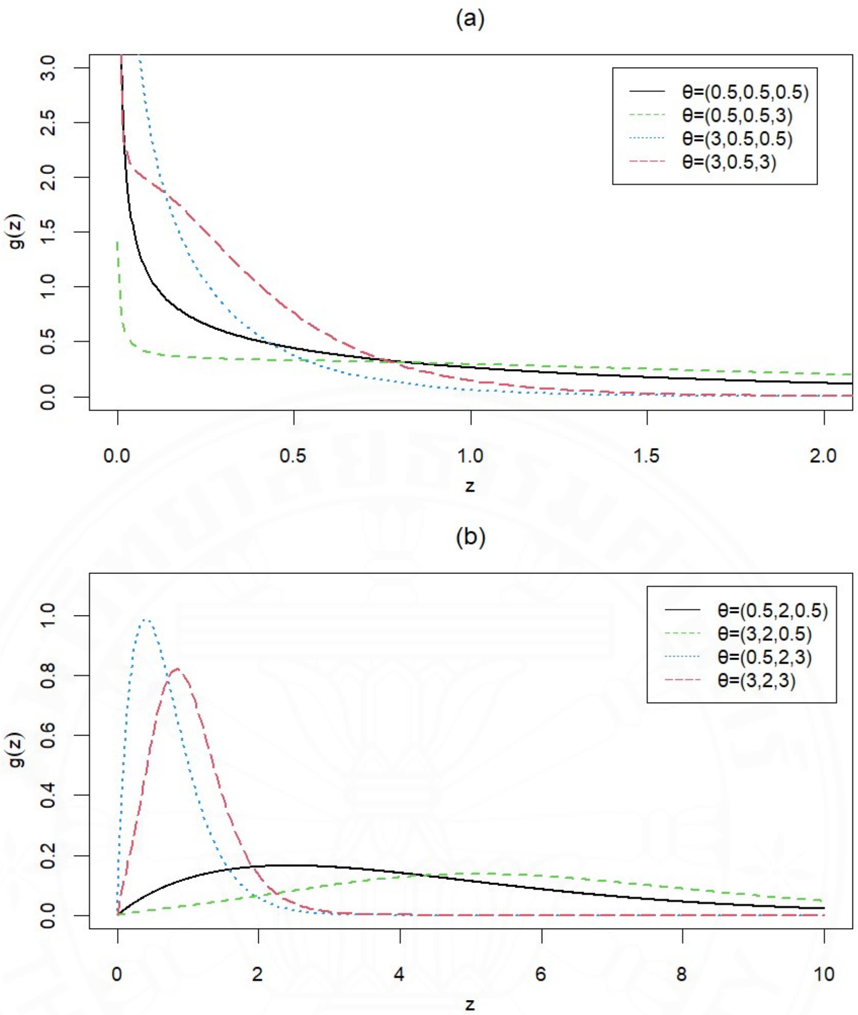

Figure 4.1. Probability density functions of the CGZTP distribution with $\boldsymbol{\theta}=(\lambda,\alpha,\beta)$, (a) $\alpha=0.5$ and (b) $\alpha=2$.

Theorem 4.1 Considering the CGZTP distribution with the pdf of equation (4.1). The CGZTP distribution reduces to a two-parameter Gamma distribution as λ approaches 0.

Proof. If λ approaches to zero, then

$$\lim_{\lambda\to 0} g(z;\boldsymbol{\theta}) = \lim_{\lambda\to 0}\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha} z^{\alpha-1} e^{-\beta z}}{\Gamma(\alpha)}\right) e^{\lambda\left(1-\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}\right)}$$

$$= \frac{\beta^{\alpha} z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda}{\left(1-e^{-\lambda}\right)}\right)\left(\lim_{\lambda\to 0} e^{-\beta z-\lambda\left(\frac{\Gamma(\alpha,Bz)}{\Gamma(\alpha)}\right)}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda}{\left(1-e^{-\lambda}\right)}\right)\left(e^{-\beta z}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda}{e^{-\lambda}\left(e^{\lambda}-1\right)}\right)\left(e^{-\beta z}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda e^{\lambda}}{\left(e^{\lambda}-1\right)}\right)\left(e^{-\beta z}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda}{\left(e^{\lambda}-1\right)}\right)\left(\lim_{\lambda\to 0}e^{\lambda}\right)\left(e^{-\beta z}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\frac{d\lambda}{d\lambda}}{\frac{d}{d\lambda}\left(e^{\lambda}-1\right)}\right)(1)\left(e^{-\beta z}\right) \qquad \text{, } \textit{L'hospital's rule}$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{1}{e^{\lambda}}\right)(1)\left(e^{-\beta z}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}}{\Gamma(\alpha)}(1)(1)\left(e^{-\beta z}\right)$$

$$=\frac{\beta^{\alpha}z^{\alpha-1}e^{-\beta z}}{\Gamma(\alpha)}.$$

Subsequently, the CGZTP distribution simplifies to a gamma distribution with two parameters.

CGZTP's cumulative distribution function is represented by the expression

$$\begin{aligned} G(z;\boldsymbol{\theta}) &= \int_0^z g(z)dz \\ &= \int_0^z \frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha}z^{\alpha-1}e^{-\beta z}}{\Gamma(\alpha)}\right)e^{\lambda\left(1-\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}\right)}dz \\ &= \frac{\lambda\beta^{\alpha}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\int_0^z z^{\alpha-1}e^{-\beta z-\lambda\frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}}dz \qquad . \end{aligned}$$

Now, solving $\int z^{\alpha-1}e^{-\beta z-\lambda\frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}}dz$ by substitute $u=\Gamma(\alpha,\beta z)$, then $\frac{du}{dz}=-\beta^{\alpha}z^{\alpha-1}e^{-\beta z}$.

Therefore, $\int z^{\alpha-1}e^{-\beta z-\lambda\frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}}dz=\int z^{\alpha-1}e^{-\beta z-\lambda\frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}}\left(-\frac{z^{1-\alpha}e^{\beta z}}{\beta^{\alpha}}du\right)=-\frac{1}{\beta^{\alpha}}\int e^{-\frac{\lambda u}{\Gamma(\alpha)}}du$.

Consider $\int e^{-\frac{\lambda u}{\Gamma(\alpha)}}du$, substitute $v=-\frac{\lambda u}{\Gamma(\alpha)}$, then $\frac{dv}{du}=-\frac{\lambda}{\Gamma(\alpha)}$ and $du=-\frac{\Gamma(\alpha)}{\lambda}dv$, and

$$\int e^{-\frac{\lambda u}{\Gamma(\alpha)}}du=-\frac{\Gamma(\alpha)}{\lambda}\int e^{v}dv=-\frac{\Gamma(\alpha)e^{v}}{\lambda}=-\frac{\Gamma(\alpha)e^{-\frac{\lambda u}{\Gamma(\alpha)}}}{\lambda}.$$

Therefore, $\int z^{\alpha-1}e^{-\beta z-\lambda\frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}}dz=-\frac{1}{\beta^{\alpha}}\int e^{-\frac{\lambda u}{\Gamma(\alpha)}}du=\frac{\Gamma(\alpha)e^{-\frac{\lambda u}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}=\frac{\Gamma(\alpha)e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}$, and then

$$G(z;\boldsymbol{\theta})=\left[\frac{\lambda\beta^{\alpha}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}+C\right)\right]_0^z=\left[\frac{e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}}{\left(1-e^{-\lambda}\right)}+C\right]_0^z$$

$$=\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}-e^{-\frac{\lambda\Gamma(\alpha,0)}{\Gamma(\alpha)}}\right]=\frac{\left(e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}-e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)}.$$

The cumulative distribution function of the CGZTP distribution is

$$G(z;\boldsymbol{\theta})=\begin{cases}\dfrac{\left(e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}-e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)} & , z>0\\ 0 & , \text{Otherwise.}\end{cases} \tag{4.2}$$

The cdf is non-decreasing function with $\lim_{z\to\infty}G(z;\boldsymbol{\theta})=1$ and $\lim_{z\to-\infty}G(z;\boldsymbol{\theta})=0$.

4.2 Some properties of distribution

4.2.1 The r^{th} quantile

The r^{th} quantile for this distribution is defined as the value z_r such that $\Gamma(\alpha,\beta z)=-\frac{\Gamma(\alpha)}{\lambda}\ln\left(r+(1-r)e^{-\lambda}\right)$.

Proof. Since the r^{th} quantile denoted by z_r and $z_r=G^{-1}(r)$. This implies that $G(z_r)=r$. As $Z\sim CGZTP$, then

$$G(z_r;\boldsymbol{\theta})=\frac{\left(e^{-\frac{\lambda\Gamma(\alpha,\beta z_r)}{\Gamma(\alpha)}}-e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)}=r$$

$$e^{-\frac{\lambda\Gamma(\alpha,\beta z_r)}{\Gamma(\alpha)}}-e^{-\lambda}=r\left(1-e^{-\lambda}\right)$$

$$e^{-\frac{\lambda\Gamma(\alpha,\beta z_r)}{\Gamma(\alpha)}}=r-re^{-\lambda}+e^{-\lambda}$$

$$-\frac{\lambda\Gamma(\alpha,\beta z_r)}{\Gamma(\alpha)}=\ln\left(r+(1-r)e^{-\lambda}\right)$$

$$\Gamma(\alpha,\beta z_r)=-\frac{\Gamma(\alpha)}{\lambda}\ln\left(r+(1-r)e^{-\lambda}\right).$$

Hence, it can be solved analytically for z_r to obtain $\Gamma(\alpha,\beta z)=-\frac{\Gamma(\alpha)}{\lambda}\ln\left(r+(1-r)e^{-\lambda}\right)$.

4.2.2 The moment generating function

The moment generating function is defined by

$$\begin{aligned}M_Z(t)=E\left(e^{tZ}\right)&=\int_0^\infty e^{tZ}g(z;\boldsymbol{\theta})dz\\&=\int_0^\infty e^{tz}\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^\alpha z^{\alpha-1}e^{-\beta z}}{\Gamma(\alpha)}\right)e^{\lambda\left(1-\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}\right)}dz\quad.\\&=\frac{\lambda\beta^\alpha}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)}\int_0^\infty z^{\alpha-1}e^{tz-\beta z-\lambda\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}dz\end{aligned}$$

The numerical values of moment can be obtained by using numerical integration. The raw moments of Z are determined from (4.1) by direct integration. the k raw moments are given by

$$E(Z^k)=\frac{d^k}{dt^k}M_Z(t)\Big|_{t=0}=\frac{\lambda\beta^\alpha}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)}\int_0^\infty z^{\alpha-1+k}e^{-\beta z-\lambda\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}dz\ ,\ k\in\mathbb{N}.$$

A direct mathematical expression for the raw moments is not obtainable. However, convergence of the proof of moment can be achieved by applying the comparison theorem to an improper integral. Suppose that $m(z)=z^{\alpha-1+k}e^{-\beta z-\lambda\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}$ and $n(z)=z^{\alpha-1+k}e^{-\beta z}$ are continuous functions with $0\le m(z)\le n(z)$ for $z\ge 0$. Since

$\int_0^\infty n(z)dz = \beta^{-(\alpha+k)}\Gamma(\alpha+k)$, which means that this integral converges, $\int_0^\infty m(z)dz$ also converges. For all values of k , it can be proved that the raw moments of the distribution converge.

4.2.3 The mean and variance

The mean and variance of z are given, respectively, by

$$E(Z) = \frac{\lambda\beta^\alpha}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)}\int_0^\infty z^\alpha e^{-\beta z-\lambda\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}dz \text{ , and}$$

$$Var(Z) = \frac{\lambda\beta^\alpha}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)}\int_0^\infty z^{\alpha+1} e^{-\beta z-\lambda\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}dz - \left[E(Z)\right]^2.$$

4.3 Survival function and Hazard function

Using (4.1) and (4.2), survival function and hazard function of the CGZTP distribution are given by

$$S(z;\boldsymbol{\theta}) = 1-G(z;\boldsymbol{\theta}) = 1-\frac{\left(e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}-e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)} = \frac{\left(1-e^{-\lambda}\right)-\left(e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}-e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)}$$

$$= \frac{1-e^{-\lambda}-e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}+e^{-\lambda}}{\left(1-e^{-\lambda}\right)} = \frac{\left(1-e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)}, \text{ and}$$

$$H(z;\boldsymbol{\theta}) = \frac{g(z;\boldsymbol{\theta})}{s(z;\boldsymbol{\theta})}$$

$$= \frac{\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^\alpha z^{\alpha-1}e^{-\beta z}}{\Gamma(\alpha)}\right)e^{\lambda\left(1-\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}\right)}}{\frac{\left(1-e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)}} = \frac{\lambda\beta^\alpha z^{\alpha-1}e^{-\beta z-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}}{\Gamma(\alpha)\left(1-e^{-\frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}\right)}$$

We define the function $\eta(z) = -\frac{g'(z;\boldsymbol{\theta})}{g(z;\boldsymbol{\theta})}$, then

$$\eta(z) = -\frac{\dfrac{\lambda\beta^{\alpha} z^{\alpha-2} e^{-\beta z - \frac{\lambda\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\left[\alpha - 1 - \beta z + \dfrac{\lambda(\beta z)^{\alpha} e^{-\beta z}}{\Gamma(\alpha)}\right]}{\dfrac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\dfrac{\beta^{\alpha} z^{\alpha-1} e^{-\beta z}}{\Gamma(\alpha)}\right) e^{\lambda\left(1-\frac{\Gamma(\alpha,\beta z)}{\Gamma(\alpha)}\right)}} = -\frac{1}{z}\left[\alpha - 1 - \beta z + \frac{\lambda(\beta z)^{\alpha} e^{-\beta z}}{\Gamma(\alpha)}\right],$$

and $\eta'(z) = \frac{1}{\Gamma(\alpha) z^2}\left[(\alpha-1)\Gamma(\alpha) + \lambda(\beta z)^{\alpha}(\beta z - \alpha + 1)e^{-\beta z}\right]$. For $\alpha = 1$, $\eta'(z) > 0$ for all z. Then CGZTP distribution has an increasing hazard function that follows from Glaser (1980). However, in cases $\alpha > 1$ or $0 < \alpha < 1$, the sign of $\eta'(z)$ relates to all parameters of the distribution. For example, when $\alpha > 1$, $\eta'(z)$ is greater than 0 if $\beta z - \alpha + 1 > 0$. This condition depends on the value of z and parameter β. And if $\beta z - \alpha + 1 < 0$, the sign of $\eta'(z)$ will follow the sign of $(\alpha-1)\Gamma(\alpha) + \lambda(\beta z)^{\alpha}(\beta z - \alpha + 1)e^{-\beta z}$. In the latter case, the shape of hazard function can bathtub. When $0 < \alpha < 1$, the sign of $\eta'(z)$ will depend on the sign of $(\alpha-1)\Gamma(\alpha) + \lambda(\beta z)^{\alpha}(\beta z - \alpha + 1)e^{-\beta z}$. There are no obvious conditions that are functions of only one parameter. Figure 4.2 depicts several shapes of the hazard function for specific values of $\boldsymbol{\theta}$.

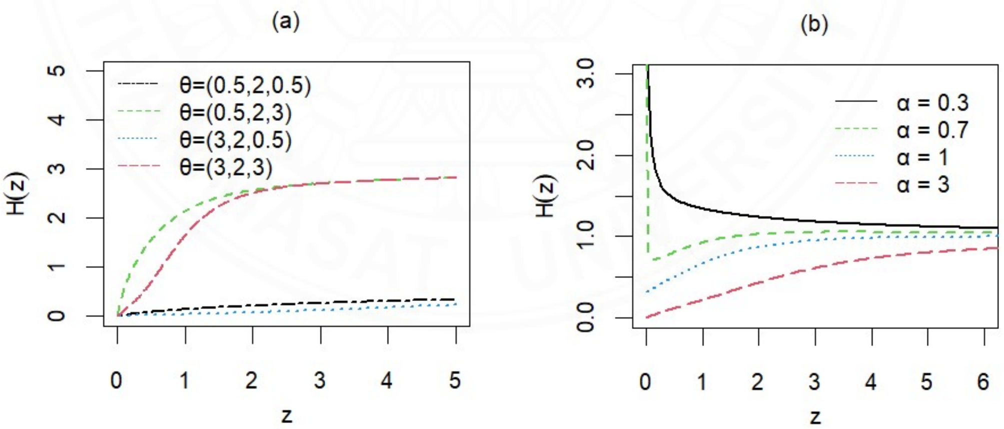

Figure 4.2. Hazard functions of the CGZTP distribution, (a) $\alpha = 2$ and (b) $\lambda = 2,\ \beta = 1$.

4.4 Random number generation

Samples from the CGZTP distribution are generated using a rejection sampling algorithm in R programming. Continuous uniform distribution $U(0,20)$ is employed as the proposal distribution $p(z)$, and the smallest c that maximizes $\frac{g(z;\boldsymbol{\theta})}{p(z)}$ is selected. The algorithm is shown as follow:

Step 1: find a constant c such that $cp(z) \geq g(z;\boldsymbol{\theta})$;

Step 2: obtain a sample z from $U(0,20)$;

Step 3: obtain a sample u from $U(0,1)$;

Step 4: check whether $cp(z)u \leq g(z;\boldsymbol{\theta})$. If this holds, accept z as a sample drawn from g. Otherwise, z will be rejected.

Figure 4.3 displays the histograms of 10,000 samples generated from specified cases of the CGZTP distribution using rejection sampling.

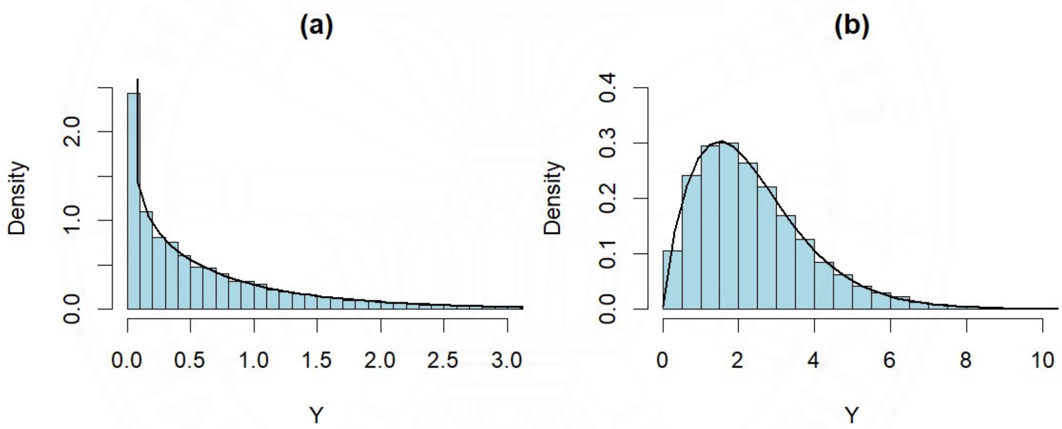

Figure 4.3 Histogram of 10,000 samples from CGZTP distribution
(a) $\lambda=1,\ \alpha=0.5,\ \beta=1$ and (b) $\lambda=1,\ \alpha=2,\ \beta=1$.

4.5 Estimation of parameters

4.5.1 Estimation by maximum likelihood

In what follows, we discuss the estimation of the parameters for CGZTP distributions. Let $Z_1, Z_2, \ldots, Z_n$ be random samples with observed values $z_1, z_2, \ldots, z_n$ from a CGZTP distributions with parameters $\boldsymbol{\theta}$. The likelihood function based on the observed random sample size of n, $w_{obs} = (z_1, z_2, \ldots, z_n)$ is given by

$$L(\boldsymbol{\theta}; w_{obs}) = \prod_{i=1}^{n} g(z_i;\ \boldsymbol{\theta}) = \left(\frac{\lambda}{1-e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^n}\right)\left(\prod_{i=1}^{n} z_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^{n} z_i\right)-\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}.$$

The corresponding log-likelihood function is

$$l(\boldsymbol{\theta}; w_{obs}) = n\left(\log\lambda - \log\left(1-e^{-\lambda}\right)\right) + n\alpha\log\beta - n\log\Gamma(\alpha) + (\alpha-1)\sum_{i=1}^{n}\log z_i$$
$$-\beta\left(\sum_{i=1}^{n} z_i\right) - \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i).$$

The first derivatives of the log-likelihood function are the following:

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial\lambda} = n\left(\frac{1}{\lambda} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\right) - \frac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)},$$

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial\alpha} = n\log\beta - n\psi_0(\alpha) + \sum_{i=1}^{n}\log z_i - \lambda\sum_{i=1}^{n}\left(\frac{\Gamma(\alpha)\dfrac{\partial\Gamma(\alpha,\beta z_i)}{\partial\alpha} - \Gamma(\alpha,\beta z_i)\dfrac{\partial\Gamma(\alpha)}{\partial\alpha}}{\left(\Gamma(\alpha)\right)^2}\right)$$

$$= n\log\beta - n\psi_0(\alpha) + \sum_{i=1}^{n}\log z_i$$
$$-\lambda\sum_{i=1}^{n}\left(\frac{\Gamma(\alpha)\left(G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{matrix}1,1\\0,0,a\end{matrix}\right) + \Gamma(\alpha,\beta z_i)\log(\beta z_i)\right) - \Gamma(\alpha,\beta z_i)\Gamma(\alpha)\psi_0(\alpha)}{\left(\Gamma(\alpha)\right)^2}\right)$$

$$= n\log\beta - n\psi_0(\alpha) + \sum_{i=1}^{n}\log z_i - \lambda\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{matrix}1,1\\0,0,a\end{matrix}\right) + \Gamma(\alpha,\beta z_i)\left(\log(\beta z_i) - \psi_0(\alpha)\right)}{\Gamma(\alpha)},$$

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial\beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i - \frac{\lambda}{\Gamma(\alpha)}\frac{\partial\left(\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)\right)}{\partial\beta}$$
$$= \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i - \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\frac{\partial\Gamma(\alpha,\beta z_i)}{\partial\beta}$$
$$= \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i - \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(-z_i^{\alpha}\beta^{\alpha-1}e^{-\beta z_i}\right)$$
$$= \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i + \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha}e^{-\beta z_i}.$$

The associated gradients are subsequently determined to be

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial \lambda} = n\left(\frac{1}{\lambda} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\right) - \frac{\sum_{i=1}^{n}\Gamma(\alpha, \beta z_i)}{\Gamma(\alpha)}, \tag{4.3}$$

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial \alpha} &= n\log\beta - n\psi_0(\alpha) + \sum_{i=1}^{n}\log z_i \\ &\quad - \lambda\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{matrix}1,1\\0,0,a\end{matrix}\right) + \Gamma(\alpha,\beta z_i)\left(\log(\beta z_i) - \psi_0(\alpha)\right)}{\Gamma(\alpha)},\end{aligned} \tag{4.4}$$

$$\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i + \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i}, \tag{4.5}$$

where $\psi_0(\alpha)$ is a digamma function that define as the 1^{st} derivative of the logarithm of gamma function and $G_{p,q}^{m,n}\left(\beta z_i \middle| \begin{matrix}a_1,\ldots a_p\\ b_1,\ldots b_q\end{matrix}\right)$ is Meijer G-function. The equation (4.5) could be solved exactly for λ as follows:

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta}; w_{obs})}{\partial \beta} &= 0 \\ \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i + \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i} &= 0 \\ \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i} &= \sum_{i=1}^{n} z_i - \frac{n\alpha}{\beta} \\ \lambda &= \frac{\Gamma(\alpha)}{\beta^{\alpha-1}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i}}\left(\sum_{i=1}^{n} z_i - \frac{n\alpha}{\beta}\right).\end{aligned}$$

Therefore, the maximum likelihood estimator of λ is $\hat{\lambda} = \frac{\Gamma(\hat{\alpha})}{\hat{\beta}^{\hat{\alpha}-1}\sum_{i=1}^{n} z_i^{\hat{\alpha}} e^{-\hat{\beta} z_i}}\left(\sum_{i=1}^{n} z_i - \frac{n\hat{\alpha}}{\hat{\beta}}\right)$, conditional upon the value of $\hat{\alpha}$ and $\hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are maximum likelihood estimates for the parameter α and β, respectively.

In the following, Theorem 4.2 expresses what the conditions are there to obtain the existence and uniqueness of the MLEs.

Theorem 4.2

(a) Let $l_1(\lambda;\alpha,\beta,w_{obs})=\dfrac{\partial l(\theta;w_{obs})}{\partial\lambda}$, If α,β are known, then $\hat{\lambda}$ is the uniquely exist root of $l_1(\lambda;\alpha,\beta,w_{obs})=0$ if $\dfrac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}<\dfrac{n}{2}$

Proof. Since $l_1(\lambda;\alpha,\beta,w_{obs})=n\left(\dfrac{1}{\lambda}-\dfrac{e^{-\lambda}}{1-e^{-\lambda}}\right)-\dfrac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}$,

$$\lim_{\lambda\to 0} l_1(\lambda;\alpha,\beta,w_{obs})=\frac{n}{2}-\frac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)},\text{ and }\lim_{\lambda\to\infty} l_1(\lambda;\alpha,\beta,w_{obs})=-\frac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}.$$

It can show that $\lim_{\lambda\to 0} l_1(\lambda;\alpha,\beta,w_{obs})=\dfrac{n}{2}-\dfrac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}>0$ as $\dfrac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}<\dfrac{n}{2}$.

Since $\dfrac{\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}>0$ for all z_i, then $\lim_{\lambda\to\infty} l_1(\lambda;\alpha,\beta,w_{obs})=-\dfrac{\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}<0$.

Therefore, there exist at least one solution of $l_1(\lambda;\alpha,\beta,w_{obs})=0$.

For the proof of uniqueness of solution, we need to show function l_1 is strictly decreasing in λ. The first derivative of l_1 is considered and given by

$$l_1'(\lambda;\alpha,\beta,w_{obs})=-\frac{n(-e^{\lambda}(\lambda^2+2)+e^{2\lambda}+1)}{(e^{\lambda}-1)^2\lambda^2}=-\frac{ne^{\lambda}(e^{-\lambda}+e^{\lambda}-(\lambda^2+2))}{(e^{\lambda}-1)^2\lambda^2}.$$

If $e^{-\lambda}+e^{\lambda}-(\lambda^2+2)>0$, then $l_1'(\lambda;\alpha,\beta,w_{obs})<0$ and l_1 is strictly decreasing in λ.

Consider $e^{\lambda}=1+\lambda+\dfrac{1}{2}\lambda^2+\dfrac{1}{3!}\lambda^3+\dots$ and $e^{-\lambda}=1-\lambda+\dfrac{1}{2}\lambda^2-\dfrac{1}{3!}\lambda^3+\dots$, then

$$e^{-\lambda}+e^{\lambda}=2+\lambda^2+\frac{2}{4!}\lambda^4+\dots>\lambda^2+2\quad\text{or}\quad e^{-\lambda}+e^{\lambda}-(\lambda^2+2)>0.$$

Therefore, $l_1'(\lambda;\alpha,\beta,w_{obs})<0$ for $\lambda>0$. This completes the proof.

(b) Let $l_3(\beta;\lambda,\alpha,w_{obs})=\dfrac{\partial l(\theta;w_{obs})}{\partial\beta}$, If λ and α are known, then there exist at least one solution of $l_3(\beta;\lambda,\alpha,w_{obs})=0$.

Proof. Since $l_3(\beta;\lambda,\alpha,w_{obs})=\dfrac{n\alpha}{\beta}-\sum_{i=1}^{n}z_i+\dfrac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}$,

$$\lim_{\beta\to 0}l_3(\beta;\lambda,\alpha,w_{obs})=\lim_{\beta\to 0}\frac{n\alpha}{\beta}-\lim_{\beta\to 0}\sum_{i=1}^{n}z_i+\lim_{\beta\to 0}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}=\infty\text{, and}$$

$$\begin{aligned}\lim_{\beta\to\infty}l_3(\beta;\lambda,\alpha,w_{obs})&=\lim_{\beta\to\infty}\frac{n\alpha}{\beta}-\lim_{\beta\to\infty}\sum_{i=1}^{n}z_i+\lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}\\&=0-\sum_{i=1}^{n}z_i+\lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}\end{aligned}$$

Consider the following:

$$\begin{aligned}\lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}&=\frac{\lambda}{\Gamma(\alpha)}\lim_{\beta\to\infty}\sum_{i=1}^{n}\beta^{\alpha-1}z_i^{\alpha}e^{-\beta z_i}\\&=\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(\lim_{\beta\to\infty}\beta^{\alpha-1}z_i^{\alpha}e^{-\beta z_i}\right)\\&=\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(z_i\lim_{\beta\to\infty}\frac{(\beta z_i)^{\alpha-1}}{e^{\beta z_i}}\right).\end{aligned}$$

Since $\lim_{\beta\to\infty}\dfrac{(\beta z_i)^{\alpha-1}}{e^{\beta z_i}}=0$, $\lim_{\beta\to\infty}\dfrac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}=0$ and $\lim_{\beta\to\infty}l_3(\beta;\lambda,\alpha,w_{obs})=-\sum_{i=1}^{n}z_i<0$.

Therefore, there exist at least one solution of $l_3(\beta;\lambda,\alpha,w_{obs})=0$.

4.5.2 Statistical inference and confidence intervals

The MLE of $\boldsymbol{\theta}$ is approximately multivariate normal with mean $\boldsymbol{\theta}$ and a variance-covariance matrix that is the inverse of expected information matrix $J(\boldsymbol{\theta})=E[I(\boldsymbol{\theta})]$, where $I(\boldsymbol{\theta})$ is the observed Fisher information matrix with elements $I_{ij}=-\partial l^2/\partial\boldsymbol{\theta}_i\partial\boldsymbol{\theta}_j$, $i,j=1,2,3$.

4.5.2.1 Case 1: All parameters are unknown

When λ,α and β are unknown, the MLEs of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}=(\hat{\lambda},\ \hat{\alpha},\ \hat{\beta})$ need to be determined numerically. By differentiating equations (4.3) – (4.5), the elements of the symmetric and second order observed information matrix, $I(\boldsymbol{\theta})$, are found as follows:

$$I_{11} = \frac{n(-e^{\lambda}(\lambda^2+2)+e^{2\lambda}+1)}{\left(e^{\lambda}-1\right)^2 \lambda^2},$$

$$I_{22} = n\psi^{(1)}(\alpha) + \lambda\sum_{i=1}^{n}\frac{\partial^2}{\partial\alpha^2}\left(\frac{\Gamma(\alpha,\beta z_i)}{\Gamma(\alpha)}\right)$$

$$= n\psi^{(1)}(\alpha) + \lambda\sum_{i=1}^{n}\left[\frac{1}{\Gamma(\alpha)}\left(\begin{array}{l} 2G_{3,4}^{4,0}\left(\beta z_i \middle| \begin{array}{c} 1,1,1 \\ 0,0,0,\alpha \end{array}\right) + 2\left(\log(\beta z_i) - \psi^{(0)}(\alpha)\right)G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{array}{c} 1,1 \\ 0,0,\alpha \end{array}\right) \\ +\Gamma(\alpha,\beta z_i)\left(-2\psi^{(0)}(\alpha)\log(\beta z_i) + \psi^{(0)}(\alpha)^2 - \psi^{(1)}(\alpha) + \log^2(\beta z_i)\right) \end{array}\right)\right],$$

$$I_{33} = \frac{n\alpha}{\beta^2} - \frac{\lambda\beta^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i}\left(\alpha - 1 - \beta z_i\right),$$

$$I_{12} = I_{21} = \sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{array}{c} 1,1 \\ 0,0,a \end{array}\right) + \Gamma(\alpha,\beta z_i)\left(\log(\beta z_i) - \psi_0(\alpha)\right)}{\Gamma(\alpha)},$$

$$I_{13} = I_{31} = -\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i},$$

$$I_{23} = I_{32} = -\frac{\partial}{\partial\alpha}\left[\frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i + \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i}\right]$$

$$= -\frac{n}{\beta} - \lambda\frac{\partial}{\partial\alpha}\left[\sum_{i=1}^{n}\frac{z_i^{\alpha} e^{-\beta z_i}\beta^{\alpha-1}}{\Gamma(\alpha)}\right]$$

$$= -\frac{n}{\beta} - \lambda\sum_{i=1}^{n} e^{-\beta z_i}\frac{\partial}{\partial\alpha}\left[\frac{z_i^{\alpha}\beta^{\alpha-1}}{\Gamma(\alpha)}\right] = -\frac{n}{\beta} - \lambda\sum_{i=1}^{n} e^{-\beta z_i}\left[\frac{z_i^{\alpha}\beta^{\alpha-1}\left(-\psi^{(0)}(\alpha) + \log(\beta) + \log(z_i)\right)}{\Gamma(\alpha)}\right].$$

The Fisher information matrix is given by

$$I(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}.$$

The inverse observed Fisher information matrix is

$$I^{-1}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} I^{11} & I^{12} & I^{13} \\ I^{21} & I^{22} & I^{23} \\ I^{31} & I^{32} & I^{33} \end{bmatrix}.$$

Therefore, the $(1-\gamma)100\%$ Wald confidence intervals for λ, α and β are $\hat{\lambda} \pm z_{1-\frac{\alpha}{2}}\sqrt{I^{11}}$, $\hat{\alpha} \pm z_{1-\frac{\alpha}{2}}\sqrt{I^{22}}$ and $\hat{\beta} \pm z_{1-\frac{\alpha}{2}}\sqrt{I^{33}}$, respectively.

4.5.2.2 Case 2: α and β are unknown

Let $Z_1, Z_2, \ldots, Z_n$ be random samples with observed values $z_1, z_2, \ldots, z_n$ from a CGZTP distributions with known parameter λ and unknown parameters α and β. The log-likelihood function based on the observed random sample size of n, $w_{obs} = \left(z_1, z_2, \ldots, z_n\right)$ is given by

$$l\left(\alpha,\beta;w_{obs}\right) = n\left(\log\lambda - \log\left(1-e^{-\lambda}\right)\right) + n\alpha\log\beta - n\log\Gamma\left(\alpha\right) + \left(\alpha-1\right)\sum_{i=1}^{n}\log z_i$$
$$-\beta\left(\sum_{i=1}^{n} z_i\right) - \frac{\lambda}{\Gamma\left(\alpha\right)}\sum_{i=1}^{n}\Gamma\left(\alpha,\beta z_i\right),$$

where $\boldsymbol{\theta} = \left(\alpha,\ \beta\right)$, and the associated gradients are found to be

$$\frac{\partial l\left(\alpha,\beta;w_{obs}\right)}{\partial\alpha} = n\log\beta - n\psi_0(\alpha) + \sum_{i=1}^{n}\log z_i$$
$$-\lambda\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{matrix}1,1\\ 0,0,a\end{matrix}\right) + \Gamma(\alpha,\beta z_i)\left(\log\left(\beta z_i\right) - \psi_0(\alpha)\right)}{\Gamma\left(\alpha\right)}, \tag{4.6}$$

$$\frac{\partial l\left(\alpha,\beta;w_{obs}\right)}{\partial\beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} z_i + \frac{\lambda\beta^{\alpha-1}}{\Gamma\left(\alpha\right)}\sum_{i=1}^{n} z_i^{\alpha} e^{-\beta z_i}. \tag{4.7}$$

By differentiating equations 4.6 – 4.7, the elements of the symmetric and second order observed information matrix, $I\left(\boldsymbol{\theta}\right)$, are found as follows:

$$I_{11} = n\psi^{(1)}(\alpha) + \lambda\sum_{i=1}^{n}\left[\frac{1}{\Gamma\left(\alpha\right)}\left(\begin{matrix}2G_{3,4}^{4,0}\left(\beta z_i \middle| \begin{matrix}1,1,1\\ 0,0,0,\alpha\end{matrix}\right) + 2\left(\log\left(\beta z_i\right) - \psi^{(0)}\left(\alpha\right)\right)G_{2,3}^{3,0}\left(\beta z_i \middle| \begin{matrix}1,1\\ 0,0,\alpha\end{matrix}\right)\\ +\Gamma\left(\alpha,\beta z_i\right)\left(-2\psi^{(0)}\left(\alpha\right)\log(\beta z_i) + \psi^{(0)}\left(\alpha\right)^2 - \psi^{(1)}\left(\alpha\right) + \log^2(\beta z_i)\right)\end{matrix}\right)\right],$$

$$I_{22}=\frac{n\alpha}{\beta^2}-\frac{\lambda\beta^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{n}z_i^{\alpha}e^{-\beta z_i}\left(\alpha-1-\beta z_i\right),$$

$$I_{12}=I_{21}=-\frac{n}{\beta}-\lambda\sum_{i=1}^{n}e^{-\beta z_i}\left[\frac{z_i^{\alpha}\beta^{\alpha-1}\left(-\psi^{(0)}(\alpha)+\log(\beta)+\log(z_i)\right)}{\Gamma(\alpha)}\right].$$

When α and β are unknown, and λ is known, the MLEs of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}=\left(\hat{\alpha},\ \hat{\beta}\right)$ also need to be determined numerically. The Fisher information matrix is given by

$$I\left(\hat{\boldsymbol{\theta}}\right)=\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}.$$

The inverse observed Fisher information matrix is

$$I^{-1}\left(\hat{\boldsymbol{\theta}}\right)=\frac{1}{\det I\left(\hat{\boldsymbol{\theta}}\right)}\begin{bmatrix} I_{22} & -I_{21} \\ -I_{12} & I_{11} \end{bmatrix},$$

where $\det I\left(\hat{\boldsymbol{\theta}}\right)=I_{11}I_{22}-I_{12}I_{21}$. Hence $(1-\gamma)100\%$ Wald confidence intervals for α and β are, $\hat{\alpha}\pm z_{1-\frac{\gamma}{2}}\sqrt{I_{22}/\left(I_{11}I_{22}-I_{12}I_{21}\right)}$ and $\hat{\beta}\pm z_{1-\frac{\gamma}{2}}\sqrt{I_{11}/\left(I_{11}I_{22}-I_{12}I_{21}\right)}$, respectively.

4.5.2.3 Case 3: λ is unknown

Let $Z_1,Z_2,\ldots,Z_n$ be random samples with observed values $z_1,z_2,\ldots,z_n$ from a CGZTP distributions with unknown parameter λ and known parameters α and β. The log-likelihood function based on the observed random sample size of n, $w_{obs}=\left(z_1,z_2,\ldots,z_n\right)$ is given by

$$l\left(\lambda;w_{obs}\right)=n\left(\log\lambda-\log\left(1-e^{-\lambda}\right)\right)+n\alpha\log\beta-n\log\Gamma\left(\alpha\right)+\left(\alpha-1\right)\sum_{i=1}^{n}\log z_i$$
$$-\beta\left(\sum_{i=1}^{n}z_i\right)-\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma\left(\alpha,\beta z_i\right),$$

and the associated gradients is

$$\frac{\partial l\left(\lambda;w_{obs}\right)}{\partial\lambda}=n\left(\frac{1}{\lambda}-\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)-\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta z_i\right)}{\Gamma(\alpha)}. \qquad (4.8)$$

By differentiating equation (4.8), the observed Fisher information is found as follows:

$$I\left(\hat{\lambda}\right)=\frac{n(-e^{\lambda}(\lambda^{2}+2)+e^{2\lambda}+1)}{\left(e^{\lambda}-1\right)^{2}\lambda^{2}}.$$

The MLE of λ has no closed form and also need to be determined numerically with $se\left(\hat{\lambda}\right)=\sqrt{1/I\left(\hat{\lambda}\right)}=\sqrt{\frac{\left(e^{\lambda}-1\right)^{2}\lambda^{2}}{n(-e^{\lambda}(\lambda^{2}+2)+e^{2\lambda}+1)}}$. Therefore, a $(1-\gamma)100\%$ Wald confidence interval for λ is $\hat{\lambda}\pm z_{1-\frac{\gamma}{2}}\sqrt{\frac{\left(e^{\lambda}-1\right)^{2}\lambda^{2}}{n(-e^{\lambda}(\lambda^{2}+2)+e^{2\lambda}+1)}}$.

4.5.3 Simulation study

The samples were generated by using the rejection sampling method, where $\lambda=0.5,1,3$, $\alpha=0.5,1,2$, and $\beta=0.5,1,3$. These values of parameters are selected such that all different shapes of distributions are represented. The MLEs of λ, α and β are numerically calculated by the simulated-annealing method via the function maxLik in R program. The maxLik package is used to calculate the MLEs, and the simulated annealing method is chosen because it gives stable solutions. The research was conducted using 1,000 simulated samples obtained from the CGZTPs, with varying sample sizes of $n=50$, 100, and 1,000. The estimation of the coverage probability (CP) and average length (AL) of the confidence intervals (CIs) can be achieved by Monte Carlo simulations.

4.5.3.1 CASE 1: All parameters are unknown

When all parameters are assumed unknown, the MLEs of λ, α and β are numerically calculated by the simulated-annealing method. Table 4.1 shows the average MLEs of λ, α, and β and their corresponding mean-squared errors (MSEs). The MSEs decrease as the sample size increases, and the bias of the MLEs is reduced for a large sample size, i.e., $n=1{,}000$. Moreover, it is observed that, with the same sample size, MSE of $\hat{\lambda}$ is larger than MSE of $\hat{\alpha}$ and $\hat{\beta}$ in most situations. When $\alpha=0.5$, $\hat{\lambda}$ is smaller than its actual value when n is large. Conversely, $\hat{\alpha}$ and $\hat{\beta}$ tend to be close to their actual values as sample sizes increase. When $\alpha = 1$ and $\alpha = 2$, all estimates become more accurate as sample sizes increase. It is observed that when the other two parameters are fixed, the MSE of each estimate tends to increase when the parameter value increase.

Wald confidence intervals are constructed for all parameters of CGZTPs. All results are presented in Table 4.2. It is found that when $\alpha = 0.5$, the CPs of α and λ are less than 0.95 when $n = 1,000$. Conversely, when $\alpha = 1$ and $\alpha = 2$, the CPs of all parameter will be closer to the nominal coverage probability, 0.95, and the ALs will decrease when the sample size (n) increases. When λ has a large value, i.e., $\lambda = 3$, the CPs of all parameter are less than 0.95 although the sample size is 1,000.

4.5.3.2 Case 2: α and β are unknown

Tables 4.3–4.4 present the results obtained from simulated data sets where the values of α and β are unknown. As sample sizes increase, estimates become more accurate and the MSE values decrease. It is noticed that the MSE of $\hat{\alpha}$ increases as α increases given that β are fixed. Similarly, when α are fixed, the MSE of $\hat{\beta}$ increases as β increases. Wald confidence intervals using observed Fisher information are constructed for α and β of CGZTPs. It is found that when $\alpha = 0.5$, the CPs of α and β tend to be less than 0.95 when $n = 1,000$. Conversely, when $\alpha = 1$ and $\alpha = 2$, the CPs of all parameter will be closer to the nominal coverage probability, 0.95, and the ALs will decrease when the sample size (n) increases. In most cases, CPs are not less than 0.95, although the sample size is only 50.

4.5.3.3 Case 3: λ is unknown

The results obtained from simulated data sets, where λ's value is unknown, are presented in Table 4.5. As the size of the samples increases, the estimates get more accurate, and the MSE values decrease in most situations. When $\lambda = 0.5$, the estimates of λ show a bias, and the bias is reduced for a large sample size, i.e., $n = 1,000$. The CPs approach the desired coverage probability of 0.95, and the average lengths decrease with increasing sample sizes. However, in cases when α is small and β is large, namely when $\alpha = 0.5$ and $\beta = 3$, the CPs are below 0.95, although the sample size being 1,000.

Table 4.1 The averages of MLEs and mean-squared errors of λ, α, and β of three-parameter CGZTP distribution.

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$			MSE		
		λ	α	β	λ	α	β
(0.5, 0.5, 0.5)	25	1.0084	0.5205	0.5978	2.7120	0.0347	0.0608
	50	0.8523	0.5032	0.5419	2.0982	0.0184	0.0209
	100	0.6025	0.5133	0.5272	0.8314	0.0095	0.0103
	1,000	0.3051	0.5317	0.5000	0.1033	0.0020	0.0009
(0.5, 0.5, 1)	25	1.0136	0.5364	1.2193	3.9316	0.0372	0.2735
	50	0.9602	0.5050	1.0956	3.2290	0.0212	0.0902
	100	0.6216	0.5193	1.0612	1.2464	0.0110	0.0419
	1,000	0.2686	0.5402	1.0060	0.1123	0.0026	0.0036
(0.5, 0.5, 3)	25	1.0062	0.5274	3.6296	3.7328	0.0422	2.2640
	50	0.7735	0.5271	3.3231	2.0395	0.0175	0.8741
	100	0.5594	0.5368	3.2070	1.3016	0.0109	0.4038
	1,000	0.1974	0.5568	3.0279	0.1322	0.0040	0.0349
(0.5, 1, 0.5)	25	1.3076	1.0255	0.5780	5.0939	0.1752	0.0440
	50	1.2281	0.9454	0.5285	3.6437	0.0976	0.0156
	100	0.9596	0.9631	0.5125	2.3558	0.0631	0.0072
	1,000	0.5450	0.9928	0.5007	0.1843	0.0073	0.0008
(0.5, 1, 1)	25	1.2740	1.0045	1.1330	4.7066	0.1584	0.1654
	50	1.2948	0.9467	1.0527	4.5115	0.1058	0.0620
	100	1.1703	0.9303	1.0234	3.3607	0.0744	0.0302
	1,000	0.5876	0.9878	0.9988	0.4647	0.0103	0.0029
(0.5, 1, 3)	25	1.3636	0.9994	3.4562	5.5612	0.1490	1.5594
	50	1.6152	0.9129	3.1174	7.7816	0.1191	0.5193
	100	1.3908	0.9052	3.0417	5.8937	0.0888	0.2849
	1,000	0.6146	0.9843	3.0007	0.5556	0.0125	0.0286
(0.5, 2, 0.5)	25	1.6442	1.9836	0.5599	7.8701	0.7411	0.0404
	50	1.4310	1.8734	0.5168	5.2615	0.4243	0.0127
	100	1.1417	1.8648	0.5015	3.0390	0.2667	0.0060
	1,000	0.5984	1.9793	0.5034	0.1845	0.0274	0.0006
(0.5, 2, 1)	25	1.7655	1.9397	1.0818	9.4854	0.7583	0.1362
	50	1.7107	1.8272	1.0269	8.4883	0.4794	0.0575
	100	1.4237	1.8104	0.9845	5.7168	0.3566	0.0286
	1,000	0.5892	1.9728	0.9979	0.3769	0.0363	0.0026
(0.5, 2, 3)	25	2.1390	1.8772	3.2653	15.2703	0.7108	1.1699
	50	1.7192	1.8591	3.0821	9.9924	0.4981	0.5286
	100	1.6607	1.7539	2.9284	7.5669	0.3970	0.2534
	1,000	0.6375	1.9658	2.9868	0.7605	0.0531	0.0306

Table 4.1 The averages of MLEs and mean-squared errors of λ, α, and β of three-parameter CGZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$			MSE		
		λ	α	β	λ	α	β
(1, 0.5, 0.5)	25	1.1814	0.5641	0.5792	2.4555	0.0493	0.0516
	50	1.1243	0.5415	0.5394	2.1188	0.0293	0.0194
	100	1.0367	0.5272	0.5129	1.3914	0.0160	0.0079
	1,000	0.7903	0.5363	0.5024	0.1666	0.0034	0.0009
(1, 0.5, 1)	25	1.1723	0.5668	1.1562	2.9336	0.0456	0.1923
	50	1.4147	0.5172	1.0550	4.3660	0.0310	0.0656
	100	1.0460	0.5342	1.0344	1.8840	0.0175	0.0339
	1,000	0.7075	0.5492	1.0017	0.1951	0.0043	0.0034
(1, 0.5, 3)	25	1.3063	0.5737	3.5074	4.7681	0.0515	1.7598
	50	1.1178	0.5527	3.2233	2.7874	0.0311	0.7812
	100	1.0299	0.5478	3.1116	2.7552	0.0205	0.3093
	1,000	0.5963	0.5692	3.0151	0.2570	0.0065	0.0318
(1, 1, 0.5)	25	1.4251	1.1029	0.5616	4.2387	0.2165	0.0382
	50	1.4922	0.9972	0.5200	3.7845	0.1228	0.0163
	100	1.3799	0.9753	0.5002	2.5360	0.0872	0.0066
	1,000	1.0708	0.9915	0.4995	0.3641	0.0142	0.0007
(1, 1, 1)	25	1.5178	1.0578	1.0889	5.1881	0.1974	0.1358
	50	1.6577	0.9916	1.0266	5.7279	0.1382	0.0548
	100	1.3580	0.9750	0.9998	2.9120	0.0794	0.0247
	1,000	1.0483	0.9965	0.9942	0.4728	0.0163	0.0028
(1, 1, 3)	25	1.6665	1.0366	3.3100	5.7686	0.2034	1.4247
	50	1.8042	0.9620	3.0754	6.8415	0.1296	0.5322
	100	1.8091	0.9234	2.9643	6.0743	0.1105	0.2436
	1,000	1.0843	0.9926	2.9829	0.8122	0.0190	0.0280
(1, 2, 0.5)	25	1.8082	2.0859	0.5453	6.8395	0.8560	0.0333
	50	1.4815	2.0293	0.5200	3.9371	0.4708	0.0145
	100	1.4199	1.9436	0.5021	2.4951	0.2840	0.0057
	1,000	1.0712	1.9881	0.5018	0.2858	0.0433	0.0006
(1, 2, 1)	25	1.6181	2.1278	1.1049	5.0222	0.8716	0.1414
	50	1.7122	1.9225	1.0064	5.3198	0.4782	0.0519
	100	1.6464	1.8901	0.9864	4.5325	0.3659	0.0272
	1,000	1.0225	2.0003	0.9965	0.3992	0.0523	0.0025
(1, 2, 3)	25	2.0513	2.0745	3.2711	11.2245	0.8716	1.1330
	50	1.9207	1.9092	3.0250	8.3001	0.5275	0.4795
	100	1.6756	1.9028	2.9712	5.1586	0.3738	0.2546
	1,000	1.0833	1.9736	2.9761	0.5676	0.0647	0.0278

Table 4.1 The averages of MLEs and mean-squared errors of λ, α, and β of three-parameter CGZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$			MSE		
		λ	α	β	λ	α	β
(3, 0.5, 0.5)	25	1.9698	0.8453	0.5947	5.4594	0.3129	0.0564
	50	2.3268	0.7150	0.5386	4.5848	0.1518	0.0165
	100	2.5139	0.6553	0.5265	3.6790	0.0981	0.0086
	1,000	2.7430	0.5561	0.5098	0.7855	0.0159	0.0010
(3, 0.5, 1)	25	2.1228	0.8357	1.1912	7.6089	0.2901	0.2108
	50	2.3501	0.7243	1.0937	5.1101	0.1554	0.0709
	100	2.6142	0.6440	1.0448	4.0090	0.0951	0.0344
	1,000	2.6340	0.5742	1.0258	0.8963	0.0197	0.0047
(3, 0.5, 3)	25	2.2920	0.8262	3.5234	7.8926	0.2940	1.7115
	50	2.5491	0.7091	3.2447	6.7270	0.1522	0.5772
	100	2.7923	0.6361	3.1207	5.1872	0.0939	0.3140
	1,000	2.6996	0.5758	3.0809	1.4488	0.0225	0.0515
(3, 1, 0.5)	25	2.3008	1.5688	0.6009	7.3162	0.9638	0.0521
	50	2.2571	1.4061	0.5503	4.4487	0.5402	0.0193
	100	2.4622	1.2759	0.5294	3.4684	0.3293	0.0092
	1,000	2.9382	1.0504	0.5049	0.8841	0.0597	0.0015
(3, 1, 1)	25	2.0214	1.5800	1.1814	5.7221	0.9246	0.1916
	50	2.4078	1.3818	1.1130	5.0509	0.5179	0.0826
	100	2.7317	1.2179	1.0453	4.0112	0.2867	0.0365
	1,000	2.9936	1.0376	1.0019	1.0469	0.0647	0.0061
(3, 1, 3)	25	2.3338	1.5617	3.5534	10.0096	0.9589	1.7750
	50	2.6882	1.3205	3.2552	7.0506	0.4663	0.6806
	100	3.0407	1.1638	3.1054	5.9256	0.2745	0.3294
	1,000	3.0482	1.0283	3.0056	1.0680	0.0628	0.0601
(3, 2, 0.5)	25	2.3920	2.9941	0.6108	7.8742	3.1684	0.0573
	50	2.2514	2.6981	0.5623	4.5572	1.6244	0.0228
	100	2.3292	2.5501	0.5466	2.9706	1.0486	0.0116
	1,000	2.7439	2.1980	0.5209	0.6988	0.2024	0.0021
(3, 2, 1)	25	2.3142	2.9577	1.2094	7.2028	3.1202	0.2347
	50	2.4634	2.6778	1.1245	5.6569	1.7602	0.1000
	100	2.6224	2.4115	1.0540	3.9765	1.0187	0.0426
	1,000	2.9881	2.0612	1.0051	0.9760	0.2062	0.0077
(3, 2, 3)	25	2.4159	2.9625	3.6486	7.5877	2.9633	1.9643
	50	2.7566	2.5534	3.3142	6.9299	1.5793	0.8248
	100	2.8142	2.3491	3.1434	4.9756	0.9221	0.3880
	1,000	3.0035	2.0508	2.9998	1.0139	0.2167	0.0752

Table 4.2 Coverage probabilities and average lengths of Wald CIs of λ, α, and β of three-parameter CGZTP distribution.

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	CP			AL		
		λ	α	β	λ	α	β
(0.5, 0.5, 0.5)	25	1.0000	0.9810	0.9750	6.0669	1.0470	0.8998
	50	0.9930	0.9790	0.9680	4.3949	0.7585	0.5986
	100	0.9930	0.9860	0.9690	3.1293	0.5607	0.4309
	1,000	0.9730	0.9420	0.9630	0.9826	0.1636	0.1310
(0.5, 0.5, 1)	25	0.9930	0.9670	0.9710	5.8870	1.0391	1.8147
	50	0.9900	0.9680	0.9700	4.6830	0.7497	1.2024
	100	0.9910	0.9810	0.9680	2.9295	0.5341	0.8397
	1,000	0.9550	0.9050	0.9630	0.9419	0.1634	0.2654
(0.5, 0.5, 3)	25	0.9940	0.9690	0.9780	5.9417	1.0475	5.4667
	50	0.9930	0.9810	0.9770	4.4744	0.8051	3.7024
	100	0.9870	0.9810	0.9600	2.9148	0.5517	2.5825
	1,000	0.9310	0.8030	0.9650	0.8713	0.1643	0.8107
(0.5, 1, 0.5)	25	0.9950	0.9540	0.9730	7.2451	2.0217	0.7697
	50	0.9910	0.9470	0.9640	5.7860	1.4525	0.5163
	100	0.9790	0.9520	0.9700	3.7979	1.0713	0.3549
	1,000	0.9810	0.9740	0.9430	1.3305	0.3405	0.1073
(0.5, 1, 1)	25	0.9930	0.9530	0.9760	6.6328	1.9279	1.4935
	50	0.9880	0.9310	0.9660	5.4972	1.4320	1.0171
	100	0.9740	0.9330	0.9570	4.1589	1.0601	0.7120
	1,000	0.9680	0.9710	0.9570	1.3670	0.3426	0.2147
(0.5, 1, 3)	25	0.9930	0.9560	0.9760	6.8469	1.9465	4.5766
	50	0.9800	0.9130	0.9570	6.0251	1.4147	3.0260
	100	0.9610	0.9090	0.9450	4.5462	1.0495	2.1257
	1,000	0.9520	0.9530	0.9450	1.3820	0.3411	0.6421
(0.5, 2, 0.5)	25	0.9950	0.9340	0.9600	8.4075	3.7120	0.7101
	50	0.9850	0.9310	0.9560	5.8251	2.7307	0.4741
	100	0.9710	0.9360	0.9560	4.3569	2.0646	0.3384
	1,000	0.9720	0.9710	0.9570	1.4844	0.6686	0.0972
(0.5, 2, 1)	25	0.9910	0.9240	0.9590	8.2951	3.5710	1.3686
	50	0.9680	0.9090	0.9440	6.4637	2.6853	0.9562
	100	0.9600	0.8990	0.9290	4.8107	1.9825	0.6590
	1,000	0.9710	0.9720	0.9530	1.4957	0.6709	0.1968
(0.5, 2, 3)	25	0.9840	0.9030	0.9500	8.7679	3.5020	4.1185
	50	0.9710	0.9020	0.9370	6.4254	2.6400	2.8444
	100	0.9460	0.8880	0.9200	5.1314	1.9795	1.9786
	1,000	0.9560	0.9550	0.9350	1.5055	0.6626	0.5861

Table 4.2 Coverage probabilities and average lengths of Wald CIs of λ, α, and β of three-parameter CGZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	CP			AL		
		λ	α	β	λ	α	β
(1, 0.5, 0.5)	25	0.9970	0.9830	0.9660	6.1253	1.1322	0.8305
	50	0.9960	0.9750	0.9670	4.7569	0.8401	0.5587
	100	0.9860	0.9740	0.9640	3.7231	0.6233	0.3832
	1,000	0.9180	0.8930	0.9470	1.3855	0.1951	0.1144
(1, 0.5, 1)	25	0.9970	0.9770	0.9830	6.0668	1.1268	1.6661
	50	0.9870	0.9490	0.9710	5.5506	0.8067	1.0991
	100	0.9780	0.9670	0.9630	4.0100	0.6283	0.7861
	1,000	0.8910	0.8580	0.9530	1.3278	0.1928	0.2311
(1, 0.5, 3)	25	0.9960	0.9650	0.9770	6.5836	1.1407	5.0537
	50	0.9910	0.9680	0.9540	4.8464	0.8425	3.3460
	100	0.9840	0.9550	0.9610	3.7011	0.6113	2.3290
	1,000	0.8220	0.7330	0.9580	1.2341	0.1893	0.7078
(1, 1, 0.5)	25	0.9960	0.9590	0.9670	6.7941	2.1538	0.7313
	50	0.9910	0.9480	0.9420	5.7255	1.5591	0.4908
	100	0.9800	0.9290	0.9540	4.5423	1.1960	0.3454
	1,000	0.9720	0.9600	0.9610	1.8174	0.4231	0.0985
(1, 1, 1)	25	0.9940	0.9460	0.9660	7.2154	2.0676	1.4189
	50	0.9810	0.9220	0.9620	6.0187	1.5421	0.9700
	100	0.9720	0.9440	0.9610	4.3738	1.1654	0.6763
	1,000	0.9400	0.9220	0.9470	1.7727	0.4158	0.1968
(1, 1, 3)	25	0.9970	0.9430	0.9700	7.3497	2.0359	4.3132
	50	0.9840	0.9120	0.9420	6.2418	1.4974	2.8772
	100	0.9580	0.8970	0.9330	5.2028	1.1463	2.0160
	1,000	0.9560	0.9450	0.9460	1.7814	0.4091	0.5866
(1, 2, 0.5)	25	0.9930	0.9320	0.9620	7.7010	3.8963	0.6817
	50	0.9890	0.9470	0.9530	5.8977	2.9892	0.4772
	100	0.9820	0.9420	0.9690	4.8472	2.3122	0.3403
	1,000	0.9700	0.9600	0.9430	1.9013	0.8183	0.0955
(1, 2, 1)	25	0.9930	0.9510	0.9610	7.3038	3.9464	1.3957
	50	0.9880	0.9260	0.9480	6.2170	2.8646	0.9335
	100	0.9770	0.9180	0.9310	5.1857	2.2396	0.6755
	1,000	0.9620	0.9430	0.9510	1.8842	0.8111	0.1913
(1, 2, 3)	25	0.9910	0.9190	0.9480	8.3682	3.9061	4.1589
	50	0.9760	0.9150	0.9420	6.6725	2.8364	2.8045
	100	0.9620	0.9110	0.9350	5.1426	2.2395	2.0096
	1,000	0.9530	0.9350	0.9430	1.9370	0.8150	0.5761

Table 4.2 Coverage probabilities and average lengths of Wald CIs of λ, α, and β of three-parameter CGZTP distribution. (Cont.)

$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	n	CP			AL		
		λ	α	β	λ	α	β
(3, 0.5, 0.5)	25	0.9860	0.9530	0.9610	7.5732	1.7128	0.7791
	50	0.9330	0.9120	0.9690	7.0909	1.2692	0.5143
	100	0.8750	0.8650	0.9530	6.0295	0.9762	0.3640
	1,000	0.8620	0.8810	0.9350	3.2441	0.4655	0.1272
(3, 0.5, 1)	25	0.9790	0.9480	0.9710	8.5693	1.7877	1.6034
	50	0.9380	0.9150	0.9610	6.9094	1.2692	1.0336
	100	0.8860	0.8700	0.9480	6.2302	0.9625	0.7184
	1,000	0.8060	0.8360	0.9150	3.0436	0.4469	0.2441
(3, 0.5, 3)	25	0.9910	0.9540	0.9750	9.2352	1.7299	4.6359
	50	0.9410	0.9120	0.9740	7.7488	1.2417	3.0687
	100	0.8700	0.8360	0.9430	6.8819	0.9947	2.2113
	1,000	0.7870	0.8180	0.8910	3.0553	0.4432	0.7293
(3, 1, 0.5)	25	0.9850	0.9340	0.9650	8.2555	3.0396	0.7596
	50	0.9440	0.8980	0.9590	6.7190	2.3077	0.5125
	100	0.8710	0.8420	0.9550	5.9865	1.8246	0.3746
	1,000	0.8910	0.8760	0.9050	3.4543	0.8829	0.1497
(3, 1, 1)	25	0.9820	0.9480	0.9760	7.7795	3.1022	1.5109
	50	0.9350	0.8950	0.9550	7.0465	2.3213	1.0518
	100	0.8950	0.8640	0.9420	6.5269	1.8057	0.7512
	1,000	0.8630	0.8510	0.8980	3.4505	0.8516	0.2965
(3, 1, 3)	25	0.9740	0.9380	0.9670	8.2036	3.0878	4.5140
	50	0.9510	0.8950	0.9520	7.8241	2.2637	3.0931
	100	0.8940	0.8570	0.9210	6.9836	1.7545	2.2353
	1,000	0.8850	0.8570	0.8790	3.6457	0.8905	0.9309
(3, 2, 0.5)	25	0.9910	0.9400	0.9650	9.1877	5.7226	0.7931
	50	0.9540	0.9100	0.9550	6.9806	4.3740	0.5540
	100	0.8920	0.8590	0.9380	5.8210	3.4131	0.4054
	1,000	0.8840	0.8800	0.8690	3.2007	1.6292	0.1690
(3, 2, 1)	25	0.9800	0.9370	0.9630	8.9002	5.7688	1.5990
	50	0.9530	0.8790	0.9340	7.5081	4.2186	1.0860
	100	0.8910	0.8340	0.9270	6.2823	3.2889	0.7947
	1,000	0.8830	0.8740	0.8920	3.5355	1.6297	0.3467
(3, 2, 3)	25	0.9880	0.9360	0.9530	9.0264	5.7778	4.8079
	50	0.9560	0.8950	0.9420	8.4645	4.2413	3.3446
	100	0.9120	0.8670	0.9270	6.6772	3.3480	2.4440
	1,000	0.8880	0.8680	0.8960	3.5955	1.6482	1.0513

Table 4.3 The averages of MLEs and mean-squared errors of α and β of of two-parameter CGZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
			α	β	α	β
0.5	(0.5, 0.5)	25	0.5608	0.5942	0.0275	0.0594
		50	0.5311	0.5444	0.0105	0.0210
		100	0.5224	0.5319	0.0047	0.0101
		1,000	0.5114	0.5106	0.0005	0.0009
	(0.5, 1)	25	0.5731	1.2156	0.0297	0.2567
		50	0.5393	1.1032	0.0113	0.0895
		100	0.5266	1.0723	0.0051	0.0427
		1,000	0.5159	1.0313	0.0007	0.0042
	(0.5, 3)	25	0.5680	3.6066	0.0259	2.1161
		50	0.5479	3.3500	0.0126	0.8705
		100	0.5378	3.2565	0.0058	0.3996
		1,000	0.5248	3.1251	0.0010	0.0461
	(1, 0.5)	25	1.1417	0.5821	0.1478	0.0449
		50	1.0605	0.5372	0.0499	0.0165
		100	1.0344	0.5184	0.0209	0.0072
		1,000	1.0029	0.5022	0.0019	0.0007
	(1, 1)	25	1.1225	1.1529	0.1251	0.1789
		50	1.0594	1.0715	0.0466	0.0647
		100	1.0360	1.0410	0.0228	0.0313
		1,000	1.0034	1.0035	0.0017	0.0022
	(1, 3)	25	1.1268	3.5054	0.1284	1.7616
		50	1.0619	3.1941	0.0478	0.5477
		100	1.0253	3.0999	0.0205	0.2518
		1,000	1.0025	3.0096	0.0018	0.0227
	(2, 0.5)	25	2.2791	0.5770	0.5593	0.0392
		50	2.1226	0.5341	0.1949	0.0130
		100	2.0505	0.5141	0.0838	0.0058
		1,000	2.0150	0.5049	0.0081	0.0006
	(2, 1)	25	2.2731	1.1462	0.5534	0.1469
		50	2.1415	1.0778	0.2206	0.0612
		100	2.0438	1.0217	0.0849	0.0231
		1,000	2.0043	1.0029	0.0080	0.0022
	(2, 3)	25	2.2563	3.4488	0.4866	1.3142
		50	2.1390	3.2143	0.2073	0.5162
		100	2.0473	3.0647	0.0817	0.1985
		1,000	2.0057	3.0071	0.0082	0.0198

Table 4.3 The averages of MLEs and mean-squared errors of α and β of of two-parameter CGZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
			α	β	α	β
1	(0.5, 0.5)	25	0.5646	0.5846	0.0311	0.0520
		50	0.5354	0.5413	0.0126	0.0183
		100	0.5243	0.5280	0.0058	0.0087
		1,000	0.5116	0.5102	0.0005	0.0008
	(0.5, 1)	25	0.5727	1.1740	0.0351	0.1982
		50	0.5347	1.0887	0.0123	0.0819
		100	0.5243	1.0563	0.0054	0.0365
		1,000	0.5137	1.0259	0.0006	0.0034
	(0.5, 3)	25	0.5754	3.5571	0.0328	1.9073
		50	0.5493	3.3279	0.0157	0.8745
		100	0.5335	3.2044	0.0062	0.3254
		1,000	0.5215	3.1053	0.0009	0.0382
	(1, 0.5)	25	1.1079	0.5642	0.1213	0.0359
		50	1.0650	0.5375	0.0579	0.0176
		100	1.0311	0.5158	0.0230	0.0066
		1,000	1.0050	0.5027	0.0020	0.0006
	(1, 1)	25	1.1446	1.1639	0.1446	0.1662
		50	1.0656	1.0645	0.0566	0.0580
		100	1.0214	1.0294	0.0213	0.0252
		1,000	1.0030	1.0013	0.0020	0.0022
	(1, 3)	25	1.1330	3.4106	0.1558	1.4490
		50	1.0608	3.1976	0.0501	0.5539
		100	1.0250	3.0856	0.0228	0.2389
		1,000	1.0024	3.0101	0.0018	0.0194
	(2, 0.5)	25	2.3289	0.5819	0.6507	0.0375
		50	2.1541	0.5404	0.2338	0.0148
		100	2.0697	0.5184	0.0914	0.0054
		1,000	2.0179	0.5056	0.0096	0.0006
	(2, 1)	25	2.3069	1.1546	0.6666	0.1637
		50	2.1053	1.0520	0.2120	0.0511
		100	2.0638	1.0322	0.0911	0.0220
		1,000	2.0096	1.0044	0.0080	0.0020
	(2, 3)	25	2.3025	3.4434	0.6219	1.3330
		50	2.1279	3.1993	0.2210	0.4829
		100	2.0744	3.1047	0.1034	0.2197
		1,000	2.0037	3.0082	0.0084	0.0189

Table 4.3 The averages of MLEs and mean-squared errors of α and β of of two-parameter CGZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
			α	β	α	β
3	(0.5, 0.5)	25	0.5901	0.5865	0.0692	0.0513
		50	0.5445	0.5417	0.0201	0.0163
		100	0.5185	0.5159	0.0073	0.0055
		1,000	0.5063	0.5051	0.0007	0.0006
	(0.5, 1)	25	0.5784	1.1408	0.0486	0.1659
		50	0.5457	1.0755	0.0205	0.0594
		100	0.5201	1.0371	0.0075	0.0260
		1,000	0.5062	1.0103	0.0006	0.0023
	(0.5, 3)	25	0.5754	3.4098	0.0492	1.3826
4		50	0.5429	3.2239	0.0179	0.5298
		100	0.5278	3.1467	0.0086	0.2698
		1,000	0.5114	3.0510	0.0008	0.0219
	(1, 0.5)	25	1.1696	0.5671	0.2343	0.0366
		50	1.0802	0.5319	0.0784	0.0126
		100	1.0284	0.5115	0.0312	0.0055
		1,000	1.0059	0.5025	0.0031	0.0006
	(1, 1)	25	1.1708	1.1436	0.2307	0.1540
		50	1.0944	1.0757	0.0918	0.0595
		100	1.0470	1.0357	0.0379	0.0268
		1,000	1.0014	1.0020	0.0029	0.0022
	(1, 3)	25	1.1688	3.4148	0.2226	1.3592
		50	1.0889	3.2140	0.0862	0.5687
		100	1.0311	3.0690	0.0339	0.2140
		1,000	1.0022	3.0064	0.0030	0.0188
	(2, 0.5)	25	2.3564	0.5727	0.9120	0.0375
		50	2.1456	0.5309	0.2887	0.0124
		100	2.0864	0.5190	0.1373	0.0058
		1,000	2.0340	0.5085	0.0135	0.0006
	(2, 1)	25	2.3010	1.1256	0.8810	0.1517
		50	2.1622	1.0670	0.3464	0.0585
		100	2.0619	1.0241	0.1226	0.0209
		1,000	2.0069	1.0031	0.0128	0.0023
	(2, 3)	25	2.2764	3.3533	0.8119	1.2332
		50	2.1360	3.1680	0.3229	0.4852
		100	2.0635	3.0706	0.1438	0.2200
		1,000	2.0094	3.0110	0.0116	0.0183

Table 4.4 Coverage probabilities and average lengths of Wald CIs of α and β of two-parameter CGZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	CP		AL	
			α	β	α	β
0.5	(0.5, 0.5)	25	0.9600	0.9620	0.5551	0.7871
		50	0.9630	0.9580	0.3695	0.5158
		100	0.9630	0.9480	0.2564	0.3575
		1,000	0.9400	0.9390	0.0792	0.1090
	(0.5, 1)	25	0.9720	0.9570	0.5687	1.6032
		50	0.9640	0.9540	0.3757	1.0413
		100	0.9600	0.9530	0.2587	0.7192
		1,000	0.9100	0.9190	0.0799	0.2197
	(0.5, 3)	25	0.9750	0.9750	0.5628	4.7692
		50	0.9650	0.9640	0.3823	3.1531
		100	0.9540	0.9550	0.2647	2.1741
		1,000	0.8160	0.8970	0.0815	0.6635
	(1, 0.5)	25	0.9570	0.9640	1.1943	0.6932
		50	0.9550	0.9550	0.7795	0.4552
		100	0.9590	0.9550	0.5365	0.3113
		1,000	0.9440	0.9400	0.1640	0.0957
	(1, 1)	25	0.9590	0.9630	1.1728	1.3745
		50	0.9520	0.9560	0.7788	0.9080
		100	0.9410	0.9340	0.5373	0.6250
		1,000	0.9510	0.9530	0.1641	0.1911
	(1, 3)	25	0.9580	0.9600	1.1796	4.1859
		50	0.9550	0.9560	0.7807	2.7062
		100	0.9470	0.9480	0.5315	1.8636
		1,000	0.9450	0.9520	0.1640	0.5733
	(2, 0.5)	25	0.9620	0.9560	2.4768	0.6564
		50	0.9560	0.9670	1.6249	0.4304
		100	0.9530	0.9630	1.1081	0.2933
		1,000	0.9440	0.9500	0.3440	0.0912
	(2, 1)	25	0.9610	0.9650	2.4693	1.3041
		50	0.9580	0.9570	1.6402	0.8685
		100	0.9560	0.9640	1.1042	0.5831
		1,000	0.9430	0.9490	0.3421	0.1811
	(2, 3)	25	0.9740	0.9680	2.4723	3.9583
		50	0.9520	0.9540	1.6391	2.5915
		100	0.9570	0.9570	1.1065	1.7492
		1,000	0.9460	0.9390	0.3426	0.5434

Table 4.4 Coverage probabilities and average lengths of Wald CIs of α and β of two-parameter CGZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	CP		AL	
			α	β	α	β
1	(0.5, 0.5)	25	0.9760	0.9620	0.5966	0.7419
		50	0.9640	0.9600	0.3975	0.4898
		100	0.9590	0.9560	0.2745	0.3390
		1,000	0.9470	0.9500	0.0845	0.1040
	(0.5, 1)	25	0.9700	0.9690	0.6064	1.4882
		50	0.9640	0.9430	0.3972	0.9856
		100	0.9520	0.9520	0.2747	0.6781
		1,000	0.9370	0.9350	0.0849	0.2090
	(0.5, 3)	25	0.9800	0.9600	0.6092	4.5014
		50	0.9560	0.9470	0.4092	2.9988
		100	0.9590	0.9610	0.2801	2.0508
		1,000	0.8740	0.9210	0.0864	0.6310
	(1, 0.5)	25	0.9610	0.9680	1.2256	0.6576
		50	0.9420	0.9390	0.8301	0.4437
		100	0.9490	0.9500	0.5669	0.3018
		1,000	0.9550	0.9630	0.1743	0.0932
	(1, 1)	25	0.9670	0.9710	1.2679	1.3529
		50	0.9480	0.9640	0.8310	0.8791
		100	0.9620	0.9640	0.5613	0.6028
		1,000	0.9490	0.9520	0.1740	0.1856
	(1, 3)	25	0.9650	0.9680	1.2716	4.0156
		50	0.9560	0.9480	0.8269	2.6412
		100	0.9500	0.9450	0.5634	1.8060
		1,000	0.9650	0.9510	0.1739	0.5580
	(2, 0.5)	25	0.9610	0.9680	2.6557	0.6538
		50	0.9580	0.9630	1.7346	0.4302
		100	0.9660	0.9630	1.1761	0.2919
		1,000	0.9440	0.9300	0.3622	0.0900
	(2, 1)	25	0.9580	0.9580	2.6326	1.2987
		50	0.9510	0.9500	1.6938	0.8379
		100	0.9590	0.9580	1.1722	0.5811
		1,000	0.9560	0.9490	0.3608	0.1789
	(2, 3)	25	0.9530	0.9560	2.6232	3.8679
		50	0.9550	0.9610	1.7151	2.5518
		100	0.9530	0.9540	1.1788	1.7480
		1,000	0.9450	0.9530	0.3597	0.5361

Table 4.4 Coverage probabilities and average lengths of Wald CIs of α and β of two-parameter CGZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	n	CP		AL	
			α	β	α	β
3	(0.5, 0.5)	25	0.9540	0.9510	0.7621	0.6855
		50	0.9530	0.9640	0.4932	0.4487
		100	0.9600	0.9800	0.3303	0.3029
		1,000	0.9460	0.9420	0.1018	0.0939
	(0.5, 1)	25	0.9670	0.9500	0.7464	1.3337
		50	0.9600	0.9550	0.4948	0.8911
		100	0.9550	0.9550	0.3317	0.6089
		1,000	0.9600	0.9510	0.1018	0.1878
	(0.5, 3)	25	0.9650	0.9610	0.7437	3.9923
		50	0.9580	0.9530	0.4927	2.6736
		100	0.9500	0.9370	0.3375	1.8465
		1,000	0.9440	0.9560	0.1030	0.5669
	(1, 0.5)	25	0.9660	0.9680	1.5608	0.6442
		50	0.9530	0.9610	1.0154	0.4261
		100	0.9550	0.9650	0.6818	0.2896
		1,000	0.9370	0.9410	0.2106	0.0899
	(1, 1)	25	0.9560	0.9660	1.5618	1.2981
		50	0.9460	0.9580	1.0295	0.8619
		100	0.9470	0.9400	0.6953	0.5866
		1,000	0.9480	0.9550	0.2095	0.1792
	(1, 3)	25	0.9590	0.9680	1.5615	3.8800
		50	0.9450	0.9470	1.0252	2.5780
		100	0.9490	0.9470	0.6836	1.7375
		1,000	0.9530	0.9560	0.2096	0.5375
	(2, 0.5)	25	0.9590	0.9620	3.1903	0.6554
		50	0.9680	0.9710	2.0350	0.4250
		100	0.9570	0.9610	1.4002	0.2937
		1,000	0.9420	0.9370	0.4313	0.0908
	(2, 1)	25	0.9600	0.9550	3.1155	1.2897
		50	0.9530	0.9580	2.0535	0.8556
		100	0.9600	0.9680	1.3827	0.5790
		1,000	0.9360	0.9370	0.4258	0.1792
	(2, 3)	25	0.9510	0.9530	3.0940	3.8499
		50	0.9550	0.9570	2.0305	2.5420
		100	0.9480	0.9400	1.3856	1.7386
		1,000	0.9520	0.9570	0.4260	0.5378

Table 4.5 The averages of MLEs, mean-squared errors, coverage probabilities, and average lengths of Wald CIs of λ of one-parameter CGZTP distribution.

λ	(α,β)	n	$AV(\hat{\lambda})$	MSE	CP	AL
0.5	(0.5, 0.5)	25	0.8416	0.4117	0.9730	2.1892
		50	0.6560	0.1866	0.9710	1.5767
		100	0.5580	0.1036	0.9680	1.1678
		1,000	0.5116	0.0110	0.9620	0.4322
	(0.5, 1)	25	0.8054	0.3819	0.9790	2.1550
		50	0.6559	0.1894	0.9650	1.5743
		100	0.5593	0.0942	0.9700	1.1767
		1,000	0.5198	0.0119	0.9540	0.4324
	(0.5, 3)	25	0.8189	0.4210	0.9690	2.1583
		50	0.6711	0.1985	0.9670	1.5869
		100	0.5877	0.1032	0.9630	1.1947
		1,000	0.5448	0.0143	0.9300	0.4326
	(1, 0.5)	25	0.8023	0.3859	0.9730	2.1495
		50	0.6485	0.1702	0.9770	1.5744
		100	0.5640	0.0966	0.9800	1.1775
		1,000	0.4977	0.0124	0.9420	0.4320
	(1, 1)	25	0.8100	0.3934	0.9660	2.1544
		50	0.6512	0.1734	0.9780	1.5773
		100	0.5664	0.1009	0.9690	1.1776
		1,000	0.5029	0.0129	0.9450	0.4321
	(1, 3)	25	0.8182	0.3852	0.9710	2.1647
		50	0.6723	0.2010	0.9650	1.5877
		100	0.5431	0.0949	0.9770	1.1615
		1,000	0.5019	0.0110	0.9680	0.4322
	(2, 0.5)	25	0.7598	0.3479	0.9780	2.1114
		50	0.6418	0.1843	0.9710	1.5639
		100	0.5452	0.0914	0.9810	1.1668
		1,000	0.4975	0.0122	0.9440	0.4320
	(2, 1)	25	0.7784	0.3877	0.9730	2.1223
		50	0.6426	0.1863	0.9690	1.5636
		100	0.5518	0.0950	0.9800	1.1688
		1,000	0.4972	0.0124	0.9490	0.4319
	(2, 3)	25	0.8347	0.4057	0.9730	2.1826
		50	0.6516	0.1821	0.9690	1.5745
		100	0.5755	0.1033	0.9570	1.1828
		1,000	0.5033	0.0126	0.9540	0.4320

Table 4.5 The averages of MLEs, mean-squared errors, coverage probabilities, and average lengths of Wald CIs of λ of one-parameter CGZTP distribution. (Cont.)

λ	(α, β)	n	$AV(\hat{\lambda})$	MSE	CP	AL
1	(0.5, 0.5)	25	1.1302	0.4214	0.9740	2.4199
		50	1.0580	0.2612	0.9640	1.8246
		100	1.0428	0.1385	0.9430	1.3692
		1,000	1.0177	0.0126	0.9450	0.4407
	(0.5, 1)	25	1.1416	0.4216	0.9820	2.4326
		50	1.0556	0.2518	0.9710	1.8274
		100	1.0350	0.1374	0.9390	1.3645
		1,000	1.0278	0.0130	0.9470	0.4409
	(0.5, 3)	25	1.1465	0.4456	0.9720	2.4280
		50	1.0615	0.2375	0.9770	1.8319
		100	1.0546	0.1245	0.9490	1.3748
		1,000	1.0348	0.0133	0.9450	0.4410
	(1, 0.5)	25	1.1507	0.4625	0.9720	2.4265
		50	1.0325	0.2271	0.9740	1.8222
		100	1.0175	0.1279	0.9460	1.3635
		1,000	1.0036	0.0127	0.9500	0.4403
	(1, 1)	25	1.1178	0.4082	0.9860	2.4110
		50	1.0621	0.2468	0.9660	1.8304
		100	0.9895	0.1185	0.9580	1.3607
		1,000	1.0077	0.0134	0.9460	0.4404
	(1, 3)	25	1.1109	0.4218	0.9830	2.4031
		50	1.0507	0.2458	0.9670	1.8245
		100	1.0012	0.1181	0.9610	1.3625
		1,000	0.9979	0.0121	0.9530	0.4402
	(2, 0.5)	25	1.1543	0.4329	0.9770	2.4400
		50	1.0353	0.2395	0.9740	1.8169
		100	1.0053	0.1261	0.9570	1.3614
		1,000	0.9962	0.0129	0.9450	0.4402
	(2, 1)	25	1.1182	0.4259	0.9750	2.4082
		50	1.0282	0.2143	0.9810	1.8234
		100	1.0117	0.1288	0.9480	1.3634
		1,000	1.0027	0.0122	0.9480	0.4403
	(2, 3)	25	1.1367	0.4666	0.9670	2.4143
		50	1.0347	0.2338	0.9650	1.8205
		100	1.0227	0.1292	0.9510	1.3655
		1,000	0.9962	0.0123	0.9560	0.4402

Table 4.5 The averages of MLEs, mean-squared errors, coverage probabilities, and average lengths of Wald CIs of λ of one-parameter CGZTP distribution. (Cont.)

λ	(α, β)	n	$AV(\hat{\lambda})$	MSE	CP	AL
3	(0.5, 0.5)	25	3.0834	0.8019	0.9510	3.3882
		50	3.0567	0.3636	0.9570	2.3732
		100	3.0328	0.1809	0.9570	1.6689
		1,000	3.0088	0.0176	0.9510	0.5248
	(0.5, 1)	25	3.0918	0.7370	0.9640	3.3875
		50	3.0566	0.3719	0.9570	2.3736
		100	3.0193	0.1976	0.9350	1.6663
		1,000	3.0130	0.0184	0.9450	0.5249
	(0.5, 3)	25	3.1620	0.8476	0.9570	3.4204
		50	3.0539	0.3491	0.9640	2.3720
		100	3.0411	0.1776	0.9520	1.6701
		1,000	3.0219	0.0186	0.9410	0.5255
	(1, 0.5)	25	3.1058	0.8320	0.9500	3.3981
		50	3.0632	0.3838	0.9550	2.3760
		100	3.0152	0.1753	0.9580	1.6647
		1,000	2.9984	0.0178	0.9450	0.5240
	(1, 1)	25	3.0871	0.7700	0.9590	3.3872
		50	3.0359	0.4110	0.9390	2.3701
		100	3.0336	0.1814	0.9540	1.6688
		1,000	3.0023	0.0176	0.9510	0.5244
	(1, 3)	25	3.0826	0.7719	0.9600	3.3861
		50	3.0367	0.3792	0.9530	2.3685
		100	2.9902	0.1792	0.9580	1.6600
		1,000	3.0074	0.0171	0.9550	0.5246
	(2, 0.5)	25	3.1013	0.8045	0.9480	3.3921
		50	3.0277	0.3751	0.9470	2.3660
		100	3.0283	0.1883	0.9500	1.6676
		1,000	3.0001	0.0172	0.9550	0.5242
	(2, 1)	25	3.0189	0.7849	0.9480	3.3603
		50	3.0489	0.3539	0.9600	2.3705
		100	3.0296	0.1861	0.9510	1.6683
		1,000	3.0002	0.0190	0.9440	0.5242
	(2, 3)	25	3.1028	0.7815	0.9500	3.3928
		50	3.0264	0.4185	0.9450	2.3678
		100	3.0317	0.1827	0.9550	1.6682
		1,000	3.0054	0.0186	0.9510	0.5245

CHAPTER 5
BAYESIAN ESTIMATION FOR GAMMA ZERO-TRUNCATED POISSON AND COMPLEMENTARY GAMMA ZERO-TRUNCATED POISSON DISTRIBUTION

In this chapter, different prior distributions are derived in a Bayesian inference of the GZTP and CGZTP distributions. The performance of the prior distributions is assessed and the Bayesian estimates and credible intervals for the unknown parameters are generated based on the simulated data sets. Samples are generated from the posterior distributions under the aforementioned priors for the Bayesian analysis using the random walk Metropolis algorithm.

5.1 Prior and posterior distributions

5.1.1 Case 1: α and β are unknown

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample with observed values $y_1, y_2, \ldots, y_n$ from a GZTP distributions with known parameter λ and unknown parameters α and β. The likelihood function based on the observed random sample size of n, $y_1, y_2, \ldots, y_n$ is given by

$$L(\alpha,\beta;y_1,y_2,\ldots,y_n)=\left(\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right)^n\left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^n}\right)\left(\prod_{i=1}^{n}y_i\right)^{\alpha-1}e^{-\beta\left(\sum_{i=1}^{n}y_i\right)+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)},$$

and let $Z_1, Z_2, \ldots, Z_n$ be random samples with observed values $z_1, z_2, \ldots, z_n$ from a CGZTP distributions with known parameter λ and unknown parameters α and β. The log-likelihood function based on the observed random sample size of n, $z_1, z_2, \ldots, z_n$ is given by

$$L(\alpha,\beta;z_1,z_2,\ldots,z_n)=\left(\frac{\lambda}{1-e^{-\lambda}}\right)^n\left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^n}\right)\left(\prod_{i=1}^{n}z_i\right)^{\alpha-1}e^{-\beta\left(\sum_{i=1}^{n}z_i\right)-\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}.$$

In the following, the priors for a Bayesian estimation of GZTP and CGZTP are presented. To estimate parameters for the GZTP or CGZTP distributions when α and

β are unknown but λ is known, we assume that α and β have priors $p_1(\cdot)$ and $p_2(\cdot)$, which correspond to $Gamma(a,\, b)$ and $Gamma(c,\, d)$, respectively, and they are independently distributed. The prior distributions for α and β are obtained as follow:

$$p_1(\alpha) = \frac{b^a \alpha^{a-1} e^{-b\alpha}}{\Gamma(a)}, \;\; \alpha > 0,\; a > 0,\; b > 0,$$

$$p_2(\beta) = \frac{d^c \beta^{c-1} e^{-d\beta}}{\Gamma(c)}, \;\; \beta > 0,\; c > 0,\; d > 0.$$

Hence, the joint posterior distributions given data $y_1, y_2, \ldots, y_n$ and data $z_1, z_2, \ldots, z_n$ are given, respectively, by:

$$\begin{aligned} p(\alpha,\beta | y_1, y_2, \ldots, y_n) &= \frac{L(\alpha,\beta; y_1, y_2, \ldots, y_n)\, p_1(\alpha)\, p_2(\beta)}{\int_0^\infty\int_0^\infty L(\alpha,\beta; y_1, y_2, \ldots, y_n)\, p_1(\alpha)\, p_2(\beta)\, d\alpha\, d\beta} \\ &\propto L(\alpha,\beta; y_1, y_2, \ldots, y_n)\, p_1(\alpha)\, p_2(\beta) \\ &\propto \left(\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\right) \left(\prod_{i=1}^n y_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^n y_i\right) + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^n \Gamma(\alpha,\beta y_i)} \\ &\quad \times \left(\frac{b^a \alpha^{a-1} e^{-b\alpha}}{\Gamma(a)}\right)\left(\frac{d^c \beta^{c-1} e^{-d\beta}}{\Gamma(c)}\right) \end{aligned}$$

and

$$\begin{aligned} p(\alpha,\beta | z_1, z_2, \ldots, z_n) &= \frac{L(\alpha,\beta; z_1, z_2, \ldots, z_n)\, p_1(\alpha)\, p_2(\beta)}{\int_0^\infty\int_0^\infty L(\alpha,\beta; z_1, z_2, \ldots, z_n)\, p_1(\alpha)\, p_2(\beta)\, d\alpha\, d\beta} \\ &\propto L(\alpha,\beta; z_1, z_2, \ldots, z_n)\, p_1(\alpha)\, p_2(\beta) \\ &\propto \left(\frac{\lambda}{1-e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\right) \left(\prod_{i=1}^n z_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^n z_i\right) - \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^n \Gamma(\alpha,\beta z_i)} \\ &\quad \times \left(\frac{b^a \alpha^{a-1} e^{-b\alpha}}{\Gamma(a)}\right)\left(\frac{d^c \beta^{c-1} e^{-d\beta}}{\Gamma(c)}\right). \end{aligned}$$

The marginal posterior distributions of α and β have no closed form. As a consequence, the MCMC method is employed to provide the Bayesian estimation.

5.1.2 Case 2: λ is unknown

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample with observed values $y_1, y_2, \ldots, y_n$ from a GZTP distributions with unknown parameter λ and known parameters α and β. The likelihood function based on the observed random sample size of n, $w_{obs} = \left(y_1, y_2, \ldots, y_n\right)$ is given by

$$L\left(\lambda; y_1, y_2, \ldots, y_n\right) = \left(\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^n}\right)\left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^{n} y_i\right)+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)},$$

and let $Z_1, Z_2, \ldots, Z_n$ be random samples with observed values $z_1, z_2, \ldots, z_n$ from a CGZTP distributions with unknown parameter λ and known parameters α and β. The likelihood function based on the observed random sample size of n, $w_{obs} = \left(z_1, z_2, \ldots, z_n\right)$ is given by

$$L\left(\lambda; z_1, z_2, \ldots, z_n\right) = \left(\frac{\lambda}{1-e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^n}\right)\left(\prod_{i=1}^{n} z_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^{n} z_i\right)-\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}.$$

In the following, different priors for a Bayesian estimation of GZTP and CGZTP are presented. To estimate parameters for the GZTP or CGZTP distributions when λ is unknown, we assume that λ has priors $p(\cdot)$ which corresponds to $Gamma(m, n)$, and the prior distribution for λ are obtained as follow:

$$p(\lambda) = \frac{n^m \lambda^{m-1} e^{-n\lambda}}{\Gamma(m)}, \ \lambda > 0, \ m > 0, \ n > 0,$$

Hence, the posterior distributions given data $y_1, y_2, \ldots, y_n$ and data $z_1, z_2, \ldots, z_n$ are given, respectively, by:

$$p\left(\lambda \middle| y_1, y_2, \ldots, y_n\right) = \frac{L\left(\lambda; y_1, y_2, \ldots, y_n\right) p(\lambda)}{\int_0^\infty L\left(\lambda; y_1, y_2, \ldots, y_n\right) p(\lambda) d\lambda}$$
$$\propto L\left(\lambda; y_1, y_2, \ldots, y_n\right) p(\lambda)$$

$$\propto \left(\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right)^{n}\left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^{n}}\right)\left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^{n} y_i\right)+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)}\left(\frac{n^m \lambda^{m-1} e^{-n\lambda}}{\Gamma(m)}\right)$$

and

$$\begin{aligned} p\left(\lambda \mid z_1, z_2, \ldots, z_n\right) &= \frac{L\left(\lambda; z_1, z_2, \ldots, z_n\right) p(\lambda)}{\int_0^{\infty} L\left(\lambda; z_1, z_2, \ldots, z_n\right) p(\lambda)\, d\lambda} \\ &\propto L\left(\lambda; z_1, z_2, \ldots, z_n\right) p(\lambda) \\ &\propto \left(\frac{\lambda}{1-e^{-\lambda}}\right)^{n}\left(\frac{\beta^{n\alpha}}{\left(\Gamma(\alpha)\right)^{n}}\right)\left(\prod_{i=1}^{n} z_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^{n} z_i\right)-\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta z_i)}\left(\frac{n^m \lambda^{m-1} e^{-n\lambda}}{\Gamma(m)}\right). \end{aligned}$$

Because the posterior distributions are complicated, the Markov Chain Monte Carlo method is used to simulate samples from them.

5.2 Simulation Study

In this section, we investigate the performance of the proposed estimators through a simulation study. The simulation study is carried out for different sample size and with different hyperparameter values. In particular, we take sample sizes $n = 25,\ 50,\ 100,\ \text{and}\ 1{,}000$, and the hyperparameter is selected based on the condition that the mean of prior distribution closely approximates the parameter value. Table 5.1–5.2 show the values of hyperparameters for selected cases of GZTP and CGZTP distributions. In the absence of an analytic form for marginal posterior distributions, the random walk Metropolis algorithm must be used to obtain such distributions and, by extension, to extract parameter characteristics including Bayes estimators and credible intervals. The chain is executed for 10,000 iterations, including a burn-in period of 1,000. A description of the random walk Metropolis algorithm used in this study is provided in the following section.

Since the density of the posterior distribution is proportional to the product of the likelihood and the density of the prior distribution, we use $L(\alpha,\beta; y_1, y_2, \ldots, y_n)\, p_1(\alpha)\, p_2(\beta)$ or $L(\alpha,\beta; z_1, z_2, \ldots, z_n)\, p_1(\alpha)\, p_2(\beta)$ as the target density for generating random samples from the joint posterior distribution of α and β. The random walk Metropolis algorithm for generating random samples from the joint posterior distributions of α and β is shown as follows:

i. Choose starting values of $\theta_0 = \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}$ and define σ,

ii. At step i, we draw $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}_i \sim MVN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ and draw a new value $\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_i = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{i-1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}_i$,

iii. The candidate $\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_i$ will be accepted with a probability given by the Metropolis ratio:

$$r\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{i-1}, \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_i\right) = \min\left(\frac{L\left(\tilde{\alpha}, \tilde{\beta}; y_1, y_2, \ldots, y_n\right) p_1\left(\tilde{\alpha}\right) p_2\left(\tilde{\beta}\right)}{L\left(\alpha, \beta; y_1, y_2, \ldots, y_n\right) p_1\left(\alpha\right) p_2\left(\beta\right)}, 1\right) \text{ for GZTP and}$$

$$r\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{i-1}, \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_i\right) = \min\left(\frac{L\left(\tilde{\alpha}, \tilde{\beta}; y_1, y_2, \ldots, y_n\right) p_1\left(\tilde{\alpha}\right) p_2\left(\tilde{\beta}\right)}{L\left(\alpha, \beta; y_1, y_2, \ldots, y_n\right) p_1\left(\alpha\right) p_2\left(\beta\right)}, 1\right) \text{ for CGZTP.}$$

Since we define $\sigma = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}$ for selected cases of GZTP and CGZTP, the trace plots of α_i and β_i from the joint posterior distributions are shown in Figures 5.1 – 5.2.

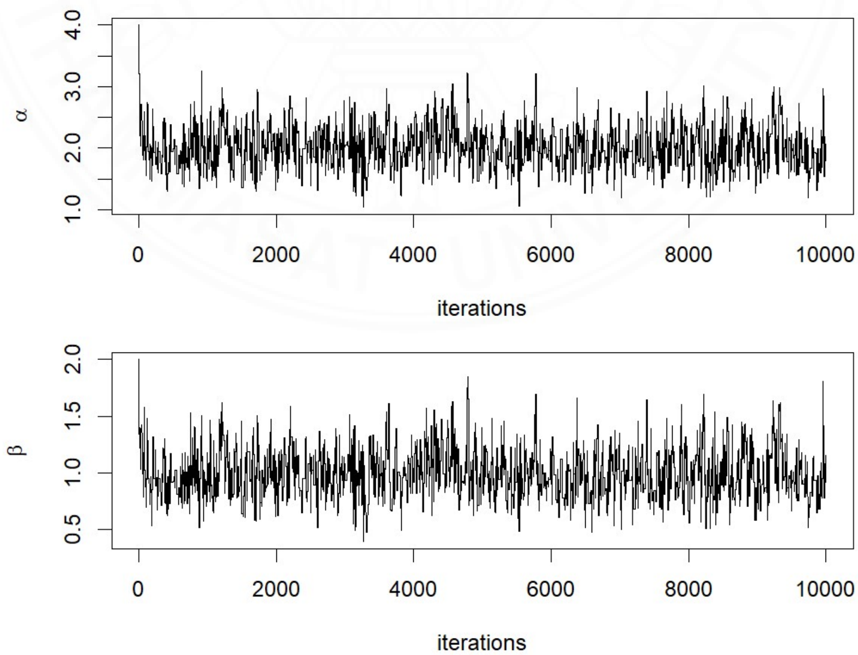

Figure 5.1 The trace plots of α_i and β_i from the joint posterior distribution corresponding to Prior 4 for GZTP with $\lambda = 1$, $\alpha = 2$, $\beta = 1$ and $n = 50$.

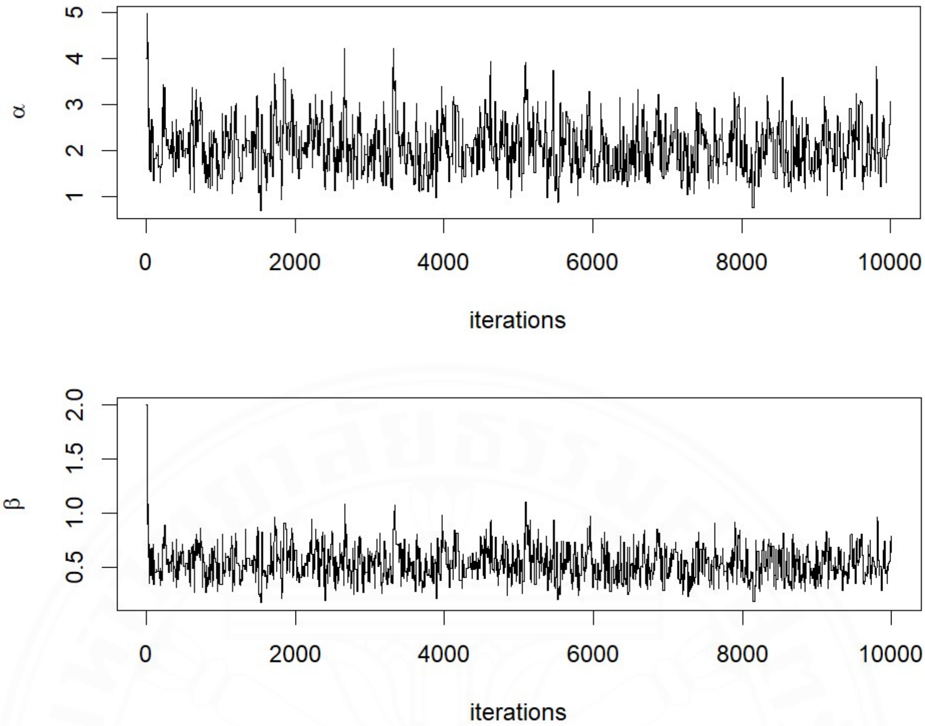

Figure 5.2 The trace plots of α_i and β_i from the joint posterior distribution corresponding to Prior 2 for for CGZTP with $\lambda = 0.5,\ \alpha = 2,\ \beta = 0.5$ and $n = 25$.

We use $L(\lambda; y_1, y_2, \ldots, y_n)\, p(\lambda)$ or $L(\lambda; z_1, z_2, \ldots, z_n)\, p(\lambda)$ as the target density for generating random samples from the posterior distribution of λ. The random walk Metropolis algorithm for generating random samples from the posterior distribution of λ are given below:

i. Choose starting values of λ_0 and define σ,

ii. At step i, we draw $\varepsilon_i \sim N(0, \sigma^2)$ and draw a new value $\tilde{\lambda}_i = \lambda_{i-1} + \varepsilon_i$,

iii. The candidate $\tilde{\lambda}_i$ will be accepted with a probability given by the Metropolis ratio:

$$r(\lambda_{i-1}, \tilde{\lambda}_i) = \min\left(\frac{L(\tilde{\lambda}; y_1, y_2, \ldots, y_n)\, p(\lambda)}{L(\lambda; y_1, y_2, \ldots, y_n)\, p(\lambda)}, 1 \right) \text{ for GZTP and}$$

$$r(\lambda_{i-1}, \tilde{\lambda}_i) = \min\left(\frac{L(\tilde{\lambda}; z_1, z_2, \ldots, z_n)\, p(\lambda)}{L(\lambda; z_1, z_2, \ldots, z_n)\, p(\lambda)}, 1 \right) \text{ for CGZTP.}$$

In this algorithm, the chosen S value for each of the selected situations is 0.5. The Figures 5.3 – 5.4 display the trace plots of λ_i obtained from the posterior distributions.

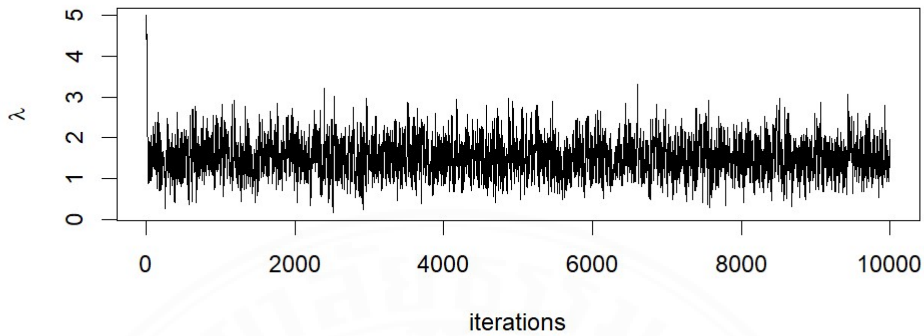

Figure 5.3 The trace plot of λ_i from posterior distribution corresponding to Prior 2 for GZTP with $\lambda = 1, \ \alpha = 1, \ \beta = 1$ and $n = 50$.

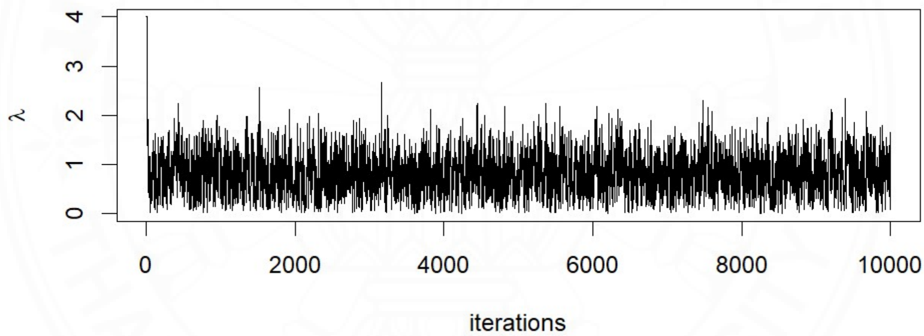

Figure 5.4 The trace plot of λ_i from posterior distribution corresponding to Prior 1 for CGZTP with $\lambda = 1, \ \alpha = 2, \ \beta = 1$ and $n = 50$.

In order to evaluate the priors' performance, point estimates for parameters are averaged on 1,000 simulated samples. Subsequently, the mean squared errors (MSEs) of the parameter estimates are calculated, and the coverage probability of 95% confidence intervals and their average lengths are determined.

Tables 5.3–5.4 present the results obtained from simulated data sets from GZTP where the values of α and β are unknown. As sample sizes increase, estimates become more accurate and the MSE values decrease. It is noticed that the MSE of $\hat{\alpha}$ increases as α increases given that β are fixed. Similarly, when α are fixed, the MSE of $\hat{\beta}$ increases as β increases. Prior 1 has the highest MSE when compared to the other priors, while Prior 4 has the lowest MSE. The 95% credible intervals for α and β are also computed using the MCMC samples. It is found that when the sample size (n) increases, the CPs will be closer to the nominal coverage probability, 0.95, and the ALs will decrease. In most cases, CPs are not less than 0.95, although the sample size is only 25; however, the CPs are below 0.95, with the sample size being 1,000. Moreover, the Prior 4 tends to have the smallest ALs with the same sample size.

From generated CGZTP data sets where the values of α and β are unknown, Tables 5.5–5.6 give the averages of Bayes estimates, MSEs, CPs and ALs of parameters. As sample sizes increase, estimates become more accurate and the MSE values decrease. It is observed that, holding β constant, the MSE of $\hat{\alpha}$ increases as α increases. Likewise, as β increases, the MSE of $\hat{\beta}$ increases when α remains constant. Applying Prior 4 results in the lowest MSE values for $\hat{\alpha}$ and $\hat{\beta}$. When sample sizes are small, the CPs approach the desired coverage probability of 0.95, and the ALs decrease with increasing sample sizes. However, the CPs are less than 0.95, with a sample size of 1,000.

Tables 5.7–5.8 display the results obtained from GZTP and CGZTP simulated data sets in which λ's value is unknown. The estimates tend to be close to their actual values, although the sample sizes are small. As sample sizes increase, the MSE values decrease for all prior. However, Prior 2 yields the lower MSE when λ is small, i.e., $\lambda = 0.5$ or $\lambda = 1$ and Prior 1 yields the lower MSE when λ is large. For all situations, the CPs approach the desired coverage probability of 0.95, and the average lengths decrease with increasing sample sizes. However, when comparing the same sample size, the ALs of Prior 1 and Prior 2 show a close value.

Table 5.1 The hyperparameters's values of α and β for the selected cases of GZTP and CGZTP distributions.

$\alpha \sim Gamma(a,b)$	$\beta \sim Gamma(c,d)$	Prior	a	b	c	d	Variance	
							α	β
2	0.5	Prior 1	1	0.5	0.5	1	high	high
		Prior 2	1	0.5	2	4	high	low
		Prior 3	2	1	0.5	1	low	high
		Prior 4	2	1	2	4	low	low
2	1	Prior 1	1	0.5	0.5	0.5	high	high
		Prior 2	1	0.5	2	2	high	low
		Prior 3	2	1	0.5	0.5	low	high
		Prior 4	2	1	2	2	low	low
1	1	Prior 1	0.5	0.5	0.5	0.5	high	high
		Prior 2	0.5	0.5	2	2	high	low
		Prior 3	2	2	0.5	0.5	low	high
		Prior 4	2	2	2	2	low	low

Table 5.2 The hyperparameters's values of λ for the selected cases of GZTP and CGZTP distributions.

$\lambda \sim Gamma(m,n)$	Prior	m	n	Variance
0.5	Prior 1	1	2	high
	Prior 2	3	6	low
1	Prior 1	1	1	high
	Prior 2	3	3	low
3	Prior 1	3	1	high
	Prior 2	1.5	0.5	low

Table 5.3 The average Bayes estimates and mean-squared errors of α and β for GZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
				α	β	α	β
0.5	(2, 0.5)	Prior 1	25	2.1328	0.5470	0.2777	0.0282
			50	2.0926	0.5299	0.1516	0.0144
			100	2.0508	0.5181	0.0701	0.0065
			1,000	2.0088	0.5031	0.0070	0.0006
		Prior 2	25	2.1057	0.5366	0.2124	0.0196
			50	2.0845	0.5267	0.1331	0.0122
			100	2.0481	0.5171	0.0663	0.0061
			1,000	2.0087	0.5031	0.0070	0.0006
		Prior 3	25	2.1171	0.5426	0.2363	0.0249
			50	2.0877	0.5285	0.1403	0.0136
			100	2.0498	0.5179	0.0677	0.0064
			1,000	2.0087	0.5031	0.0070	0.0006
		Prior 4	25	2.0948	0.5338	0.1854	0.0179
			50	2.0800	0.5256	0.1235	0.0115
			100	2.0470	0.5168	0.0639	0.0059
			1,000	2.0086	0.5030	0.0069	0.0006
0.5	(2, 1)	Prior 1	25	2.1530	1.0977	0.2941	0.1151
			50	2.0762	1.0521	0.1517	0.0593
			100	2.0391	1.0241	0.0701	0.0263
			1,000	2.0017	1.0018	0.0065	0.0024
		Prior 2	25	2.1242	1.0764	0.2224	0.0804
			50	2.0674	1.0456	0.1322	0.0499
			100	2.0372	1.0226	0.0661	0.0243
			1,000	2.0016	1.0017	0.0064	0.0024
		Prior 3	25	2.1350	1.0877	0.2467	0.1000
			50	2.0715	1.0495	0.1399	0.0557
			100	2.0378	1.0233	0.0676	0.0256
			1,000	2.0016	1.0018	0.0064	0.0024
		Prior 4	25	2.1123	1.0701	0.1933	0.0724
			50	2.0635	1.0436	0.1230	0.0472
			100	2.0363	1.0221	0.0638	0.0237
			1,000	2.0016	1.0018	0.0064	0.0024

Table 5.3 The average Bayes estimates and mean-squared errors of α and β for GZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
				α	β	α	β
1	(1, 1)	Prior 1	25	1.0639	1.1393	0.0626	0.1780
			50	1.0359	1.0743	0.0306	0.0773
			100	1.0133	1.0283	0.0126	0.0320
			1,000	1.0018	1.0023	0.0013	0.0031
		Prior 2	25	1.0496	1.1026	0.0479	0.1105
			50	1.0319	1.0642	0.0269	0.0618
			100	1.0126	1.0264	0.0119	0.0289
			1,000	1.0016	1.0020	0.0013	0.0030
		Prior 3	25	1.0540	1.1263	0.0503	0.1572
			50	1.0339	1.0719	0.0278	0.0726
			100	1.0129	1.0281	0.0121	0.0314
			1,000	1.0017	1.0023	0.0013	0.0030
		Prior 4	25	1.0428	1.0949	0.0398	0.1005
			50	1.0297	1.0613	0.0246	0.0583
			100	1.0120	1.0255	0.0114	0.0283
			1,000	1.0016	1.0020	0.0013	0.0030
1	(2, 1)	Prior 1	25	2.0983	1.0889	0.2705	0.1277
			50	2.0942	1.0662	0.1412	0.0609
			100	2.0233	1.0191	0.0658	0.0277
			1,000	2.0010	1.0008	0.0057	0.0025
		Prior 2	25	2.0729	1.0670	0.2013	0.0864
			50	2.0848	1.0584	0.1230	0.0506
			100	2.0212	1.0174	0.0617	0.0255
			1,000	2.0010	1.0009	0.0057	0.0025
		Prior 3	25	2.0843	1.0800	0.2298	0.1116
			50	2.0892	1.0632	0.1308	0.0574
			100	2.0231	1.0190	0.0639	0.0271
			1,000	2.0009	1.0008	0.0057	0.0025
		Prior 4	25	2.0633	1.0615	0.1757	0.0782
			50	2.0806	1.0562	0.1149	0.0480
			100	2.0207	1.0172	0.0596	0.0248
			1,000	2.0009	1.0008	0.0056	0.0024

Table 5.4 Coverage probabilities and average lengths of credible intervals of α and β for GZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	CP		AL	
				α	β	α	β
0.5	(2, 0.5)	Prior 1	25	0.9690	0.9640	2.0294	0.6273
			50	0.9500	0.9480	1.4353	0.4397
			100	0.9460	0.9460	1.0019	0.3077
			1,000	0.9340	0.9260	0.3083	0.0937
		Prior 2	25	0.9820	0.9670	1.9098	0.5801
			50	0.9540	0.9620	1.3915	0.4233
			100	0.9530	0.9540	0.9892	0.3024
			1,000	0.9310	0.9270	0.3077	0.0936
		Prior 3	25	0.9810	0.9620	1.9597	0.6098
			50	0.9520	0.9470	1.4070	0.4330
			100	0.9480	0.9540	0.9931	0.3056
			1,000	0.9320	0.9310	0.3079	0.0938
		Prior 4	25	0.9840	0.9710	1.8566	0.5675
			50	0.9580	0.9560	1.3681	0.4172
			100	0.9500	0.9550	0.9777	0.3004
			1,000	0.9330	0.9320	0.3075	0.0936
0.5	(2, 1)	Prior 1	25	0.9700	0.9600	2.0519	1.2565
			50	0.9580	0.9480	1.4253	0.8743
			100	0.9420	0.9480	0.9976	0.6098
			1,000	0.9360	0.9440	0.3098	0.1897
		Prior 2	25	0.9770	0.9730	1.9272	1.1600
			50	0.9640	0.9570	1.3800	0.8400
			100	0.9510	0.9490	0.9829	0.5975
			1,000	0.9390	0.9430	0.3096	0.1896
		Prior 3	25	0.9750	0.9640	1.9754	1.2200
			50	0.9570	0.9530	1.4002	0.8638
			100	0.9450	0.9490	0.9910	0.6063
			1,000	0.9350	0.9440	0.3097	0.1897
		Prior 4	25	0.9810	0.9750	1.8675	1.1329
			50	0.9640	0.9640	1.3588	0.8314
			100	0.9520	0.9490	0.9753	0.5949
			1,000	0.9410	0.9430	0.3097	0.1897

Table 5.4 Coverage probabilities and average lengths of credible intervals of α and β for GZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	CP		AL	
				α	β	α	β
1	(1, 1)	Prior 1	25	0.9600	0.9550	0.9296	1.4939
			50	0.9420	0.9570	0.6427	1.0209
			100	0.9560	0.9530	0.4450	0.7008
			1,000	0.9310	0.9410	0.1378	0.2159
		Prior 2	25	0.9690	0.9680	0.8728	1.3412
			50	0.9460	0.9660	0.6243	0.9694
			100	0.9580	0.9600	0.4383	0.6844
			1,000	0.9300	0.9390	0.1373	0.2151
		Prior 3	25	0.9640	0.9640	0.8875	1.4471
			50	0.9520	0.9590	0.6309	1.0099
			100	0.9560	0.9530	0.4409	0.6983
			1,000	0.9360	0.9410	0.1377	0.2155
		Prior 4	25	0.9700	0.9720	0.8376	1.3081
			50	0.9470	0.9660	0.6119	0.9580
			100	0.9610	0.9620	0.4338	0.6791
			1,000	0.9360	0.9420	0.1372	0.2153
1	(2, 1)	Prior 1	25	0.9630	0.9600	1.9375	1.2959
			50	0.9490	0.9560	1.3946	0.9193
			100	0.9490	0.9550	0.9575	0.6301
			1,000	0.9470	0.9480	0.2987	0.1962
		Prior 2	25	0.9680	0.9750	1.8157	1.1905
			50	0.9600	0.9650	1.3502	0.8810
			100	0.9510	0.9620	0.9417	0.6176
			1,000	0.9470	0.9470	0.2985	0.1959
		Prior 3	25	0.9640	0.9650	1.8746	1.2626
			50	0.9500	0.9590	1.3703	0.9058
			100	0.9490	0.9600	0.9512	0.6269
			1,000	0.9450	0.9490	0.2985	0.1962
		Prior 4	25	0.9740	0.9750	1.7639	1.1647
			50	0.9550	0.9640	1.3289	0.8700
			100	0.9520	0.9610	0.9363	0.6140
			1,000	0.9460	0.9480	0.2983	0.1958

Table 5.5 The average Bayes estimates and mean-squared errors of α and β for CGZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
				α	β	α	β
0.5	(2, 0.5)	Prior 1	25	2.1328	0.5417	0.2987	0.0234
			50	2.0704	0.5232	0.1592	0.0112
			100	2.0598	0.5173	0.0833	0.0058
			1,000	2.0161	0.5056	0.0085	0.0006
		Prior 2	25	2.1063	0.5336	0.2345	0.0171
			50	2.0631	0.5211	0.1413	0.0097
			100	2.0570	0.5164	0.0786	0.0054
			1,000	2.0152	0.5054	0.0092	0.0007
		Prior 3	25	2.1133	0.5372	0.2492	0.0203
			50	2.0652	0.5221	0.1458	0.0104
			100	2.0582	0.5169	0.0795	0.0056
			1,000	2.0167	0.5055	0.0084	0.0006
		Prior 4	25	2.0922	0.5305	0.1994	0.0151
			50	2.0581	0.5200	0.1306	0.0091
			100	2.0551	0.5160	0.0756	0.0052
			1,000	2.0082	0.5037	0.0075	0.0005
0.5	(2, 1)	Prior 1	25	2.1589	1.1014	0.3628	0.1104
			50	2.0898	1.0490	0.1648	0.0464
			100	2.0476	1.0264	0.0887	0.0246
			1,000	2.0050	1.0020	0.0078	0.0022
		Prior 2	25	2.1254	1.0811	0.2767	0.0800
			50	2.0820	1.0443	0.1461	0.0400
			100	2.0450	1.0249	0.0839	0.0229
			1,000	2.0051	1.0020	0.0078	0.0022
		Prior 3	25	2.1354	1.0905	0.2955	0.0947
			50	2.0836	1.0462	0.1497	0.0430
			100	2.0461	1.0258	0.0854	0.0239
			1,000	2.0047	1.0018	0.0077	0.0022
		Prior 4	25	2.1103	1.0743	0.2345	0.0710
			50	2.0767	1.0418	0.1345	0.0374
			100	2.0436	1.0243	0.0805	0.0222
			1,000	2.0049	1.0019	0.0077	0.0021

Table 5.5 The average Bayes estimates and mean-squared errors of α and β for CGZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	$AV(\hat{\boldsymbol{\theta}})$		MSE	
				α	β	α	β
1	(1, 1)	Prior 1	25	1.0926	1.1075	0.1016	0.1165
			50	1.0491	1.0549	0.0427	0.0515
			100	1.0189	1.0224	0.0189	0.0227
			1,000	1.0012	0.9998	0.0021	0.0023
		Prior 2	25	1.0667	1.0701	0.0781	0.0811
			50	1.0329	1.0342	0.0400	0.0408
			100	1.0187	1.0201	0.0191	0.0222
			1,000	1.0044	1.0045	0.0022	0.0025
		Prior 3	25	1.0601	1.0685	0.0755	0.0933
			50	1.0412	1.0463	0.0409	0.0500
			100	1.0197	1.0194	0.0191	0.0222
			1,000	1.0024	1.0024	0.0020	0.0023
		Prior 4	25	1.0535	1.0754	0.0617	0.0749
			50	1.0481	1.0521	0.0365	0.0440
			100	1.0174	1.0222	0.0180	0.0208
			1,000	1.0016	1.0022	0.0020	0.0023
1	(2, 1)	Prior 1	25	2.1333	1.0704	0.3636	0.0914
			50	2.0823	1.0440	0.1880	0.0446
			100	2.0313	1.0128	0.0861	0.0213
			1,000	2.0031	1.0008	0.0087	0.0022
		Prior 2	25	2.1062	1.0556	0.2809	0.0668
			50	2.0745	1.0399	0.1684	0.0387
			100	2.0299	1.0121	0.0815	0.0199
			1,000	2.0029	1.0007	0.0087	0.0021
		Prior 3	25	2.1099	1.0605	0.2907	0.0765
			50	2.0765	1.0417	0.1699	0.0411
			100	2.0300	1.0123	0.0823	0.0206
			1,000	2.0032	1.0008	0.0087	0.0022
		Prior 4	25	2.0895	1.0490	0.2332	0.0582
			50	2.0683	1.0373	0.1528	0.0360
			100	2.0281	1.0114	0.0779	0.0192
			1,000	2.0031	1.0008	0.0086	0.0022

Table 5.6 Coverage probabilities and average lengths of credible intervals of α and β for CGZTP distribution.

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	CP		AL	
				α	β	α	β
0.5	(2, 0.5)	Prior 1	25	0.9720	0.9740	2.1980	0.5895
			50	0.9530	0.9600	1.5412	0.4122
			100	0.9510	0.9470	1.0954	0.2906
			1,000	0.8770	0.8820	0.3057	0.0806
		Prior 2	25	0.9800	0.9790	2.0767	0.5489
			50	0.9630	0.9660	1.5017	0.3982
			100	0.9530	0.9420	1.0833	0.2858
			1,000	0.8600	0.8590	0.3030	0.0812
		Prior 3	25	0.9780	0.9740	2.1080	0.5695
			50	0.9630	0.9630	1.5130	0.4055
			100	0.9530	0.9470	1.0860	0.2884
			1,000	0.8750	0.8830	0.3079	0.0816
		Prior 4	25	0.9840	0.9820	2.0040	0.5340
			50	0.9660	0.9710	1.4722	0.3924
			100	0.9540	0.9470	1.0699	0.2832
			1,000	0.8910	0.9010	0.3072	0.0820
0.5	(2, 1)	Prior 1	25	0.9690	0.9610	2.2273	1.1997
			50	0.9690	0.9620	1.5563	0.8241
			100	0.9430	0.9330	1.0910	0.5780
			1,000	0.9130	0.9080	0.3256	0.1718
		Prior 2	25	0.9740	0.9720	2.0964	1.1129
			50	0.9690	0.9660	1.5142	0.7970
			100	0.9430	0.9440	1.0767	0.5681
			1,000	0.9110	0.9120	0.3247	0.1716
		Prior 3	25	0.9730	0.9640	2.1300	1.1578
			50	0.9680	0.9700	1.5265	0.8126
			100	0.9470	0.9370	1.0796	0.5736
			1,000	0.9170	0.9110	0.3252	0.1718
		Prior 4	25	0.9790	0.9750	2.0198	1.0820
			50	0.9730	0.9720	1.4867	0.7857
			100	0.9480	0.9410	1.0680	0.5644
			1,000	0.9110	0.9150	0.3246	0.1715

Table 5.6 Coverage probabilities and average lengths of credible intervals of α and β for CGZTP distribution. (Cont.)

λ	$\boldsymbol{\theta}=(\alpha,\beta)$	Prior	n	CP		AL	
				α	β	α	β
1	(1, 1)	Prior 1	25	0.9490	0.9600	1.1538	1.2331
			50	0.9580	0.9460	0.7954	0.8469
			100	0.9460	0.9480	0.5481	0.5874
			1,000	0.8860	0.8870	0.1581	0.1687
		Prior 2	25	0.9600	0.9660	1.0849	1.1257
			50	0.9490	0.9610	0.7675	0.8096
			100	0.9530	0.9430	0.5430	0.5768
			1,000	0.8890	0.8980	0.1608	0.1707
		Prior 3	25	0.9580	0.9570	1.0596	1.1496
			50	0.9520	0.9500	0.7702	0.8305
			100	0.9590	0.9530	0.5392	0.5791
			1,000	0.9050	0.9020	0.1611	0.1717
		Prior 4	25	0.9710	0.9670	1.0215	1.1009
			50	0.9660	0.9530	0.7598	0.8078
			100	0.9510	0.9560	0.5337	0.5721
			1,000	0.9060	0.8980	0.1597	0.1707
1	(2, 1)	Prior 1	25	0.9440	0.9470	2.2804	1.1363
			50	0.9460	0.9520	1.6069	0.8008
			100	0.9460	0.9470	1.1287	0.5572
			1,000	0.9050	0.9080	0.3395	0.1687
		Prior 2	25	0.9550	0.9680	2.1590	1.0650
			50	0.9570	0.9620	1.5767	0.7788
			100	0.9520	0.9460	1.1156	0.5500
			1,000	0.9050	0.9070	0.3390	0.1683
		Prior 3	25	0.9550	0.9610	2.1749	1.0976
			50	0.9560	0.9500	1.5804	0.7889
			100	0.9420	0.9490	1.1136	0.5520
			1,000	0.9080	0.9070	0.3400	0.1690
		Prior 4	25	0.9660	0.9630	2.0762	1.0340
			50	0.9540	0.9610	1.5427	0.7682
			100	0.9460	0.9440	1.0986	0.5438
			1,000	0.9070	0.9060	0.3388	0.1683

Table 5.7 The averages of Bayes estimate, mean-squared errors, coverage probabilities, and average lengths of credible intervals of λ of GZTP distribution.

λ	(α,β)	Prior	n	$AV(\hat{\lambda})$	MSE	CP	AL
0.5	(2, 0.5)	Prior 1	25	0.4693	0.0518	0.9980	1.3225
			50	0.4396	0.0492	0.9930	1.1184
			100	0.4326	0.0482	0.9850	0.9505
			1,000	0.4754	0.0123	0.9480	0.4307
		Prior 2	25	0.5034	0.0104	1.0000	1.0042
			50	0.4898	0.0135	1.0000	0.9139
			100	0.4785	0.0177	1.0000	0.8064
			1,000	0.4846	0.0095	0.9580	0.4037
0.5	(2, 1)	Prior 1	25	0.4748	0.0460	1.0000	1.3397
			50	0.4571	0.0499	0.9980	1.1470
			100	0.4418	0.0470	0.9790	0.9643
			1,000	0.4724	0.0118	0.9530	0.4308
		Prior 2	25	0.4967	0.0099	1.0000	0.9945
			50	0.4862	0.0142	1.0000	0.9077
			100	0.4806	0.0171	0.9990	0.8090
			1,000	0.4791	0.0095	0.9580	0.4035
1	(1, 1)	Prior 1	25	0.8853	0.1970	0.9820	1.9691
			50	0.8710	0.1611	0.9540	1.6183
			100	0.9120	0.1111	0.9570	1.3058
			1,000	0.9906	0.0126	0.9490	0.4399
		Prior 2	25	0.9811	0.0736	0.9990	1.6606
			50	0.9910	0.0844	0.9890	1.4316
			100	0.9497	0.0629	0.9780	1.1528
			1,000	0.9851	0.0122	0.9580	0.4332
1	(2, 1)	Prior 1	25	0.8926	0.1999	0.9740	1.9752
			50	0.8936	0.1605	0.9610	1.6380
			100	0.9093	0.1002	0.9430	1.3070
			1,000	0.9877	0.0127	0.9490	0.4400
		Prior 2	25	0.9670	0.0782	0.9950	1.6444
			50	0.9568	0.0726	0.9850	1.4057
			100	0.9512	0.0639	0.9820	1.1519
			1,000	0.9933	0.0109	0.9580	0.4338
3	(2, 1)	Prior 1	25	2.9575	0.5360	0.9560	3.0281
			50	2.9849	0.2807	0.9670	2.2532
			100	2.9879	0.1660	0.9590	1.6262
			1,000	3.0017	0.0179	0.9490	0.5231
		Prior 2	25	2.9295	0.6923	0.9340	3.2293
			50	2.9770	0.3199	0.9590	2.3274
			100	2.9860	0.1769	0.9560	1.6525
			1,000	3.0017	0.0181	0.9500	0.5243

Table 5.8 The averages of Bayes estimate, mean-squared errors, coverage probabilities, and average lengths of credible intervals of λ of CGZTP distribution.

λ	(α,β)	Prior	n	$AV(\hat{\lambda})$	MSE	CP	AL
0.5	(2, 0.5)	Prior 1	25	0.4791	0.0505	1.0000	1.3436
			50	0.4518	0.0507	0.9970	1.1372
			100	0.4331	0.0413	0.9850	0.9591
			1,000	0.4673	0.0137	0.9510	0.4299
		Prior 2	25	0.4969	0.0098	1.0000	0.9944
			50	0.4908	0.0164	1.0000	0.9137
			100	0.4868	0.0187	0.9990	0.8150
			1,000	0.4818	0.0097	0.9580	0.4035
0.5	(2, 1)	Prior 1	25	0.4659	0.0459	1.0000	1.3163
			50	0.4535	0.0461	0.9990	1.1441
			100	0.4416	0.0449	0.9770	0.9645
			1,000	0.4710	0.0127	0.9480	0.4300
		Prior 2	25	0.4944	0.0107	1.0000	0.9891
			50	0.4927	0.0156	1.0000	0.9164
			100	0.4932	0.0186	0.9990	0.8224
			1,000	0.4788	0.0097	0.9610	0.4037
1	(1, 1)	Prior 1	25	0.8908	0.1929	0.9820	1.9761
			50	0.8759	0.1559	0.9590	1.6258
			100	0.9003	0.1070	0.9470	1.3019
			1,000	0.9857	0.0134	0.9400	0.4399
		Prior 2	25	0.9700	0.0738	0.9960	1.6479
			50	0.9737	0.0822	0.9970	1.4200
			100	0.9525	0.0616	0.9700	1.1551
			1,000	0.9921	0.0133	0.9590	0.4341
1	(2, 1)	Prior 1	25	0.8949	0.1924	0.9890	1.9854
			50	0.8925	0.1564	0.9660	1.6416
			100	0.9169	0.1120	0.9470	1.3040
			1,000	0.9946	0.0127	0.9460	0.4391
		Prior 2	25	0.9814	0.0789	1.0000	1.6575
			50	0.9669	0.0814	0.9940	1.4134
			100	0.9532	0.0629	0.9850	1.1542
			1,000	0.9956	0.0110	0.9600	0.4335
3	(2, 1)	Prior 1	25	2.9543	0.4952	0.9650	3.0312
			50	2.9486	0.3152	0.9580	2.2427
			100	2.9724	0.1639	0.9570	1.6232
			1,000	2.9993	0.0190	0.9540	0.5225
		Prior 2	25	2.9260	0.6385	0.9570	3.2327
			50	2.9385	0.3590	0.9500	2.3197
			100	2.9699	0.1748	0.9530	1.6493
			1,000	2.9992	0.0191	0.9490	0.5235

CHAPTER 6
COMPARISON OF ESTIMATION METHODS

This chapter compares the performances of Bayes estimates of the unknown parameters using MCMC procedure under different priors with the maximum likelihood estimators (MLEs).

6.1 Estimation on GZTP distribution

6.1.1 Case 1: α and β are unknown

In Tables 3.3, 3.4, 5.3, and 5.4, it is observed that the Bayes estimates obtained using the random walk Metropolis and MLE procedures are quite similar in nature. The MSEs and average lengths decrease as the sample size increases. When we use the informative prior (Prior 4), the performance of the Bayes estimates is better than the corresponding MLEs in terms of the shorter average lengths and fewer MSEs as n is small. Figures 6.1–6.3 illustrate MSEs, CP, and ALs for selected case of GZTP.

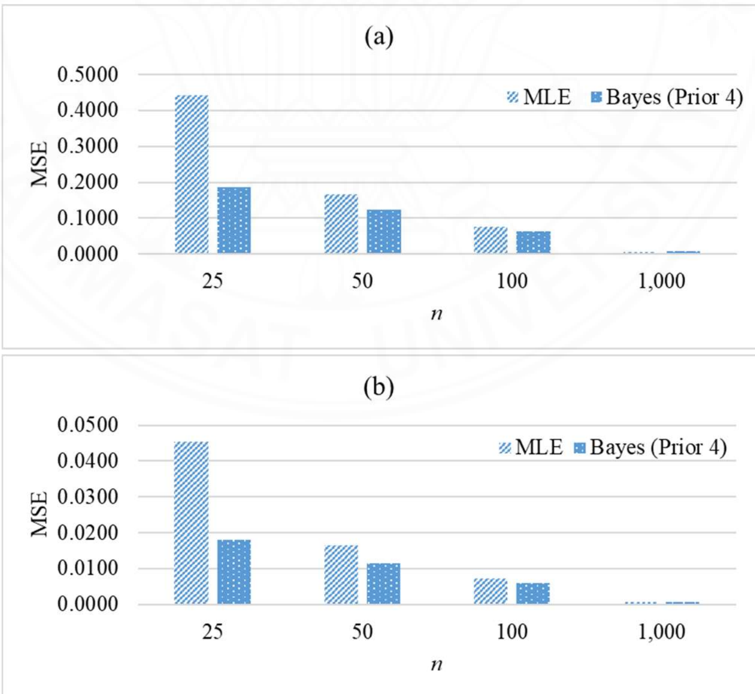

Figure 6.1 The MSE of maximum likelihood estimators (MLE) and Bayes estimators under Prior 4 for the two-parameter GZTP distribution with $\lambda = 0.5, \alpha = 2$ and $\beta = 0.5$, (a) MSE of $\hat{\alpha}$, and (b) MSE of $\hat{\beta}$.

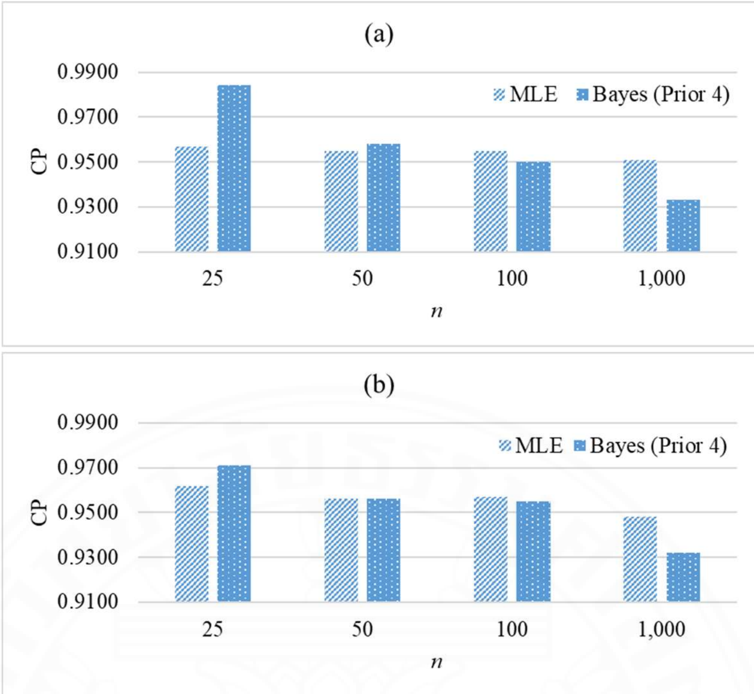

Figure 6.2 Coverage probabilities of Wald CIs and credible intervals of α and β of the two-parameter GZTP distribution with $\alpha = 2$ and $\beta = 0.5$, (a) CP of α, and (b) CP of β.

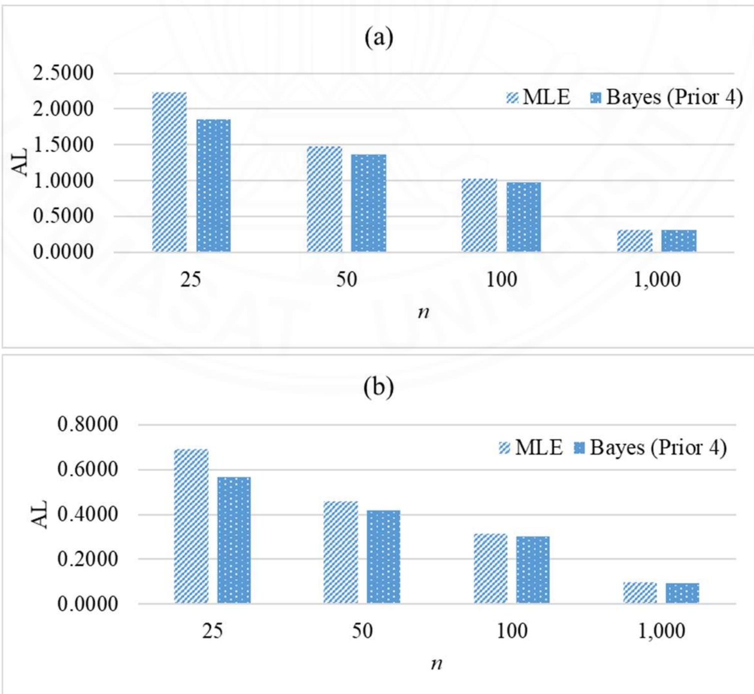

Figure 6.3 The average lengths of Wald CIs and credible intervals of α and β of the two-parameter GZTP distribution with $\lambda = 0.5,\ \alpha = 2$ and $\beta = 0.5$, (a) CP of α, and (b) CP of β.

6.1.2 Case 2: λ is unknown

The behavior (MSEs and ALs) of the Bayes estimates under Prior 2 in Tables 3.5 and 5.7 is very similar to the corresponding behavior of the MLEs. As the size of the sample increases, both the MSEs and average lengths decrease. When using the informative Prior 2, the MSEs of Bayes estimates are lower than the corresponding MLEs. Since n is small, credible intervals perform at shorter average lengths despite having larger CPs. Figures 6.4 display the MSEs, CPs, and ALs for specific cases of GZTP.

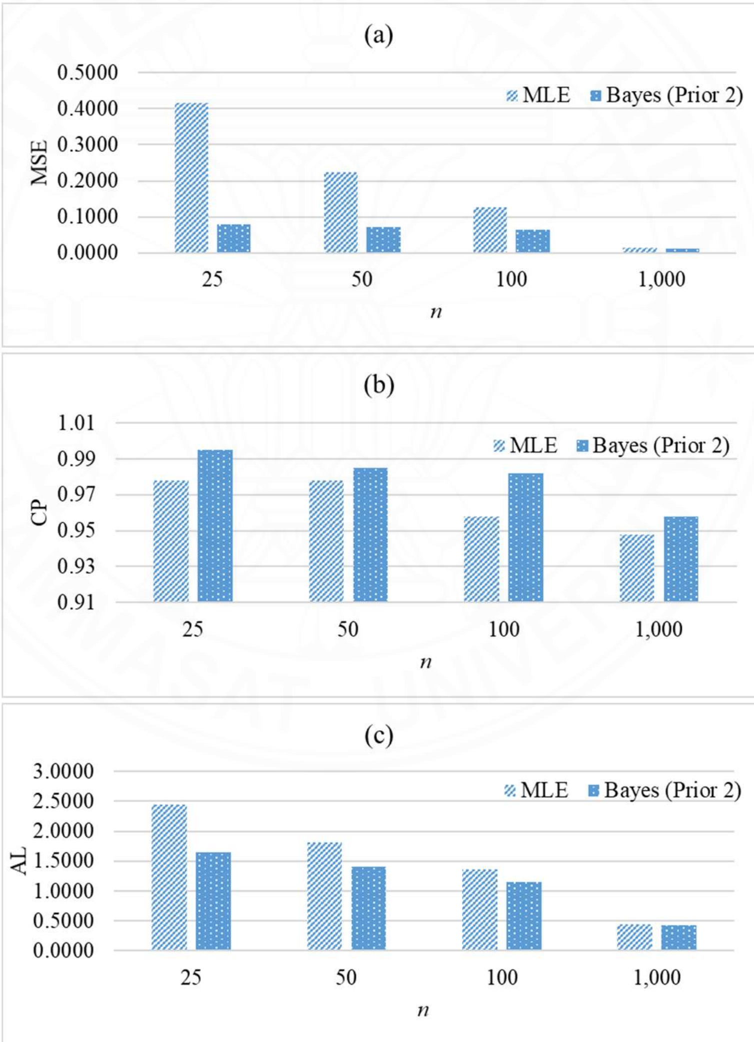

Figure 6.4 The mean-squared errors, coverage probabilities, and average lengths of Wald CIs and the credible interval of the one-parameter GZTP distribution with $\lambda = 1$, $\alpha = 2$ and $\beta = 1$.

6.2 Estimation on CGZTP distribution

6.2.1 Case 1: α and β are unknown

Tables 4.3, 4.4, 5.5, and 5.6 show that MSEs and average lengths decrease with sample size. With the informative Prior 4, Bayes estimate MSEs are lower than MLEs, and credible intervals perform better at shorter average lengths when n is small. MSEs, CPs, and ALs for selected case of CGZTP are shown in Figures 6.5–6.7.

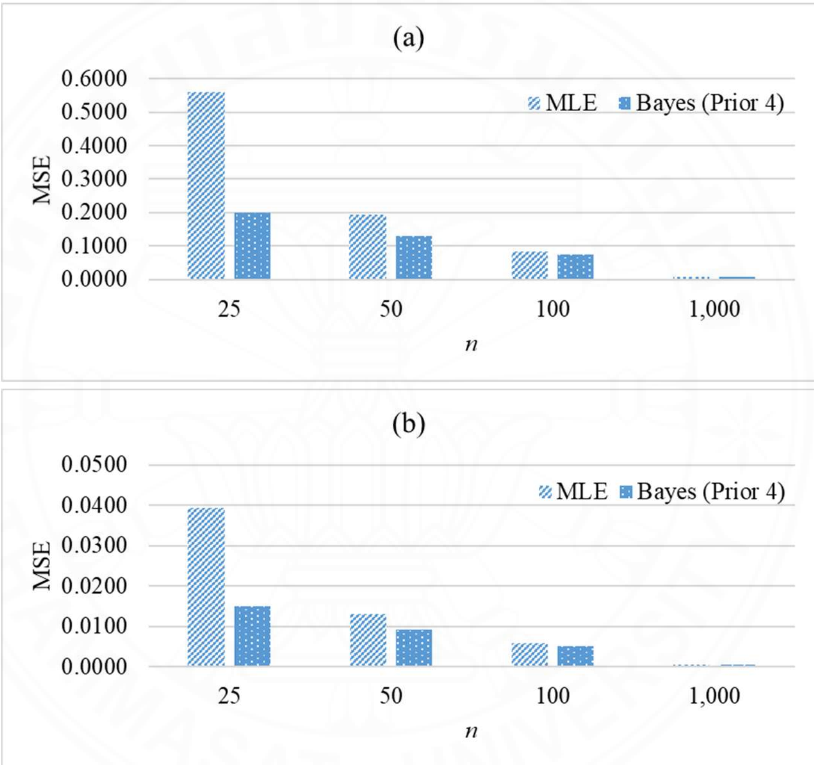

Figure 6.5 The MSE of maximum likelihood estimators (MLE) and Bayes estimators under Prior 4 for the two-parameter CGZTP distribution with $\lambda = 0.5$, $\alpha = 2$ and $\beta = 0.5$, (a) MSE of $\hat{\alpha}$, and (b) MSE of $\hat{\beta}$.

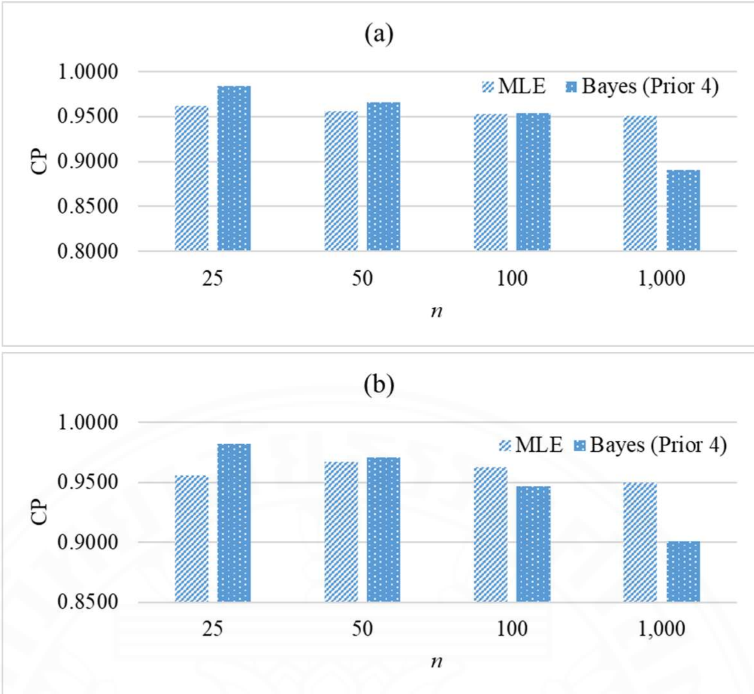

Figure 6.6 Coverage probabilities of Wald CIs and credible intervals of α and β of the two-parameter CGZTP distribution with $\lambda = 0.5,\ \alpha = 2$ and $\beta = 0.5$, (a) CP of α, and (b) CP of β.

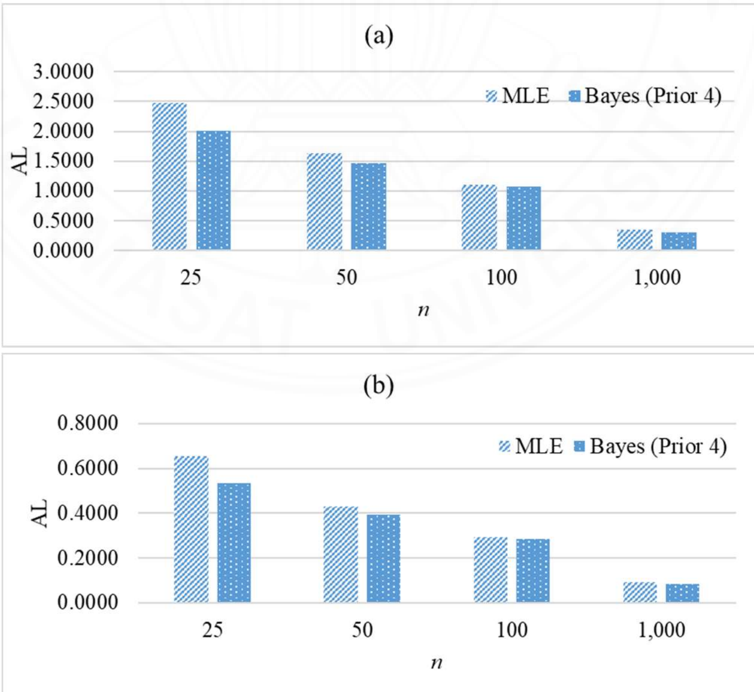

Figure 6.7 The average lengths of Wald CIs and credible intervals of α and β of the two-parameter CGZTP distribution with $\lambda = 0.5,\ \alpha = 2$ and $\beta = 0.5$, (a) AL of α, and (b) AL of β.

6.2.2 Case 2: λ is unknown

In Tables 3.5 and 5.7, the behavior (MSEs and ALs) of the Bayes estimates under Prior 2 is quite similar to that of the corresponding MLEs. Both the mean squared errors (MSEs) and average lengths exhibit a decrease as the sample size increases. By employing the informative Prior 2, Bayes estimates yield MSEs that are less than the corresponding MLEs. Given the small value of n, credible intervals have shorter average lengths. For specific case of CGZTP, the MSEs, CPs, and ALs are illustrated in Figure 6.8.

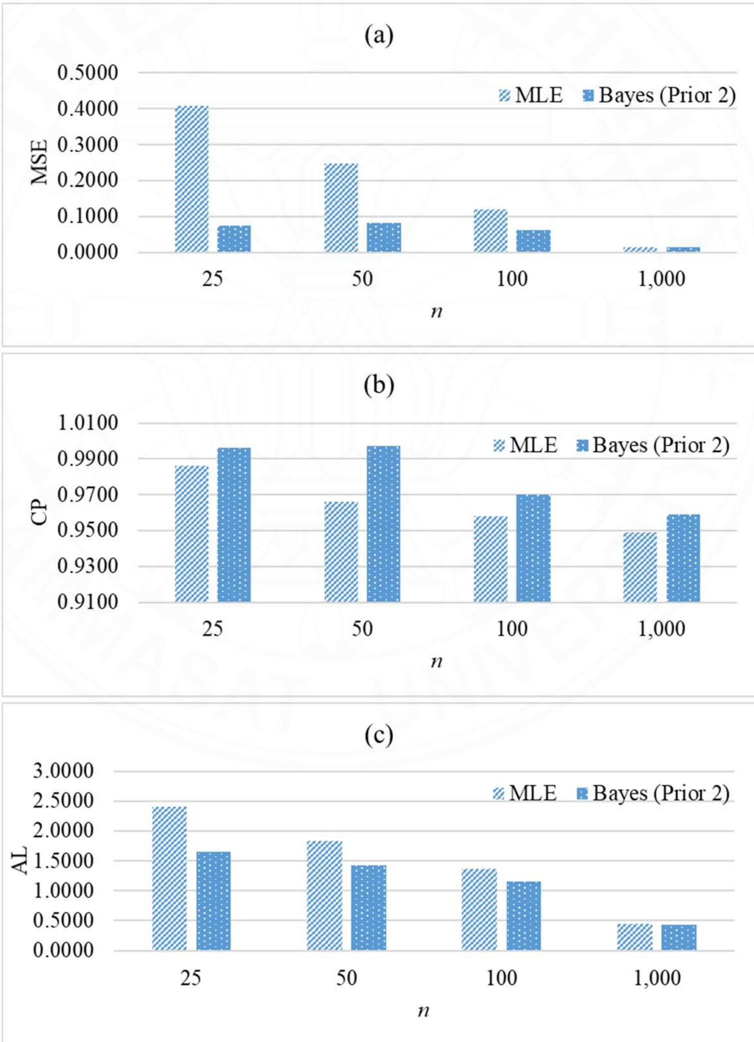

Figure 6.8 The mean-squared errors, coverage probabilities, and average lengths of Wald CIs and the credible interval of the one-parameter CGZTP distribution with $\lambda = 1,\ \alpha = 1$ and $\beta = 1$.

CHAPTER 7
APPLICATIONS

In this chapter, the real datasets are used to illustrate the use of the proposed GZTP and CGZTP distributions. The remission time of bladder cancer patients, march precipitation, and the number of successive failures are considered. For these distributions, the simulated-annealing method is used for the numerical computation of MLEs, and the model comparison is conducted using the Kolmogorov-Smirnov (K-S) test and Akaike's information criterion (AIC). Those comparative pdfs are given, respectively:

WP: $$f(y;\boldsymbol{\theta})=\frac{\alpha\beta\lambda y^{\alpha-1}}{1-e^{-\lambda}}e^{-\lambda-\beta y^{\alpha}+\lambda e^{-\beta y^{\alpha}}},\ y>0,\ \boldsymbol{\theta}=(\lambda,\alpha,\beta)$$

Gamma: $$f(y;\boldsymbol{\theta})=\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)},\ y>0,\ \boldsymbol{\theta}=(\alpha,\beta)$$

7.1 Dataset 1: Remission time of bladder cancer patients

The dataset consists of the number of months that 128 patients with bladder cancer spent in remission, as reported by Lee and Wang (2003). The lists of some descriptive statistics are shown in Table 7.1. Table 7.2 displays the MLEs, Kolmogorov–Smirnov (K–S) statistics with their corresponding p-values, and AIC values for the GZTP, CGZTP, WP, and gamma models. The results show that all distributions can be used to model the data at a significant level 0.05. However, the K–S test statistic takes the smallest value and the largest p-value under the GZTP distribution. The P–P plots, given in Figure 7.1, confirm the fit of the GZTP, CGZTP, Gamma, and WP distributions to the dataset.

Table 7.1 Descriptive statistics of remission time.

n	minimum	maximum	median	mean	skewness	SD
128	0.080	79.050	6.395	9.366	3.326	10.508

Table 7.2 Maximum likelihood estimates, goodness-of-fit testing and AIC for remission time dataset.

Distribution	Estimates	K-S	p-value	AIC
GZTP	$\hat{\boldsymbol{\theta}} = (3.9201,\ 1.4169,\ 0.0623)$	0.03598	0.9964	825.6098
CGZTP	$\hat{\boldsymbol{\theta}} = (0.0238,\ 1.1455,\ 0.1247)$	0.06657	0.6431	832.8822
Gamma	$\hat{\boldsymbol{\theta}} = (1.1748,\ 0.1255)$	0.07319	0.4992	830.7359
WP	$\hat{\boldsymbol{\theta}} = (4.0130,\ 1.2744,\ 0.0171)$	0.04550	0.9536	826.4022

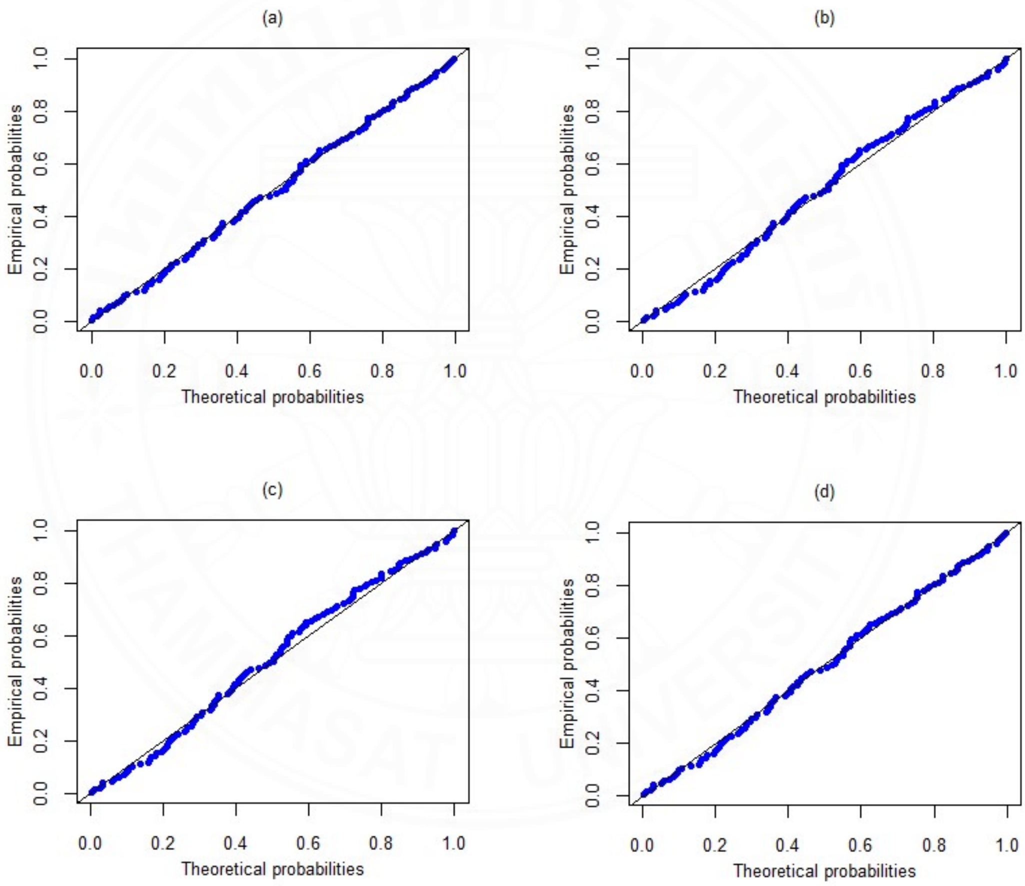

Figure 7.1 P-P plot for the remission time for (a) GZTP, (b) CGZTP, (c) Gamma, and (d) WP distributions.

7.2 Dataset 2: March Precipitation

This dataset is obtained from Hinkley (1977), and it is made up of 30 consecutive measurements of the amount of precipitation that fell in March in Minneapolis/ St. Paul. The lists of some descriptive statistics are shown in Table 7.3. The MLEs and statistics for model selections are summarized in Table 7.4 and suggest that all distributions can be used to model the data at a significant level of 0.05. The K–S test statistic has the smallest value and the largest *p*-value under the CGZTP distribution, while the AIC of the gamma model is the lowest. However, the best model depends on the choice of criteria; however, all these distributions are still useful, and the P–P plots, given in Figure 7.2, confirm the fit of the CGZTP, Gamma, and WP distributions to the dataset.

Table 7.3 Descriptive statistics of March precipitation.

n	minimum	maximum	median	mean	skewness	SD
30	0.320	4.750	1.470	1.675	1.1447	1.0006

Table 7.4 Maximum likelihood estimates, goodness-of-fit testing and AIC for March precipitation dataset.

Distribution	Estimates	K-S	*p*-value	AIC
GZTP	$\hat{\boldsymbol{\theta}} = (0.3811,\ 3.1587,\ 1.7838)$	0.05708	0.9999736	82.19037
CGZTP	$\hat{\boldsymbol{\theta}} = (0.0794,\ 2.8988,\ 1.7343)$	0.05480	0.9999906	82.22071
Gamma	$\hat{\boldsymbol{\theta}} = (2.9677,\ 1.7718)$	0.05552	0.9999868	80.19691
WP	$\hat{\boldsymbol{\theta}} = (2.1745,\ 2.1041,\ 0.1358)$	0.05709	0.9999734	82.50643

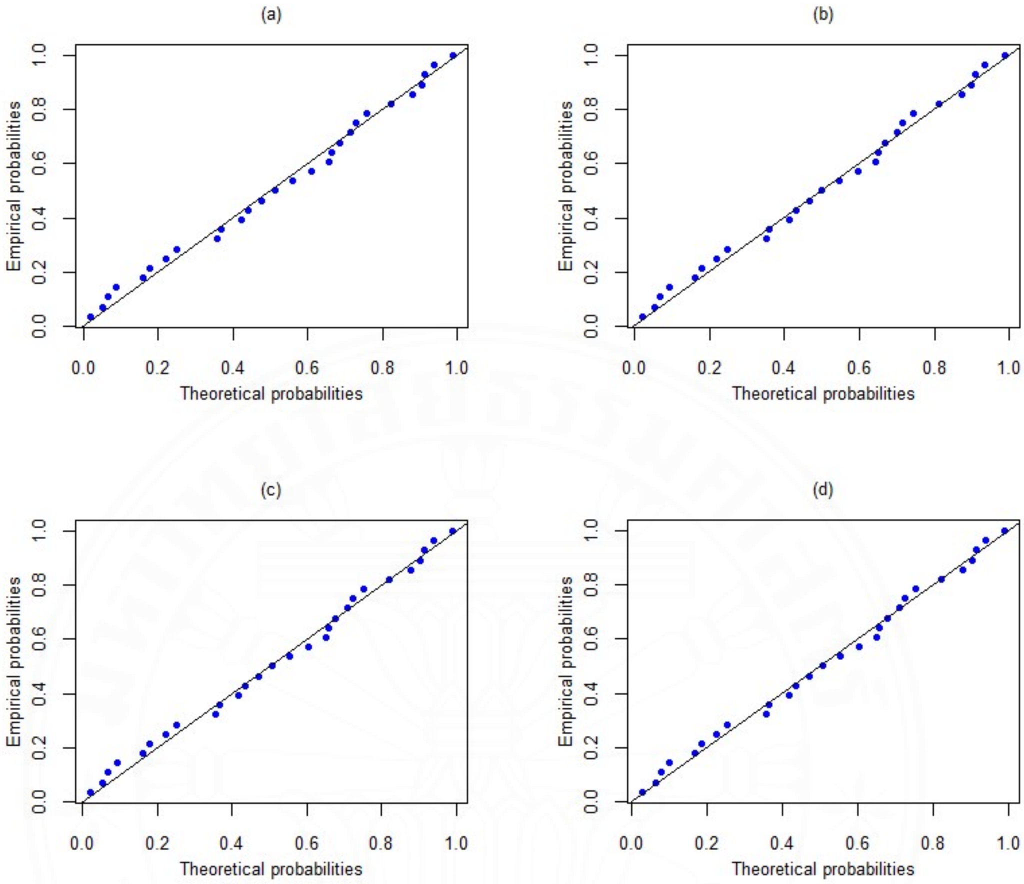

Figure 7.2 P-P plot for the March precipitation for (a) GZTP, (b) CGZTP, (c) Gamma, and (d) WP distributions.

7.3 Dataset 3: The number of successive failures

The dataset is obtained from Proschan (1963), and it is made up of 213 observations about how many times the air conditioning system on each of 13 Boeing 720 jet planes failed in a row. The lists of some descriptive statistics are shown in Table 7.5. The K–S test statistic for GZTP and CGZTP are lower than that of the gamma distribution. This suggests that both GZTP and CGZTP distributions are useful for the dataset. Also, the P-P plot, given in Figure 7.3, shows that most points lie not far from a straight diagonal line from the bottom left to the top right of the plot.

Table 7.5 Descriptive statistics of the number of successive failures.

n	minimum	maximum	median	mean	skewness	SD
213	1.00	603.00	57.00	93.14	1.66655	106.7636

Table 7.6 Maximum likelihood estimates, goodness-of-fit test, and AIC for the number of successive failure dataset.

Distribution	Estimates	K-S	p-value	AIC
GZTP	$\hat{\theta}=(0.6413,\ 1.0448,\ 0.0096)$	0.0530	0.5890	2359.026
CGZTP	$\hat{\theta}=(0.1096, 0.8411, 0.0096)$	0.0561	0.5136	2364.206
Gamma	$\hat{\theta}=(0.9048, 0.0098)$	0.0575	0.4826	2360.642
WP	$\hat{\theta}=(1.4579,\ 1.0814,\ 0.0049)$	0.0414	0.8579	2357.321

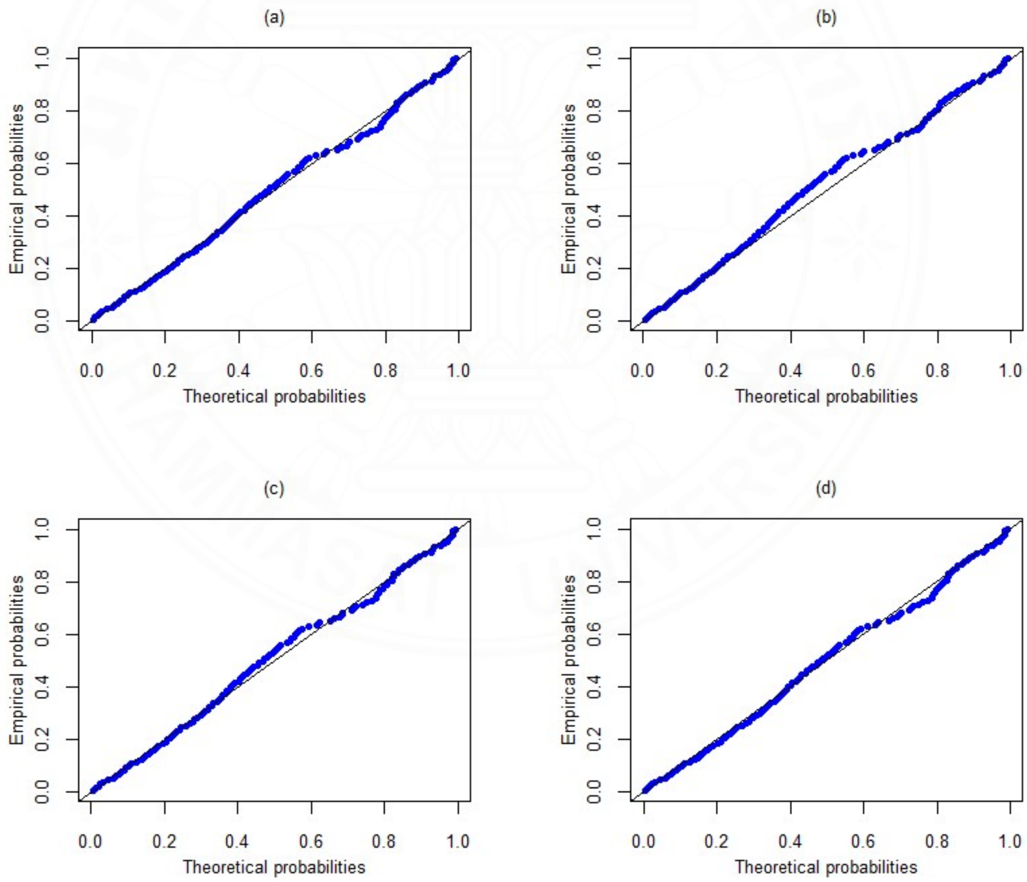

Figure 7.3 P-P plot for the number of successive failures for (a) GZTP, (b) CGZTP, (c) Gamma, and (d) WP distributions.

CHAPTER 8
CONCLUSIONS AND RECOMMENDATIONS

The GZTP distribution is newly constructed by compounding the gamma and zero-truncated Poisson distributions with the minimum compound function. The plots of the probability density function and hazard function were presented to show the flexibility of this distribution. The maximum likelihood estimators were studied, and it was found that some MLEs have no closed form. The formula of asymptotic variance-covariance matrix of the MLEs was also explicitly derived. Simulations were performed to demonstrate the behavior of MLEs in the GZTP distribution. The results show that as sample sizes increase, both MSEs and average lengths decrease; however, $\hat{\lambda}$ tends to have a higher bias value than $\hat{\alpha}$ and $\hat{\beta}$. This means that the confidence intervals do not cover the true parameter of λ, and the CPs do not reach the nominal coverage probability. However, the situation appears only in the GZTP when all parameters are unknown.

The gamma and zero-truncated Poisson distributions are compounded by using maximum function to create the CGZTP distribution. Its basic statistical properties are established in this work. The plots of hazard functions show the flexibility of this distribution, as they can be increasing, decreasing, or bathtub-shaped. The MLEs and the corresponding variance-covariance matrix are mathematically derived, and some proofs of their existence and uniqueness are provided. Furthermore, a simulation study was also conducted to show the ability of parameter estimation and the quality of estimation in some case studies. The Wald confidence intervals are useful, although the samples are not large. In a few cases, large sample sizes are required to achieve the nominal level. Finally, the CGZTP model was applied to real data to demonstrate the distribution's utility.

In this study, we have considered the Bayesian inference of the unknown parameters of the two-parameter and one-parameter GZTP and CGZTP distributions. In cases where two parameters are unknown, it is assumed that all parameters have a gamma distribution and are independently distributed. In cases where one parameter is unknown, it is also assumed to have a gamma distribution. The assumed priors are quite flexible in nature. We obtain the Bayes estimates and the corresponding credible intervals using the random walk Metropolis procedure. Simulation results suggest that the Bayes estimates behave much better than the maximum likelihood estimates, as n is small. However, the estimation when all parameters are unknown is not included in this study because the Markov chains of estimates appear to be a nonstationary process.

In future research, Bayes estimation on different prior distributions on parameters of GZTP and CGZTP will be considered, especially in the case of all parameters are unknown. Other frequentist estimation procedures for the parameters of the GZTP and CGZTP distributions will be considered, such as method of moments, ordinary and weighted least-squares, percentile, maximum product of spacings, and minimum distance estimators

REFERENCES

Books and Book Articles

Erdinç, O. (2017). Optimization in renewable energy systems : recent perspectives. Butterworth-Heinemann.

Hogg, R. V., Mckean, J. W., & Craig, A. T. (2019). Introduction to mathematical statistics. Pearson.

Jeffreys, H. (1961). Theory of Probability. Claredon Press, Oxford, UK

Lee, E. T., & John Wenyu Wang. (2013). Statistical methods for survival data analysis. Wiley.

Robert, C. P. and G. Casella (2013). Monte Carlo Statistical Methods. Springer Verlag.

Articles

Adamidis, K., Dimitrakopoulou, T., & Loukas, S. (2005). On an extension of the exponential-geometric distribution. Statistics & Probability Letters, 73(3), 259–269. https://doi.org/10.1016/j.spl.2005.03.013.

Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. Statistics & Probability Letters, 39(1), 35–42. https://doi.org/10.1016/s0167-7152(98)00012-1

Alkarni, S., & Oraby, A. (2012). A compound class of Poisson and lifetime distributions. Journal of Statistics Applications & Probability, 1(1), 45–51. https://doi.org/10.12785/jsap/010106.

Barreto-Souza, W., de Morais, A. L., & Cordeiro, G. M. (2011). The Weibull-geometric distribution. Journal of Statistical Computation and Simulation, 81(5), 645–657. https://doi.org/10.1080/00949650903436554.

Barreto-Souza, W., & Silva, R. B. (2013). A likelihood ratio test to discriminate exponential–Poisson and gamma distributions. Journal of Statistical Computation and Simulation, 85(4), 802–823. https://doi.org/10.1080/00949655.2013.847097.

Bidram, H., & Alavi, S. M. (2014). A Note on Exponentiated F-geometric Distributions. Journal of Modern Mathematics Frontier, 3(1), 18. https://doi.org/10.14355/jmmf.2014.0301.03.

Cancho, V. G., Louzada-Neto, F., & Barriga, G. D. C. (2011). The Poisson-exponential lifetime distribution. Computational Statistics & Data Analysis, 55(1), 677–686. https://doi.org/10.1016/j.csda.2010.05.033.

Ciumara, R. and Preda, V. (2009). The Weibull-logarithmic distribution in lifetime analysis and its properties. In conference of ASMDA-2009. June 30-July 3, Vilnius, LITHUANIA.

Glaser, R. E. (1980). Bathtub and Related Failure Rate Characterizations. Journal of the American Statistical Association, 75(371), 667–672. https://doi.org/10.1080/01621459.1980.10477530.

Gui, W., Zhang, H., & Guo, L. (2017). The Complementary Lindley-Geometric Distribution and Its Application in Lifetime Analysis. Sankhya B, 79(2), 316–335. https://doi.org/10.1007/s13571-017-0142-1.

Gui, W., Zhang, S., & Xin-You Lü. (2014). The Lindley-Poisson distribution in lifetime analysis and its properties. Hacettepe Journal of Mathematics and Statistics, 43(2), 1–1. https://doi.org/10.15672/hjms.201427453.

Hemmati, F., Khorram, E., & Saeid Rezakhah. (2011). A new three-parameter ageing distribution. Journal of Statistical Planning and Inference, 141(7), 2266–2275. https://doi.org/10.1016/j.jspi.2011.01.007.

Henningsen, A., & Toomet, O. (2010). maxLik: A package for maximum likelihood estimation in R. Computational Statistics, 26(3), 443–458. https://doi.org/10.1007/s00180-010-0217-1.

Hinkley, D. (1977). On Quick Choice of Power Transformation. Applied Statistics, 26(1), 67. https://doi.org/10.2307/2346869.

Ismail, E. (2016). The Complementary Compound Truncated Poisson-Weibull Distribution for Pricing Catastrophic Bonds for Extreme Earthquakes. British Journal of Economics, Management & Trade 14, 1–9 .

Kirkpatric, S., Gelatt, CD. and Vecchi, MP. (1983). Optimization by simulated annealing. Science 220, 671–680.

Kuş, C. (2007). A new lifetime distribution. Computational Statistics & Data Analysis, 51(9), 4497–4509. https://doi.org/10.1016/j.csda.2006.07.017.

Laplace, P. S. (1986). Memoir on the Probability of the Causes of Events. Statistical Science, 1(3). https://doi.org/10.1214/ss/1177013621.

Louzada, F., Maris, E., & Aparecida, M. (2012). On the Distribution of the Minimum or Maximum of a Random Number of i.i.d. Lifetime Random Variables. Applied Mathematics, 03(04), 350–353. https://doi.org/10.4236/am.2012.34054.

Louzada, F., Ramos, P. L., & Ferreira, P. H. (2018). Exponential-Poisson distribution: estimation and applications to rainfall and aircraft data with zero occurrence. Communications in Statistics - Simulation and Computation, 1–20.

Louzada, F., Ramos, P. L., & Perdoná, G. S. C. (2016). Different Estimation Procedures for the Parameters of the Extended Exponential Geometric Distribution for Medical Data. Computational and Mathematical Methods in Medicine, 2016, 1–12.

Lu, W., & Shi, D. (2012). A new compounding life distribution: the Weibull–Poisson distribution. Journal of Applied Statistics, 39(1), 21–38. https://doi.org/10.1080/02664763.2011.575126.

Meijer, C. S. (1936). Über Whittakersche bzw. Besselsche Funktionen und deren Produkte. Nieuw Archief Voor Wiskunde. Tweede Serie, 18(4), 10–39. https://zbmath.org/?format=complete&q=an:62.0421.02.

Proschan, F. (2000). Theoretical Explanation of Observed Decreasing Failure Rate. Technometrics, 42(1), 7–11. https://doi.org/10.1080/00401706.2000.10485974.

Tahmasbi, R., & Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. Computational Statistics & Data Analysis, 52(8), 3889–3901. https://doi.org/10.1016/j.csda.2007.12.002.

Tojeiro, C., Louzada, F., Roman, M., & Borges, P. (2012). The complementary Weibull geometric distribution. Journal of Statistical Computation and Simulation, 84(6), 1345–1362. https://doi.org/10.1080/00949655.2012.744406.

Xu, B., Guo, Y., & Zhu, N. (2016). The parameter bayesian estimation of two-parameter exponential-poisson distribution and its optimal property. Journal of Interdisciplinary Mathematics, 19(4), 697–707. https://doi.org/10.1080/09720502.2016.1179489.

Zakerzadeh, H. and Mahmoudi, E. (2012). A new two parameter lifetime distribution: model and properties. ArXiv (Cornell University).

APPENDICES

APPENDIX A

R CODE FOR FINDING THE MLES AND CONTRUCT WALD CIS

```
#--GZTP distribution
library(pracma)
library(gsl)
library(maxLik)

#target pdf
f<-function(x){
  return((lambda*exp(-lambda)/(1-exp(-lambda)))*
           ((beta^alpha)*(x^(alpha-1))*exp(-beta*x)/gamma(alpha))*
        exp(lambda*(1as.numeric(gammainc(beta*x,alpha)[1])/gamma(alpha))))
}

#proposal pdf
g<-function(x){
  return(1/20)
}

#Gen x
Genx.rej<-function(n){
  res<-numeric(n)
  i<-0
  while(i<n){
    u<-runif(1,0,1)
    X<-runif(1,0,20)
    #X<-rexp(1,1)
    if(c*g(X)*u<=f(X)){
      i<-i+1
      res[i]<-X
    }
  }
  return(res)
}

#calculate CP and AL
Wald.CP<-function(dat){
  M<-nrow(dat)
  z<-qnorm(0.975)
  est<-dat[,2:4]
  var<-dat[,5:7]
  L<-est-z*sqrt(var)
  U<-est+z*sqrt(var)
  temp<-matrix(NA,nrow = M,ncol = 3)
  temp2<-matrix(NA,nrow = M,ncol = 3)
  dif<-matrix(NA,nrow = M,ncol = 3)
  dif2<-matrix(NA,nrow = M,ncol = 3)
  i<-1
  while(i<=M)
  {
    if(lambda>=L[i,1]&&lambda<=U[i,1]){temp[i,1]<-1}
    else{temp[i,1]<-0}
    dif[i,1]<-U[i,1]-L[i,1]
    if(alpha>=L[i,2]&&alpha<=U[i,2]){temp[i,2]<-1}
    else{temp[i,2]<-0}
    dif[i,2]<-U[i,2]-L[i,2]
    if(beta>=L[i,3]&&beta<=U[i,3]){temp[i,3]<-1}
```

```
    else{temp[i,3]<-0}
    dif[i,3]<-U[i,3]-L[i,3]

    if(lambda>=max(c(0,L[i,1]))&&lambda<=U[i,1]){temp2[i,1]<-1}
    else{temp2[i,1]<-0}
    dif2[i,1]<-U[i,1]-max(c(0,L[i,1]))
    if(alpha>=max(c(0,L[i,2]))&&alpha<=U[i,2]){temp2[i,2]<-1}
    else{temp2[i,2]<-0}
    dif2[i,2]<-U[i,2]-max(c(0,L[i,2]))
    if(beta>=max(c(0,L[i,3]))&&beta<=U[i,3]){temp2[i,3]<-1}
    else{temp2[i,3]<-0}
    dif2[i,3]<-U[i,3]-max(c(0,L[i,3]))
    i<-i+1
  }
  CP<-colMeans(temp)
  CP2<-colMeans(temp2)
  len<-colMeans(dif)
  len2<-colMeans(dif2)
  out<-c(CP2,len2)
  return(out)
}


#---case1---
lambda<-0.5
alpha<-0.5
beta<-0.5
n<-50
m<-1500
dat<-matrix(NA,nrow = m,ncol = 10)

# maximize f(x)/g(x)
r<-function(x){f(x)/g(x)}
optimum<-optimize(r,interval=c(0,20),maximum=T)
c<-optimum$objective

#----------------
for(t in 1:m){

  Y<-Genx.rej(n)

  llike2 <- function(theta){
    a <- n*(log(theta[1])-theta[1]-log(1-
           exp(theta[1])))+n*theta[2]*log(theta[3])-
           n*log(gamma(theta[2]))+(theta[2]-1)*sum(log(Y))-
           theta[3]*sum(Y)+(theta[1]/gamma(theta[2]))*sum(gamma_inc(theta[2
           ],theta[3]*Y))
    return(a)
  }
  solution1 <- maxLik(llike2, start = c(0.5,0.5,0.5),method = "SANN")
  var.mat<-solve(-solution1$hessian)
  dat[t,]<c(solution1$code,solution1$estimate,var.mat[1,1],
        var.mat[2,2],var.mat[3,3],var.mat[1,2],var.mat[1,3],var.mat[2,3])
}
dat<-subset(dat, dat[,1]==0)
dat<-subset(dat,dat[,5]>0&dat[,6]>0&dat[,7]>0)
dat<-dat[1:1000,]
Hes.var<-c(mean(dat[,5],na.rm = T),mean(dat[,6],na.rm=T),
    mean(dat[,7],na.rm = T),mean(dat[,8],na.rm = T),
    mean(dat[,9],na.rm = T),mean(dat[,10],na.rm = T))
V<-cov(dat[,2:4])
simu.var<-c(V[1,1],V[2,2],V[3,3],V[1,2],V[1,3],V[2,3])
mean.est<-colMeans(dat[,2:4])
```

```
true.par<-c(lambda,alpha,beta)
se<-sqrt(simu.var[1:3])
bias.est<-mean.est-true.par
mse<-c(sum((true.par[1]-dat[,2])^2)/nrow(dat),
       sum((true.par[2]-dat[,3])^2)/nrow(dat),
       sum((true.par[3]-dat[,4])^2)/nrow(dat))
Wald.CI<-Wald.CP(dat)
All<-c(mean.est,se,mse,bias.est,simu.var,Hes.var,Wald.CI)
```

```
#---CGZTP distribution
library(pracma)
library(gsl)
library(maxLik)

#target pdf
f<-function(x){
  return((lambda*exp(-lambda)/(1-exp(-lambda)))*
           ((beta^alpha)*(x^(alpha-1))*exp(-beta*x)/gamma(alpha))*
           exp(lambda*as.numeric(gammainc(beta*x,alpha)[1])/gamma(alpha)))
}

#proposal pdf
g<-function(x){
  return(1/20)
}

#Gen x
Genx.rej<-function(n){
  res<-numeric(n)
  i<-0
  while(i<n){
    u<-runif(1,0,1)
    X<-runif(1,0,20)
    if(c*g(X)*u<=f(X)){
      i<-i+1
      res[i]<-X
    }
  }
  return(res)
}

#calculate CP and AL
Wald.CP<-function(dat){
  M<-nrow(dat)
  z<-qnorm(0.975)
  est<-dat[,2:4]
  var<-dat[,5:7]
  L<-est-z*sqrt(var)
  U<-est+z*sqrt(var)
  temp<-matrix(NA,nrow = M,ncol = 3)
  temp2<-matrix(NA,nrow = M,ncol = 3)
  dif<-matrix(NA,nrow = M,ncol = 3)
  dif2<-matrix(NA,nrow = M,ncol = 3)
  i<-1
  while(i<=M)
  {
    if(lambda>=L[i,1]&&lambda<=U[i,1]){temp[i,1]<-1}
    else{temp[i,1]<-0}
    dif[i,1]<-U[i,1]-L[i,1]
    if(alpha>=L[i,2]&&alpha<=U[i,2]){temp[i,2]<-1}
```

```
    else{temp[i,2]<-0}
    dif[i,2]<-U[i,2]-L[i,2]
    if(beta>=L[i,3]&&beta<=U[i,3]){temp[i,3]<-1}
    else{temp[i,3]<-0}
    dif[i,3]<-U[i,3]-L[i,3]

    if(lambda>=max(c(0,L[i,1]))&&lambda<=U[i,1]){temp2[i,1]<-1}
    else{temp2[i,1]<-0}
    dif2[i,1]<-U[i,1]-max(c(0,L[i,1]))
    if(alpha>=max(c(0,L[i,2]))&&alpha<=U[i,2]){temp2[i,2]<-1}
    else{temp2[i,2]<-0}
    dif2[i,2]<-U[i,2]-max(c(0,L[i,2]))
    if(beta>=max(c(0,L[i,3]))&&beta<=U[i,3]){temp2[i,3]<-1}
    else{temp2[i,3]<-0}
    dif2[i,3]<-U[i,3]-max(c(0,L[i,3]))
    i<-i+1
  }
  CP<-colMeans(temp)
  CP2<-colMeans(temp2)
  len<-colMeans(dif)
  len2<-colMeans(dif2)
  out<-c(CP2,len2)
  return(out)
}


#---case1---
lambda<-0.5
alpha<-0.5
beta<-0.5
n<-50
m<-1500
dat<-matrix(NA,nrow = m,ncol = 10)

# maximize f(x)/g(x)
r<-function(x){f(x)/g(x)}
optimum<-optimize(r,interval=c(0,20),maximum=T)
c<-optimum$objective

#----------------
for(t in 1:m){

  Y<-Genx.rej(n)

  llike2 <- function(theta){
    a <- n*(log(theta[1])-log(1-exp(-theta[1])))+n*theta[2]*log(theta[3])-
   n*log(gamma(theta[2]))+(theta[2]-1)*sum(log(Y))-theta[3]*sum(Y)-
   (theta[1]/gamma(theta[2]))*sum(gamma_inc(theta[2],theta[3]*Y))
    return(a)
  }
  solution1 <- maxLik(llike2, start = c(0.5,0.5,0.5),method = "SANN")
  var.mat<-solve(-solution1$hessian)
  dat[t,]<-c(solution1$code,solution1$estimate,var.mat[1,1],
          var.mat[2,2],var.mat[3,3],var.mat[1,2],
          var.mat[1,3],var.mat[2,3])
}
dat<-subset(dat, dat[,1]==0)
dat<-subset(dat,dat[,5]>0&dat[,6]>0&dat[,7]>0)
dat<-dat[1:1000,]
Hes.var<-c(mean(dat[,5],na.rm = T),mean(dat[,6],na.rm = T),
   mean(dat[,7],na.rm = T),mean(dat[,8],na.rm = T),
   mean(dat[,9],na.rm = T),mean(dat[,10],na.rm = T))
V<-cov(dat[,2:4])
```

```
simu.var<-c(V[1,1],V[2,2],V[3,3],V[1,2],V[1,3],V[2,3])
mean.est<-colMeans(dat[,2:4])
true.par<-c(lambda,alpha,beta)
se<-sqrt(simu.var[1:3])
bias.est<-mean.est-true.par
mse<-c(sum((true.par[1]-dat[,2])^2)/nrow(dat),sum((true.par[2]-
       dat[,3])^2)/nrow(dat),sum((true.par[3]-dat[,4])^2)/nrow(dat))
Wald.CI<-Wald.CP(dat)
All<-c(mean.est,se,mse,bias.est,simu.var,Hes.var,Wald.CI)
```

APPENDIX B

R CODE FOR COMPUTING BAYES ESTIMATE AND CREDIBLE INTERVAL

```
library(pracma)
library(gsl)
library(MASS)
library(nimble)
library(matrixStats)
library(openxlsx)
library(Metrics)

#----- Rejection sampling----------
#target pdf
f<-function(x){
  return((lambda*exp(-lambda)/(1-exp(-lambda)))*
      ((beta^alpha)*(x^(alpha-1))*exp(-beta*x)/gamma(alpha))*
    exp(lambda*(1as.numeric(gammainc(beta*x,alpha)[1])/gamma(alpha))))
}
#proposal pdf
g<-function(x){
  return(1/20)
}
#Gen x
Genx.rej<-function(n){
  res<-numeric(n)
  i<-0
  while(i<n){
    u<-runif(1,0,1)
    X<-runif(1,0,20)
    #X<-rexp(1,1)
    if(c*g(X)*u<=f(X)){
      i<-i+1
      res[i]<-X
    }
  }
  return(res)
}
```

```
#--------Metropolis Hastings----------------------------
alpha.fun <- function(vec.par, dat.obs){
  a <- 1; b <- 0.5; c <-2; d <-4;
  alpha.old <- vec.par[1]
  beta.old <- vec.par[2]
  alpha.cur <- vec.par[3]
  beta.cur <- vec.par[4]
  n <- length(dat.obs)
  loglik <- function(theta){
        a <- n*(log(lambda)-lambda-log(1-exp(-
        lambda)))+n*theta[1]*log(theta[2])-n*log(gamma(theta[1]))+
        (theta[1]-1)*sum(log(dat.obs))theta[2]*sum(dat.obs)+(
        lambda/gamma(theta[1])*sum(gamma_inc(theta[1],theta[2]*dat.obs))
    return(a)
  }

  logprior<-function(theta){
    lp1 <- dgamma(theta[1],shape=a,rate=b,log = TRUE )
    lp2 <- dgamma(theta[2],shape=c,rate=d,log = TRUE )
    return(lp1+lp2)
  }

  logpost<-function(theta){
    return(loglik(theta)+logprior(theta))
  }

  if(alpha.cur>0 & beta.cur>0){ratio<-
            exp(logpost(c(alpha.cur,beta.cur))-
            logpost(c(alpha.old,beta.old)))} else{ratio <- 0}
  return(min(ratio,  1))
}

GenerateMcMCSample <- function(N, sigma, data){
  X <- matrix(NA,  nrow = N, ncol = 2)
  X[1,] <- c(5,5)
  for(i in 2:N){
    par.cur <- X[i-1,] + mvrnorm(n=1,mu=c(0,0),sigma)
    alpha.val <- alpha.fun(c(X[i-1,], par.cur),  data)
    U <- runif(1)
    if(U < alpha.val){ X[i,] <- par.cur }
```

```
    else{X[i,] <- X[i-1,]}
  }
  return(X)
}

#----case 1---------
lambda<-0.5
alpha<-2
beta<-0.5
n<-25
m<-1000
est<-L<-U<-cp<-matrix(NA,nrow = m,ncol = 2)

# maximize f(x)/g(x)
r<-function(x){f(x)/g(x)}
optimum<-optimize(r,interval=c(0,20),maximum=T)
c<-optimum$objective
i<-1
while(i<=m){
  data <- Genx.rej(n)
  sigma <- matrix(c(0.3,0,0,0.1),ncol=2)
  sample<- GenerateMcMCSample(10000, sigma, data)
  true.par<-c(alpha,beta)
  if(mean(sample[-(1:1000),1])>0 & mean(sample[-(1:1000),2])>0){
    est[i,]<-colMeans(sample[-(1:1000),])
    L[i,]<-colQuantiles(sample[-(1:1000),],probs=0.05/2)
    U[i,]<-colQuantiles(sample[-(1:1000),],probs=1-0.05/2)
    cp[i,]<-(true.par>=L[i,]&true.par<=U[i,])*1
    i<-i+1}
  else{i<-i}
}
plot(1:10000,sample[,1],type = "l")
plot(1:10000,sample[,2],type = "l")

#---result--
mean.est<-colMeans(est)
true.par<-c(alpha,beta)
v<-cov(est)
simu.var<-c(v[1,1],v[2,2],v[1,2])
se<-sqrt(c(v[1,1],v[2,2]))
```

```
mse<-c(mse(true.par[1],est[,1]),mse(true.par[2],est[,2]))
bias.est<-mean.est-true.par
mean_L<-colMeans(L)
mean_U<-colMeans(U)
cp.est<-colMeans(cp)
AL<-colMeans(U-L)
All<-c(mean.est,se,mse,bias.est,simu.var,mean_L,mean_U,cp.est,AL)
```

APPENDIX C
R COODE FOR FINDING K-S TEST, AIC, AND BUILDING THE P-P PLOT

```
library(pracma)
library(maxLik)
library(gsl)
library(MASS)
library(fitdistrplus)
library(EnvStats)

ks.2sided<-function(x,Fx,alpha){
  n <- length(x)
  i <- numeric(n)
  D.plus <- numeric(n)
  D.minus <- numeric(n)
  j <- numeric(n)
  K <- numeric(n)
  j <- seq(1,n,1)
  K <- sort(x)
  for(i in 1:n){
    D.plus[i] <- j[i]/n-Fx(K[i])
    D.minus[i] <- Fx(K[i])-(j[i]-1)/n
  }
  D.n <- max(max(D.minus),max(D.plus))
  P.value <- 2*exp(-2*n*(D.n)^2)
  Cri.value <-
    sqrt(- (log(alpha/2))/2)/(sqrt(n)+0.12+(0.11/sqrt(n)))
  cat("Statistics(D) is ", D.n, "\nApproximated P-value: ",
      P.value, "\nApproximated Critical Value: " ,
      Cri.value, "\n")
  if(D.n < Cri.value)
    cat("Accept null hypothesis")
  else
    cat("Reject null hypothesis")
}


library(MixtureInf)
library(fitdistrplus)

#data1:remission data
Y<-c(0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20,
     2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06,
     7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64,5.09, 7.26, 9.47,
     14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81,
     2.62, 3.82, 5.32,7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,
     10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96,
     36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75,
     4.26, 5.41, 7.63,17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25,
     17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36,1.40, 3.02, 4.34,
     5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76,
     3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03,
     20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36,6.93, 8.65, 12.63,
     22.69)


#data2:successive values of March precipitation
```

```
# pdf unimodal, hazard increasing, compare gamma, WP
Y<-c(0.77,1.74,0.81,1.20,1.95,1.20,0.47,1.43,
     3.37,2.20,3.00,3.09,1.51,2.10,0.52,1.62,
     1.31,0.32,0.59,0.81,2.81,1.87,1.18,1.35,
     4.75,2.48,0.96,1.89,0.90,2.05)

library(MixtureInf)
library(fitdistrplus)
#data1:Lu&shi succesive failure
#pdf, hazard decrease, can use, but less p-value
data("timesoffailure")
Y<-timesoffailure$x

#---GZTP---
n<-length(Y)
llike2 <- function(theta){
  a <- n*(log(theta[1])-theta[1]-log(1-exp(-
       theta[1])))+n*theta[2]*log(theta[3])-
       n*log(gamma(theta[2]))+(theta[2]-1)*sum(log(Y))-
       theta[3]*sum(Y)+(theta[1]/gamma(theta[2]))*
       sum(gamma_inc(theta[2],theta[3]*Y))
  return(a)
}
solution1 <- maxLik(llike2, start = c(1,1,1),method = "SANN")
lambda<-solution1$estimate[1]
alpha<-solution1$estimate[2]
beta<-solution1$estimate[3]
solution1$estimate
AIC(solution1)

Fx1<-function(x){
  return((1-exp(lambda*as.numeric(
     gammainc(beta*x,alpha)[2])/gamma(alpha)-lambda))/(1-exp(-lambda)))
}
H<-Vectorize(Fx1)
ks.2sided(Y,Fx1,0.05)
ks.test(Y,H)
ks.test(Y,H)$p.value

Y<-sort(unique(Y))
n<-length(Y)
empCDF <- data.frame(x = sort(unique(Y)), p = seq_len(n) / n)
trueCDF <- H(Y)
plot(trueCDF, empCDF$p,xlab ="",
ylab = "",pch=20,col="blue",cex=1,cex.axis=0.8,cex.lab=0.8,lwd=1.5)
title("(a)",font.main=1,cex.main=0.8,line = 1)
title(ylab="Empirical probabilities",line=2,cex.lab=0.8)
title(xlab="Theoretical probabilities",line=2,cex.lab=0.8)
abline(0,1)

#--CGZTP--
n<-length(Y)
llike2 <- function(theta){
  a <- n*(log(theta[1])-log(1-exp(-theta[1])))+
     n*theta[2]*log(theta[3])-n*log(gamma(theta[2]))+
    (theta[2]-1)*sum(log(Y))-theta[3]*sum(Y)-
    (theta[1]/gamma(theta[2]))*sum(gamma_inc(theta[2],theta[3]*Y))
  return(a)
}
solution1 <- maxLik(llike2, start = c(1,1,1),method = "SANN")
lambda<-solution1$estimate[1]
alpha<-solution1$estimate[2]
beta<-solution1$estimate[3]
```

```
solution1$estimate
AIC(solution1)

Fx1<-function(x){
  return((exp(-lambda*as.numeric(
  gammainc(beta*x,alpha)[2])/gamma(alpha))-exp(-lambda))/(1-exp(-lambda)))
}
H<-Vectorize(Fx1)
ks.2sided(Y,Fx1,0.05)
ks.test(Y,H)
ks.test(Y,H)$p.value

Y<-sort(unique(Y))
n<-length(Y)
empCDF <- data.frame(x = sort(unique(Y)), p = seq_len(n) / n)
trueCDF <- H(Y)
plot(trueCDF, empCDF$p,xlab ="",
ylab = "",pch=20,col="blue",cex=1,cex.axis=0.8,cex.lab=0.8,lwd=1.5)
title("(b)",font.main=1,cex.main=0.8,line = 1)
title(ylab="Empirical probabilities",line=2,cex.lab=0.8)
title(xlab="Theoretical probabilities",line=2,cex.lab=0.8)
abline(0,1)

#---gamma
n<-length(Y)
llike2 <- function(theta){
  a<-n*theta[1]*log(theta[2])+(theta[1]-1)*
     sum(log(Y))-theta[2]*sum(Y)-n*log(gamma(theta[1]))
  return(a)
}
solution1 <- maxLik(llike2, start = c(1,1),method = "SANN")
alpha<-solution1$estimate[1]
beta<-solution1$estimate[2]
solution1$estimate
AIC(solution1)

Fx1<-function(x){
  return(1-as.numeric(gammainc(beta*x,alpha)[2])/gamma(alpha))
}
H<-Vectorize(Fx1)
ks.2sided(Y,Fx1,0.05)
ks.test(Y,H)
ks.test(Y,H)$p.value

Y<-sort(unique(Y))
n<-length(Y)
empCDF <- data.frame(x = sort(unique(Y)), p = seq_len(n) / n)
trueCDF <- H(Y)
plot(trueCDF, empCDF$p,xlab ="",
ylab = "",pch=20,col="blue",cex=1,cex.axis=0.8,cex.lab=0.8,lwd=1.5)
title("(c)",font.main=1,cex.main=0.8,line = 1)
title(ylab="Empirical probabilities",line=2,cex.lab=0.8)
title(xlab="Theoretical probabilities",line=2,cex.lab=0.8)
abline(0,1)

#---weibull poisson
n<-length(Y)
llike2 <- function(theta){
  a <- n*log(theta[1]*theta[2]*theta[3])+
  (theta[2]-1)*sum(log(Y))+theta[1]*sum(exp(-theta[3]*Y^theta[2]))
  -theta[3]*sum(Y^theta[2])-n*log(exp(theta[1])-1)
  return(a)
}
```

```
solution1 <- maxLik(llike2, start = c(1,1,1),method = "SANN")
lambda<-solution1$estimate[1]
alpha<-solution1$estimate[2]
beta<-solution1$estimate[3]
solution1$estimate
AIC(solution1)

Fx1<-function(x){
  return((exp(lambda*exp(-beta*x^alpha))-exp(lambda))*(1-exp(lambda))^-1)
}
H<-Vectorize(Fx1)
ks.2sided(Y,Fx1,0.05)
ks.test(Y,H)
ks.test(Y,H)$p.value

Y<-sort(unique(Y))
n<-length(Y)
empCDF <- data.frame(x = sort(unique(Y)), p = seq_len(n) / n)
trueCDF <- H(Y)
plot(trueCDF, empCDF$p,xlab ="",
ylab = "",pch=20,col="blue",cex.axis=0.8,cex.lab=0.8,lwd=1.5)
title("(d)",font.main=1,cex.main=0.8,line = 1)
title(ylab="Empirical probabilities",line=2,cex.lab=0.8)
title(xlab="Theoretical probabilities",line=2,cex.lab=0.8)
abline(0,1)
```

BIOGRAPHY

Name	Ausaina Niyomdecha
Educational Attainment	2017: M.Sc. in Applied Statistics, Thammasat University, Thailand
	2015: B.Sc. in Mathematics, Mahidol University, Thailand
Scholarship	2012-2023: Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST)

Publications

Niyomdecha, A., Srisuradetchai, P., & Tulyanitikul, B. (2023). Gamma Zero-Truncated Poisson Distribution with the Minimum Compounded Function. Thailand Statistician, 21(4), 863–886.

Niyomdecha, A., and Srisuradetchai, P. (2023). Complementary Gamma Zero-Truncated Poisson Distribution and Its Application. Mathematics 11, no. 11: 2584. https://doi.org/10.3390/math11112584