TOTAL AND PAIRED DOMINATION NUMBERS AND γ-TOTAL AND γ-PAIRED DOMINATING GRAPHS OF SOME GRAPHS

BY

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A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (MATHEMATICS)

DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
THAMMASAT UNIVERSITY

ACADEMIC YEAR 2023

THAMMASAT UNIVERSITY
FACULTY OF SCIENCE AND TECHNOLOGY

DISSERTATION

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ENTITLED

TOTAL AND PAIRED DOMINATION NUMBERS AND γ-TOTAL AND γ-PAIRED DOMINATING GRAPHS OF SOME GRAPHS

was approved as partial fulfillment of the requirements for
the degree of Doctor of Philosophy (Mathematics)

on June 6, 2023

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Dissertation Title	Total and Paired Domination Numbers and γ-Total and γ-Paired Dominating Graphs of Some Graphs
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Degree	Doctor of Philosophy (Mathematics)
Department /Faculty/University	Mathematics and Statistics Faculty of Science and Technology Thammasat University
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Academic Year	2023

ABSTRACT

Let G be a graph without isolated vertices. A total dominating set of G is a set $D \subseteq V(G)$ such that every vertex of G is adjacent to at least one vertex in D. A paired dominating set of G is a total dominating set whose induced subgraph contains a perfect matching. The total (paired) domination number of G is the minimum cardinality of a total (paired) dominating set of G. The γ-total (γ-paired) dominating graph of G is the graph whose vertex set contains all minimum total (paired) dominating sets of G, and two vertices of this graph are adjacent if they differ by exactly one vertex. In this dissertation, we determine the total domination numbers and the paired domination numbers of some cylinders, some wheel related graphs, windmill graphs, lollipop graphs, umbrella graphs, and coconut graphs. We also determine the γ-total dominating graphs and the γ-paired dominating graphs of some families of graphs including some graphs mentioned above.

Keywords: total domination number, paired domination number, total dominating graph, paired dominating graph, gamma graph

ACKNOWLEDGEMENTS

I would like to express my gratitude to the following individuals and organizations for their invaluable support during my Ph.D. journey. Without them, accomplishing my dissertation would not have been possible.

First and foremost, I am deeply indebted to my advisor, Dr. Nantapath Trakultraipruk, for providing me with invaluable guidance, insight, and unwavering support throughout the entire dissertation process. I have learned an immense amount from him, in terms of research and various other aspects. It has been a true honor and privilege to have had such an excellent advisor over the past five years, and I sincerely hope that he takes as much pride in this dissertation as I do.

I would also like to extend my gratitude to all members of the dissertation committee, consisting of Dr. Chalermpong Worawannotai, Dr. Khajee Jantarakhajorn, Dr. Yanisa Chaiya, and Dr. Siripong Sirisuk, for actively engaging with my dissertation and providing constructive comments and suggestions.

The completion of this dissertation would have been impossible without the generous financial support from the National Research Council of Thailand under the grant number N41A660292 and the Thammasat University's Research Administration Division (Ph.D. Scholarship, 1/2018). Additionally, I am immensely grateful to Global Affairs Canada for providing the Canada-ASEAN Scholarships and Educational Exchanges for Development (SEED) 2021-2022. This opportunity allowed me to conduct research with Dr. Kieka Mynhardt at the University of Victoria, Canada. I extend special thanks to Kieka for her kindness, assistance, and invaluable advice on my research.

I would like to express my appreciation to all the teachers who provided the necessary and vital equipment, without which I would not have been able to work efficiently on this dissertation. Furthermore, I wish to extend my gratitude to all the staff at Thammasat University, particularly those in the Department of Mathematics and Statistics, for their generosity and unwavering willingness to provide assistance when needed.

Last but certainly not least, I would like to express my heartfelt thanks to my family for their unwavering support, encouragement, and unconditional love. They have always been there to boost my confidence when I felt weak or discouraged, and I consider myself extremely fortunate to have them in my life.

Finally, I extend my gratitude to my roommates and friends at Thammasat University, Saint Gabriel's College, the University of Victoria, and Global Village Victoria, who have been a constant source of emotional support. Thank you for helping me maintain a semblance of sanity during the pandemic. I feel incredibly lucky to have such a wonderful network of friends who love and support me.

Pannawat Eakawinrujee

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CHAPTER 1
INTRODUCTION

Graph theory, which is a branch of discrete mathematics, concerns with the relationship between vertices and edges. It has been used in our daily life, for example, using GPS to determine a shortest or quickest route (destinations and their connections are considered as vertices and edges, respectively) and designing a bus route to pick up students to deliver to the school (each bus stop and each route are viewed as a vertex and an edge, respectively, so a Hamiltonian path represents one of the possible routes containing all bus stops).

Graph theory has been used as a tool to solve mathematical problems for years. It is also widely used to study and model various applications in different areas such as chemistry, biology, computer science, etc. In particular, during the pandemic, graph can be used to find the possible spread of COVID-19 [11, 43]. Consequently, graph theory has become more popular as people realized its significant advantages.

We first introduce several fundamental concepts in graph theory such as neighborhoods, subgraphs, graph operations, and isomorphisms. Moreover, we provide the definitions for various families of graphs, including paths, cycles, complete graphs, bipartite graphs, and complete bipartite graphs. For notations and terminologies, we in general follow [66].

Formally, a *graph* G comprises a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its *endpoints*. If an edge e has endpoints u and v, then we denote that by $e = uv$; moreover, u and v are said to be *adjacent*, and u and e are said to be *incident*. We write $|V(G)|$ and $|E(G)|$ for the number of vertices and edges, respectively, in G.

An edge is called a *loop* if its endpoints are the same. Edges are said to be *multiple* if they have the same endpoints. A graph with no loops or multiple edges is said to be *simple*.

Let G be a simple graph. The *degree* of a vertex $v \in V(G)$, denoted by $d(v)$, is the number of edges incident to v. An *isolated vertex* is a vertex with degree zero. A vertex of degree one is called a *leaf*, and a vertex adjacent to a leaf is called a *support vertex*.

For any vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G) : uv \in E(G)\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set $D \subseteq V(G)$, the *open neighborhood* of D is $N(D) = \bigcup_{v \in D} N(v)$, and the *closed neighborhood* of D is $N[D] = N(D) \cup D$.

A *matching* in a graph is a set of edges with no shared endpoints. A *perfect matching* in a graph is a matching such that every vertex of the graph is incident to exactly one edge in the matching.

A graph H is a *subgraph* of a graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set $D \subseteq V(G)$, the *induced subgraph* $G[D]$ is the graph whose vertex set is D and whose egde set consists of all edges in $E(G)$ that have both endpoints in D. The induced subgraph $G[D]$ may also be called the *subgraph of G induced by D*. A *spanning subgraph* of a graph G is a subgraph with vertex set $V(G)$.

A *path* with p vertices, denoted by P_p, is a sequence of p vertices where any two consecutive vertices in the sequence are adjacent in the graph. The first and the last vertices in the sequence are called the *endpoints*. A *cycle* with p vertices, denoted by C_p, is the graph obtained from a path P_p by adding the edge joining two endpoints. We use $P_p = (v_1, v_2, \ldots, v_p)$ to represent the path with $V(P_p) = \{v_1, v_2, \ldots, v_p\}$ and $E(P_p) = \{v_i v_{i+1} : 1 \leq i \leq p-1\}$. Similarly, $C_p = (v_1, v_2, \ldots, v_p)$ represents the cycle with $V(C_p) = \{v_1, v_2, \ldots, v_p\}$ and $E(C_p) = \{v_i v_{i+1} : 1 \leq i \leq p-1\} \cup \{v_p v_1\}$.

A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*. A *component* of a graph G is a connected subgraph of G that is not contained in any other connected subgraph of G. An *odd component* of a graph is a component with odd vertices.

Let $G - D$ be the graph obtained from G by deleting all vertices in $D \subseteq V(G)$ and edges incident with them.

The *complement* of a graph G is the graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and two distinct vertices of $\overline{G}$ are adjacent if they are not adjacent in G.

The *union* of graphs $G_1, G_2, \ldots, G_p$, denoted by $G_1 \cup G_2 \cup \cdots \cup G_p$, is the graph with the vertex set $\bigcup_{i=1}^{p} V(G_i)$ and the edge set $\bigcup_{i=1}^{p} E(G_i)$.

The *disjoint union* of graphs G and H, denoted by $G + H$, is the graph obtained by taking the union of G and H with disjoint vertex sets. In general, pG is the graph consisting of p pairwise disjoint copies of G.

The *join* of graphs G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edge uv for all $u \in V(G)$ and $v \in V(H)$.

The *cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H)$, and two vertices (u, v) and (u', v') are adjacent in $G \square H$ if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. If G and H are both paths (respectively, both cycles), then $G \square H$ is called a *grid* (respectively, a *toroidal mesh*). If one of G and H is a path and the other is a cycle, then $G \square H$ is called a *cylinder*.

Let G and H be graphs. An *isomorphism* from G to H is a bijection f : $V(G) \to V(H)$ such that any two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. If there exists an isomorphism from G to H, then G is *isomorphic to* H, denoted by $G \cong H$.

A *complete graph* K_p with p vertices is the graph whose vertices are pairwise adjacent.

A *fan graph* $F_{p,q}$ is the join $\overline{K_p} \vee P_q$. If $p = 1$, then the vertex of degree q is called the *central vertex*.

A *bipartite graph* is the graph whose vertex set can be partitioned into two independent sets (sets of pairwise nonadjacent vertices) called *partite sets*.

A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if they are in different partite sets. We use $K_{p,q}$ to denote a complete bipartite graph with partite sets of cardinalities p and q.

A *double star* $S_{p,q}$ is the graph obtained from $K_{1,p}$ and $K_{1,q}$ by adding the edge joining the two support vertices.

We next discuss domination and its variations in graphs, which are a wide and well-studied field of graph theory. A *dominating set* of G is a set $D \subseteq V(G)$ such that every vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set of G with cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. Domination was introduced formally by Berge [3] in 1958, and the domination number of a graph was referred to as the "coefficient of external stability". In 1962, Ore [54] first used the term "domination number". For detailed literature on domination in graphs, see Haynes *et al.* [25, 26]. Applications of dominating sets include security models where each vertex in a dominating set represents the location of a guard capable of protecting every vertex it dominates.

In addition to usual domination, there are many well-known domination parameters such as total domination and paired domination, which are studied in this dissertation. A set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ is adjacent to some vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-*set*. Total domination in graphs was introduced by Cockayne *et al.* [9] and extensively studied by Henning and Teo [29]. Total domination plays a role in the problem of placing monitoring devices in a system. Every site in the system, including the monitors, is adjacent to a monitor site. If a monitor is damaged, then an adjacent monitor can still protect the system.

A set $D \subseteq V(G)$ is a *paired dominating set* of G if it is a dominating set of G and the subgraph of G induced by D contains a perfect matching M. If an edge $uv \in M$, then $\{u, v\}$ is said to be *paired*. The *paired domination number* of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of G. A paired dominating set of cardinality $\gamma_{pr}(G)$ is called a $\gamma_{pr}(G)$-*set*. Paired domination in graphs was introduced by Haynes and Slater [27]. For more details on this topic, see [13]. Paired domination can be a model for assigning backups to police officers. To ensure the safety of each officer, it is common practice that officers are dispatched in pairs, that is, they are assigned partners so each can back up the other.

Note that $\gamma_{pr}(G) \geq \gamma_t(G)$ since a paired dominating set of G is also a total dominating set of G, and the paired domination number is an even integer.

This dissertation is organized as follows. Chapter 2 recalls some previous results and provides a brief summary of new findings. Chapter 3 demonstrates the total domination numbers and the paired domination numbers of various graphs. The γ-total dominating graphs and the γ-paired dominating graphs of several graphs are contained in Chapter 4. Finally, Chapter 5 provides the conclusions and discussions.

CHAPTER 2
REVIEW OF LITERATURE

In this chapter, we state some results from the literature and give an overview of new results. We begin with some results on domination numbers, total domination numbers, and paired domination numbers of some graphs. We then review the results on γ-graphs (defined below), including several versions of γ-graphs.

2.1 Domination Numbers

The study of domination numbers and other related domination parameters of graphs serves several important purposes and has various applications in graph theory and related fields. Domination numbers have received attention from scholars for many reasons. For example, in facility location problems, businesses and organizations need to decide where to place facilities such as hospitals, police stations, and service centers to provide coverage. The domination number helps in determining the minimum number of locations needed to serve a population effectively. Due to the reasons mentioned above, there were many scholars researching domination numbers of graphs, with a particular focus on a grid $P_p\square P_q$, a toroidal mesh $C_p\square C_q$, and a cylinder $P_p\square C_q$, where p and q are both positive integers. The domination number of $P_p\square P_q$ was determined by Jacobson and Kinch [32] for $p \in \{1, 2, 3, 4\}$ and all q, and by Chang and Clark [7] for $p \in \{5, 6\}$ and all q. Chang [6] devoted his dissertation to study the domination number of $P_p\square P_q$. He also provided its upper bound for $7 \leq p \leq q$ and a conjecture $\gamma(P_p\square P_q) = \lfloor\frac{(p+2)(q+2)}{5}\rfloor - 4$ for $16 \leq p \leq q$. Fisher [18] used programming algorithms to compute the domination number of $P_p\square P_q$ for $p \leq 21$ and all q. This computation comfirmed that the Chang's conjecture holds for $16 \leq p \leq 21$. For $p \in \{22, 23\}$, Gonçalves *et al.* [21] mentioned that it can also be proved by using the Fisher's method. Moreover, they proved the Chang's conjecture for $24 \leq p \leq q$.

Klavžar and Seifter [35] computed the domination number of $C_p\square C_q$ for $p \in \{3, 4, 5\}$ and $q \geq p$, except when $p = 5$ and $q \equiv 3 \pmod 5$. Klavžar and Žerovnik [36] gave the value for this exceptional case. Pavlič and Žerovnik [55] showed the domination number of $C_p\square C_q$ for $p \in \{6, 7\}$ and $q \geq p$.

The domination number of $P_p \square C_q$ was studied by Nandi *et al.* [53]. They determined the exact value of $\gamma(P_p \square C_q)$ for $p \in \{2, 3, 4\}$ and $q \geq 3$, as well as its bounds for $p = 5$. Pavlič and Žerovnik [55] completed the case for $p = 5$ and also provided the exact value for $p \in \{6, 7\}$ and $q \geq 3$. Moreover, they determined the domination number of $P_p \square C_q$ for all p and $q \in \{3, 4, \ldots, 11\}$.

For the Jahangir graph $J_{p,q}$ with $q \geq 3$, the domination number of $J_{p,q}$ was computed by different researchers. Specifically, Mojdeh and Ghameshlou [46], Shaheen *et al.* [59], and Mtarneh *et al.* [48] computed the domination number of $J_{p,q}$ for $p = 2$, $p = 3$, and $p \geq 4$, respectively. Later, Shaheen *et al.* [60] identified inaccuracies in the results from [48] for some values of q and sebsequently corrected them.

Several scholars have also determined the total and the paired domination numbers of graphs for various reasons, such as facility location problems, surveillance systems, and security applications. The total and the paired domination numbers of graphs, particularly $P_p \square P_q$, $C_p \square C_q$, and $P_p \square C_q$, have been studied. Gravier [22] and Proffitt *et al.* [56] determined the total domination number and the paired domination number, respectively, of $P_p \square P_q$ for $p \in \{2, 3, 4\}$ and $q \geq 2$. Klobučar [37] computed the total domination number of $P_p \square P_q$ for $p \in \{5, 6\}$ and $q \geq 2$. Kuziak *et al.* [41] showed that Klobučar's result was false when $p = 6$ and $q \equiv 0, 4, 5, 6 \pmod 7$, and they then corrected that result. The paired domination number of $P_p \square P_q$ for $p \in \{5, 6\}$ and $q \geq p$ is investigated in Section 3. Hu and Xu [31] determined the total domination number and the paired domination number of $C_p \square C_q$ for $p \in \{3, 4\}$ and $q \geq 3$, and they provided some upper bounds for $p, q \geq 5$. Hu *et al.* [30] provided the total domination number and the paired domination number of $P_p \square C_q$ for $p \geq 2$ and $q \in \{3, 4\}$. We extend the previous results by presenting the total domination number and the paired domination number of $P_p \square C_q$ for $p \in \{2, 3, 4\}$ and $q \geq 5$ in Subsection 3.3.1. We also give their upper and lower bounds for the other values of p and q in Subsections 3.3.2 and 3.3.3, respectively.

Mojdeh and Ghameshlou [46] determined the total domination number of the Jahangir graph $J_{2,q}$ for $q \geq 3$, while Mtarneh *et al.* [48] gave the total domination number of the Jahangir graph $J_{p,q}$ for $p, q \geq 3$. In Section 3.2, we demonstrate that, for $p \geq 4$, the previously mentioned result is incorrect for some values of q, and we subsequently correct this mistake.

Apart from the results mentioned above, further results on total domination numbers can be found in [47] for 3-regular Knödel graphs, in [38, 45] for hexagonal grids, in [8, 33] for central graphs, in [12] for splitting graphs, in [34] for middle graphs, in [39] for line graphs, in [44] for total graphs, and in [4, 61] for other graph classes.

2.2 γ-Graphs

Some graphs have many minimum dominating sets, so it is worth to ask a question that, for a graph G, whether a $\gamma(G)$-set can be obtained from another $\gamma(G)$-set by deleting and adding a single vertex. This problem has motivated many authors to define a new class of graphs called *γ-graphs*, which have two versions. First defined by Subramanian and Sridharan [64] in 2008, the *γ-graph* of a graph G, denoted by $\gamma \cdot G$, is the graph whose vertices are $\gamma(G)$-sets, and two vertices D_1 and D_2 of $\gamma \cdot G$ are adjacent if they satisfy the following condition:

$$D_2 = (D_1 \setminus \{u\}) \cup \{v\} \text{ for some } u \in D_1 \text{ and } v \notin D_1. \tag{2.2.1}$$

In other words, D_1 and D_2 differ by exactly one vertex. This version of the γ-graph is also refered to as the *jump adjacency model*. For additional results on $\gamma \cdot G$, see [42, 62, 63].

In 2011, Fricke *et al.* [19] independently defined a different *γ-graph* of G, denoted by $G(\gamma)$. The vertex set of $G(\gamma)$ is the same as $\gamma \cdot G$, and two vertices of $G(\gamma)$ are adjacent if they satisfy the condition (2.2.1) and uv is an edge in G. This version of the γ-graph is referred to as the *slide adjacency model*. Observe that $G(\gamma)$ is a spanning subgraph of $\gamma \cdot G$. The γ-graph $G(\gamma)$ has been further studied in [5, 10, 16].

Mynhardt and Roux [49, 50] and Mynhardt and Teshima [52] defined and studied the graphs with the slide adjacency model for many domination parameters. There are also many graphs using other domination variants with the jump adjacency model as discussed below. The *γ-total dominating graph $TD_\gamma(G)$* of G is the graph whose vertices are $\gamma_t(G)$-sets and is defined by Wongsriya and Trakultraipruk [67] in 2017. These two authors presented the γ-total dominating graphs of paths and cycles. This dissertation also presents the γ-total dominating graphs of other classes of graphs appeared in Sections 4.1, 4.2, 4.4, and 4.5. In 2019, Samanmoo *et al.* [57] introducted the

γ*-independent dominating graph* of G, which is the graph whose vertices are $\gamma_i(G)$-sets (the minimum independent dominating sets). The authors provided the γ-independent dominating graphs of all paths and all cycles. Sanguanpong and Trakultraipruk [58] in 2022 presented the graph whose vertices are $\gamma_{ip}(G)$-sets (the minimum induced-paired dominating sets), and this graph is called the γ*-induced-paired dominating graph* of G. The γ-induced-paired dominating graphs of all paths and all cycles were investigated. The γ*-paired dominating graph* $PD_\gamma(G)$ of G, which is defined by Eakawinrujee and Trakultraipruk [15] in 2022, is the graph having $\gamma_{pr}(G)$-sets as its vertices. The authors determined the γ-paired dominating graphs of paths. In this dissertation, we present the γ-paired dominating graphs of cycles in Section 4.3, and the ones of other classes of graphs are determined in Sections 4.1, 4.2, 4.4, and 4.5.

Another class of graphs having a vertex set consisting of all dominating sets which are not necessarily minimum was introduced by Haas and Seyffarth [23]. The k*-dominating graph* $D_k(G)$ of G is the graph whose vertices are dominating sets with cardinality at most k. Two vertices of $D_k(G)$ are adjacent if they differ by either adding or deleting a single vertex of G. Further results on $D_k(G)$ can be found in [1, 14, 24, 51, 65]. The k*-total dominating graph* [2] and the k*-independent dominating graph* [17] are defined analogously by using total dominating sets and independent dominating sets, respectively, instead of dominating sets.

CHAPTER 3
TOTAL AND PAIRED DOMINATION NUMBERS

In this chapter, we first recall some definitions and useful results that are used in our main proofs. We next determine the total and the paired domination numbers of wheel graphs, flower graphs, helm graphs, sunflower graphs, Jahangir graphs, some cylinders, some closed helm graphs, some web graphs, lollipop graphs, umbrella graphs, and coconut graphs. Moreover, we present some upper and lower bounds for the total and the paired domination numbers of the other cylinders. We also provide some upper bounds for the total and the paired domination numbers of the other closed helm graphs and web graphs.

Let D be a total (paired) dominating set of G. We say that a vertex $u \in D$ *dominates* a vertex v if they are adjacent in G. In addition, a vertex $u \in D$ *dominates* a set $S \subseteq V(G)$ if u is adjacent to every vertex in S. Note that every leaf must be dominated by its support vertex, so we get the following observation.

Observation 3.0.1. *Every support vertex of a graph G is in every total dominating set of G and in every paired dominating set of G.*

Henning [28] and Haynes and Slater [27] determined the total domination numbers and the paired domination numbers, respectively, of paths and cycles, which are shown in the following lemmas.

Lemma 3.0.2 ([28])**.** *For any integer* $p \geq 3$, $\gamma_t(P_p) = \gamma_t(C_p) = \lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor$.

Lemma 3.0.3 ([27])**.** *For any integer* $p \geq 3$, $\gamma_{pr}(P_p) = \gamma_{pr}(C_p) = 2\lceil \frac{p}{4} \rceil$.

Gavlas and Schultz [20] and Proffitt *et al.* [56] defined efficient total domination and efficient paired domination, respectively. A set D is an *efficient total* (*paired*) *dominating set* of G if D is a total (paired) dominating set of G and $|N(v) \cap D| = 1$ for every $v \in V(G)$.

Lemma 3.0.4 ([40, 56])**.** *If D is an efficient total* (*paired*) *dominating set of G, then* $\gamma_t(G) = |D|$ ($\gamma_{pr}(G) = |D|$).

3.1 Wheel Graphs, Helm Graphs, Flower Graphs, and Sunflower Graphs

We first provide the definitions of these four graphs, and we then determine their total and paired domination numbers. For an integer $p \geq 3$,

1. the *wheel graph* W_p is the join $K_1 \vee C_p$, where $V(K_1) = \{c\}$ and $V(C_p) = \{u_1, u_2, \ldots, u_p\}$,
2. the *helm graph* H_p is obtained from the wheel graph W_p by adding the vertices $v_1, v_2, \ldots, v_p$ and the edge $u_i v_i$ for all $i \in \{1, 2, \ldots, p\}$,
3. the *flower graph* Fl_p is obtained from the helm graph H_p by adding the edge cv_i for all $i \in \{1, 2, \ldots, p\}$, and
4. the *sunflower graph* Sf_p is obtained from the helm graph H_p by adding the edges $v_p u_1$ and $v_i u_{i+1}$ for all $i \in \{1, 2, \ldots, p-1\}$.

The wheel graph W_p and the helm graph H_p are shown in Figure 3.1, while the flower graph Fl_6 and the sunflower graph Sf_6 are depicted in Figure 3.2.

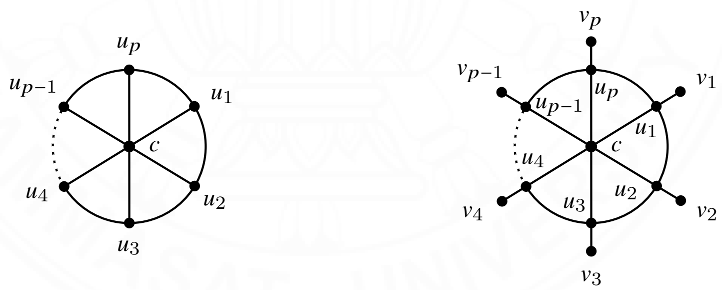

Figure 3.1 The wheel graph W_p (left) and the helm graph H_p (right)

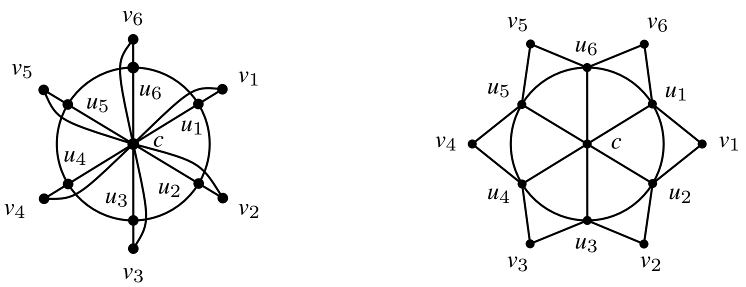

Figure 3.2 The flower graph Fl_6 (left) and the sunflower graph Sf_6 (right)

Note that $\{c, u_1\}$ is a minimum total (paired) dominating set of wheel graphs and flower graphs, so we obtain the following obvious result.

Theorem 3.1.1. *For any integer $p \geq 3$, $\gamma_t(W_p) = \gamma_{pr}(W_p) = \gamma_t(Fl_p) = \gamma_{pr}(Fl_p) = 2$.*

The total and the paired domination numbers of helm graphs are calculated in the following theorem.

Theorem 3.1.2. *For any integer $p \geq 3$,*

$$\gamma_t(H_p) = p \ \ \text{and} \ \ \gamma_{pr}(H_p) = \begin{cases} p & \text{if } p \text{ is even;} \\ p+1 & \text{if } p \text{ is odd.} \end{cases}$$

Proof. By Observation 3.0.1, the vertices $u_1, u_2, \ldots, u_p$ are in every $\gamma_t(H_p)$-set. Hence, $\gamma_t(H_p) \geq p$. Since $D = \{u_1, u_2, \ldots, u_p\}$ is a total dominating set of H_p, $\gamma_t(H_p) \leq |D| = p$. It follows that $\gamma_t(H_p) = p$. Moreover, D is a paired dominating set of H_p if p is even, and $D \cup \{c\}$ is a paired dominating set of H_p if p is odd. Thus, $\gamma_{pr}(H_p) \leq p$ if p is even, and $\gamma_{pr}(H_p) \leq p+1$ if p is odd. Since $\gamma_{pr}(H_p) \geq \gamma_t(H_p) = p$ and $\gamma_{pr}(H_p)$ is even, we get that $\gamma_{pr}(H_p) = p$ if p is even, and $\gamma_{pr}(H_p) = p+1$ if p is odd. □

We next determine the total and the paired domination numbers of sunflower graphs as shown below.

Theorem 3.1.3. *For any integer $p \geq 3$,*

$$\gamma_t(Sf_p) = \begin{cases} 2 & \text{if } p = 3; \\ \lceil \frac{p}{2} \rceil + 1 & \text{if } p \geq 4; \end{cases} \ \ \text{and} \ \ \gamma_{pr}(Sf_p) = 2\lceil \frac{p}{3} \rceil.$$

Proof. Clearly, $\gamma_t(Sf_3) = 2$. Let $p \geq 4$ and D be a $\gamma_t(Sf_p)$-set. To dominate $v_1, v_2, \ldots, v_p$, D contains at least $\lceil \frac{p}{2} \rceil$ vertices from $\{u_1, u_2, \ldots, u_p\}$. If $|D| = \lceil \frac{p}{2} \rceil$, then, without loss of generality, $D = \{u_i : i \equiv 1 \pmod 2\}$. We can see that there is some vertex in D that is not dominated. This contradicts the fact that D is a total dominating set of Sf_p, so $|D| \geq \lceil \frac{p}{2} \rceil + 1$. Since $\{c\} \cup \{u_i : i \equiv 1 \pmod 2\}$ is a total dominating set of Sf_p with cardinality $\lceil \frac{p}{2} \rceil + 1$, we get that $\gamma_t(Sf_p) = \lceil \frac{p}{2} \rceil + 1$.

Note that any two adjacent vertices can dominate at most three vertices in $\{v_1, v_2, \ldots, v_p\}$, so $\gamma_{pr}(Sf_p) \geq 2\lceil \frac{p}{3} \rceil$. If $p \equiv 0, 2 \pmod 3$, let $D = \{u_i, u_{i+1} : i \equiv 1 \pmod 3\}$; otherwise, let $D = \{u_i, u_{i+1} : i \equiv 1 \pmod 3, i \neq p\} \cup \{u_{p-1}, u_p\}$. Then D is a paired dominating set of Sf_p with cardinality $2\lceil \frac{p}{3} \rceil$, so $\gamma_{pr}(Sf_p) = 2\lceil \frac{p}{3} \rceil$. □

3.2 Jahangir Graphs

Let $p \geq 1$ and $q \geq 3$ be integers. Let C_{pq} be the cycle with the vertex set $V(C_{pq}) = \{u_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ and the edge set $E(C_{pq}) = \{u_{i,j}u_{i+1,j} : 1 \leq i \leq p-1, 1 \leq j \leq q\} \cup \{u_{p,j}u_{1,j+1} : 1 \leq j \leq q-1\} \cup \{u_{p,q}u_{1,1}\}$. The *Jahangir graph* $J_{p,q}$ is obtained from the cycle C_{pq} by adding the vertex c and the edge $cu_{1,j}$ for all $j \in \{1, 2, \ldots, q\}$. Note that $J_{1,q}$ is the wheel graph W_q and $J_{2,q}$ is known as the *gear graph*. Figure 3.3 illustrates the Jahangir graphs $J_{2,q}$ and $J_{3,q}$.

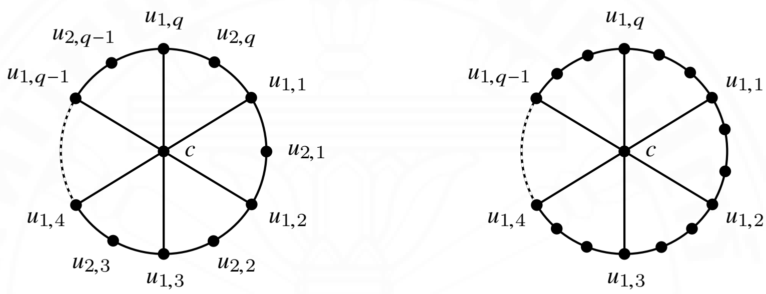

Figure 3.3 The Jahangir graphs $J_{2,q}$ (left) and $J_{3,q}$ (right)

For any integer $q \geq 3$, the total domination number of the Jahangir graph $J_{p,q}$ was determined by Mojdeh and Ghameshlou [46] when $p = 2$ and by Mtarneh *et al.* [48] when $p \geq 3$ is an integer.

Lemma 3.2.1. *For any integer* $q \geq 3$,

1. [46] $\gamma_t(J_{2,q}) = \lceil \frac{q}{2} \rceil + 1$ *and*

2. [48] $\gamma_t(J_{3,q}) = q + 1$ *and* $\gamma_t(J_{p,q}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ *for any integer* $p \geq 4$.

Consider the Jahangir graph $J_{7,3}$. Lemma 3.2.1(2) gives that $\gamma_t(J_{7,3}) = 11$, but $\{c, u_{1,1}, u_{4,1}, u_{5,1}, u_{1,2}, u_{4,2}, u_{5,2}, u_{1,3}, u_{4,3}, u_{5,3}\}$ is a total dominating set of $J_{7,3}$ with cardinality 10. It seems that, for $p \equiv 1, 2, 3 \pmod 4$ and $p \geq 4$, the value $\gamma_t(J_{p,q}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ does not hold for some values of q. We then correct this mistake in the next theorem.

Theorem 3.2.2. *Let $p \geq 4$ and $q \geq 3$ be integers. Then*

$$\gamma_t(J_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod 4; \\ \frac{(p-1)q}{2} + 2 & \text{if } p \equiv 1 \pmod 4; \\ \lceil \frac{(p-1)q}{2} \rceil + 1 & \text{if } p \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. If $p \equiv 0 \pmod 4$, then by Lemma 3.2.1(2), we have $\gamma_t(J_{p,q}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ can be simplified as $\gamma_t(J_{p,q}) = \frac{pq}{2}$.

Let $p = 4k + 1$ for some $k \geq 1$ and $D = \{c, u_{1,1}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, 1 \leq j \leq q\}$ (see Figure 3.4 for $p = 5$ and $q = 4$). We can check that D is a

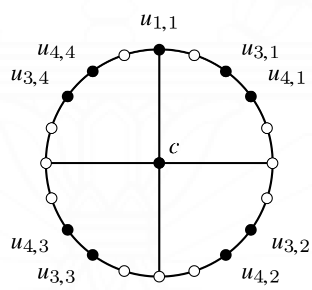

Figure 3.4 The total dominating set (bold vertices) of $J_{5,4}$

total dominating set of $J_{p,q}$ with $|D| = \frac{(p-1)q}{2} + 2$. By the constraction of D, we have that $|D| \leq |S|$ for every total dominating set S with $c \in S$ and $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap S| = 1$. We claim that among all total dominating sets containing the vertex c, D is minimum. Suppose that D' is a total dominating set containing c with $|D'| < |D|$. Then $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'| \geq 2$. Without loss of generality, we may assume that $u_{1,1}, u_{1,j} \in D'$ for some $j \in \{2, 3, \ldots, q\}$. Note that D' contains at least $2k$ vertices from $\{u_{2,j-1}, u_{3,j-1}, \ldots, u_{p,j-1}\}$ and at least $2k$ vertices from $\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}$. Thus, $D'' = (D' \setminus \{u_{2,j-1}, \ldots, u_{p,j-1}, u_{1,j}, u_{2,j}, \ldots, u_{p,j}\}) \cup \{u_{i,j-1}, u_{i+1,j-1}, u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4\}$ is a total dominating set of $J_{p,q}$ with $|D''| < |D'|$ and $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| < |\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'|$. We repeat this process until we get that $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| = 1$. We get a contradiction since $|D| \leq |D''| < |D'| < |D|$, so the claim holds. Next, we show that D is a $\gamma_t(J_{p,q})$-set. Assume that $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$. Then $c \notin \widehat{D}$ and $\widehat{D}$ is also a total dominating set of C_{pq}, so we get that

$|\widehat{D}| \geq \gamma_t(C_{pq}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$ by Lemma 3.0.2. Then

$$|\widehat{D}| \geq \begin{cases} 8kl + 2l \geq 8kl + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l, l \geq 1; \\ 8kl + 2k + 2l + 1 > 8kl + 2k + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l+1, l \geq 1; \\ 8kl + 4k + 2l + 2 > 8kl + 4k + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l+2, l \geq 1; \\ 8kl + 6k + 2l + 2 \geq 8kl + 6k + 2 = \frac{(p-1)q}{2} + 2 = |D| & \text{if } q = 4l+3, l \geq 0; \end{cases}$$

contradicting with the assumption $|\widehat{D}| < |D|$. Therefore, $\gamma_t(J_{p,q}) = \frac{(p-1)q}{2} + 2$.

Let $p = 4k + 2$ for some $k \geq 1$ and $D = \{c\} \cup \{u_{1,j} : j \equiv 1 \pmod 2\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4, j \equiv 1 \pmod 2\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, j \equiv 0 \pmod 2\}$ (see Figure 3.5 for $p = 6$ and $q = 4$). We can check that D is a total dominating

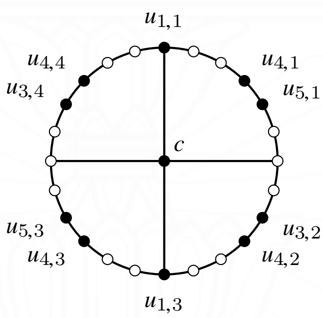

Figure 3.5 The total dominating set (bold vertices) of $J_{6,4}$

set of $J_{p,q}$ with $|D| = \lceil \frac{(p-1)q}{2} \rceil + 1$. We show that among all total dominating sets containing the vertex c, D is minimum. Suppose that D' is total dominating set with $c \in D'$ and $|D'| < |D|$. If $u_{1,j} \in D'$ for all $j \in \{1, 2, \ldots, q\}$, then $|\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\} \cap D'| \geq 2k$, so $|D'| \geq 1 + q + 2kq = 1 + (1 + 2k)q = 1 + \lceil \frac{(4k+2)q}{2} \rceil \geq 1 + \lceil \frac{(4k+1)q}{2} \rceil = 1 + \lceil \frac{(p-1)q}{2} \rceil = |D|$, a contradiction. Thus, $u_{1,j} \notin D'$ for some $j \in \{1, 2, \ldots, q\}$. If $u_{1,j+1} \notin D'$, then $|\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\} \cap D'| \geq 2k+1$. Hence, $D'' = (D' \setminus \{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}) \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4\} \cup \{u_{1,j+1}\}$ is a total dominating set of $J_{p,q}$ with $|D''| \leq |D'|$, so we can assume that if $u_{1,j} \notin D'$, then $u_{1,j+1} \in D'$. This implies that $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'| \geq \lceil \frac{q}{2} \rceil$ and $|\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\} \cap D'| \geq 2k$ for each $j \in \{1, 2, \ldots, q\}$. Therefore, $|D'| \geq 1 + \lceil \frac{q}{2} \rceil + 2kq = 1 + \lceil \frac{q}{2} + \frac{(p-2)q}{2} \rceil = 1 + \lceil \frac{(p-1)q}{2} \rceil = |D|$, a contradiction. The claim follows. Next, we prove that D is a $\gamma_t(J_{p,q})$-set. We assume that $\widehat{D}$ is a total dominating set of $J_{p,q}$ with $|\widehat{D}| < |D|$, so $c \notin \widehat{D}$. We note that $|\widehat{D}| \geq \gamma_t(C_{pq}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor$.

Then

$$|\widehat{D}| \geq \begin{cases} 4kl + 2l > 4kl + l + 1 = \lceil \frac{(p-1)q}{2} \rceil + 1 = |D| & \text{if } q = 2l, l \geq 2; \\ 4kl + 2k + 2l + 2 > 4kl + 2k + l + 2 = \lceil \frac{(p-1)q}{2} \rceil + 1 = |D| & \text{if } q = 2l+1, l \geq 1; \end{cases}$$

which is a contradiction. It follows that $\gamma_t(J_{p,q}) = \lceil \frac{(p-1)q}{2} \rceil + 1$.

Let $p = 4k+3$ for some $k \geq 1$ and $D = \{c\} \cup \{u_{1,j} : 1 \leq j \leq q\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4), 1 \leq j \leq q\}$ (see Figure 3.6 for $p = 7$ and $q = 4$). Then D is a total

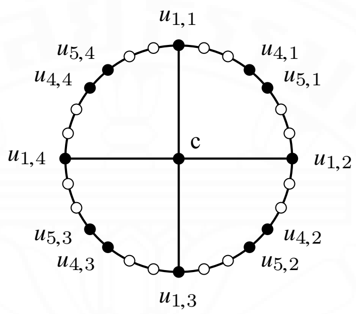

Figure 3.6 The total dominating set (bold vertices) of $J_{7,4}$

dominating set of $J_{p,q}$ with $|D| = \lceil \frac{(p-1)q}{2} \rceil + 1$. Note that among all total dominating sets containing c and $u_{1,1}, u_{1,2}, \ldots, u_{1,q}$, D is minimum. If D' is a total dominating set containing c and $|D'| < |D|$, then $u_{1,j} \notin D'$ for some $j \in \{1, 2, \ldots, q\}$. We can check that D' contains at least $2k + 1$ vertices from $\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}$ if $u_{1,j+1} \in D'$ and at least $2k+2$ vertices from $\{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}$ if $u_{1,j+1} \notin D'$. If $u_{1,j+1} \in D'$, let $D'' = (D' \setminus \{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}) \cup \{u_{1,j}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4)\}$; otherwise, let $D'' = (D' \setminus \{u_{2,j}, u_{3,j}, \ldots, u_{p,j}\}) \cup \{u_{1,j}, u_{1,j+1}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4)\}$. Thus, D'' is a total dominating set with $|D''| \leq |D'|$ and $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| > |\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D'|$. We repeat this process until we obtain that $|\{u_{1,1}, u_{1,2}, \ldots, u_{1,q}\} \cap D''| = q$. We see that $|D| \leq |D''| \leq |D'| < |D|$, a contradiction. This follows that among all total dominating sets containing c, D is minimum. We next prove that D is a $\gamma_t(J_{p,q})$-set. If $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$, then $c \notin \widehat{D}$. We get a contradiction because $|\widehat{D}| \geq \gamma_t(C_{pq}) = \lfloor \frac{pq+2}{4} \rfloor + \lfloor \frac{pq+3}{4} \rfloor \geq \lfloor \frac{pq+2}{4} + \frac{pq+3}{4} \rfloor - 1 \geq \lfloor \frac{pq}{2} \rfloor \geq \lfloor \frac{pq}{2} - \frac{q}{2} + 1 \rfloor = \lfloor \frac{(p-1)q}{2} \rfloor + 1 = \lceil \frac{(p-1)q}{2} \rceil + 1 = |D|$. This completes the proof. □

According to the proofs of Theorem 3.2.2, we observe that, for all $p \geq 4$, the results of Theorem 3.2.2 give the smaller values than the results of Lemma 3.2.1(2) when $p \equiv 1 \pmod 4$ and $q \geq 5$, $p \equiv 2 \pmod 4$ and $q \geq 3$, and $p \equiv 3 \pmod 4$ and $q \geq 3$.

Next, we calculate the paired domination number of $J_{p,q}$ for any integers $p \geq 2$ and $q \geq 3$.

Theorem 3.2.3. *Let $p \geq 2$ and $q \geq 3$ be integers. Then*

$$\gamma_{pr}(J_{p,q}) = \begin{cases} 2\lceil \frac{pq}{4} \rceil & \text{if } p \equiv 0, 2 \pmod 4; \\ \frac{(p-1)q}{2} + 2 & \text{if } p \equiv 1 \pmod 4; \\ 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. We first determine $\gamma_{pr}(J_{p,q})$ for $p \equiv 0, 1 \pmod 4$. By the fact that $\gamma_{pr}(J_{p,q}) \geq \gamma_t(J_{p,q})$ and Theorem 3.2.2, we immediately obtain the lower bounds of $\gamma_{pr}(J_{p,q})$ for $p \equiv 0, 1 \pmod 4$. For upper bounds, we observe that $\{u_{i,j}, u_{i+1,j} : i \equiv 1 \pmod 4, 1 \leq j \leq q\}$ is a paired dominating set of $J_{p,q}$ with cardinality $2\lceil \frac{pq}{4} \rceil$ if $p \equiv 0 \pmod 4$, and $\{c, u_{1,1}\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, 1 \leq j \leq q\}$ is a paired dominating set of $J_{p,q}$ with cardinality $\frac{(p-1)q}{2} + 2$ if $p \equiv 1 \pmod 4$.

Let $p = 4k+2$ for some $k \geq 0$. We can check that $\{u_{1,j}, u_{2,j}, u_{5,j}, u_{6,j} : j \equiv 1 \pmod 2\} \cup \{u_{3,j}, u_{4,j} : j \equiv 0 \pmod 2\}$ (see Figure 3.7 for $p = 6$ and $q = 4$) is a paired

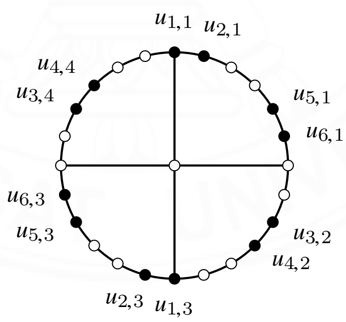

Figure 3.7 The paired dominating set (bold vertices) of $J_{6,4}$

dominating set of $J_{p,q}$ with cardinality $2\lceil \frac{pq}{4} \rceil$, so $\gamma_{pr}(J_{p,q}) \leq 2\lceil \frac{pq}{4} \rceil$. Next, we show that $\gamma_{pr}(J_{p,q}) \geq 2\lceil \frac{pq}{4} \rceil$. Let D be a $\gamma_{pr}(J_{p,q})$-set. If $c \notin D$, then D is also a paired dominating set of C_{pq}, and thus $|D| \geq \gamma_{pr}(C_{pq}) = 2\lceil \frac{pq}{4} \rceil$. Next, without loss of generality, we assume that the pair $\{c, u_{1,1}\} \subseteq D$. Then $|\{u_{3,1}, u_{4,1}, \ldots, u_{p,1}\} \cap D| \geq 2k$. To dominate $u_{2,2j}, \ldots, u_{p,2j}, u_{1,2j+1}, u_{2,2j+1}, \ldots, u_{p,2j+1}$ for each $1 \leq j \leq \lfloor \frac{q-2}{2} \rfloor$, D contains at least $4k + 2$ vertices from them. If $q = 2l$ for some $l \geq 2$, then $|\{u_{2,q}, u_{3,q}, \ldots, u_{p-1,q}\} \cap$

$D| \geq 2k$, and hence $|D| \geq 2 + (2k) + \lfloor \frac{q-2}{2} \rfloor (4k+2) + (2k) = 4kl + 2l = 2\lceil \frac{pq}{4} \rceil$. If $q = 2l+1$ for some $l \geq 1$, then $|\{u_{2,q-1}, \ldots, u_{p,q-1}, u_{1,q}, u_{2,q}, \ldots, u_{p-1,q}\} \cap D| \geq 4k+2$, so $|D| \geq 2 + (2k) + \lfloor \frac{q-2}{2} \rfloor (4k+2) + (4k+2) = 4kl + 2l + 2k + 2 = 2\lceil \frac{pq}{4} \rceil$.

Let $p = 4k+3$ for some $k \geq 0$. Define the set $E = \{c\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 3 \pmod 4, i < p, j \equiv 0 \pmod 3\} \cup \{u_{p,j} : j \equiv 0 \pmod 3\} \cup \{u_{1,j} : j \equiv 1 \pmod 3\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 0 \pmod 4, j \equiv 1 \pmod 3\} \cup \{u_{i,j}, u_{i+1,j} : i \equiv 1 \pmod 4, j \equiv 2 \pmod 3\}$ (see Figure 3.8 for $p = 7$ and $q = 4$). If $q \equiv 1, 2 \pmod 3$ (respectively, $q \equiv 0$

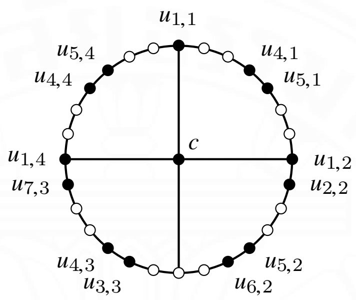

Figure 3.8 The paired dominating set (bold vertices) of $J_{7,4}$

$\pmod 3$)), then $D = E$ (respectively, $D = E \cup \{u_{p-1,q}\}$) is a paired dominating set of $J_{p,q}$ with $|D| = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. We claim that among all paired dominating sets containing c, D is minimum. Let D' be any paired dominating set of $J_{p,q}$ containing c. Without loss of generality, let $\{c, u_{1,1}\}$ be paired in D'. Then $|\{u_{3,1}, \ldots, u_{p,1}, u_{1,2}, u_{2,2}, \ldots, u_{p,2}\} \cap D'| \geq 4k+2$. To dominate $u_{2,3j}, \ldots, u_{p,3j}, u_{1,3j+1}, u_{2,3j+1}, \ldots, u_{p,3j+1}, u_{1,3j+2}, u_{2,3j+2}, \ldots, u_{p,3j+2}$ for each $1 \leq j \leq \lfloor \frac{q-3}{3} \rfloor$, D' contains at least $6k+4$ vertices from them. If $q = 3l$ for some $l \geq 1$, then $|\{u_{2,q}, u_{3,q}, \ldots, u_{p-1,q}\} \cap D'| \geq 2k+2$, and hence $|D'| \geq 2 + (4k+2) + \lfloor \frac{q-3}{3} \rfloor (6k+4) + (2k+2) = 6kl + 4l + 2 = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. If $q = 3l+1$ for some $l \geq 1$, then $|\{u_{2,q-1}, \ldots, u_{p,q-1}, u_{2,q}, \ldots, u_{p-1,q}\} \cap D'| \geq 4k+2$, so $|D'| \geq 2 + (4k+2) + \lfloor \frac{q-3}{3} \rfloor (6k+4) + (4k+2) = 6kl + 2k + 4l + 2 = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. If $q = 3l+2$ for some $l \geq 1$, then $|\{u_{2,q-2}, \ldots, u_{p,q-2}, u_{2,q-1}, \ldots, u_{p,q-1}, u_{2,q}, \ldots, u_{p-1,q}\} \cap D'| \geq 6k+4$, so $|D'| \geq 2 + (4k+2) + \lfloor \frac{q-3}{3} \rfloor (6k+4) + (6k+4) = 6kl + 4k + 4l + 4 = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. Now, the claim holds. We next prove that D is a $\gamma_{pr}(J_{p,q})$-set. If D'' is a paired dominating set with $|D''| < |D|$, then $c \notin D''$. Note that D'' is also a paired dominating set of C_{pq}, so $|D''| \geq \gamma_{pr}(C_{pq}) = 2\lceil \frac{pq}{4} \rceil \geq 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil = |D|$, a contradiction. We conclude that $\gamma_{pr}(J_{p,q}) = 2\lceil \frac{pq - \lceil \frac{q-3}{3} \rceil}{4} \rceil$. □

3.3 Cylinders

In this section, we compute the total and the paired domination numbers of a cylinder $P_p \Box C_q$ for $p \in \{2,3,4\}$ and $q \geq 5$. We also provide their upper and lower bounds for the other values of p and q.

Gravier [22] and Proffitt *et al.* [56] determined the total domination number and the paired domination number, respectively, of $P_p \Box P_q$ for $p \in \{2,3,4\}$ and $q \geq p$, as stated in the following lemmas.

Lemma 3.3.1 ([22, 56]). *For any integer $q \geq 2$, $\gamma_t(P_2 \Box P_q) = \gamma_{pr}(P_2 \Box P_q) = 2\lceil \frac{q}{3} \rceil$.*

Lemma 3.3.2 ([22, 56]). *For any integer $q \geq 3$, $\gamma_t(P_3 \Box P_q) = q$ and*

$$\gamma_{pr}(P_3 \Box P_q) = \begin{cases} q & \text{if } q \text{ is even;} \\ q+1 & \text{if } q \text{ is odd.} \end{cases}$$

Lemma 3.3.3 ([22, 56]). *For any integer $q \geq 4$,*

$$\gamma_t(P_4 \Box P_q) = \gamma_{pr}(P_4 \Box P_q) = \begin{cases} \lfloor \frac{6q+8}{5} \rfloor & \text{if } q \equiv 1,2,4 \pmod 5; \\ \lfloor \frac{6q+12}{5} \rfloor & \text{if } q \equiv 0,3 \pmod 5). \end{cases}$$

Klobučar [37] computed the total domination number of $P_5 \Box P_q$ for $q \geq 5$.

Lemma 3.3.4 ([37]). *For any integer $q \geq 5$,* $\gamma_t(P_5 \Box P_q) = \begin{cases} 10 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \neq 6. \end{cases}$

Klobučar also published a result on the total domination number of $P_6 \Box P_q$; however, Kuziak *et al.* [41] later showed that this result was false. Their improved result is shown in the following lemma.

Lemma 3.3.5 ([41]). *For any integer $q \geq 6$,*

$$\gamma_t(P_6 \Box P_q) = \begin{cases} \lfloor \frac{12q}{7} \rfloor + 2 & \text{if } q \equiv 0,4,6 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 3 & \text{if } q \equiv 1,2,3 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 4 & \text{if } q \equiv 5 \pmod 7). \end{cases}$$

Let $P_p = (1, 2, \ldots, p)$ and $P_q = (1, 2, \ldots, q)$ be two paths with p and q vertices, respectively. We use $v_{i,j}$ to denote the vertex in $P_p \Box P_q$ corresponding to vertex (i, j) for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$. We consider the vertices of $P_p \Box P_q$ as the entries in a matrix.

We next present the paired domination number of $P_p \Box P_q$ for $p \in \{5, 6\}$ and $q \geq p$ in the following two theorems.

Theorem 3.3.6. *For any integer $q \geq 5$,*

$$\gamma_{pr}(P_5 \Box P_q) = \begin{cases} 10 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 0, 3 \pmod 4; \\ \lfloor \frac{3q}{2} \rfloor + 3 & \text{if } q \equiv 1, 2 \pmod 4 \text{ and } q \neq 6. \end{cases}$$

Proof. Since $\gamma_{pr}(P_5 \Box P_q) \geq \gamma_t(P_5 \Box P_q)$ and $\gamma_{pr}(P_5 \Box P_q)$ is even, by Lemma 3.3.4, we get that

$$\gamma_{pr}(P_5 \Box P_q) \geq \begin{cases} 10 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 0, 3 \pmod 4; \\ \lfloor \frac{3q}{2} \rfloor + 3 & \text{if } q \equiv 1, 2 \pmod 4 \text{ and } q \neq 6. \end{cases}$$

We check that $\{v_{1,2}, v_{2,2}, v_{1,5}, v_{2,5}, v_{4,1}, v_{4,2}, v_{4,3}, v_{4,4}, v_{4,5}, v_{4,6}\}$ is a paired dominating set of $P_5 \Box P_6$, so $\gamma_{pr}(P_5 \Box P_6) = 10$. Let $q \neq 6$ and $D = \{v_{1,i} : i \equiv 3, 4, 7, 10, 11, 14, 15 \pmod{16}\} \cup \{v_{2,i} : i \equiv 1, 7 \pmod{16}\} \cup \{v_{3,i} : i \equiv 1, 4, 5, 9, 12, 13 \pmod{16}\} \cup \{v_{4,i} : i \equiv 9, 15 \pmod{16}\} \cup \{v_{5,i} : i \equiv 2, 3, 6, 7, 11, 12, 15 \pmod{16}\}$ (see Figure 3.9 for $q = 19$).

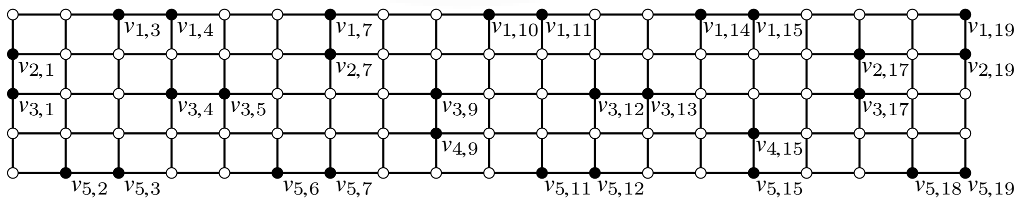

Figure 3.9 The paired dominating set (black vertices) of $P_5 \Box P_{19}$

If $q \equiv 7, 15 \pmod{16}$, then D is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$.

If $q \equiv 3, 4, 12 \pmod{16}$, then $D \cup \{v_{2,q}\}$ is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$.

If $q \equiv 0, 9, 13 \pmod{16}$, then $D \cup \{v_{1,q}, v_{2,q}\}$ is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$ if $q \equiv 0 \pmod{16}$, and $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 3$ if $q \equiv 9, 13 \pmod{16}$.

If $q \equiv 2, 6 \pmod{16}$, then $D \cup \{v_{1,q}, v_{2,q}, v_{4,q}\}$ is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 3$.

If $q \equiv 11 \pmod{16}$, then $D \cup \{v_{4,q}\}$ is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$.

If $q \equiv 1, 5, 8 \pmod{16}$, then $D \cup \{v_{4,q}, v_{5,q}\}$ is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 3$ if $q \equiv 1, 5 \pmod{16}$, and $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$ if $q \equiv 8 \pmod{16}$.

If $q \equiv 10, 14 \pmod{16}$, then $D \cup \{v_{2,q}, v_{4,q}, v_{5,q}\}$ is a paired dominating set of $P_5 \Box P_q$, so $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 3$.

Clearly, if $q \equiv 0, 4, 8, 12 \pmod{16}$, then $q \equiv 0 \pmod{4}$ and $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$. If $q \equiv 1, 5, 9, 13 \pmod{16}$, then $q \equiv 1 \pmod{4}$ and $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 3$. If $q \equiv 2, 6, 10, 14 \pmod{16}$, then $q \equiv 2 \pmod{4}$ and $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 3$. If $q \equiv 3, 7, 11, 15 \pmod{16}$, then $q \equiv 3 \pmod{4}$ and $\gamma_{pr}(P_5 \Box P_q) \leq \lfloor \frac{3q}{2} \rfloor + 2$. The theorem follows. □

Theorem 3.3.7. *For any integer $q \geq 6$,*

$$\gamma_{pr}(P_6 \Box P_q) = \begin{cases} \lfloor \frac{12q}{7} \rfloor + 2 & \text{if } q \equiv 0, 4, 6 \pmod{7}; \\ \lfloor \frac{12q}{7} \rfloor + 3 & \text{if } q \equiv 1, 2, 3 \pmod{7}; \\ \lfloor \frac{12q}{7} \rfloor + 4 & \text{if } q \equiv 5 \pmod{7}. \end{cases}$$

Proof. By the fact that $\gamma_{pr}(P_6 \Box P_q) \geq \gamma_t(P_6 \Box P_q)$ and Lemma 3.3.5, we obtain the lower bound for $\gamma_{pr}(P_6 \Box P_q)$ immediately.

Let $D = \{v_{1,i}, v_{6,i} : i \equiv 2, 3, 6 \pmod{7}\} \cup \{v_{2,i}, v_{5,i} : i \equiv 6 \pmod{7}\} \cup \{v_{3,i}, v_{4,i} : i \equiv 1, 4 \pmod{7}\}$ (see Figure 3.10 for $q = 11$).

If $q \equiv 4, 6 \pmod{7}$, then D is a paired dominating set of $P_6 \Box P_q$, and thus $\gamma_{pr}(P_6 \Box P_q) \leq \lfloor \frac{12q}{7} \rfloor + 2$.

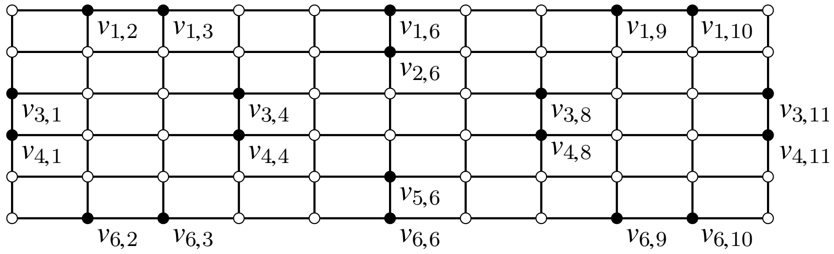

Figure 3.10 The paired dominating set (black vertices) of $P_6\square P_{11}$

If $q \equiv 0, 3 \pmod 7$, then $D\cup\{v_{3,q}, v_{4,q}\}$ is a paired dominating set of $P_6\square P_q$, and thus $\gamma_{pr}(P_6\square P_q) \le \lfloor\frac{12q}{7}\rfloor + 2$ if $q \equiv 0 \pmod 7$ and $\gamma_{pr}(P_6\square P_q) \le \lfloor\frac{12q}{7}\rfloor + 3$ if $q \equiv 3 \pmod 7$.

If $q \equiv 1, 2 \pmod 7$, then $D\cup\{v_{2,q}, v_{5,q}\}$ is a paired dominating set of $P_6\square P_q$, so $\gamma_{pr}(P_6\square P_q) \le \lfloor\frac{12q}{7}\rfloor + 3$.

If $q \equiv 5 \pmod 7$, then $D \cup \{v_{1,q}, v_{2,q}, v_{5,q}, v_{6,q}\}$ is a paired dominating set of $P_6\square P_q$, so $\gamma_{pr}(P_6\square P_q) \le \lfloor\frac{12q}{7}\rfloor + 4$.

This completes the proof. □

3.3.1 Total and Paired Domination Numbers of Some Cylinders

Hu *et al.* [30] provided the total and the paired domination numbers of $P_p\square C_q$ for $p \ge 2$ and $q \in \{3, 4\}$.

Theorem 3.3.8 ([30]). *For any integer $p \ge 2$,*

$$\gamma_t(P_p\square C_3) = \begin{cases} \lceil\frac{4p}{5}\rceil + 1 & \text{if } p \equiv 0, 1 \pmod 5; \\ \lceil\frac{4p}{5}\rceil & \text{if } p \equiv 2, 3, 4 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(P_p\square C_3) = \begin{cases} \lceil\frac{4p}{5}\rceil + 2 & \text{if } p \equiv 0 \pmod 5; \\ \lceil\frac{4p}{5}\rceil + 1 & \text{if } p \equiv 1, 3 \pmod 5; \\ \lceil\frac{4p}{5}\rceil & \text{if } p \equiv 2, 4 \pmod 5. \end{cases}$$

Theorem 3.3.9 ([30]). *For any integer $p \ge 2$,*

$$\gamma_t(P_p\square C_4) = \gamma_{pr}(P_p\square C_4) = \begin{cases} p + 1 & \text{if } p \text{ is odd}; \\ p + 2 & \text{if } p \text{ is even}. \end{cases}$$

Let $P_p = (1, 2, \ldots, p)$ be the path with p vertices and $C_q = (1, 2, \ldots, q)$ the cycle with q vertices. We denote the vertex of $P_p \square C_q$ corresponding to vertex (i, j) as $v_{i,j}$ for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$. We think of $v_{i,j}$ as being in row i and column j of $P_p \square C_q$. For each $j \in \{1, 2, \ldots, q\}$, let $Y_j = \{v_{i,j} : 1 \leq i \leq p\}$.

We now investigate the total and the paired domination numbers of $P_p \square C_q$ for $p \in \{2, 3, 4\}$ and $q \geq 5$.

Theorem 3.3.10. *For any integer $q \geq 5$, $\gamma_t(P_2 \square C_q) = \gamma_{pr}(P_2 \square C_q) = 2\lceil \frac{q}{3} \rceil$.*

Proof. Let D be a $\gamma_t(P_2 \square C_q)$-set. Let $f(l)$ be the cardinality of D in the first l column of $P_2 \square C_q$ for any $5 \leq l \leq q$. We claim that $f(l+3) \geq f(l) + 2$ for $5 \leq l \leq q$. Consider the graph $P_2 \square C_{l+3}$. To dominate all vertices in $Y_{l+1} \cup Y_{l+2} \cup Y_{l+3}$, we need at least two vertices. Then these two vertices do not dominate any vertices in $\bigcup_{i=1}^{l} Y_i$, so we need at least $f(l)$ vertices to dominate $\bigcup_{i=1}^{l} Y_i$. The claim follows. Next, we prove that $f(q) \geq 2\lceil \frac{q}{3} \rceil$. We prove by induction on q. It is easy to check that $f(5) \geq 4$. Let $q > 5$ and suppose that the result holds for all values less than q. Then $f(q) \geq f(q-3)+2 \geq 2\lceil \frac{q-3}{3} \rceil + 2 = 2\lceil \frac{q}{3} \rceil$. Hence, $\gamma_t(P_2 \square C_q) = |D| = f(q) \geq 2\lceil \frac{q}{3} \rceil$.

Let $D = \{v_{1,i}, v_{2,i} : i \equiv 2 \pmod 3\}$. If $q \equiv 0 \pmod 3$, then D is a paired dominating set of $P_2 \square C_q$ and $\gamma_{pr}(P_2 \square C_q) \leq 2\lceil \frac{q}{3} \rceil$. If $q \equiv 1, 2 \pmod 3$, then $D \cup \{v_{1,q}, v_{2,q}\}$ is a paired dominating set of $P_2 \square C_q$, so $\gamma_{pr}(P_2 \square C_q) \leq 2\lceil \frac{q}{3} \rceil$. Therefore, $2\lceil \frac{q}{3} \rceil \leq \gamma_t(P_2 \square C_q) \leq \gamma_{pr}(P_2 \square C_q) \leq 2\lceil \frac{q}{3} \rceil$, so we are done. □

Theorem 3.3.11. *For any integer $q \geq 5$,*

$$\gamma_t(P_3 \square C_q) = q$$

and

$$\gamma_{pr}(P_3 \square C_q) = \begin{cases} q & \text{if } q \text{ is even;} \\ q + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. It is easy to check that $D = \{v_{2,i} : 1 \leq i \leq q\}$ is a total dominating set of $P_3 \square C_q$ with $|D| = q$. We show that D is a $\gamma_t(P_3 \square C_q)$-set. Clearly, this claim holds for $q = 3$ and then we let $q \geq 4$. Assume on the contrary that there is a total dominating set D' such that $|D'| < |D|$. It follows that there exists a set Y_i such that $Y_i \cap D' = \emptyset$; such Y_i is called a *zero column*. Among all total dominating sets, let D' have the fewest zero columns.

Let i be the smallest index such that $Y_i \cap D' = \emptyset$, and we may assume that $i \notin \{1, q\}$ since $P_3 \square C_q$ is symmetric. Then $|Y_j \cap D'| \geq 1$ for each $1 \leq j < i$. If $|Y_j \cap D'| \geq 2$ for some $j < i$, then $D'' = (D' \setminus \{v_{1,k}, v_{3,k}\}) \cup \{v_{2,k}\}$ for $1 \leq k \leq i$ is a total dominating set having fewer zero columns than D', which contradicts our choice of D'. Thus, we may assume that $|Y_j \cap D'| = 1$ for each $1 \leq j < i$, which implies that $Y_j \cap D' = \{v_{2,j}\}$. We see that $v_{2,i-1}$ dominates $v_{2,i}$. Hence, to dominate $v_{1,i}$ and $v_{3,i}$, D' must contain the vertices $v_{1,i+1}$ and $v_{3,i+1}$. If $v_{2,i+1} \in D'$, then $(D' \setminus \{v_{1,i+1}, v_{3,i+1}\}) \cup \{v_{2,i}, v_{2,i+2}\}$ is a total dominating set having fewer zero columns than D', contradicting our choice of D'. If $v_{2,i+1} \notin D'$, then $v_{1,i+2}, v_{3,i+2} \in D'$, and thus $(D' \setminus \{v_{1,i+1}, v_{1,i+2}, v_{3,i+1}, v_{3,i+2}\}) \cup \{v_{2,i}, v_{2,i+1}, v_{2,i+2}, v_{2,i+3}\}$ is a total dominating set having fewer zero columns than D', again contradicting our choice of D'. Now, we can conclude that D is a $\gamma_t(P_3 \square C_q)$-set. Thus, $\gamma_t(P_3 \square C_q) = |D| = q$. We also get that $\gamma_{pr}(P_3 \square C_q) \geq q$ if q is even, and $\gamma_{pr}(P_3 \square C_q) \geq q + 1$ if q is odd. To complete the proof, we observe that $\{v_{2,i} : 1 \leq i \leq q\}$ (respectively, $\{v_{2,i} : 1 \leq i \leq q\} \cup \{v_{1,q}\}$) is a paired dominating set of $P_3 \square C_q$ if q is even (respectively, q is odd). Thus, $\gamma_{pr}(P_3 \square C_q) \leq q$ if q is even, and $\gamma_{pr}(P_3 \square C_q) \leq q + 1$ if q is odd. □

Theorem 3.3.12. *For any integer $q \geq 5$,*

$$\gamma_t(P_4 \square C_q) = \gamma_{pr}(P_4 \square C_q) = \begin{cases} \lfloor \frac{6q+8}{5} \rfloor - 1 & \text{if } q \equiv 0 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor & \text{if } q \equiv 1, 2, 4 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 3 \pmod 5. \end{cases}$$

Proof. Let $D = \{v_{2,i}, v_{3,i}, v_{1,i+2}, v_{1,i+3}, v_{4,i+2}, v_{4,i+3} : i \equiv 1 \pmod 5)\}$ (see Figure 3.11 for $q = 10$). If $q \equiv 0 \pmod 5$, then D is efficient total and efficient paired dominating sets of $P_4 \square C_q$, and by Lemma 3.0.4, $\gamma_t(P_4 \square C_q) = \gamma_{pr}(P_4 \square C_q) = \frac{6q}{5} = \lfloor \frac{6q+8}{5} \rfloor - 1$. If $q \equiv 1, 4 \pmod 5$, then D is a paired dominating set of $P_4 \square C_q$ and $\gamma_{pr}(P_4 \square C_q) \leq \lfloor \frac{6q+8}{5} \rfloor$. If $q \equiv 2, 3 \pmod 5$, then $D \cup \{v_{2,q}, v_{3,q}\}$ is a paired dominating set of $P_4 \square C_q$, so $\gamma_{pr}(P_4 \square C_q) \leq \lfloor \frac{6q+8}{5} \rfloor$ if $q \equiv 2 \pmod 5$, and $\gamma_{pr}(P_4 \square C_q) \leq \lfloor \frac{6q+8}{5} \rfloor + 1$ if $q \equiv 3 \pmod 5$.

To complete the proof, we only show the lower bound of $\gamma_t(P_4 \square C_q)$ for $q \equiv 1, 2, 3, 4 \pmod 5$. We first let $q = 5k + 3$, where $k \geq 1$. Note that D is an efficient total dominating set of $P_4 \square C_{5k}$ with $|D| = 6k$. These $6k$ vertices dominate only the vertices in $\bigcup_{i=1}^{5k-1} Y_i \cup \{v_{1,5k}, v_{4,5k}, v_{2,5k+3}, v_{3,5k+3}\}$ of $P_4 \square C_{5k+3}$. To dominate the remaining vertices in $Y_{5k+1} \cup Y_{5k+2} \cup \{v_{2,5k}, v_{3,5k}, v_{1,5k+3}, v_{4,5k+3}\}$, we need six vertices.

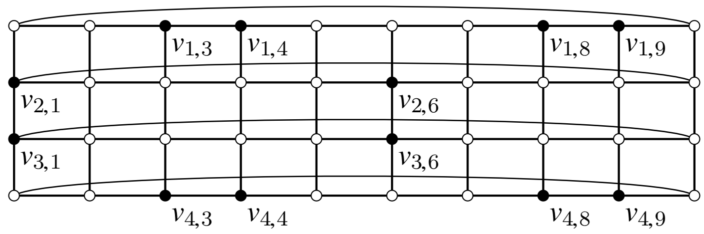

Figure 3.11 The total and paired dominating sets (black vertices) of $P_4\Box C_{10}$

Thus, $\gamma_t(P_4\Box C_q) \geq 6k+6 = \lfloor\frac{6q+8}{5}\rfloor + 1$ for $q = 5k+3$. We next consider the case $q \equiv 1, 2, 4 \pmod 5$. To dominate all vertices in $Y_{q-2} \cup Y_{q-1}$, we need at least four vertices. These four vertices do not dominate any vertices in $\bigcup_{i=1}^{q-4} Y_i$. Note that the induced subgraph $P_4\Box C_q[\bigcup_{i=1}^{q-4} Y_i]$ is $P_4\Box P_{q-4}$. Therefore, $\gamma_t(P_4\Box C_q) \geq 4 + \gamma_t(P_4\Box P_{q-4})$. By Lemma 3.3.3, we get that

$$\gamma_t(P_4\Box C_q) \geq \begin{cases} 4 + \lfloor\frac{6(5k-3)+8}{5}\rfloor = 6k+2 = \lfloor\frac{6(5k+1)+8}{5}\rfloor = \lfloor\frac{6q+8}{5}\rfloor & \text{if } q = 5k+1, k \geq 1; \\ 4 + \lfloor\frac{6(5k-2)+12}{5}\rfloor = 6k+4 = \lfloor\frac{6(5k+2)+8}{5}\rfloor = \lfloor\frac{6q+8}{5}\rfloor & \text{if } q = 5k+2, k \geq 1; \\ 4 + \lfloor\frac{6(5k)+12}{5}\rfloor = 6k+6 = \lfloor\frac{6(5k+4)+8}{5}\rfloor = \lfloor\frac{6q+8}{5}\rfloor & \text{if } q = 5k+4, k \geq 1. \end{cases}$$

This completes the proof. □

3.3.2 Upper Bounds of $\gamma_t(P_p\Box C_q)$ and $\gamma_{pr}(P_p\Box C_q)$

We recently determine the exact values of $\gamma_t(P_p\Box C_q)$ and $\gamma_{pr}(P_p\Box C_q)$ for $p \in \{2, 3, 4\}$ and $q \geq 5$. Now, we present the upper bounds of $\gamma_t(P_p\Box C_q)$ and $\gamma_{pr}(P_p\Box C_q)$ for the other values of p and q. In the next three lemmas, we first give their upper bounds for small values of p and q, that is, $p \geq 5$ and $q = 5$ [30]; $p = 5$ and $q \geq 6$; $p = 6$ and $q \geq 6$. We then provide upper bounds for $p \geq 7$ and $q \geq 6$ in Theorem 3.3.19 (below).

Lemma 3.3.13 ([30]). *For any integer $p \geq 5$,*

$$\gamma_t(P_p\Box C_5) \leq \begin{cases} \lceil\frac{9p}{7}\rceil & \textit{if } p \equiv 4 \pmod 7; \\ \lceil\frac{9p}{7}\rceil + 1 & \textit{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(P_p \square C_5) \leq \begin{cases} \lceil \frac{4p}{3} \rceil & \textit{if } p \equiv 1 \pmod 3; \\ \lceil \frac{4p}{3} \rceil + 1 & \textit{if } p \equiv 2 \pmod 3; \\ \lceil \frac{4p}{3} \rceil + 2 & \textit{if } p \equiv 0 \pmod 3. \end{cases}$$

Note that $\gamma_t(P_p \square C_q) \leq \gamma_t(P_p \square P_q)$ and $\gamma_{pr}(P_p \square C_q) \leq \gamma_{pr}(P_p \square P_q)$. By Lemma 3.3.4 and Theorem 3.3.6, we immediately get the upper bounds of $\gamma_t(P_5 \square C_q)$ and $\gamma_{pr}(P_5 \square C_q)$ for $q \equiv 1, 2, 3 \pmod 4$ in Lemma 3.3.14; however, if $q \equiv 0 \pmod 4$, then we have $\gamma_t(P_5 \square C_q) = \gamma_{pr}(P_5 \square C_q) = \frac{3q}{2}$ by Lemma 3.3.16. By Lemma 3.3.5 and Theorem 3.3.7, we have the results of Lemma 3.3.15 for $q \equiv 1, 2, 4, 5, 6 \pmod 7$). If $q \equiv 0, 3 \pmod 7$, let $D = \{v_{1,i}, v_{6,i} : i \equiv 2, 3, 6 \pmod 7\} \cup \{v_{2,i}, v_{5,i} : i \equiv 6 \pmod 7\} \cup \{v_{3,i}, v_{4,i} : i \equiv 1, 4 \pmod 7\}$ as defined in the proof of Theorem 3.3.7. Then D is a paired dominating set of $P_6 \square C_q$ with cardinality $\lfloor \frac{12q}{7} \rfloor$ (respectively, $\lfloor \frac{12q}{7} \rfloor + 1$) if $q \equiv 0 \pmod 7$ (respectively, $q \equiv 3 \pmod 7$).

Lemma 3.3.14. *For any integer $q \geq 6$,*

$$\gamma_t(P_5 \square C_q) \leq \begin{cases} 10 & \textit{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \textit{if } q \equiv 1, 2, 3 \pmod 4 \textit{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(P_5 \square C_q) \leq \begin{cases} 10 & \textit{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 3 & \textit{if } q \equiv 1, 2 \pmod 4 \textit{ and } q \neq 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \textit{if } q \equiv 3 \pmod 4. \end{cases}$$

Lemma 3.3.15. *For any integer $q \geq 6$,*

$$\gamma_t(P_6 \square C_q) \leq \gamma_{pr}(P_6 \square C_q) \leq \begin{cases} \lfloor \frac{12q}{7} \rfloor & \textit{if } q \equiv 0 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 3 & \textit{if } q \equiv 1, 2 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 1 & \textit{if } q \equiv 3 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 2 & \textit{if } q \equiv 4, 6 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 4 & \textit{if } q \equiv 5 \pmod 7. \end{cases}$$

Before we establish the upper bounds of $\gamma_t(P_p \square C_q)$ and $\gamma_{pr}(P_p \square C_q)$ for $p \geq 7$ and $q \geq 6$, we need the following lemmas.

Lemma 3.3.16. *If* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 4$*, then* $\gamma_t(P_p \square C_q) = \gamma_{pr}(P_p \square C_q) = \frac{(p+1)q}{4}$.

Proof. Let $D = \{v_{i,j}, v_{i,j+1}, v_{i+2,j+2}, v_{i+2,j+3} : i, j \equiv 1 \pmod 4\}$. It is easy to check that D is both an efficient total and an efficient paired dominating set of $P_p \square C_q$ with $|D| = \frac{(p+1)q}{4}$, so the theorem follows. □

Lemma 3.3.17 ([30]). *For any integers* $p \geq 1$ *and* $q \geq 4$*,* $\gamma_t(P_p \square C_q) \leq \gamma_t(P_{p+1} \square C_q)$ *and* $\gamma_{pr}(P_p \square C_q) \leq \gamma_{pr}(P_{p+1} \square C_q)$.

Lemma 3.3.18. *For any integers* $p \geq 1$ *and* $q \geq 4$*,* $\gamma_t(P_p \square C_q) \leq \gamma_t(P_p \square C_{q+1})$ *and* $\gamma_{pr}(P_p \square C_q) \leq \gamma_{pr}(P_p \square C_{q+1})$.

Proof. Let D be a $\gamma_t(P_p \square C_{q+1})$-set or a $\gamma_{pr}(P_p \square C_{q+1})$-set. If $Y_{q+1} \cap D = \emptyset$, then D is both a total and a paired dominating set of $P_p \square C_q$, so $\gamma_t(P_p \square C_q) \leq |D|$ and $\gamma_{pr}(P_p \square C_q) \leq |D|$. Next, we assume that $Y_{q+1} \cap D \neq \emptyset$. Let $A = \{i : v_{i,q+1} \in D\}$, $B = \{i : v_{i,q} \in D\}$, and $D' = D \setminus \{v_{i,q+1} : i \in A \cap B\} \cup \{v_{i,q-1} : i \in A \cap B\} \setminus \{v_{i,q+1} : i \in A \setminus B\} \cup \{v_{i,q} : i \in A \setminus B\}$. We can check that D' is a total dominating set of $P_p \square C_q$. Hence, $\gamma_t(P_p \square C_q) \leq |D'| \leq |D|$. If the induced subgraph $P_p \square C_q[D']$ does not contain odd components, then D' is a paired dominating set of $P_p \square C_q$, so $\gamma_t(P_p \square C_q) \leq |D|$. Thus, we assume that $P_p \square C_q[D']$ contains $k \geq 1$ odd component. By the construction of D' from D, $|D'| \leq |D| - k$. Let D'' be the set obtained from D' by adding k vertices such that $P_p \square C_q[D'']$ does not contain odd components. Therefore, D'' is a paired dominating set of $P_p \square C_q$, so $\gamma_{pr}(P_p \square C_q) \leq |D''| = |D'| + k \leq (|D| - k) + k = |D|$. □

We next provide the upper bounds for the total and the paired domination numbers of $P_p \square C_q$ for $p \geq 7$ and $q \geq 6$.

Theorem 3.3.19. *For any integers* $p \geq 7$ *and* $q \geq 6$*,*

$$\gamma_t(P_p \square C_q) \leq \gamma_{pr}(P_p \square C_q) \leq 2\lceil \frac{p+1}{2} \rceil \lceil \frac{q}{4} \rceil.$$

Proof. We only show the upper bound of $\gamma_{pr}(P_p \square C_q)$. Let $p = 2k_1 + 1 - i$ and $q = 4k_2 - j$, where k_1 and k_2 are positive integers, $i \in \{0, 1\}$, and $j \in \{0, 1, 2, 3\}$. By Lemma 3.3.16, $\gamma_{pr}(P_{2k_1+1} \square C_{4k_2}) = \frac{(2k_1+2)(4k_2)}{4} = 2\lceil \frac{p+1}{2} \rceil \lceil \frac{q}{4} \rceil$. Lemmas 3.3.17 and 3.3.18 show that $\gamma_{pr}(P_p \square C_q) \leq \gamma_{pr}(P_{2k_1+1} \square C_{4k_2}) = 2\lceil \frac{p+1}{2} \rceil \lceil \frac{q}{4} \rceil$. □

Next, we give some upper bounds of $\gamma_t(P_p \Box C_q)$ and $\gamma_{pr}(P_p \Box C_q)$ which are better than those in Theorem 3.3.19 for some special values of p and q. For Theorems 3.3.20, 3.3.21, 3.3.22, and 3.3.23, we let

$$D = \{v_{i,j}, v_{i,j+1}, v_{i+2,j+2}, v_{i+2,j+3} : i, j \equiv 1 \pmod 4\}.$$

Theorem 3.3.20. *If* $7 \le p \equiv 1 \pmod 2$ *and* $9 \le q \equiv 1 \pmod 4$*, then* $\gamma_t(P_p \Box C_q) \le \frac{(p+1)(q+1)}{4}$.

Proof. For $p \equiv 1, 3 \pmod 4$, let $D_t = D \cup \{v_{i,q} : i \equiv 3 \pmod 4\}$ (see Figure 3.12). It is easy to check that D_t is a total dominating set of $P_p \Box C_q$ with $|D_t| = \frac{(p+1)(q+1)}{4}$. Thus, $\gamma_t(P_p \Box C_q) \le \frac{(p+1)(q+1)}{4}$. □

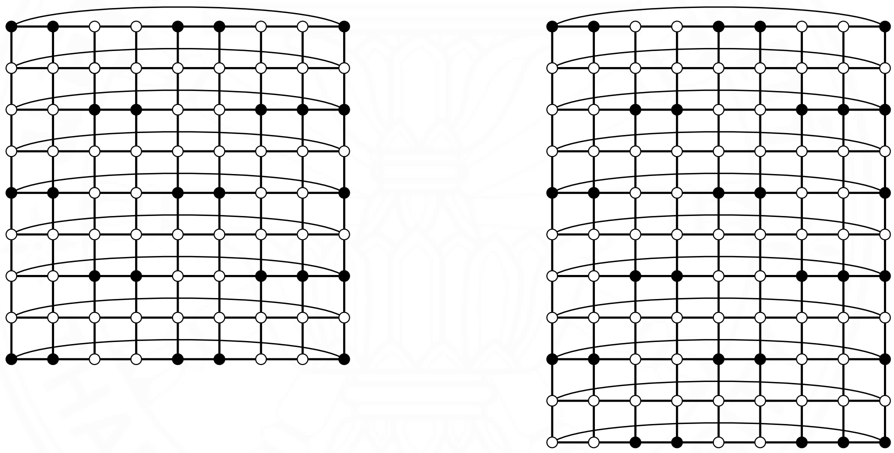

Figure 3.12 The total dominating sets (black vertices) of $P_9 \Box C_9$ (left) and $P_{11} \Box C_9$ (right)

Let $p = 2k_1 + 1$ and $q = 4k_2 + 1$, where $k_1 \ge 3$ and $k_2 \ge 2$. We get that $\gamma_t(P_p \Box C_q) \le 2(k_1 + 1)(k_2 + \frac{1}{2})$ by Theorem 3.3.20, which is better than that of Theorem 3.3.19, providing $\gamma_t(P_p \Box C_q) \le 2(k_1 + 1)(k_2 + 1)$, for these values of p and q.

Theorem 3.3.21. *If* $8 \le p \equiv 0 \pmod 2$ *and* $9 \le q \equiv 1 \pmod 4$*, then* $\gamma_t(P_p \Box C_q) \le \frac{(p+2)(q+1)}{4} - 2$ *and* $\gamma_{pr}(P_p \Box C_q) \le \frac{(p+2)(q+3)}{4} - 4$.

Proof. For $p \equiv 0 \pmod 4$, let $D_t = D \cup \{v_{i,q} : i \equiv 3 \pmod 4, i < p-1\} \cup \{v_{p,j}, v_{p,j+1} : j \equiv 1 \pmod 4, j < q\}$ (see Figure 3.13 (left) for $p = 8$ and $q = 9$) and $D_p = D_t \cup \{v_{i,q} : i \equiv 0, 2 \pmod 4, i < p\}$. Then D_t is a total dominating set of $P_p \Box C_q$ with $|D_t| = \frac{(p+2)(q+1)}{4} - 2$ and D_p is a paired dominating set of $P_p \Box C_q$ with $|D_p| = \frac{(p+2)(q+3)}{4} - 4$.

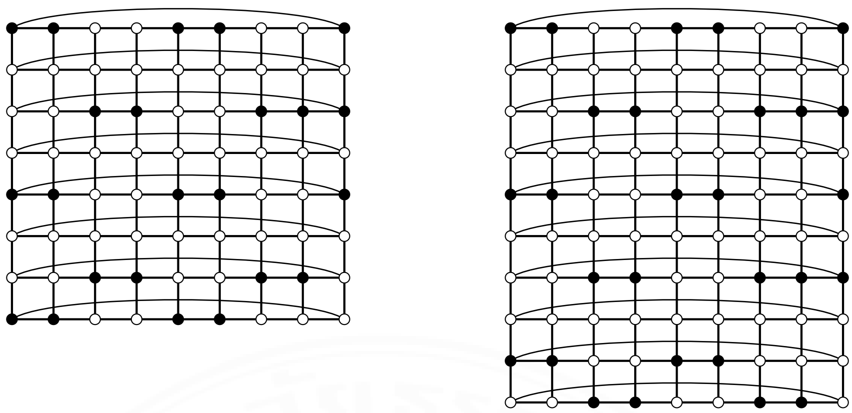

Figure 3.13 The total dominating sets (black vertices) of $P_8\square C_9$ (left) and $P_{10}\square C_9$ (right)

For $p \equiv 2 \pmod 4$, let $D_t = (D \cup \{v_{i,q} : i \equiv 3 \pmod 4\} \cup \{v_{p,j}, v_{p,j+1} : j \equiv 3 \pmod 4), j < q\}) \setminus \{v_{p-1,q}\}$ (see Figure 3.13 (right) for $p = 10$ and $q = 9$) and $D_p = D_t \cup \{v_{i,q} : i \equiv 0, 2 \pmod 4), i < p\}$. Then D_t is a total dominating set of $P_p\square C_q$ with $|D_t| = \frac{(p+2)(q+1)}{4} - 2$ and D_p is a paired dominating set of $P_p\square C_q$ with $|D_p| = \frac{(p+2)(q+3)}{4} - 4$. □

Let $p = 2k_1$ and $q = 4k_2 + 1$, where $k_1 \geq 4$ and $k_2 \geq 2$. Theorem 3.3.21 gives that $\gamma_t(P_p\square C_q) \leq 2(k_1 + 1)(k_2 + \frac{1}{2}) - 2$ and $\gamma_{pr}(P_p\square C_q) \leq 2(k_1 + 1)(k_2 + 1) - 4$, which are better than those of Theorem 3.3.19, showing that $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq 2(k_1 + 1)(k_2 + 1)$, for these specific values of p and q.

Theorem 3.3.22. *If* $8 \leq p \equiv 0 \pmod 2$ *and* $6 \leq q \equiv 2 \pmod 4$*, then* $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq \frac{(p+2)(q+2)}{4} - 4$.

Proof. If $p \equiv 0 \pmod 4$, let $D_p = (D \cup \{v_{i,1}, v_{i,q} : i \equiv 3 \pmod 4), i < p - 1\} \cup \{v_{p,j}, v_{p,j+1} : j \equiv 1 \pmod 4)\}) \setminus \{v_{p,2}, v_{p,q-1}\}$ (see Figure 3.14 (left) for $p = 8$ and $q = 10$). If $p \equiv 2 \pmod 4$, let $D_p = (D \cup \{v_{i,1}, v_{i,q} : i \equiv 3 \pmod 4)\} \cup \{v_{p,j}, v_{p,j+1} : j \equiv 1 \pmod 4)\}) \setminus \{v_{p-1,1}, v_{p-1,q}, v_{p,1}, v_{p,q}\}$ (see Figure 3.14 (right) for $p = 10$ and $q = 10$). Then D_p is a paired dominating set of $P_p\square C_q$ with $|D_p| = \frac{(p+2)(q+2)}{4} - 4$. □

Let $p = 2k_1$ and $q = 4k_2 + 2$, where $k_1 \geq 4$ and $k_2 \geq 1$. Theorem 3.3.22 shows that $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq 2(k_1+1)(k_2+1)-4$, which are better than those

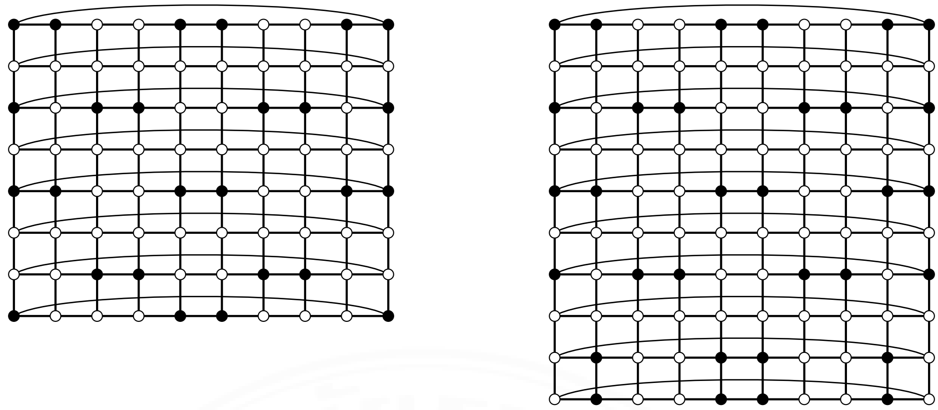

Figure 3.14 The paired dominating sets (black vertices) of $P_8\square C_{10}$ (left) and $P_{10}\square C_{10}$ (right)

of Theorem 3.3.19, providing that $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq 2(k_1+1)(k_2+1)$, for these specific values of p and q.

Theorem 3.3.23. *If* $8 \leq p \equiv 0 \pmod 2$ *and* $7 \leq q \equiv 3 \pmod 4$*, then* $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq \frac{(p+2)(q+1)}{4} - 2$.

Proof. If $p \equiv 0 \pmod 4$, let $D_p = D \cup \{v_{i,1} : i \equiv 3 \pmod 4, i < p-1\} \cup \{v_{p,j}, v_{p,j+1} : j \equiv 3 \pmod 4\}$ (see Figure 3.15 (left) for $p = 8$ and $q = 11$). If $p \equiv 2 \pmod 4$, let $D_p = (D \cup \{v_{i,1} : i \equiv 3 \pmod 4\} \cup \{v_{p,j}, v_{p,j+1} : j \equiv 1 \pmod 4\}) \setminus \{v_{p-1,1}, v_{p,1}\}$ (see Figure 3.15 (right) for $p = 10$ and $q = 11$). Then D_p is a paired dominating set of $P_p\square C_q$ with $|D_p| = \frac{(p+2)(q+1)}{4} - 2$. □

Let $p = 2k_1$ and $q = 4k_2 + 3$, where $k_1 \geq 4$ and $k_2 \geq 1$. Theorem 3.3.23 gives that $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq 2(k_1+1)(k_2+1) - 2$, which are better than those of Theorem 3.3.19, showing that $\gamma_t(P_p\square C_q) \leq \gamma_{pr}(P_p\square C_q) \leq 2(k_1+1)(k_2+1)$, for these specific values of p and q.

3.3.3 Lower Bounds of $\gamma_t(P_p\square C_q)$ and $\gamma_{pr}(P_p\square C_q)$

Previously, we obtain the exact values of $\gamma_t(P_p\square C_q)$ and $\gamma_{pr}(P_p\square C_q)$ for $p \in \{2, 3, 4\}$ and $q \geq 5$ and their upper bounds for $p, q \geq 5$. We next provide the lower bounds of $\gamma_t(P_p\square C_q)$ and $\gamma_{pr}(P_p\square C_q)$ for $p, q \geq 5$. We first need the following two lemmas.

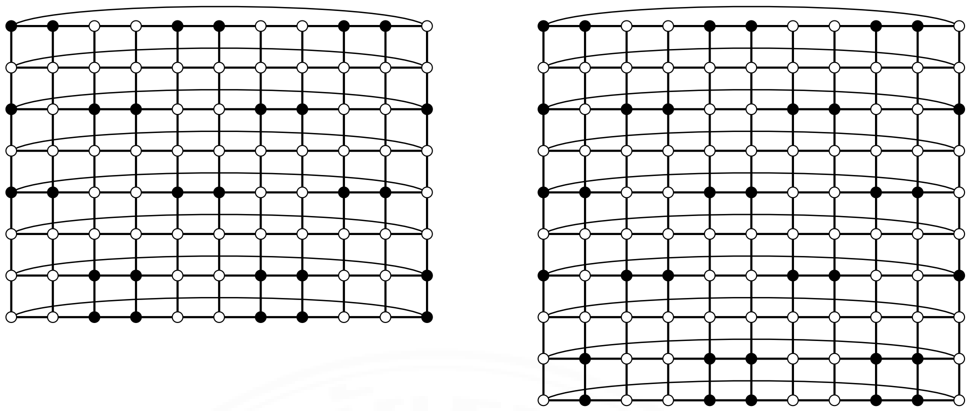

Figure 3.15 The paired dominating sets (black vertices) of $P_8\square C_{11}$ (left) and $P_{10}\square C_{11}$ (right)

Lemma 3.3.24. *For any integers $p \geq 1$ and $q \geq 4$, $\gamma_t(P_p\square C_q) \geq \gamma_t(P_p\square P_{q-2})$.*

Proof. Let D be a $\gamma_t(P_p\square C_q)$-set, $A = \{i : v_{i,q} \in D\}$, and $B = \{i : v_{i,q-1}\}$. Define the set $D' = D \setminus \{v_{i,q} : i \in A\} \cup \{v_{i,2} : i \in A\} \setminus \{v_{i,q-1} : i \in B\} \cup \{v_{i,q-3} : i \in B\}$. We can verify that D' is a total dominating set of $P_p\square P_{q-2}$, so $\gamma_t(P_p\square P_{q-2}) \leq |D'| \leq |D| = \gamma_t(P_p\square C_q)$. □

Lemma 3.3.25 ([22])**.** *For any integers $p \geq 17$ and $q \geq 19$,*

$$\gamma_t(P_p\square P_{q-2}) \geq \frac{3p(q-2)+2(p+q-2)}{12} - 1.$$

By the fact that $\gamma_{pr}(P_p\square C_q) \geq \gamma_t(P_p\square C_q)$ and Lemmas 3.3.24 and 3.3.25, we get the second result in Theorem 3.3.26. For the first result of Theorem 3.3.26, we can get that by applying Lemma 3.3.17 and Theorem 3.3.12.

Theorem 3.3.26. *Let p and q be integers.*

1. *If $5 \leq p \leq 16$ and $5 \leq q \leq 18$, then*

$$\gamma_{pr}(P_p\square C_q) \geq \gamma_t(P_p\square C_q) \geq \begin{cases} \lfloor \frac{6q+8}{5} \rfloor - 1 & \text{if } q \equiv 0 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor & \text{if } q \equiv 1,2,4 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 3 \pmod 5. \end{cases}$$

2. *If $p \geq 17$ and $q \geq 19$, then $\gamma_{pr}(P_p\square C_q) \geq \gamma_t(P_p\square C_q) \geq \frac{3p(q-2)+2(p+q-2)}{12} - 1$.*

3.4 Closed Helm Graphs and Web Graphs

Let $P_p = (1, 2, \ldots, p)$ be the path with p vertices and $C_q = (1, 2, \ldots, q)$ be the cycle with q vertices. We use $u_{i,j}$ to denote the vertex of $P_p \square C_q$ corresponding to the vertex (i, j) for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$. For any integers $p \geq 1$ and $q \geq 3$,

1. the *closed helm graph* $CH_{p,q}$ is obtained from $P_p \square C_q$ by adding the vertex c and the edge $cu_{1,j}$ for all $j \in \{1, 2, \ldots, q\}$, and
2. the *web graph* $W_{p,q}$ is obtained from $CH_{p,q}$ by adding the vertices $v_1, v_2, \ldots, v_q$ and the edge $u_{p,j}v_j$ for all $j \in \{1, 2, \ldots, q\}$.

We observe that $CH_{1,q} \cong W_q$ and $W_{1,q} \cong H_q$ for all $q \geq 3$. The closed helm graph $CH_{p,q}$ and the web graph $W_{p,q}$ are illustrated in Figure 3.16.

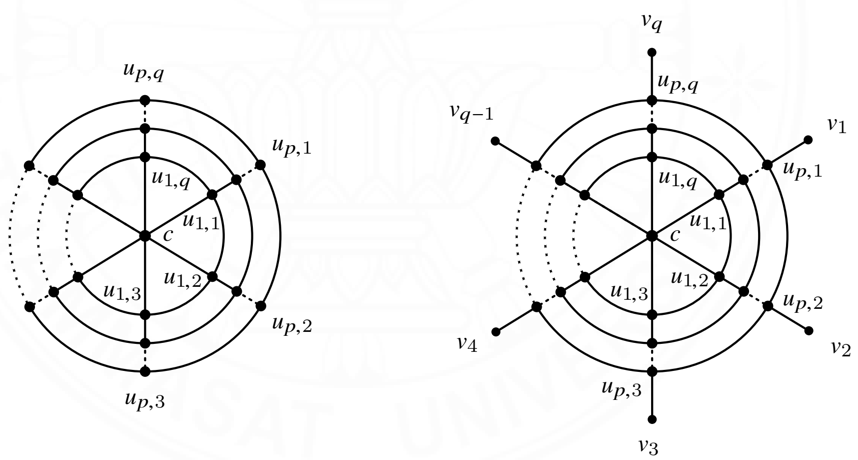

Figure 3.16 The closed helm graph $CH_{p,q}$ (left) and the web graph $W_{p,q}$ (right)

Throughout this section, for each $i \in \{1, 2, \ldots, p\}$, let $C^i = \{u_{i,j} : 1 \leq j \leq q\}$ be the set of vertices in i^{th} cycle of $CH_{p,q}$ and $W_{p,q}$.

In this section, we determine the total and the paired domination numbers of $CH_{p,q}$ and $W_{p,q}$ for some values of p and q. We then provide their upper bounds for the other cases.

3.4.1 Total and Paired Domination Numbers of Some Closed Helm Graphs

We compute the total and the paired domination numbers of $CH_{p,q}$, where the values of p and q are divided as follows: $p \geq 2$ and $q = 3$; $p \geq 2$ and $q = 4$; $p = 2$ and $q \geq 5$; $p = 3$ and $q \geq 5$; $p = 4$ and $q \geq 5$.

Theorem 3.4.1. *For any integer $p \geq 2$,*

$$\gamma_t(CH_{p,3}) = \begin{cases} \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 0, 1 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 3, 4 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(CH_{p,3}) = \begin{cases} \lceil \frac{4p}{5} \rceil + 2 & \text{if } p \equiv 0 \pmod 5; \\ \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 1, 3 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 4 \pmod 5. \end{cases}$$

Proof. Let D be a $\gamma_t(CH_{p,3})$-set. If $c \in D$, then, without loss of generality, $u_{1,1} \in D$ to dominate c. It follows that $u_{2,1} \notin D$; otherwise, $D \setminus \{c\}$ is a total dominating set of $CH_{p,3}$ with cardinality less than D. Hence, $D' = (D \setminus \{c\}) \cup \{u_{2,1}\}$ is a total dominating set of $CH_{p,3}$ with $|D'| = |D|$, so we can assume that $c \notin D$. Then D is also a total dominating set of $P_p \Box C_3$, and thus $|D| \geq \gamma_t(P_p \Box C_3)$. By Theorem 3.3.8, we get

$$\gamma_t(CH_{p,3}) = |D| \geq \gamma_t(P_p \Box C_3) = \begin{cases} \lceil \frac{4p}{5} \rceil + 1 & \text{if } p \equiv 0, 1 \pmod 5; \\ \lceil \frac{4p}{5} \rceil & \text{if } p \equiv 2, 3, 4 \pmod 5. \end{cases}$$

Since $\gamma_{pr}(CH_{p,3}) \geq \gamma_t(CH_{p,3})$ and $\gamma_{pr}(CH_{p,3})$ is even, we get that $\gamma_{pr}(CH_{p,3}) \geq \lceil \frac{4p}{5} \rceil$ if $p \equiv 2, 4 \pmod 5$, $\gamma_{pr}(CH_{p,3}) \geq \lceil \frac{4p}{5} \rceil + 1$ if $p \equiv 1, 3 \pmod 5$, and $\gamma_{pr}(CH_{p,3}) \geq \lceil \frac{4p}{5} \rceil + 2$ if $p \equiv 0 \pmod 5$.

To obtain the upper bounds of $\gamma_t(CH_{p,3})$ and $\gamma_{pr}(CH_{p,3})$, we let

$$D = \{u_{i,2} : i \equiv 1, 2 \pmod 5\} \cup \{u_{i,1}, u_{i,3} : i \equiv 4 \pmod 5\}.$$

If $p \equiv 2, 4 \pmod 5$, then D is a total dominating set of $CH_{p,3}$, so $\gamma_t(CH_{p,3}) \leq |D| = \lceil \frac{4p}{5} \rceil$. If $p \equiv 3 \pmod 5$, then $D \cup \{u_{p,2}\}$ is a total dominating set of $CH_{p,3}$, so $\gamma_t(CH_{p,3}) \leq |D| + 1 = \lceil \frac{4p}{5} \rceil$. If $p \equiv 0, 1 \pmod 5$, then $D \cup \{u_{p-1,2}\}$ is a total dominating set of $CH_{p,3}$, so $\gamma_t(CH_{p,3}) \leq |D| + 1 = \lceil \frac{4p}{5} \rceil + 1$. Now, the theorem holds for $\gamma_t(CH_{p,3})$.

If $p \equiv 2, 4 \pmod 5$, then D is a paired dominating set of $CH_{p,3}$, and thus $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil$. If $p \equiv 1 \pmod 5$, then $D \cup \{u_{p,3}\}$ is a paired dominating set of $CH_{p,3}$, so $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil + 1$. If $p \equiv 0, 3 \pmod 5$, then $D \cup \{u_{p,1}, u_{p,3}\}$ is a paired dominating set of $CH_{p,3}$, so $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil + 2$ if $p \equiv 0 \pmod 5$, and $\gamma_{pr}(CH_{p,3}) \leq \lceil \frac{4p}{5} \rceil + 1$ if $p \equiv 3 \pmod 5$. The theorem follows. □

Theorem 3.4.2. *For any integer $p \geq 2$,*

$$\gamma_t(CH_{p,4}) = \gamma_{pr}(CH_{p,4}) = \begin{cases} p+1 & \text{if } p \text{ is odd;} \\ p+2 & \text{if } p \text{ is even.} \end{cases}$$

Proof. Let D be a $\gamma_t(CH_{p,4})$-set. If $c \in D$, then, without loss of generality, $u_{1,1} \in D$, and thus $u_{1,2} \notin D$. Therefore, $D' = (D \setminus \{c\}) \cup \{u_{1,2}\}$ is a total dominating set of $CH_{p,4}$ with $|D'| = |D|$, so we can assume that $c \notin D$. Then D is also a total dominating set of $P_p \square C_4$. By Theorem 3.3.9, we have

$$\gamma_t(CH_{p,4}) = |D| \geq \gamma_t(P_p \square C_4) = \begin{cases} p+1 & \text{if } p \text{ is odd;} \\ p+2 & \text{if } p \text{ is even.} \end{cases}$$

Note that $\gamma_{pr}(CH_{p,4}) \geq \gamma_t(CH_{p,4})$. To complete this theorem, we only determine the upper bound of $\gamma_{pr}(CH_{p,4})$. Let $D = \{u_{i,1}, u_{i,2} : i \equiv 1 \pmod 4\} \cup \{u_{i,3}, u_{i,4} : i \equiv 3 \pmod 4\}$. Then D is a paired dominating set of $CH_{p,4}$ with cardinality $p+1$ if p is odd, and $D \cup \{u_{p,1}, u_{p,2}\}$ is a paired dominating set of $CH_{p,4}$ with cardinality $p+2$ if p is even. □

Theorem 3.4.3. *For any integer $q \geq 5$,*

$$\gamma_t(CH_{2,q}) = \begin{cases} 4 & \text{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil & \text{if } q \equiv 0, 1 \pmod 4; \\ 2\lceil \frac{q+3}{4} \rceil - 1 & \text{if } q \equiv 2, 3 \pmod 4 \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(CH_{2,q}) = \begin{cases} 4 & \text{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil & \text{otherwise.} \end{cases}$$

Proof. Obviously, $\{u_{1,1}, u_{2,1}, u_{1,4}, u_{2,4}\}$ is a $\gamma_t(CH_{2,6})$-set and a $\gamma_{pr}(CH_{2,6})$-set, so let $q \neq 6$. If $q \equiv 0, 1 \pmod 4$, let $D = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 3 \pmod 4\}$; otherwise, let $D = \{c, u_{1,1}, u_{2,1}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 0 \pmod 4\}$. Then D is a total dominating set of $CH_{2,q}$ with $|D| = 2\lceil \frac{q+3}{4} \rceil$ if $q \equiv 0, 1 \pmod 4$, and $|D| = 2\lceil \frac{q+3}{4} \rceil - 1$ if $q \equiv 2, 3 \pmod 4$. We first show that among all total dominating sets containing c, D is minimum. Suppose on the contrary that there is a total dominating set D' containing c such that $|D'| < |D|$. By the construction of D, we have $|D| \le |S|$ for every total dominating set S with $c \in S$ and $|S \cap C^1| = 1$. Without loss of generality, we assume that D' contains at least two vertices $u_{1,1}$ and $u_{1,l}$ of C^1 for some $l \neq 1$. Hence, $D'' = (D' \setminus \{u_{1,l}\}) \cup \{u_{2,l+1}\}$ is a total dominating set with $|D''| \le |D'|$ and $|D'' \cap C^1| < |D' \cap C^1|$. We can repeat this process until we obtain $|D'' \cap C^1| = 1$. Therefore, $|D| \le |D''| \le |D'| < |D|$, a contradiction.

We next claim that D is a $\gamma_t(CH_{2,q})$-set. Suppose that $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$. Then $c \notin \widehat{D}$, and thus $\widehat{D}$ is a total dominating set of $P_2 \square C_q$. By Theorem 3.3.10, $|\widehat{D}| \ge \gamma_t(P_2 \square C_q) = 2\lceil \frac{q}{3} \rceil$. Then

$$|\widehat{D}| \ge \begin{cases} 2\lceil \frac{4k}{3} \rceil = 2(k + \lceil \frac{k}{3} \rceil) \ge 2(k+1) = 2\lceil \frac{q+3}{4} \rceil = |D| & \text{if } q = 4k, k \ge 2; \\ 2\lceil \frac{4k+1}{3} \rceil = 2(k + \lceil \frac{k+1}{3} \rceil) \ge 2(k+1) = 2\lceil \frac{q+3}{4} \rceil = |D| & \text{if } q = 4k+1, k \ge 1; \\ 2\lceil \frac{4k+2}{3} \rceil = 2(k + \lceil \frac{k+2}{3} \rceil) > 2(k+2) - 1 = 2\lceil \frac{q+3}{4} \rceil - 1 = |D| & \text{if } q = 4k+2, k \ge 2; \\ 2\lceil \frac{4k+3}{3} \rceil = 2(k + \lceil \frac{k+3}{3} \rceil) > 2(k+2) - 1 = 2\lceil \frac{q+3}{4} \rceil - 1 = |D| & \text{if } q = 4k+3, k \ge 1; \end{cases}$$

contradicting the assumption $|\widehat{D}| < |D|$, so our claim holds.

Since $\gamma_{pr}(CH_{2,q}) \ge \gamma_t(CH_{2,q})$ and $\gamma_{pr}(CH_{2,q})$ is even, we get $\gamma_{pr}(CH_{2,q}) \ge 2\lceil \frac{q+3}{4} \rceil$. If $q \equiv 0, 1 \pmod 4$, then let $D = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 3 \pmod 4\}$; otherwise, let $D = \{c, u_{1,1}, u_{2,1}, u_{2,2}\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 0 \pmod 4\}$. Then D is a paired dominating set of $CH_{2,q}$ with cardinality $2\lceil \frac{q+3}{4} \rceil$, and thus $\gamma_{pr}(CH_{2,q}) = 2\lceil \frac{q+3}{4} \rceil$. □

Theorem 3.4.4. *For any integer $q \ge 5$,*

$$\gamma_t(CH_{3,q}) = \begin{cases} 2\lceil \frac{q+2}{3} \rceil & \textit{if } q \equiv 0, 2, 3 \pmod 6; \\ 2\lceil \frac{q+3}{3} \rceil - 1 & \textit{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(CH_{3,q}) = 2\lceil \frac{q+3}{3} \rceil.$$

Proof. Let $E_1 = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{3,j} : j \equiv 0 \pmod 3\}$, $E_2 = \{c, u_{1,1}\} \cup \{u_{2,j}, u_{3,j} : j \equiv 2 \pmod 3\}$, and $E_3 = \{c, u_{1,1}, u_{2,1}\} \cup \{u_{3,j}, u_{3,j+1} : j \equiv 3 \pmod 6\} \cup \{u_{2,j}, u_{2,j+1} : j \equiv 0$

$\pmod 6\}$. If $q \equiv 0, 3 \pmod 6$ (respectively, $q \equiv 2 \pmod 6$ and $q \equiv 1, 4, 5 \pmod 6$), then $D = E_1$ (respectively, $D = E_2$ and $D = E_3$) is a total dominating set of $CH_{3,q}$ with cardinality $2\lceil \frac{q+2}{3} \rceil$ (respectively, $2\lceil \frac{q+2}{3} \rceil$ and $2\lceil \frac{q+3}{3} \rceil - 1$). We show that among all total dominating sets containing the vertex c, D is minimum. Assume that D' is a total dominating set with $c \in D'$ and $|D'| < |D|$. Let $S = \{v : v \notin N(c)\}$. Then the induced subgraph $CH_{3,q}[S]$ contains $P_2 \square C_q$ and the vertex c. Thus, D' contains at least $2\lceil \frac{q}{3} \rceil$ vertices to dominate $P_2 \square C_q$ (by Theorem 3.3.10) and one vertex of C^1 to dominate c. Hence, $|D'| \geq 2\lceil \frac{q}{3} \rceil + 2 = 2\lceil \frac{q+3}{3} \rceil \geq |D|$, a contradiction.

We next show that D is a $\gamma_t(CH_{3,q})$-set. Suppose that $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$. Then $c \notin \widehat{D}$, and hence $\widehat{D}$ is also a total dominating set of $P_3 \square C_q$. By Theorem 3.3.11, $|\widehat{D}| \geq \gamma_t(P_3 \square C_q) = q$. Then

$$|\widehat{D}| \geq \begin{cases} 6k \geq 4k+2 = 2(2k+1) = 2\lceil \frac{q+2}{3} \rceil = |D| & \text{if } q = 6k, k \geq 1; \\ 6k+1 \geq 4k+3 = 2(2k+2)-1 = 2\lceil \frac{q+3}{3} \rceil - 1 = |D| & \text{if } q = 6k+1, k \geq 1; \\ 6k+2 \geq 4k+4 = 2(2k+2) = 2\lceil \frac{q+2}{3} \rceil = |D| & \text{if } q = 6k+2, k \geq 1; \\ 6k+3 > 4k+4 = 2(2k+2) = 2\lceil \frac{q+2}{3} \rceil = |D| & \text{if } q = 6k+3, k \geq 1; \\ 6k+4 > 4k+5 = 2(2k+3)-1 = 2\lceil \frac{q+3}{3} \rceil - 1 = |D| & \text{if } q = 6k+4, k \geq 1; \\ 6k+5 \geq 4k+5 = 2(2k+3)-1 = 2\lceil \frac{q+3}{3} \rceil - 1 = |D| & \text{if } q = 6k+5, k \geq 0, \end{cases}$$

contradicting with our assumption $|\widehat{D}| < |D|$.

Note that $\gamma_{pr}(CH_{3,q}) \geq \gamma_t(CH_{3,q})$. If $q \equiv 0, 2, 3 \pmod 6$, then $\gamma_{pr}(CH_{3,q}) \geq 2\lceil \frac{q+2}{3} \rceil = 2\lceil \frac{q+3}{3} \rceil$. Since $\gamma_{pr}(CH_{3,q})$ is even, we get $\gamma_{pr}(CH_{3,q}) \geq 2\lceil \frac{q+3}{3} \rceil$ for $q \equiv 1, 4, 5 \pmod 6$. Consider the sets E_1, E_2, and E_3 as defined above. We observe that the set E_1 (respectively, E_2) is a paired dominating set with cardinality $2\lceil \frac{q+2}{3} \rceil = 2\lceil \frac{q+3}{3} \rceil$ if $q \equiv 0, 3 \pmod 6$ (respectively, $q \equiv 2 \pmod 6$). If $q \equiv 1, 4, 5 \pmod 6$, then $E_3 \cup \{u_{3,1}\}$ is a paired dominating set with cardinality $2\lceil \frac{q+3}{3} \rceil$. This completes the proof. □

Theorem 3.4.5. *For any integer $q \geq 5$,*

$$\gamma_t(CH_{4,q}) = \begin{cases} 6 & \text{if } q = 5; \\ q+2 & \text{if } q \geq 6; \end{cases}$$

and

$$\gamma_{pr}(CH_{4,q}) = \begin{cases} 6 & \text{if } q = 5; \\ q+2 & \text{if } q \text{ is even}; \\ q+3 & \text{if } q \text{ is odd and } q \neq 5. \end{cases}$$

Proof. If $q = 5$, then it is clear that $\{u_{2,1}, u_{3,1}, u_{1,3}, u_{1,4}, u_{4,3}, u_{4,4}\}$ is both a $\gamma_t(CH_{4,q})$-set and a $\gamma_{pr}(CH_{4,q})$-set, so we let $q \geq 6$. Let $D = \{c, u_{1,q}\} \cup \{u_{3,j} : 1 \leq j \leq q\}$, which is a total dominating set of $CH_{4,q}$ with $|D| = q + 2$. Let D' be a total dominating set containing c. Then the induced subgraph $CH_{4,q}[\{v : v \notin N(c)\}]$ consists of $P_3 \square C_q$ and the vertex c. By Theorem 3.3.11, D' contains at least q vertices to dominate $P_3 \square C_q$. It also contains one vertex of C^1 to dominate c. Hence, $|D'| \geq q + 2$. We conclude that among all total dominating sets of $CH_{4,q}$ containing c, D is minimum.

Next, we show that D is a $\gamma_t(CH_{4,q})$-set. If $\widehat{D}$ is a total dominating set with $|\widehat{D}| < |D|$, then $c \notin \widehat{D}$, so $\widehat{D}$ is also a total dominating set of $P_4 \square C_q$. Theorem 3.3.12 gives that

$$|\widehat{D}| \geq \gamma_t(P_4 \square C_q) = \begin{cases} \lfloor \frac{6q+8}{5} \rfloor - 1 & \text{if } q \equiv 0 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor & \text{if } q \equiv 1, 2, 4 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 3 \pmod 5. \end{cases}$$

It is easy to check that $|\widehat{D}| \geq |D|$ for all $q \geq 6$, a contradiction. Therefore, $\gamma_t(CH_{4,q}) = q + 2$.

Since $\gamma_{pr}(CH_{4,q}) \geq \gamma_t(CH_{4,q})$ and $\gamma_{pr}(CH_{4,q})$ is even, $\gamma_{pr}(CH_{4,q}) \geq q + 2$ if q is even, and $\gamma_{pr}(CH_{4,q}) \geq q + 3$ if q is odd. Note that D (respectively, $D \cup \{u_{2,1}\}$) defined above is a paired dominating set if q is even (respectively, odd). The theorem follows. □

3.4.2 Upper Bounds of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$

Recently, we determine the exact values of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$ for some values of p and q. Next, we present their upper bounds for $p, q \geq 5$. Before we can achieve these bounds, we need the following result.

Let H be the graph obtained from $CH_{p,q}$ by deleting the vertices c and $u_{1,j}$ for all $j \in \{1, 2, \ldots, q\}$. Note that the union of $\{c, u_{1,1}\}$ and a $\gamma_t(H)$-set (respectively, a $\gamma_{pr}(H)$-set) is a total dominating set (respectively, a paired dominating set) of $CH_{p,q}$. Then we have the following lemma where we replace H by $P_{p-1}\Box C_q$.

Lemma 3.4.6. *If $p \geq 2$ and $q \geq 3$ are integers, then $\gamma_t(CH_{p,q}) \leq \gamma_t(P_{p-1}\Box C_q) + 2$ and $\gamma_{pr}(CH_{p,q}) \leq \gamma_{pr}(P_{p-1}\Box C_q) + 2$.*

By using Lemma 3.4.6 and Lemma 3.3.13 (respectively, Theorem 3.3.12), we can easily get Theorem 3.4.7 (respectively, Theorem 3.4.8).

Theorem 3.4.7. *For any integer $p \geq 5$,*

$$\gamma_t(CH_{p,5}) \leq \begin{cases} \lceil \frac{9(p-1)}{7} \rceil + 2 & \text{if } p \equiv 5 \pmod 7; \\ \lceil \frac{9(p-1)}{7} \rceil + 3 & \text{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(CH_{p,5}) \leq \begin{cases} \lceil \frac{4(p-1)}{3} \rceil + 3 & \text{if } p \equiv 0 \pmod 3; \\ \lceil \frac{4(p-1)}{3} \rceil + 4 & \text{if } p \equiv 1 \pmod 3; \\ \lceil \frac{4(p-1)}{3} \rceil + 2 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Theorem 3.4.8. *For any integer $q \geq 6$,*

$$\gamma_t(CH_{5,q}) \leq \gamma_{pr}(CH_{5,q}) \leq \begin{cases} \lfloor \frac{6q+8}{5} \rfloor + 1 & \text{if } q \equiv 0 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor + 2 & \text{if } q \equiv 1, 2, 4 \pmod 5; \\ \lfloor \frac{6q+8}{5} \rfloor + 3 & \text{if } q \equiv 3 \pmod 5. \end{cases}$$

In the next theorem, if $q \equiv 0 \pmod 4$, then $\gamma_t(CH_{6,q}) \leq \gamma_{pr}(CH_{6,q}) \leq \gamma_{pr}(P_5\Box C_q) + 2 = \frac{6q}{4} + 2 = \lfloor \frac{3q}{2} \rfloor + 2$ by Lemmas 3.4.6 and 3.3.16. For the other cases, we can get the results from Lemmas 3.4.6 and 3.3.14.

Theorem 3.4.9. *For any integer $q \geq 6$,*

$$\gamma_t(CH_{6,q}) \leq \begin{cases} 12 & \text{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \text{if } q \equiv 0 \pmod 4; \\ \lfloor \frac{3q}{2} \rfloor + 4 & \text{if } q \equiv 1, 2, 3 \pmod 4 \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(CH_{6,q}) \leq \begin{cases} 12 & \textit{if } q = 6; \\ \lfloor \frac{3q}{2} \rfloor + 2 & \textit{if } q \equiv 0 \pmod 4; \\ \lfloor \frac{3q}{2} \rfloor + 5 & \textit{if } q \equiv 1, 2 \pmod 4) \textit{ and } q \neq 6; \\ \lfloor \frac{3q}{2} \rfloor + 4 & \textit{if } q \equiv 3 \pmod 4. \end{cases}$$

Theorems 3.4.10, 3.4.11, and 3.4.12 can be got from Lemma 3.4.6 with Lemma 3.3.15, Theorem 3.3.19, and Theorem 3.3.20, respectively. Theorem 3.4.13 can also be obtained from Lemma 3.4.6 and Theorems 3.3.21 - 3.3.23.

Theorem 3.4.10. *For any integer $q \geq 6$,*

$$\gamma_t(CH_{7,q}) \leq \gamma_{pr}(CH_{7,q}) \leq \begin{cases} \lfloor \frac{12q}{7} \rfloor + 2 & \textit{if } q \equiv 0 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 5 & \textit{if } q \equiv 1, 2 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 3 & \textit{if } q \equiv 3 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 4 & \textit{if } q \equiv 4, 6 \pmod 7; \\ \lfloor \frac{12q}{7} \rfloor + 6 & \textit{if } q \equiv 5 \pmod 7. \end{cases}$$

Theorem 3.4.11 gives the upper bounds of $\gamma_t(CH_{p,q})$ and $\gamma_{pr}(CH_{p,q})$ for $p \geq 8$ and $q \geq 6$, while Theorems 3.4.12 and 3.4.13 give some better bounds than those of Theorem 3.4.11 for some special values of p and q.

Theorem 3.4.11. *For any integers $p \geq 8$ and $q \geq 6$,*

$$\gamma_t(CH_{p,q}) \leq \gamma_{pr}(CH_{p,q}) \leq 2\lceil \frac{p}{2} \rceil \lceil \frac{q}{4} \rceil + 2.$$

Theorem 3.4.12. *If $8 \leq p \equiv 0 \pmod 2$ and $9 \leq q \equiv 1 \pmod 4$, then $\gamma_t(CH_{p,q}) \leq \frac{p(q+1)}{4} + 2$.*

Theorem 3.4.13. *If $9 \leq p \equiv 1 \pmod 2$ and $q \geq 6$, then*

$$\gamma_t(CH_{p,q}) \leq \begin{cases} \frac{(p+1)(q+1)}{4} & \textit{if } q \equiv 1, 3 \pmod 4; \\ \frac{(p+1)(q+2)}{4} - 2 & \textit{if } q \equiv 2 \pmod 4; \end{cases}$$

and

$$\gamma_{pr}(CH_{p,q}) \leq \begin{cases} \frac{(p+1)(q+3)}{4} - 2 & \textit{if } q \equiv 1 \pmod 4; \\ \frac{(p+1)(q+2)}{4} - 2 & \textit{if } q \equiv 2 \pmod 4; \\ \frac{(p+1)(q+1)}{4} & \textit{if } q \equiv 3 \pmod 4. \end{cases}$$

3.4.3 Total and Paired Domination Numbers of Some Web Graphs

We present the total and the paired domination numbers of a web graph $W_{p,q}$, where the values of p and q are divided as follows: $p \in \{2, 3\}$ and $q \geq 3$; $p \geq 4$ and $q \in \{3, 4\}$; $p \in \{4, 5, 6\}$ and $q \geq 5$.

Theorem 3.4.14. *For any integer $q \geq 3$,*

$$\gamma_t(W_{2,q}) = q + 1 \text{ and } \gamma_{pr}(W_{2,q}) = \begin{cases} q + 1 & \text{if } q \text{ is odd;} \\ q + 2 & \text{if } q \text{ is even.} \end{cases}$$

Proof. Note that each $\gamma_t(W_{2,q})$-set must contain q support vertices and one vertex of C^1, so $\gamma_t(W_{2,q}) \geq q+1$. Since $\gamma_{pr}(W_{2,q})$ is even, $\gamma_{pr}(W_{2,q}) \geq q+1$ if q is odd, and $\gamma_{pr}(W_{2,q}) \geq q + 2$ if q is even. Note that $\{u_{1,1}\} \cup C^2$ is a total dominating set with cardinality $q + 1$, and it is also a paired dominating set if q is odd. If q is even, then $\{c, u_{1,1}\} \cup C^2$ is a paired dominating set with cardinality $q + 2$. □

Theorem 3.4.15. *For any integer $q \geq 3$,*

$$\gamma_t(W_{3,q}) = q + 2 \text{ and } \gamma_{pr}(W_{3,q}) = \begin{cases} q + 2 & \text{if } q \text{ is even;} \\ q + 3 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Note that all q support vertices of C^3 are in every $\gamma_t(W_{3,q})$-set. Let $S = \{v : v \notin N(C^3)\}$. Then the induced subgraph $W_{3,q}[S] \cong W_q$. Since $\gamma_t(W_q) = 2$, $\gamma_t(W_{3,q}) \geq q+2$. We also get that $\gamma_{pr}(W_{3,q}) \geq q + 2$ if q is even, and $\gamma_{pr}(W_{3,q}) \geq q + 3$ if q is odd. We note that $D = \{c, u_{1,1}\} \cup C^3$ is a total dominating set. If q is even, then D is also a paired dominating set; otherwise, $D \cup \{u_{2,1}\}$ is a paired dominating set. □

Before we prove the other results, we need the following lemma which shows that the total (paired) domination number of $W_{p,q}$ can be calculated from the total (paired) domination number of $CH_{p-2,q}$.

Lemma 3.4.16. *For any integers $p \geq 4$ and $q \geq 3$,*

$$\gamma_t(W_{p,q}) = \gamma_t(CH_{p-2,q}) + q$$

and

$$\gamma_{pr}(W_{p,q}) = \begin{cases} \gamma_{pr}(CH_{p-2,q}) + q & \text{if } q \text{ is even;} \\ \gamma_{pr}(CH_{p-2,q}) + q + 1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Note that the union of a $\gamma_t(CH_{p-2,q})$-set and C^p is a total dominating set of $W_{p,q}$. Futhermore, if $|C^p| = q$ is even, then the union of a $\gamma_{pr}(CH_{p-2,q})$-set and C^p is a paired dominating set of $W_{p,q}$; otherwise, the union of a $\gamma_{pr}(CH_{p-2,q})$-set and $C^p \cup \{v_1\}$ is a paired dominating set of $W_{p,q}$. Now, we get the upper bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$.

We know that all q support vertices of C^p are in every $\gamma_t(W_{p,q})$-set and every $\gamma_{pr}(W_{p,q})$-set. Let $S = \{v : v \notin N(C^p)\}$. Then the induced subgraph $W_{p,q}[S]$ is $CH_{p-2,q}$. Therefore, $\gamma_t(W_{p,q}) \geq \gamma_t(CH_{p-2,q}) + q$. Similarly, $\gamma_{pr}(W_{p,q}) \geq \gamma_{pr}(CH_{p-2,q}) + q$ if q is even, and $\gamma_{pr}(W_{p,q}) \geq \gamma_{pr}(CH_{p-2,q}) + q + 1$ if q is odd. □

By Lemma 3.4.16 together with Theorems 3.4.1 - 3.4.5, we obtain Theorems 3.4.17 - 3.4.21, respectively.

Theorem 3.4.17. *For any integer $p \geq 4$,*

$$\gamma_t(W_{p,3}) = \begin{cases} \lceil \frac{4(p-2)}{5} \rceil + 3 & \textit{if } p \equiv 0, 1, 4 \pmod 5; \\ \lceil \frac{4(p-2)}{5} \rceil + 4 & \textit{if } p \equiv 2, 3 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(W_{p,3}) = \begin{cases} \lceil \frac{4(p-2)}{5} \rceil + 5 & \textit{if } p \equiv 0, 3 \pmod 5; \\ \lceil \frac{4(p-2)}{5} \rceil + 4 & \textit{if } p \equiv 1, 4 \pmod 5; \\ \lceil \frac{4(p-2)}{5} \rceil + 6 & \textit{if } p \equiv 2 \pmod 5. \end{cases}$$

Theorem 3.4.18. *For any integer $p \geq 4$,*

$$\gamma_t(W_{p,4}) = \gamma_{pr}(W_{p,4}) = \begin{cases} p + 3 & \textit{if } p \textit{ is odd}; \\ p + 4 & \textit{if } p \textit{ is even}. \end{cases}$$

Theorem 3.4.19. *For any integer $q \geq 5$,*

$$\gamma_t(W_{4,q}) = \begin{cases} 10 & \textit{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil + q & \textit{if } q \equiv 0, 1 \pmod 4; \\ 2\lceil \frac{q+3}{4} \rceil + q - 1 & \textit{if } q \equiv 2, 3 \pmod 4 \textit{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(W_{4,q}) = \begin{cases} 10 & \textit{if } q = 6; \\ 2\lceil \frac{q+3}{4} \rceil + q & \textit{if } q \textit{ is even and } q \neq 6; \\ 2\lceil \frac{q+3}{4} \rceil + q + 1 & \textit{if } q \textit{ is odd}. \end{cases}$$

Theorem 3.4.20. *For any integer $q \geq 5$,*

$$\gamma_t(W_{5,q}) = \begin{cases} 2\lceil \frac{q+2}{3} \rceil + q & \text{if } q \equiv 0, 2, 3 \pmod 6; \\ 2\lceil \frac{q+3}{3} \rceil + q - 1 & \text{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(W_{5,q}) = \begin{cases} 2\lceil \frac{q+3}{3} \rceil + q & \text{if } q \text{ is even}; \\ 2\lceil \frac{q+3}{3} \rceil + q + 1 & \text{if } q \text{ is odd}. \end{cases}$$

Theorem 3.4.21. *For any integer $q \geq 5$,*

$$\gamma_t(W_{6,q}) = \begin{cases} 11 & \text{if } q = 5; \\ 2q + 2 & \text{if } q \geq 6; \end{cases}$$

and

$$\gamma_{pr}(W_{6,q}) = \begin{cases} 12 & \text{if } q = 5; \\ 2q + 2 & \text{if } q \text{ is even}; \\ 2q + 4 & \text{if } q \text{ is odd and } q \neq 5. \end{cases}$$

3.4.4 Upper Bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$

Previously, we show the exact values of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for small values of p and q. We now present the upper bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for $p \geq 7$ and $q \geq 5$. By using Lemma 3.4.16 together with Theorems 3.4.7 - 3.4.13, we easily get Theorems 3.4.22 - 3.4.28, respectively.

Theorem 3.4.22. *For any integer $p \geq 7$,*

$$\gamma_t(W_{p,5}) \leq \begin{cases} \lceil \frac{9(p-3)}{7} \rceil + 7 & \text{if } p \equiv 0 \pmod 7; \\ \lceil \frac{9(p-3)}{7} \rceil + 8 & \text{otherwise}; \end{cases}$$

and

$$\gamma_{pr}(W_{p,5}) \leq \begin{cases} \lceil \frac{4(p-3)}{3} \rceil + 10 & \text{if } p \equiv 0 \pmod 3; \\ \lceil \frac{4(p-3)}{3} \rceil + 8 & \text{if } p \equiv 1 \pmod 3; \\ \lceil \frac{4(p-3)}{3} \rceil + 9 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Theorem 3.4.23. *For any integer $q \geq 6$,*

$$\gamma_t(W_{7,q}) \leq \begin{cases} \lfloor \frac{11q+8}{5} \rfloor + 1 & \text{if } q \equiv 0 \pmod 5; \\ \lfloor \frac{11q+8}{5} \rfloor + 2 & \text{if } q \equiv 1,2,4 \pmod 5; \\ \lfloor \frac{11q+8}{5} \rfloor + 3 & \text{if } q \equiv 3 \pmod 5; \end{cases}$$

and

$$\gamma_{pr}(W_{7,q}) \leq \begin{cases} \lfloor \frac{11q+8}{5} \rfloor + 1 & \text{if } q \equiv 0 \pmod{10}; \\ \lfloor \frac{11q+8}{5} \rfloor + 3 & \text{if } q \equiv 1,7,8,9 \pmod{10}; \\ \lfloor \frac{11q+8}{5} \rfloor + 2 & \text{if } q \equiv 2,4,5,6 \pmod{10}; \\ \lfloor \frac{11q+8}{5} \rfloor + 4 & \text{if } q \equiv 3 \pmod{10}. \end{cases}$$

Theorem 3.4.24. *For any integer $q \geq 6$,*

$$\gamma_t(W_{8,q}) \leq \begin{cases} 18 & \text{if } q = 6; \\ \lfloor \frac{5q}{2} \rfloor + 2 & \text{if } q \equiv 0 \pmod 4; \\ \lfloor \frac{5q}{2} \rfloor + 4 & \text{if } q \equiv 1,2,3 \pmod 4 \text{ and } q \neq 6; \end{cases}$$

and

$$\gamma_{pr}(W_{8,q}) \leq \begin{cases} 18 & \text{if } q = 6; \\ \lfloor \frac{5q}{2} \rfloor + 2 & \text{if } q \equiv 0 \pmod 4; \\ \lfloor \frac{5q}{2} \rfloor + 6 & \text{if } q \equiv 1 \pmod 4; \\ \lfloor \frac{5q}{2} \rfloor + 5 & \text{if } q \equiv 2,3 \pmod 4 \text{ and } q \neq 6. \end{cases}$$

Theorem 3.4.25. *For any integer $q \geq 6$,*

$$\gamma_t(W_{9,q}) \leq \begin{cases} \lfloor \frac{19q}{7} \rfloor + 2 & \text{if } q \equiv 0 \pmod 7; \\ \lfloor \frac{19q}{7} \rfloor + 5 & \text{if } q \equiv 1,2 \pmod 7; \\ \lfloor \frac{19q}{7} \rfloor + 3 & \text{if } q \equiv 3 \pmod 7; \\ \lfloor \frac{19q}{7} \rfloor + 4 & \text{if } q \equiv 4,6 \pmod 7; \\ \lfloor \frac{19q}{7} \rfloor + 6 & \text{if } q \equiv 5 \pmod 7; \end{cases}$$

and

$$\gamma_{pr}(W_{9,q}) \leq \begin{cases} \lfloor \frac{19q}{7} \rfloor + 2 & \textit{if } q \equiv 0 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 6 & \textit{if } q \equiv 1, 9, 12 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 5 & \textit{if } q \equiv 2, 8, 11, 13 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 4 & \textit{if } q \equiv 3, 4, 6 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 7 & \textit{if } q \equiv 5 \pmod{14}; \\ \lfloor \frac{19q}{7} \rfloor + 3 & \textit{if } q \equiv 7, 10 \pmod{14}. \end{cases}$$

The following theorem gives the upper bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ for $p \geq 10$ and $q \geq 6$.

Theorem 3.4.26. *For any integers $p \geq 10$ and $q \geq 6$,*

$$\gamma_t(W_{p,q}) \leq 2\lceil \frac{p-2}{2} \rceil \lceil \frac{q}{4} \rceil + q + 2$$

and

$$\gamma_{pr}(W_{p,q}) \leq \begin{cases} 2\lceil \frac{p-2}{2} \rceil \lceil \frac{q}{4} \rceil + q + 2 & \textit{if } q \textit{ is even}; \\ 2\lceil \frac{p-2}{2} \rceil \lceil \frac{q}{4} \rceil + q + 3 & \textit{if } q \textit{ is odd}. \end{cases}$$

The next two theorems provide some better bounds of $\gamma_t(W_{p,q})$ and $\gamma_{pr}(W_{p,q})$ than ones in Theorem 3.4.26 for some special values of p and q.

Theorem 3.4.27. *If $10 \leq p \equiv 0 \pmod{2}$ and $9 \leq q \equiv 1 \pmod{4}$, then $\gamma_t(W_{p,q}) \leq \frac{(p+2)(q+1)}{4} + 1$.*

Theorem 3.4.28. *If $11 \leq p \equiv 1 \pmod{2}$ and $q \geq 6$, then*

$$\gamma_t(W_{p,q}) \leq \begin{cases} \frac{(p+3)(q+1)}{4} - 1 & \textit{if } q \equiv 1, 3 \pmod{4}; \\ \frac{(p+3)(q+2)}{4} - 4 & \textit{if } q \equiv 2 \pmod{4}; \end{cases}$$

and

$$\gamma_{pr}(W_{p,q}) \leq \begin{cases} \frac{(p+3)(q+3)}{4} - 4 & \textit{if } q \equiv 1 \pmod{4}; \\ \frac{(p+3)(q+2)}{4} - 4 & \textit{if } q \equiv 2 \pmod{4}; \\ \frac{(p+3)(q+1)}{4} & \textit{if } q \equiv 3 \pmod{4}. \end{cases}$$

3.5 Windmill Class of Graphs

The windmill class of graphs consists of the French windmill class of graphs and the Dutch windmill class of graphs. In Subsection 3.5.1, we determine the total domination numbers and the paired domination numbers for the French windmill class of graphs. The total domination numbers and the paired domination numbers for the Dutch windmill class of graphs appear in Subsection 3.5.2

3.5.1 French Windmill Class of Graphs

We first introduce French windmill graphs, French star windmill graphs, French complete windmill graphs, and French cycle windmill graphs, which belong to the French windmill class of graphs. Then the total domination numbers and the paired domination numbers of these four graphs are computed.

For any integers $p, q \geq 1$, let qK_p have the vertex set $V(qK_p) = \{v_i^j : 1 \leq i \leq p, 1 \leq j \leq q\}$ and the edge set $E(qK_p) = \{v_i^j v_{i'}^j : i \neq i', 1 \leq j \leq q\}$.

The *French windmill graph* $W_{p,q}$ is obtained from qK_p by adding the vertex c and the edge cv_i^j for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$.

The *French star windmill graph* $SW_{p,q}$ is obtained from qK_p by adding the vertex c and the edge cv_1^j for all $j \in \{1, 2, \ldots, q\}$.

The *French complete windmill graph* $KW_{p,q}$ is obtained from qK_p by adding the edge $v_1^j v_1^{j'}$ for all $j \neq j'$.

The *French cycle windmill graph* $CW_{p,q}$ is obtained from qK_p by adding the edges $v_1^1 v_1^q$ and $v_1^j v_1^{j+1}$for all $j \in \{1, 2, \ldots, q-1\}$.

For instance, the French windmill graph $W_{4,4}$ and the French star windmill graph $SW_{5,4}$ are shown in Figure 3.17, and the French complete windmill graph $KW_{5,4}$ and the French cycle windmill graph $CW_{5,4}$ are illustrated in Figure 3.18.

We roughly say that French star windmill graphs, French complete windmill graphs, and French cycle windmill graphs are obtained by replacing the shared vertex c of French windmill graphs with a star, a complete graph, and a cycle, respectively.

We now note that $\{c, v_1^1\}$ is a paired dominating set of $W_{p,q}$ for all $p, q \geq 1$, so $\gamma_{pr}(W_{p,q}) \leq 2$. Combining the fact that $2 \leq \gamma_t(W_{p,q}) \leq \gamma_{pr}(W_{p,q})$, we can easily get the following result.

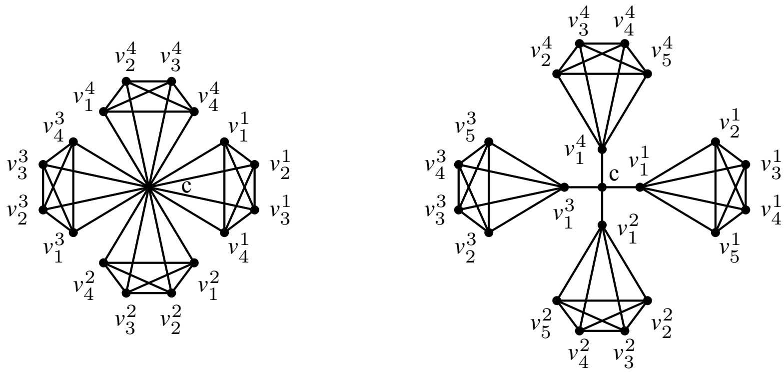

Figure 3.17 The French windmill graph $W_{4,4}$ (left) and the French star windmill graph $SW_{5,4}$ (right)

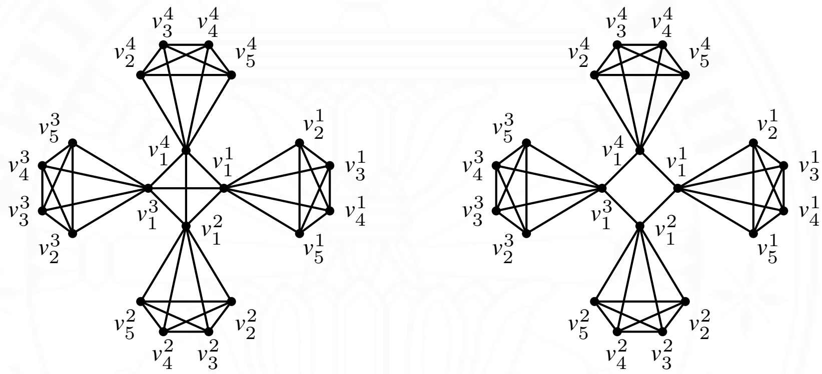

Figure 3.18 The French complete windmill graph $KW_{5,4}$ (left) and the French cycle windmill graph $CW_{5,4}$ (right)

Lemma 3.5.1. *Let p and q be positive integers. Then $\gamma_t(W_{p,q}) = \gamma_{pr}(W_{p,q}) = 2$.*

We observe that $SW_{1,q} \cong K_{1,q}$ for all $q \geq 1$, so it is obvious that $\gamma_t(SW_{1,q}) = 2 = \gamma_{pr}(SW_{1,q})$. For any integers $p \geq 2$ and $q \geq 1$, we can have the results on $\gamma_t(SW_{p,q})$ and $\gamma_{pr}(SW_{p,q})$ in the following theorem.

Theorem 3.5.2. *Let $p \geq 2$ and $q \geq 1$ be integers. Then $\gamma_t(SW_{p,q}) = q + 1$ and $\gamma_{pr}(SW_{p,q}) = 2q$.*

Proof. If $p = 2$ or $q = 1$, then it is easy to check that $\{c, v_1^1, v_1^2, \ldots, v_1^q\}$ is a $\gamma_t(SW_{p,q})$-set and $\{v_1^j, v_2^j : 1 \leq j \leq q\}$ is a $\gamma_{pr}(SW_{p,q})$-set, so we are done in this case. Let $p \geq 3$, $q \geq 2$, and D be a $\gamma_t(SW_{p,q})$-set. We claim that $c \in D$. If $c \notin D$, then D must contain exactly two vertices from $\{v_1^j, v_2^j, \ldots, v_p^j\}$ for each $j \in \{1, 2, \ldots, q\}$, whereas $\{c, v_1^1, v_1^2, \ldots, v_1^q\}$

is a total dominating set with cardinality $q + 1 < 2q = |D|$, a contradiction. Next, we assume on the contrary that $v_1^j \notin D$ for some $j \in \{1, 2, \ldots, q\}$. Then two vertices of $\{v_2^j, v_3^j, \ldots, v_p^j\}$ are in D. Hence, $(D \setminus \{v_2^j, v_3^j, \ldots, v_p^j\}) \cup \{v_1^j\}$ is a total dominating set with cardinality less than D. Thus, D contains the vertices $c, v_1^1, v_1^2, \ldots, v_1^q$, implying that $\gamma_t(SW_{p,q}) = |D| \geq q + 1$. Note that $\{c, v_1^1, v_1^2, \ldots, v_1^q\}$ is a total dominating set of $SW_{p,q}$, so $\gamma_t(SW_{p,q}) \leq q + 1$. Now, we can conclude that $\gamma_t(SW_{p,q}) = q + 1$.

Let D be a $\gamma_{pr}(SW_{p,q})$-set. We show that $|D| \geq 2q$. If $c \notin D$, then D contains two vertices from $\{v_1^j, v_2^j, \ldots, v_p^j\}$ for each $j \in \{1, 2, \ldots, q\}$, so $|D| \geq 2q$. Next, we assume that $c \in D$. Without loss of generality, we may assume that $\{c, v_1^1\}$ is paired in D. Let $S = \{v : v \notin N(\{c, v_1^1\})\}$. Then the induced subgraph $SW_{p,q}[S] \cong (q-1)K_{p-1}$. Obviously, D contains two vertices from each K_{p-1}, so $|D| \geq 2 + 2(q-1) = 2q$. We can check that $\{v_1^j, v_2^j : 1 \leq j \leq q\}$ is paired dominating set of $SW_{p,q}$, so $\gamma_{pr}(SW_{p,q}) \leq 2q$. We can conclude that $\gamma_{pr}(SW_{p,q}) = 2q$. □

Obviously, the total and the paired domination numbers of $KW_{1,1} \cong K_1$ are not defined. We next note that $KW_{p,1} \cong K_p$ for all $p \geq 2$ and $KW_{1,q} \cong K_q$ for all $q \geq 2$, so $\gamma_t(KW_{p,1}) = 2 = \gamma_{pr}(KW_{p,1})$ and $\gamma_t(KW_{1,q}) = 2 = \gamma_{pr}(KW_{1,q})$. In the following theorem, we provide the total and the paired domination numbers of $KW_{p,q}$ for all $p, q \geq 2$.

Theorem 3.5.3. *Let $p, q \geq 2$ be integers. Then*

$$\gamma_t(KW_{p,q}) = q \text{ and } \gamma_{pr}(KW_{p,q}) = \begin{cases} q & \text{if } q \text{ is even;} \\ q+1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. If $p = 2$, then Observation 3.0.1 implies that the q support vertices form a $\gamma_t(KW_{p,q})$-set. Let $p \geq 3$. Similar to the proof of Theorem 3.5.2, we can verify that for any $\gamma_t(KW_{p,q})$-set D, D contains only the vertices $v_1^1, v_1^2, \ldots, v_1^q$. Therefore, $\gamma_t(KW_{p,q}) = q$.

Since $\gamma_{pr}(KW_{p,q}) \geq \gamma_t(KW_{p,q})$ and $\gamma_{pr}(KW_{p,q})$ is even, $\gamma_{pr}(KW_{p,q}) \geq q$ if q is even, and $\gamma_{pr}(KW_{p,q}) \geq q+1$ if q is odd. Let $D = \{v_1^j : 1 \leq j \leq q\}$. If q is even, then D is a paired dominating set of $KW_{p,q}$, so $\gamma_{pr}(KW_{p,q}) = q$. If q is odd, then $D \cup \{v_2^1\}$ is a paired dominating set of $KW_{p,q}$, so $\gamma_{pr}(KW_{p,q}) = q + 1$. □

Clearly, $\gamma_t(CW_{1,1})$ and $\gamma_{pr}(CW_{1,1})$ are not defined, and $\gamma_t(CW_{p,1}) = 2 = \gamma_{pr}(CW_{p,1})$ for all $p \geq 2$ because $CW_{p,1} \cong K_p$. Note that $CW_{1,2} \cong P_2$ and $CW_{1,q} \cong C_q$ for all $q \geq 3$, so we obtain the exact values of $\gamma_t(CW_{1,q})$ and $\gamma_{pr}(CW_{1,q})$ for all $q \geq 2$ by Lemmas 3.0.2 and 3.0.3, respectively. For any integers $p, q \geq 2$, we have the following theorem.

Theorem 3.5.4. *Let $p, q \geq 2$ be integers. Then*

$$\gamma_t(CW_{p,q}) = q \text{ and } \gamma_{pr}(CW_{p,q}) = \begin{cases} q & \text{if } q \text{ is even;} \\ q+1 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Note that every total dominating set of $CW_{p,q}$ is also a total dominating set of $KW_{p,q}$, so we get that $\gamma_t(CW_{p,q}) \geq \gamma_t(KW_{p,q})$. Likewise, we obtain that $\gamma_{pr}(CW_{p,q}) \geq \gamma_{pr}(KW_{p,q})$. Theorem 3.5.3 gives the lower bounds of $\gamma_t(CW_{p,q})$ and $\gamma_{pr}(CW_{p,q})$. Let $D = \{v_1^j : 1 \leq j \leq q\}$. Then D is a total dominating set of $CW_{p,q}$, so $\gamma_t(CW_{p,q}) = q$. Moreover, D is a paired dominating set of $CW_{p,q}$ if q is even, and $D \cup \{v_2^1\}$ is a paired dominating set of $CW_{p,q}$ if q is odd. Thus, $\gamma_{pr}(CW_{p,q}) = q$ if q is even, and $\gamma_{pr}(CW_{p,q}) = q + 1$ if q is odd. □

3.5.2 Dutch Windmill Class of Graphs

The Dutch windmill class of graphs contains Dutch windmill graphs, Dutch star windmill graphs, Dutch complete windmill graphs, and Dutch cycle windmill graphs, of which the definitions are provided below. We then determine the total and the paired domination numbers of these four graphs.

For any integers $p, q \geq 1$, let qP_p have the vertex set $V(qP_p) = \{v_i^j : 1 \leq i \leq p, 1 \leq j \leq q\}$ and the edge set $E(qP_p) = \{v_i^j v_{i+1}^j : 1 \leq i \leq p-1, 1 \leq j \leq q\}$.

The *Dutch windmill graph $D_{p,q}$* is obtained from qP_p by adding the vertex c and the edges cv_1^j and cv_p^j for all $j \in \{1, 2, \ldots, q\}$.

For any integers $p \geq 3$ and $q \geq 1$, let qC_p have the vertex set $V(qC_p) = \{v_i^j : 1 \leq i \leq p, 1 \leq j \leq q\}$ and the edge set $E(qC_p) = \{v_i^j v_{i+1}^j : 1 \leq i \leq p-1, 1 \leq j \leq q\} \cup \{v_1^j v_p^j : 1 \leq j \leq q\}$.

The *Dutch star windmill graph $SD_{p,q}$* is obtained from qC_p by adding the vertex c and the edge cv_1^j for all $j \in \{1, 2, \ldots, q\}$.

The *Dutch complete windmill graph* $KD_{p,q}$ is obtained from qC_p by adding the edge $v_1^j v_1^{j'}$ for all $j \neq j'$.

The *Dutch cycle windmill graph* $CD_{p,q}$ is obtained from qC_p by adding the edges $v_1^1 v_1^q$ and $v_1^j v_1^{j+1}$ for all $j \in \{1, 2, \ldots, q-1\}$.

We illustrate the Dutch windmill graph $D_{4,4}$ and the Dutch star windmill graph $SD_{5,4}$ in Figure 3.19, and the Dutch complete windmill graph $KD_{5,4}$ and the Dutch cycle windmill graph $CD_{5,4}$ in Figure 3.20.

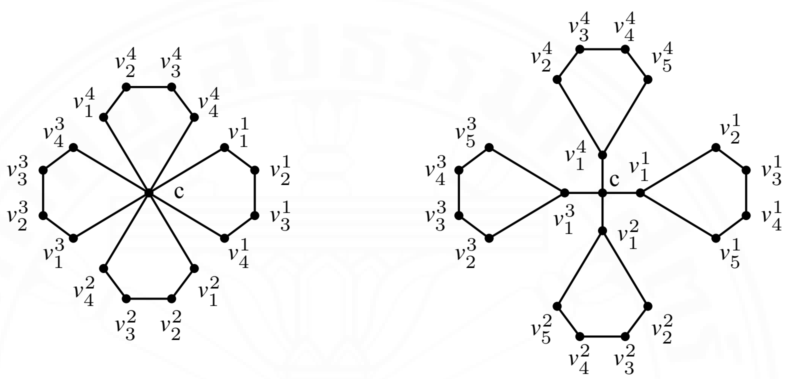

Figure 3.19 The Dutch windmill graph $D_{4,4}$ (left) and the Dutch star windmill graph $SD_{5,4}$ (right)

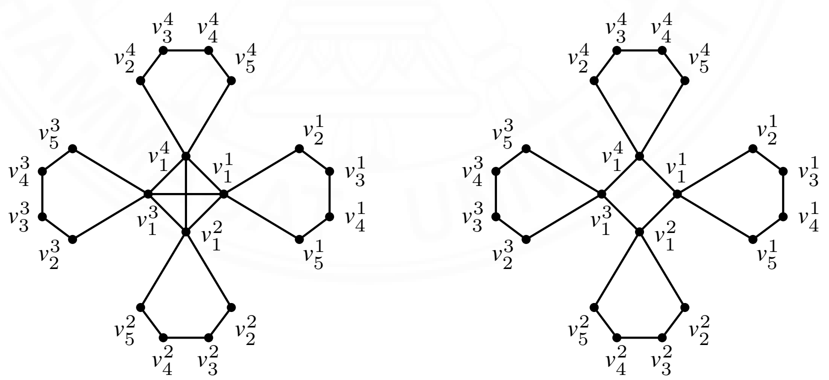

Figure 3.20 The Dutch complete windmill graph $KD_{5,4}$ (left) and the Dutch cycle windmill graph $CD_{5,4}$ (right)

We roughly say that Dutch star windmill graphs, Dutch complete windmill graphs, and Dutch cycle windmill graphs are obtained by replacing the shared vertex c of Dutch windmill graphs with a star, a complete graph, and a cycle, respectively.

We first determine the total and the paired domination numbers of $D_{p,q}$ for all $p, q \geq 1$.

Theorem 3.5.5. *Let p and q be positive integers. Then*

$$\gamma_t(D_{p,q}) = \begin{cases} 2q\lceil \frac{p-2}{4} \rceil + 1 & \text{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-2}{4} \rceil + 2 & \text{if } p \equiv 1, 2 \pmod 4; \\ 2q\lceil \frac{p-2}{4} \rceil + 1 - q & \text{if } p \equiv 3 \pmod 4; \end{cases}$$

and

$$\gamma_{pr}(D_{p,q}) = \begin{cases} 2q\lceil \frac{p-2}{4} \rceil + 2 & \text{if } p \equiv 0, 1, 2 \pmod 4; \\ 2q\lceil \frac{p-2}{4} \rceil & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. It is easy to verify that $\gamma_t(D_{p,q}) = 2 = \gamma_{pr}(D_{p,q})$ for $p \in \{1, 2\}$. Next, we let $p \geq 3$. We first construct a total dominating set of $D_{p,q}$ containing the vertex c. Let $E = \{c\} \cup \{v_i^j, v_{i+1}^j : i \equiv 3 \pmod 4, i \neq p, 1 \leq j \leq q\}$. If $p \equiv 0 \pmod 4$, then we can check that $D = E$ is a total dominating set of $D_{p,q}$ with $|D| = 2q\lceil \frac{p-2}{4} \rceil + 1$. If $p \equiv 1, 2 \pmod 4$, then $D = E \cup \{v_1^1\}$ is a total dominating set of $D_{p,q}$ with $|D| = 2q\lceil \frac{p-2}{4} \rceil + 2$. If $p \equiv 3 \pmod 4$, then $D = E \cup \{v_p^j : 1 \leq j \leq q\}$ is a total dominating set of $D_{p,q}$ with $|D| = 2q\lceil \frac{p-2}{4} \rceil + 1 - q$.

Next, we show that D is a $\gamma_t(D_{p,q})$-set. Suppose on the contrary that there is a total dominating set D' of $D_{p,q}$ such that $|D'| < |D|$. By the construction of D, among all total dominating sets of $D_{p,q}$ containing the vertex c, D is minimum. We can conclude that $c \notin D'$. Without loss of generality, we may assume that $v_1^1 \in D'$ and $v_2^1 \in D'$ to dominate c and v_1^1, respectively. Let $S = \{v : v \notin N(\{v_1^1, v_2^1\})\}$. Then the induced subgraph $D_{p,q}[S]$ contains P_{p-3} and $(q-1)P_p$. Thus, $|D'| \geq 2 + \gamma_t(P_{p-3}) + (q-1)\gamma_t(P_p)$. By Lemma 3.0.2, we get that $|D'| \geq 2 + (\lfloor \frac{p-1}{4} \rfloor + \lfloor \frac{p}{4} \rfloor) + (q-1)(\lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor)$. Then we consider the following four cases.

Case 1: Let $p = 4k$, where $k \geq 1$. Then $|D'| \geq 2 + (\lfloor \frac{4k-1}{4} \rfloor + \lfloor \frac{4k}{4} \rfloor) + (q-1)(\lfloor \frac{4k+2}{4} \rfloor + \lfloor \frac{4k+3}{4} \rfloor) = 2qk + 1 = 2q\lceil \frac{4k-2}{4} \rceil + 1 = 2q\lceil \frac{p-2}{4} \rceil + 1 = |D|$.

Case 2: Let $p = 4k + 1$, where $k \geq 1$. Then $|D'| \geq 2 + (\lfloor \frac{4k}{4} \rfloor + \lfloor \frac{4k+1}{4} \rfloor) + (q-1)(\lfloor \frac{4k+3}{4} \rfloor + \lfloor \frac{4k+4}{4} \rfloor) = 2qk + q + 1 \geq 2qk + 2 = 2q\lceil \frac{4k-1}{4} \rceil + 2 = 2q\lceil \frac{p-2}{4} \rceil + 2 = |D|$.

Case 3: Let $p = 4k + 2$, where $k \geq 1$. Then $|D'| \geq 2 + (\lfloor \frac{4k+1}{4} \rfloor + \lfloor \frac{4k+2}{4} \rfloor) + (q-1)(\lfloor \frac{4k+4}{4} \rfloor + \lfloor \frac{4k+5}{4} \rfloor) = 2qk + 2q \geq 2qk + 2 = 2q\lceil \frac{4k}{4} \rceil + 2 = 2q\lceil \frac{p-2}{4} \rceil + 2 = |D|$.

Case 4: Let $p = 4k+3$, where $k \geq 0$. Then $|D'| \geq 2+(\lfloor\frac{4k+2}{4}\rfloor+\lfloor\frac{4k+3}{4}\rfloor)+(q-1)(\lfloor\frac{4k+5}{4}\rfloor+\lfloor\frac{4k+6}{4}\rfloor) = 2qk+2q \geq 2qk+q+1 = 2q\lceil\frac{4k+1}{4}\rceil+1-q = 2q\lceil\frac{p-2}{4}\rceil+1-q = |D|$.

We can see that all cases contradict with the assumption $|D'| < |D|$, so D is a $\gamma_t(D_{p,q})$-set, and we then obtain the total domination numbers of $D_{p,q}$.

Consider the set E as defined above. We can check that $E \cup \{v_1^1\}$ is a paired dominating set of $D_{p,q}$ if $p \equiv 0, 1, 2 \pmod 4$, and $E \cup \{v_p^1\} \cup \{v_{p-1}^j, v_p^j : 2 \leq j \leq q\}$ is a paired dominating set of $D_{p,q}$ if $p \equiv 3 \pmod 4$. Therefore, $\gamma_{pr}(D_{p,q}) \leq 2q\lceil\frac{p-2}{4}\rceil + 2$ if $p \equiv 0, 1, 2 \pmod 4$, and $\gamma_{pr}(D_{p,q}) \leq 2q\lceil\frac{p-2}{4}\rceil$ if $p \equiv 3 \pmod 4$. Note that $\gamma_{pr}(D_{p,q}) \geq \gamma_t(D_{p,q})$ and $\gamma_{pr}(D_{p,q})$ is even, and thus $\gamma_{pr}(D_{p,q}) = 2q\lceil\frac{p-2}{4}\rceil+2$ for $p \equiv 0, 1, 2 \pmod 4$.

Finally, we show that $\gamma_{pr}(D_{p,q}) \geq 2q\lceil\frac{p-2}{4}\rceil$ if $p \equiv 3 \pmod 4$. Let D be a $\gamma_{pr}(D_{p,q})$-set. If $c \in D$, then, without loss of generality, $v_1^1 \in D$. Let $S = \{v : v \notin N(\{c, v_1^1\})\}$ and then the induced subgraph $D_{p,q}[S]$ contains P_{p-3} and $(q-1)P_{p-2}$. By Lemma 3.0.3, we obtain that $|D| \geq 2 + \gamma_{pr}(P_{p-3}) + (q-1)\gamma_{pr}(P_{p-2}) = 2 + 2\lceil\frac{p-3}{4}\rceil + 2(q-1)\lceil\frac{p-2}{4}\rceil = 2\lceil\frac{p+1}{4}\rceil + 2(q-1)\lceil\frac{p-2}{4}\rceil \geq 2\lceil\frac{p-2}{4}\rceil + 2(q-1)\lceil\frac{p-2}{4}\rceil = 2q\lceil\frac{p-2}{4}\rceil$. We next assume that $c \notin D$. Similar to the second paragraph in this proof, we get that $|D| \geq 2+\gamma_{pr}(P_{p-3})+(q-1)\gamma_{pr}(P_p)$. Lemma 3.0.3 shows that $|D| \geq 2+2\lceil\frac{p-3}{4}\rceil+2(q-1)\lceil\frac{p}{4}\rceil = 2\lceil\frac{p+1}{4}\rceil + 2(q-1)\lceil\frac{p}{4}\rceil \geq 2\lceil\frac{p-2}{4}\rceil + 2(q-1)\lceil\frac{p-2}{4}\rceil = 2q\lceil\frac{p-2}{4}\rceil$. This completes the proof. □

Next, we compute the total and the paired domination numbers of $SD_{p,q}$. If $q = 1$, then, in the following theorem, we prove that $\gamma_t(SD_{p,q}) = \gamma_t(C_p)$ and $\gamma_{pr}(SD_{p,q}) = \gamma_{pr}(C_p)$. For any integer $q \geq 2$, we provide the exact values of $\gamma_t(SD_{p,q})$ and $\gamma_{pr}(SD_{p,q})$ in Theorem 3.5.7 (below).

Theorem 3.5.6. *Let $p \geq 3$ be an integer. Then $\gamma_t(SD_{p,1}) = \lfloor\frac{p+2}{4}\rfloor + \lfloor\frac{p+3}{4}\rfloor$ and $\gamma_{pr}(SD_{p,1}) = 2\lceil\frac{p}{4}\rceil$.*

Proof. Let D be a $\gamma_t(SD_{p,1})$-set. We show that $|D| \geq \gamma_t(C_p)$. Note that $v_1^1 \in D$. If $c \notin D$, then D is also a total dominating set of C_p, so $|D| \geq \gamma_t(C_p)$. Next, we assume that $c \in D$. Then $v_2^1 \notin D$; otherwise, $D \setminus \{c\}$ is a total dominating set of $SD_{p,1}$ with cardinality less than D, a contradiction. Thus, $D' = (D \setminus \{c\}) \cup \{v_2^1\}$ is a total dominating set of $SD_{p,1}$ with $|D'| = |D|$. Moreover, D' is a total dominating set of C_p, so $|D| = |D'| \geq \gamma_t(C_p)$. Hence, $\gamma_t(SD_{p,1}) = |D| \geq \gamma_t(C_p) = \lfloor\frac{p+2}{4}\rfloor + \lfloor\frac{p+3}{4}\rfloor$ by Lemma 3.0.2. By using the similar technique of this proof, we can verify that $\gamma_{pr}(SD_{p,1}) \geq \gamma_{pr}(C_p)$. Lemma 3.0.3 shows that $\gamma_{pr}(SD_{p,1}) \geq 2\lceil \frac{p}{4} \rceil$.

To prove the upper bounds of $\gamma_t(SD_{p,1})$ and $\gamma_{pr}(SD_{p,1})$, let $E = \{v_i^1, v_{i+1}^1 : i \equiv 1 \pmod 4, i \neq p\}$. If $p \equiv 0, 2, 3 \pmod 4$, then E is a paired dominating set of $SD_{p,1}$, and hence $\gamma_t(SD_{p,1}) \leq \gamma_{pr}(SD_{p,1}) \leq 2\lceil \frac{p}{4} \rceil = \lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor$. If $p \equiv 1 \pmod 4$, then $E \cup \{v_p^1\}$ is a total dominating set of $SD_{p,1}$ and $E \cup \{v_{p-1}^1, v_p^1\}$ is a paired dominating set of $SD_{p,1}$. Hence, $\gamma_t(SD_{p,1}) \leq 2\lceil \frac{p}{4} \rceil - 1 = \lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor$ and $\gamma_{pr}(SD_{p,1}) \leq 2\lceil \frac{p}{4} \rceil$. The theorem follows. □

Theorem 3.5.7. *Let $p \geq 3$ and $q \geq 2$ be integers. Then*

$$\gamma_t(SD_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + 2 & \text{if } p \equiv 1 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + 1 - q & \text{if } p \equiv 2, 3 \pmod 4; \end{cases}$$

and

$$\gamma_{pr}(SD_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + 2 & \text{if } p \equiv 1 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil & \text{if } p \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. If $p \equiv 0 \pmod 4$, then $\{v_i^1, v_{i+1}^1 : i \equiv 1 \pmod 4\} \cup \{v_i^j, v_{i+1}^j : i \equiv 2 \pmod 4, 2 \leq j \leq q\}$ is both an efficient total dominating set and an efficient paired dominating set of $SD_{p,q}$ with cardinality $\frac{pq}{2}$. By Lemma 3.0.4, we conclude that $\gamma_t(SD_{p,q}) = \frac{pq}{2} = \gamma_{pr}(SD_{p,q})$.

Let $p = 4k + 1$ for some $k \geq 1$. Then $D = \{c, v_1^1\} \cup \{v_i^j, v_{i+1}^j : i \equiv 3 \pmod 4, 1 \leq j \leq q\}$ is a total dominating set of $SD_{p,q}$ with $|D| = \frac{(p-1)q}{2} + 2 = 2q\lceil \frac{p-1}{4} \rceil + 2$. We claim that among all total dominating sets containing the vertex c, D is minimum. Let D' be any total dominating set containing c. To dominate c, D' contains $l \geq 1$ vertices from $\{v_1^1, v_1^2, \ldots, v_1^q\}$. If $v_1^j \in D'$ for $j \in \{1, 2, \ldots, q\}$, then, by Lemma 3.0.2, D' contains at least $\lfloor \frac{p-1}{4} \rfloor + \lfloor \frac{p}{4} \rfloor$ vertices to dominate $v_3^j, v_4^j, \ldots, v_{p-1}^j$; otherwise, D' contains at least $\lfloor \frac{p+1}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor$ vertices to dominate $v_2^j, v_3^j, \ldots, v_p^j$. Hence, $|D'| \geq 1 + l + l(\lfloor \frac{p-1}{4} \rfloor + \lfloor \frac{p}{4} \rfloor) + (q - l)(\lfloor \frac{p+1}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor) = 1 + l + l(\lfloor \frac{4k}{4} \rfloor + \lfloor \frac{4k+1}{4} \rfloor) + (q - l)(\lfloor \frac{4k+2}{4} \rfloor + \lfloor \frac{4k+3}{4} \rfloor) = 2qk + l + 1 \geq 2qk + 2 = 2q\lceil \frac{p-1}{4} \rceil + 2 = |D|$ since $p = 4k + 1$ and $l \geq 1$, so the claim holds. We next show that D is a $\gamma_t(SD_{p,q})$-set. Assume that D'' is a total dominating set with $|D''| < |D|$. The previous claim implies that $c \notin D''$. Without loss of generality, $v_1^1, v_2^1 \in D''$. Then the induced subgraph $SD_{p,q}[\{v : v \notin N(\{v_1^1, v_2^1\})\}]$ contains P_{p-4} and $(q-1)C_p$. We can check that $|D''| \geq 2 + \gamma_t(P_{p-4}) + (q-1)\gamma_t(C_p) = 2 + (\lfloor \frac{p-2}{4} \rfloor + \lfloor \frac{p-1}{4} \rfloor) + (q-1)(\lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor) = 2qk + q \geq 2qk + 2 = 2q\lceil \frac{p-1}{4} \rceil + 2 = |D|$, a contradiction. Hence, $\gamma_t(SD_{p,q}) = 2q\lceil \frac{p-1}{4} \rceil + 2$ and $\gamma_{pr}(SD_{p,q}) \geq 2q\lceil \frac{p-1}{4} \rceil + 2$. Since D (defined above) is also a paired dominating set of $SD_{p,q}$, we conclude that $\gamma_{pr}(SD_{p,q}) = 2q\lceil \frac{p-1}{4} \rceil + 2$.

Let $p \equiv 2, 3 \pmod 4$. Then $D = \{c\} \cup \{v_1^j : 1 \leq j \leq q\} \cup \{v_i^j, v_{i+1}^j : i \equiv 0 \pmod 4, 1 \leq j \leq q\}$ is a total dominating set of $SD_{p,q}$ with $|D| = 2q\lceil \frac{p-1}{4} \rceil + 1 - q$. We show that among all total dominating sets containing the vertex c, D is minimum. Suppose that D' is a total dominating set containing c. Then D' contains $l \leq q$ vertices from $\{v_1^1, v_1^2, \ldots, v_1^q\}$. Thus, $|D'| \geq 1 + l + l(\lfloor \frac{p-1}{4} \rfloor + \lfloor \frac{p}{4} \rfloor) + (q-l)(\lfloor \frac{p+1}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor) \geq 2q\lceil \frac{p-1}{4} \rceil + 1 - q = |D|$. Next, we prove that D is a $\gamma_t(SD_{p,q})$-set. If D'' is a total dominating set with $|D''| < |D|$, then $c \notin D''$. It is easy to verify that $|D''| \geq 2 + \gamma_t(P_{p-4}) + (q-1)\gamma_t(C_p) = 2 + (\lfloor \frac{p-2}{4} \rfloor + \lfloor \frac{p-1}{4} \rfloor) + (q-1)(\lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor) > 2q\lceil \frac{p-1}{4} \rceil + 1 - q = |D|$, a contradiction. Therefore, $\gamma_t(SD_{p,q}) = 2q\lceil \frac{p-1}{4} \rceil + 1 - q$. We can check that $D \cup \{v_2^j : 2 \leq j \leq q\}$ is a paired dominating set of $SD_{p,q}$ with cardinality $2q\lceil \frac{p-1}{4} \rceil$, and thus $\gamma_{pr}(SD_{p,q}) \leq 2q\lceil \frac{p-1}{4} \rceil$. Similar to the proof of Theorem 3.5.5 (last paragraph), we can get that $\gamma_{pr}(SD_{p,q}) \geq 2q\lceil \frac{p-1}{4} \rceil$. The theorem follows. □

If $q = 1$, then $KD_{p,q} \cong C_p$ for all $p \geq 3$, so we obtain $\gamma_t(KD_{p,1}) = \lfloor \frac{p+2}{4} \rfloor + \lfloor \frac{p+3}{4} \rfloor$ and $\gamma_{pr}(KD_{p,1}) = 2\lceil \frac{p}{4} \rceil$ by Lemmas 3.0.2 and 3.0.3, respectively. In the following theorem, we compute the exact values of $\gamma_t(KD_{p,q})$ and $\gamma_{pr}(KD_{p,q})$ for all $p \geq 3$ and $q \geq 2$.

Theorem 3.5.8. *Let $p \geq 3$ and $q \geq 2$ be integers. Then*

$$\gamma_t(KD_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + 1 & \text{if } p \equiv 1 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil - q & \text{if } p \equiv 2, 3 \pmod 4; \end{cases}$$

and

$$\gamma_{pr}(KD_{p,q}) = \begin{cases} \frac{pq}{2} & \textit{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + 2 & \textit{if } p \equiv 1 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil - q & \textit{if } p \equiv 2,3 \pmod 4 \textit{ and } q \textit{ is even}; \\ 2q\lceil \frac{p-1}{4} \rceil - q + 1 & \textit{if } p \equiv 2,3 \pmod 4 \textit{ and } q \textit{ is odd.} \end{cases}$$

Proof. If $p \equiv 0 \pmod 4$, then we get that $\{v_i^j, v_{i+1}^j : i \equiv 2 \pmod 4, 1 \leq j \leq q\}$ is both an efficient total and an efficient paired dominating set of $KD_{p,q}$, and thus $\gamma_t(KD_{p,q}) = \frac{pq}{2} = \gamma_{pr}(KD_{p,q})$.

Let $p \equiv 1 \pmod 4$ and $D = \{v_i^1, v_{i+1}^1 : i \equiv 1 \pmod 4, i \neq p\} \cup \{v_p^1\} \cup \{v_i^j, v_{i+1}^j : i \equiv 3 \pmod 4, 2 \leq j \leq q\}$. Then D is a total dominating set of $KD_{p,q}$ with $|D| = 2q\lceil \frac{p-1}{4} \rceil + 1$. We show that D is a $\gamma_t(KD_{p,q})$-set. Suppose that D' is a total dominating set with $|D'| < |D|$. Note that among all total dominating sets containing precisely one vertex of $\{v_1^1, v_1^2, \ldots, v_1^q\}$, D is minimum. Thus, either $|\{v_1^1, v_1^2, \ldots, v_1^q\} \cap D'| = 0$ or $|\{v_1^1, v_1^2, \ldots, v_1^q\} \cap D'| \geq 2$. If $|\{v_1^1, v_1^2, \ldots, v_1^q\} \cap D'| = 0$, then we have D' contains, without loss of generality, v_2^j and v_3^j for all $j \in \{1, 2, \ldots, q\}$. Thus, $|D'| \geq 2q + q\gamma_t(P_{p-4}) = 2q + q(\lfloor \frac{p-2}{4} \rfloor + \lfloor \frac{p-1}{4} \rfloor) > 2q\lceil \frac{p-1}{4} \rceil + 1 = |D|$ since $p \equiv 1 \pmod 4$ and $q \geq 2$, a contradiction. If D' contains $l \geq 2$ vertices from $\{v_1^1, v_1^2, \ldots, v_1^q\}$, then $|D'| \geq l + l\gamma_t(P_{p-3}) + (q-l)\gamma_t(P_{p-1}) = l + l(\lfloor \frac{p-1}{4} \rfloor + \lfloor \frac{p}{4} \rfloor) + (q-l)(\lfloor \frac{p+1}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor) > 2q\lceil \frac{p-1}{4} \rceil + 1 = |D|$, a contradiction again. Hence, $\gamma_t(KD_{p,q}) = 2q\lceil \frac{p-1}{4} \rceil + 1$, and we then also get that $\gamma_{pr}(KD_{p,q}) \geq 2q\lceil \frac{p-1}{4} \rceil + 2$. Since $D \cup \{v_{p-1}^1\}$ is a paired dominating set of $KD_{p,q}$ with cardinality $2q\lceil \frac{p-1}{4} \rceil + 2$, we get that $\gamma_{pr}(KD_{p,q}) = 2q\lceil \frac{p-1}{4} \rceil + 2$.

Let $p \equiv 2, 3 \pmod 4$ and $D = \{v_1^j : 1 \leq j \leq q\} \cup \{v_i^j, v_{i+1}^j : i \equiv 0 \pmod 4, 1 \leq j \leq q\}$. Then D is a total dominating set of $KD_{p,q}$ with $|D| = 2q\lceil \frac{p-1}{4} \rceil - q$. We show that D is a $\gamma_t(KD_{p,q})$-set. Suppose that D' is a total dominating set with $|D'| < |D|$. Among all total dominating sets containing all vertices $v_1^1, v_1^2, \ldots, v_1^q$, the set D is minimum. Then D' contains $l \leq q - 1$ vertices from $\{v_1^1, v_1^2, \ldots, v_1^q\}$, and thus $|D'| \geq l + l\gamma_t(P_{p-3}) + (q-l)\gamma_t(P_{p-1}) = l + l(\lfloor \frac{p-1}{4} \rfloor + \lfloor \frac{p}{4} \rfloor) + (q-l)(\lfloor \frac{p+1}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor) \geq 2q\lceil \frac{p-1}{4} \rceil - q = |D|$, a contradiction. Hence, $\gamma_t(KD_{p,q}) = 2q\lceil \frac{p-1}{4} \rceil - q$. We also get that $\gamma_{pr}(KD_{p,q}) \geq 2q\lceil \frac{p-1}{4} \rceil - q$ if q is even, and $\gamma_{pr}(KD_{p,q}) \geq 2q\lceil \frac{p-1}{4} \rceil - q + 1$ if q is odd. If q is even, then D is a paired dominating set of $KD_{p,q}$. If q is odd, then $D \cup \{v_2^1\}$ is a paired dominating set of $KD_{p,q}$. The theorem follows. □

Note that $CD_{p,1} \cong C_p$ for all $p \geq 3$. Lemmas 3.0.2 and 3.0.3 provide $\gamma_t(CD_{p,1})$ and $\gamma_{pr}(CD_{p,1})$, respectively. For any integer $q \geq 2$, we examine the exact values of $\gamma_t(CD_{p,q})$ and $\gamma_{pr}(CD_{p,q})$ in the next theorem. For each $j \in \{1, 2, \ldots, q\}$, let $C^j = \{v_i^j \in V(CD_{p,q}) : 1 \leq i \leq p\}$.

Theorem 3.5.9. *Let $p \geq 3$ and $q \geq 2$ be integers. Then*

$$\gamma_t(CD_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + \lceil \frac{q}{3} \rceil & \text{if } p \equiv 1 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil - q & \text{if } p \equiv 2, 3 \pmod 4; \end{cases}$$

and

$$\gamma_{pr}(CD_{p,q}) = \begin{cases} \frac{pq}{2} & \text{if } p \equiv 0 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil + 2\lceil \frac{q}{4} \rceil & \text{if } p \equiv 1 \pmod 4; \\ 2q\lceil \frac{p-1}{4} \rceil - q & \text{if } p \equiv 2, 3 \pmod 4 \text{ and } q \text{ is even}; \\ 2q\lceil \frac{p-1}{4} \rceil - q + 1 & \text{if } p \equiv 2, 3 \pmod 4 \text{ and } q \text{ is odd}. \end{cases}$$

Proof. If $p \equiv 0 \pmod 4$, then $\{v_i^j, v_{i+1}^j : i \equiv 2 \pmod 4, 1 \leq j \leq q\}$ is both an efficient total and an efficient paired dominating set of $CD_{p,q}$, so we obtain that $\gamma_t(CD_{p,q}) = \frac{pq}{2} = \gamma_{pr}(CD_{p,q})$.

Let $p = 4k + 1$ for some $k \geq 1$ and D be a $\gamma_t(CD_{p,q})$-set. Let $q = 2$. If $v_1^1, v_1^2 \in D$, then $|\{v_2^1, v_3^1, \ldots, v_p^1\} \cap D| = |\{v_2^2, v_3^2, \ldots, v_p^2\} \cap D| \geq 2k$. If $v_1^1, v_1^2 \notin D$, then $|\{v_2^1, v_3^1, \ldots, v_p^1\} \cap D| = |\{v_2^2, v_3^2, \ldots, v_p^2\} \cap D| \geq 2k + 1$. In both cases, $|D| \geq 4k + 2$. We observe that $\{v_1^1\} \cup \{v_{4l+2}^1, v_{4l+3}^1 : 0 \leq l \leq k-1\} \cup \{v_{4l+3}^2, v_{4l+4}^2 : 0 \leq l \leq k-1\}$ is a total dominating set having cardinality $4k+1 < 4k+2 \leq |D|$, a contradiction. It implies that D contains only one vertex of v_1^1 and v_1^2. Without loss of generality, let $v_1^1, v_2^1 \in D$. We can check that D contains at least $2k - 1$ vertices to dominate $v_4^1, v_5^1, \ldots, v_{p-1}^1$ and at least $2k$ vertices to dominate $v_2^2, v_3^2, \ldots, v_p^2$. Hence, $|D| \geq 2 + (2k-1) + 2k = 4k + 1 = 2q\lceil \frac{p-1}{4} \rceil + \lceil \frac{q}{3} \rceil$. Next, we let $q \geq 3$. If $v_1^{j-1}, v_1^j, v_1^{j+1} \notin D$ for some $j \in \{1, 2, \ldots, q\}$, then $|\{v_2^j, v_3^j, \ldots, v_p^j\} \cap D| = 2k + 1$ and $D' = (D \setminus C^j) \cup \{v_1^j\} \cup \{v_{4l+2}^j, v_{4l+3}^j : 0 \leq l \leq k - 1\}$ is a total dominating set of $CD_{p,q}$ with $|D'| = |D|$, so we can assume that, for any three consecutive vertices of $\{v_1^1, v_1^2, \ldots, v_1^q\}$, D contains at least one vertex. This implies that D contains at least $\lceil \frac{q}{3} \rceil$ vertices from $\{v_1^1, v_1^2, \ldots, v_1^q\}$. If $v_1^j \in D$ for some $j \in \{1, 2, \ldots, q\}$, then $|\{v_2^j, v_3^j, \ldots, v_p^j\} \cap D| \geq 2k$. If $v_1^j \notin D$ for some $j \in \{1, 2, \ldots, q\}$, then it is dominated by v_1^{j-1} or v_1^{j+1}, so $|\{v_2^j, v_3^j, \ldots, v_p^j\} \cap D| \geq 2k$. Therefore, $|D| \geq \lceil \frac{q}{3} \rceil + q(2k) = 2q\lceil \frac{4k}{4} \rceil + \lceil \frac{q}{3} \rceil = 2q\lceil \frac{p-1}{4} \rceil + \lceil \frac{q}{3} \rceil$. It is easy to verify that $\{v_1^j : j \equiv 1 \pmod 3\} \cup \{v_i^j, v_{i+1}^j : i \equiv 2 \pmod 4, j \equiv 1 \pmod 3\} \cup \{v_i^j, v_{i+1}^j : i \equiv 3 \pmod 4, j \equiv 0, 2 \pmod 3\}$ is a total dominating set of $CD_{p,q}$ with cardinality $2q\lceil \frac{p-1}{4} \rceil + \lceil \frac{q}{3} \rceil$.

Next, we determine $\gamma_{pr}(CD_{p,q})$ for $p = 4k+1$. If $q = 2$, then $\gamma_{pr}(CD_{p,q}) \geq 2q\lceil \frac{p-1}{4} \rceil + \lceil \frac{q}{3} \rceil + 1 = 2q\lceil \frac{p-1}{4} \rceil + 2\lceil \frac{q}{4} \rceil$ since $\gamma_{pr}(CD_{p,q}) \geq \gamma_t(CD_{p,q})$ and $\gamma_{pr}(CD_{p,q})$ is even. Let $q \geq 3$ and D be a $\gamma_{pr}(CD_{p,q})$-set. If v_1^j is not dominated by v_1^{j-1} and v_1^{j+1} for some $j \in \{1, 2, \ldots, q\}$, then $|C^j \cap D| = 2k+2$. Then $D' = (D \setminus C^j) \cup \{v_1^j, v_1^{j+1}\} \cup \{v_{4l+3}^j, v_{4l+4}^j : 0 \leq l \leq k-1\}$ is a paired dominating set with $|D'| = |D|$, so we can assume that every vertex of $\{v_1^1, v_1^2, \ldots, v_1^q\}$ is dominated by some vertex in the same set. If $v_1^j \in D$ is paired with either v_2^j or v_p^j for some $j \in \{1, 2, \ldots, q\}$, then $|\{v_2^j, v_3^j, \ldots, v_p^j\} \cap D| = 2k+1$. The vertices v_1^{j-1} and v_1^{j+1} are not in D; otherwise, $(D \setminus C^j) \cup \{v_{4l+3}^j, v_{4l+4}^j : 0 \leq l \leq k-1\}$ is a paired dominating set with cardinality less than D, a contradiction. Hence, $D' = (D \setminus \{v_2^j, v_3^j, \ldots, v_p^j\}) \cup \{v_1^{j+1}\} \cup \{v_{4l+3}^j, v_{4l+4}^j : 0 \leq l \leq k-1\}$ is a paired dominating set with $|D'| = |D|$, so we can also assume that the vertices in $\{v_1^1, v_1^2, \ldots, v_1^q\} \cap D$ are paired. Note that any two adjacent vertices of $\{v_1^1, v_1^2, \ldots, v_1^q\} \cap D$ can dominate at most four vertices in $\{v_1^1, v_1^2, \ldots, v_1^q\}$, so $|\{v_1^1, v_1^2, \ldots, v_1^q\} \cap D| \geq 2\lceil \frac{q}{4} \rceil$. If v_1^j and v_1^{j+1} are paired in D, then $|\{v_2^j, v_3^j, \ldots, v_p^j\} \cap D| = |\{v_2^{j+1}, v_3^{j+1}, \ldots, v_p^{j+1}\} \cap D| \geq 2k$. If $v_1^j \notin D$, then $|\{v_2^j, v_3^j, \ldots, v_p^j\} \cap D| \geq 2k$. Therefore, $|D| \geq 2\lceil \frac{q}{4} \rceil + q(2k) = 2q\lceil \frac{p-1}{4} \rceil + 2\lceil \frac{q}{4} \rceil$. Let $E = \{v_1^j : j \equiv 1, 2 \pmod 4\} \cup \{v_i^j, v_{i+1}^j : i \equiv 3 \pmod 4, 1 \leq j \leq q\}$. Then E is a paired dominating set of $CD_{p,q}$ with cardinality $2q\lceil \frac{p-1}{4} \rceil + 2\lceil \frac{q}{4} \rceil$ If $q \equiv 0, 2, 3 \pmod 4$, and $E \cup \{v_1^{q-1}\}$ is a paired dominating set of $CD_{p,q}$ with cardinality $2q\lceil \frac{p-1}{4} \rceil + 2\lceil \frac{q}{4} \rceil$ if $q \equiv 1 \pmod 4$, so we are done.

Let $p \equiv 2, 3 \pmod 4$ and $D = \{v_1^j : 1 \leq j \leq q\} \cup \{v_i^j, v_{i+1}^j : i \equiv 0 \pmod 4, 1 \leq j \leq q\}$. Then D is a total dominating set of $CD_{p,q}$, so $\gamma_t(CD_{p,q}) \leq 2q\lceil \frac{p-1}{4} \rceil - q$. If q is even, then D is a paired dominating set of $CD_{p,q}$. If q is odd, then $D \cup \{v_2^1\}$ is a paired dominating set of $CD_{p,q}$. Thus, $\gamma_{pr}(CD_{p,q}) \leq 2q\lceil \frac{p-1}{4} \rceil - q$ if q is even, and $\gamma_{pr}(CD_{p,q}) \leq 2q\lceil \frac{p-1}{4} \rceil - q + 1$ if q is odd. Note that every total dominating set of $CD_{p,q}$ is a total dominating set of $KD_{p,q}$, yielding that $\gamma_t(CD_{p,q}) \geq \gamma_t(KD_{p,q})$. Similarly, $\gamma_{pr}(CD_{p,q}) \geq \gamma_{pr}(KD_{p,q})$. We obtain the lower bounds of $\gamma_t(CD_{p,q})$ and $\gamma_{pr}(CD_{p,q})$ by Theorem 3.5.8. This completes the proof. □

3.6 Lollipop Graphs, Umbrella Graphs, and Coconut Graphs

In this section, we begin by defining lollipop graphs, umbrella graphs, and coconut graphs. Then we calculate the total and the paired domination numbers of these three classes of graphs.

Let p and q be both positive integers. A *lollipop graph* $L_{p,q}$ is obtained by appending an endpoint of a path P_p to a vertex of a complete graph K_q. Throughout this dissertation, we refer to the vertices of $L_{p,q}$ as depicted in Figure 3.21.

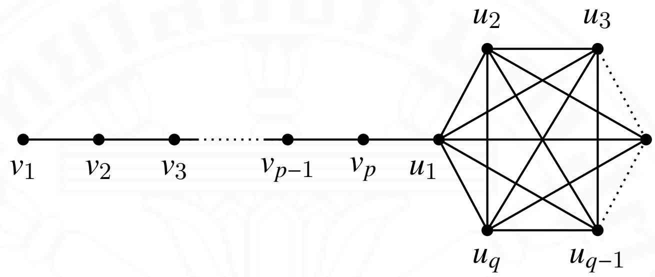

Figure 3.21 The lollipop graph $L_{p,q}$

An *umbrella graph* $U_{p,q}$ is obtained by appending an endpoint of a path P_p to the central vertex of a fan graph $K_1 \vee P_{q-1}$. A *coconut graph* $C_{p,q}$ is obtained by appending an endpoint of a path P_p to the support vertex of a complete bipartite graph $K_{1,q-1}$. We let the vertices of $U_{p,q}$ and $C_{p,q}$ be shown in Figures 3.22 and 3.23, respectively.

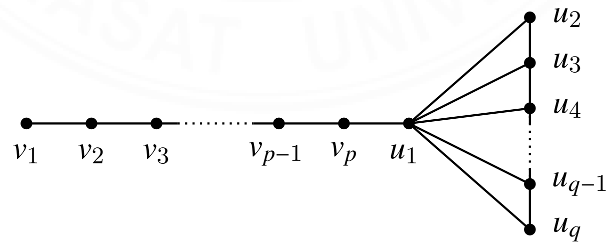

Figure 3.22 The umbrella graph $U_{p,q}$

Note that $L_{p,1} \cong U_{p,1} \cong C_{p,1} \cong P_{p+1}$ for all $p \geq 1$. By Lemmas 3.0.2 and 3.0.3, we have $\gamma_t(L_{p,1}) = \gamma_t(U_{p,1}) = \gamma_t(C_{p,1}) = \lfloor \frac{p+3}{4} \rfloor + \lfloor \frac{p+4}{4} \rfloor$ and $\gamma_{pr}(L_{p,q}) = \gamma_{pr}(U_{p,q}) = \gamma_{pr}(C_{p,q}) = 2\lceil \frac{p+1}{4} \rceil$, respectively. For any integer $q \geq 2$, we obtain the following theorem.

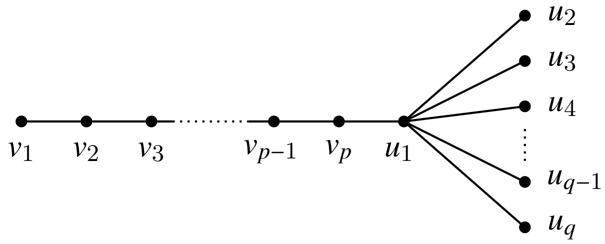

Figure 3.23 The coconut graph $C_{p,q}$

Theorem 3.6.1. *Let $p \geq 1$ and $q \geq 2$ be integers. Then*

1. $\gamma_t(L_{p,q}) = \gamma_t(U_{p,q}) = \gamma_t(C_{p,q}) = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$ *and*

2. $\gamma_{pr}(L_{p,q}) = \gamma_{pr}(U_{p,q}) = \gamma_{pr}(C_{p,q}) = 2\lceil \frac{p+2}{4} \rceil$.

Proof. If $q = 2$, then $L_{p,q} \cong P_{p+2}$, so we get $\gamma_t(L_{p,2}) = \gamma_t(P_{p+2}) = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$ by Lemma 3.0.2. Let $q \geq 3$ and P' be the graph obtained from $L_{p,q}$ by deleting the vertices $u_3, u_4, \ldots, u_q$. Clearly, $P' \cong P_{p+2}$ and then $\gamma_t(P') = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$. Let D be a $\gamma_t(L_{p,q})$-set. We next show that $|D| \geq \gamma_t(P')$. If $u_1 \in D$, then, to dominate u_1, D contains either v_p or, without loss of generality, u_2. In both cases, the set D is a total dominating set of P', and thus $|D| \geq \gamma_t(P')$. Next, we assume that $u_1 \notin D$. Since D is a $\gamma_t(L_{p,q})$-set, D contains exactly two vertices, without loss of generality, u_2 and u_3 from $\{u_2, u_3, \ldots, u_q\}$. Then $D' = (D \setminus \{u_3\}) \cup \{u_1\}$ is a total dominating set of P', and hence $|D| = |D'| \geq \gamma_t(P')$. Therefore, $\gamma_t(L_{p,q}) = |D| \geq \gamma_t(P')$. Note that $U_{p,q}$ and $C_{p,q}$ are spanning subgraphs of $L_{p,q}$, so $\gamma_t(U_{p,q}) \geq \gamma_t(L_{p,q})$ and $\gamma_t(C_{p,q}) \geq \gamma_t(L_{p,q})$. We observe that $E = \{v_i, v_{i+1} : i \equiv 2 \pmod 4, i < p\} \cup \{v_p, u_1\}$ is a total dominating set of $L_{p,q}, U_{p,q}$, and $C_{p,q}$ with $|E| = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor$, and hence we obtain the exact values of $\gamma_t(L_{p,q}), \gamma_t(U_{p,q})$, and $\gamma_t(C_{p,q})$.

We can easily check that $\gamma_{pr}(L_{p,q}) \geq \gamma_t(L_{p,q}) = \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor = 2\lceil \frac{p+2}{4} \rceil$ for $p \equiv 0, 1, 2 \pmod 4$. If $p \equiv 3 \pmod 4$, then $\gamma_{pr}(L_{p,q}) \geq \lfloor \frac{p+4}{4} \rfloor + \lfloor \frac{p+5}{4} \rfloor + 1 = 2\lceil \frac{p+2}{4} \rceil$ since $\gamma_{pr}(L_{p,q})$ is even. Similarly, $\gamma_{pr}(U_{p,q}) \geq 2\lceil \frac{p+2}{4} \rceil$ and $\gamma_{pr}(C_{p,q}) \geq 2\lceil \frac{p+2}{4} \rceil$. If $p \equiv 0, 1, 2 \pmod 4$ (respectively, $p \equiv 3 \pmod 4$), then E (respectively, $E \cup \{v_{p-2}\}$) is a paired dominating set of $L_{p,q}, U_{p,q}$, and $C_{p,q}$ with cardinality $2\lceil \frac{p+2}{4} \rceil$. The theorem follows. □

CHAPTER 4
γ-TOTAL AND γ-PAIRED DOMINATING GRAPHS

In this chapter, we first recall the definitions of γ-total dominating graphs, which were defined by Wongsriya and Trakultraipruk [67], and γ-paired dominating graphs, introduced by Eakawinrujee and Trakultraipruk [15]. We next determine the γ-total and the γ-paired dominating graphs of double stars, complete graphs, complete bipartite graphs, fan graphs, cycles, and some classes of graphs appearing in Chapter 3, including wheel graphs, helm graphs, flower graphs, lollipop graphs, umbrella graphs, and coconut graphs.

The *γ-total dominating graph* of a graph G, denoted by $TD_\gamma(G)$, is the graph whose vertices are $\gamma_t(G)$-sets, and two vertices D_1 and D_2 of $TD_\gamma(G)$ are adjacent if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. The *γ-paired dominating graph* $PD_\gamma(G)$ of G is defined analogously by using $\gamma_{pr}(G)$-sets as its vertices. Let $P_p = (1, 2, \ldots, p)$ be the path with p vertices. It is easy to check that $\{2, 3\}$ is the only $\gamma_t(P_4)$-set and the only $\gamma_{pr}(P_4)$-set, so $TD_\gamma(P_4) \cong P_1 \cong PD_\gamma(P_4)$. Figures 4.1 and 4.2 show the γ-total dominating graphs and the γ-paired dominating graphs, respectively, of P_5 and P_{10}.

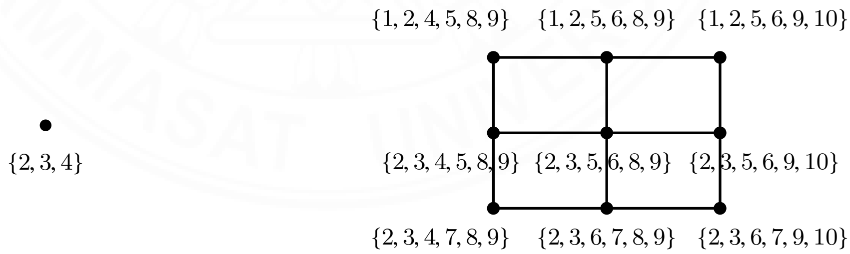

Figure 4.1 The γ-total dominating graph of P_5 (left) and P_{10} (right)

4.1 Double Stars, Complete Graphs, Complete Bipartite Graphs, and Fan Graphs

Note that the two support vertices of a double star $S_{p,q}$ form the only $\gamma_t(S_{p,q})$-set and the only $\gamma_{pr}(S_{p,q})$-set, so we get the following theorem immediately.

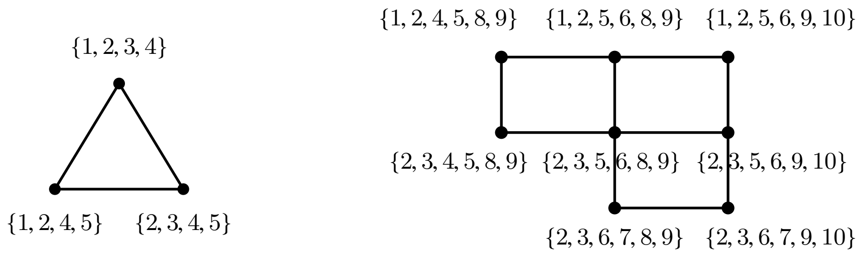

Figure 4.2 The γ-paired dominating graph of P_5 (left) and P_{10} (right)

Theorem 4.1.1. *Let $p \geq 1$ and $q \geq 1$ be integers. Then $TD_\gamma(S_{p,q}) \cong K_1 \cong PD_\gamma(S_{p,q})$.*

The *Johnson graph* $J(p, q)$ is the graph whose vertices correspond to the q-element subsets of $\{1, 2, \ldots, p\}$, where two vertices are adjacent when they meet in a $(q-1)$-element set. Clearly, $J(p, q)$ has $\binom{p}{q}$ vertices. In Figure 4.3, we show the Johnson graph $J(4, 2)$.

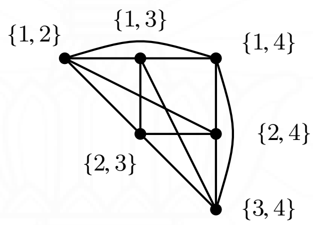

Figure 4.3 The Johnson graph $J(4, 2)$

We recall that K_p is a complete graph with p vertices. Note that $\gamma_t(K_p) = 2 = \gamma_{pr}(K_p)$ for all $p \geq 2$. It follows from the definition that the γ-total and the γ-paired dominating graphs of K_p are both precisely the Johnson graph $J(p, 2)$, as stated in the following theorem.

Theorem 4.1.2. *Let $p \geq 2$ be an integer. Then $TD_\gamma(K_p) \cong J(p, 2) \cong PD_\gamma(K_p)$.*

We then provide the γ-total and the γ-paired dominating graphs of complete bipartite graphs as follows.

Theorem 4.1.3. *Let $p \geq 1$ and $q \geq 1$ be integers. Then $TD_\gamma(K_{p,q}) \cong K_p \square K_q \cong PD_\gamma(K_{p,q})$.*

Proof. Let $K_{p,q}$ be the complete bipartite graph with the partite sets $X = \{u_1, u_2, \ldots, u_p\}$ and $Y = \{v_1, v_2, \ldots, v_q\}$. Note that $\gamma_t(K_{p,q}) = 2 = \gamma_{pr}(K_{p,q})$, and each $\gamma_t(K_{p,q})$-set and each $\gamma_{pr}(K_{p,q})$-set must contain one vertex from X and another one from Y. Therefore, all $\{u_x, v_y\}$'s with $1 \leq x \leq p$ and $1 \leq y \leq q$ are the only $\gamma_t(K_{p,q})$-sets and the only $\gamma_{pr}(K_{p,q})$-sets, and they form the Cartesian product of K_p and K_q (see Figure 4.4). □

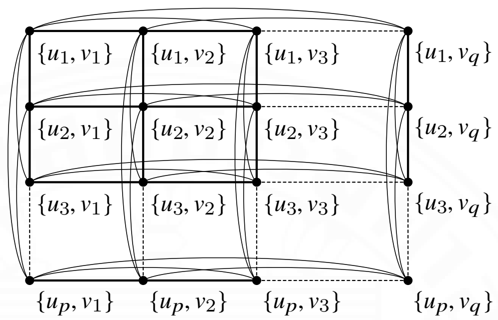

Figure 4.4 The γ-total and γ-paired dominating graphs of $K_{p,q}$

The *fan graph* $F_{p,q}$ is the join $\overline{K_p} \vee P_q$, where $V(\overline{K_p}) = \{u_1, u_2, \ldots, u_p\}$ and $V(P_q) = \{v_1, v_2, \ldots, v_q\}$. If $q = 1$, then $F_{p,q} \cong K_{p,1}$. By Theorem 4.1.3, $TD_\gamma(F_{p,1}) \cong K_p$. Let $q \geq 2$ be an integer. Note that $\gamma_t(F_{p,q}) = 2$, so a $\gamma_t(F_{p,q})$-set is one of the following types:

(i) $\{u_x, v_y\}$ for some $x \in \{1, 2, \ldots, p\}$ and $y \in \{1, 2, \ldots, q\}$;

(ii) $\{v_y, v_{y'}\}$ for some distinct $y, y' \in \{1, 2, \ldots, q\}$.

Note that all $\gamma_t(F_{p,q})$-sets of type (i) form a graph $K_p \square K_q$. We consider a $\gamma_t(F_{p,q})$-set $\{u_x, v_y\}$ in $K_p \square K_q$ as the entry in the row x and the column y.

If $q = 2$, then $\{v_1, v_2\}$ is the only $\gamma_t(F_{p,q})$-set of type (ii), and it is adjacent to every set in $K_p \square K_2$. Thus, $TD_\gamma(F_{p,2}) \cong (K_p \square K_2) \vee K_1$. For $q = 3$, $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are the only $\gamma_t(F_{p,q})$-sets of type (ii). Clearly, $\{v_1, v_2\}$ is adjacent to every set in the first two columns of $K_p \square K_3$, and $\{v_2, v_3\}$ is adjacent to every set in the last two columns of $K_p \square K_3$. If $q = 4$, then $\{v_2, v_3\}$ is the only $\gamma_t(F_{p,q})$-set of type (ii), and it is adjacent to every set in the columns two and three of $K_p \square K_4$. If $q \geq 5$, then there is no $\gamma_t(F_{p,q})$-set of type (ii), so $TD_\gamma(F_{p,q}) \cong K_p \square K_q$ (Theorem 4.1.4). It is easy to check that $PD_\gamma(F_{p,q})$ is the same as $TD_\gamma(F_{p,q})$ for all $p, q \geq 1$.

Theorem 4.1.4. *Let $p \geq 1$ and $q \geq 5$ be integers. Then $TD_\gamma(F_{p,q}) \cong K_p \square K_q \cong PD_\gamma(F_{p,q})$.*

4.2 Wheel Graphs, Helm Graphs, and Flower Graphs

If $p \in \{3, 4\}$, then the γ-total and the γ-paired dominating graphs of a wheel graph W_p are shown in Figure 4.5. For $p \geq 5$, it is clear that $\{c, u_1\}, \{c, u_2\}, \ldots, \{c, u_p\}$ are the only $\gamma_t(W_p)$-sets and the only $\gamma_{pr}(W_p)$-sets, and they form a complete graph with p vertices, so we get the result in Theorem 4.2.1.

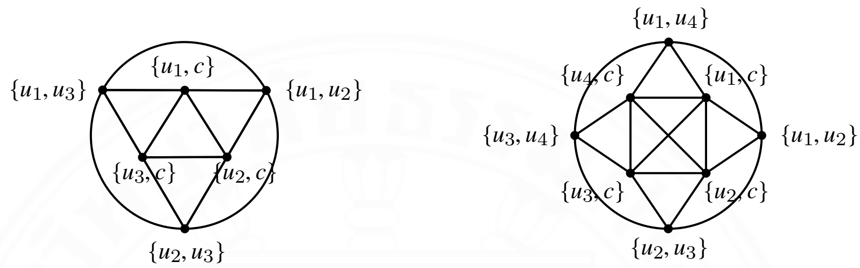

Figure 4.5 The γ-total and γ-paired dominating graphs of W_3 (left) and W_4 (right)

Theorem 4.2.1. *Let $p \geq 5$ be an integer. Then $TD_\gamma(W_p) \cong K_p \cong PD_\gamma(W_p)$.*

Theorem 4.2.2. *Let $p \geq 3$ be an integer. Then*

$$TD_\gamma(H_p) \cong K_1$$

and

$$PD_\gamma(H_p) \cong \begin{cases} K_1 & \text{if } p \text{ is even;} \\ K_{p+1} & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Clearly, the support vertices of H_p form the only one $\gamma_t(H_p)$-set, and they also form the only one $\gamma_{pr}(H_p)$-set if p is even. Hence, $TD_\gamma(H_p) \cong K_1$, and if p is even, then $PD_\gamma(H_p) \cong K_1$. Let p be odd. Then a $\gamma_{pr}(H_p)$-set must contain all p support vertices and one vertex from $\{c, v_1, v_2, \ldots, v_p\}$. Thus, there are exactly $k + 1$ $\gamma_{pr}(H_p)$-sets and they are all adjacent, so $PD_\gamma(H_p) \cong K_{p+1}$. □

We observe that $\gamma_t(Fl_p) = 2 = \gamma_{pr}(Fl_p)$, and each $\gamma_t(Fl_p)$-set and each $\gamma_{pr}(Fl_p)$-set must contain the vertex c and another vertex from $\{u_1, \ldots, u_p, v_1, \ldots, v_p\}$, so we get the following theorem.

Theorem 4.2.3. *Let $p \geq 3$ be an integer. Then $TD_\gamma(Fl_p) \cong K_{2p} \cong PD_\gamma(Fl_p)$.*

4.3 Paths and Cycles

We first review the results on the γ-total dominating graphs of paths and cycles, as well as the γ-paired dominating graphs of paths and their properties. In this section, we mainly determine the γ-paired dominating graphs of cycles.

Wongsriya and Trakultraipruk [67] studied the γ-total dominating graphs of paths and cycles, of which results are provided below.

Theorem 4.3.1 ([67]). *Let $k \geq 1$ be an integer. Then $TD_\gamma(P_{4k}) \cong P_1$.*

Theorem 4.3.2 ([67]). *Let $k \geq 1$ be an integer. Then $TD_\gamma(P_{4k-1}) \cong P_{k+1}$.*

Theorem 4.3.3 ([67]). *Let $k \geq 1$ be an integer. Then $TD_\gamma(P_{4k-2}) \cong P_k \square P_k$.*

Theorem 4.3.4 ([67]). *Let $k \geq 2$ be an integer. Then $TD_\gamma(P_{4k-3}) \cong P_{k-1}$.*

Theorem 4.3.5 ([67]). *Let $k \geq 1$ be an integer. Then* $TD_\gamma(C_{4k}) \cong \begin{cases} C_4 & \text{if } k = 1; \\ 4P_1 & \text{if } k \geq 2. \end{cases}$

Theorem 4.3.6 ([67]). *Let $k \geq 1$ be an integer. Then $TD_\gamma(C_{4k-1}) \cong C_{4k-1}$.*

Theorem 4.3.7 ([67]). *Let $k \geq 2$ be an integer. Then $TD_\gamma(C_{4k-2}) \cong C_{2k-1} \square C_{2k-1}$.*

Theorem 4.3.8 ([67]). *Let $k \geq 2$ be an integer. Then $TD_\gamma(C_{4k-3}) \cong C_{4k-3}$.*

Let $P_p = (u_1, u_2, \ldots, u_p)$ and $P_q = (v_1, v_2, \ldots, v_q)$ be two paths with p and q vertices, respectively. Fricke *et al.* [19] defined a *stepgrid* $SG_{p,q}$ to be the subgraph of $P_p \square P_q$ induced by $\{(u_x, v_y) \in V(P_p \square P_q) : x - y \leq 1\}$. We call the vertex (u_x, v_y) in the stepgrid as the *vertex at the position* (x, y). For example, the stepgrids $SG_{1,1}$, $SG_{2,2}$, and $SG_{4,3}$ are shown in Figure 4.6.

Let $P_p = (u_1, u_2, \ldots, u_p)$, $P_q = (v_1, v_2, \ldots, v_q)$, and $P_r = (w_1, w_2, \ldots, w_r)$ be three paths with p, q, and r vertices, respectively. In [15], a *stepgrid* $SG_{p,q,r}$ is the graph satisfying the following two conditions:

- It is the subgraph of $P_p \square P_q \square P_r$ induced by $\{(u_x, v_y, w_z) \in V(P_p \square P_q \square P_r) : x \leq y, z \leq y, x - z \leq 1\}$.
- It has additional edges $(u_x, v_x, w_x)(u_{x+1}, v_{x+1}, w_x)$ for all $x \in \{1, 2, \ldots, p - 1\}$.

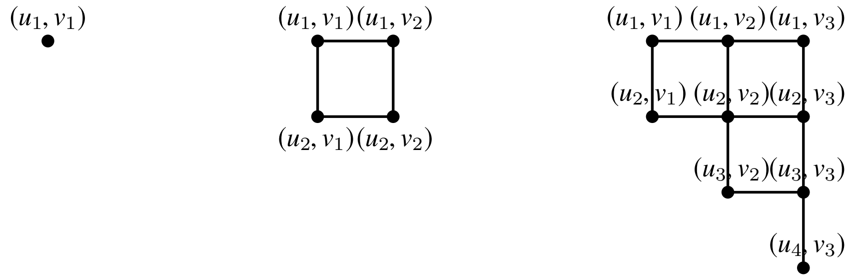

Figure 4.6 The stepgrids $SG_{1,1}$ (left), $SG_{2,2}$ (middle), and $SG_{4,3}$ (right)

The vertex (u_x, v_y, w_z) is called the *vertex at the position* (x, y, z) in $SG(p, q, r)$. For example, the stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$ are shown in Figure 4.7, and the stepgrid $SG_{4,4,3}$ is shown in Figure 4.8, where we write (x, y, z) for (u_x, v_y, w_z).

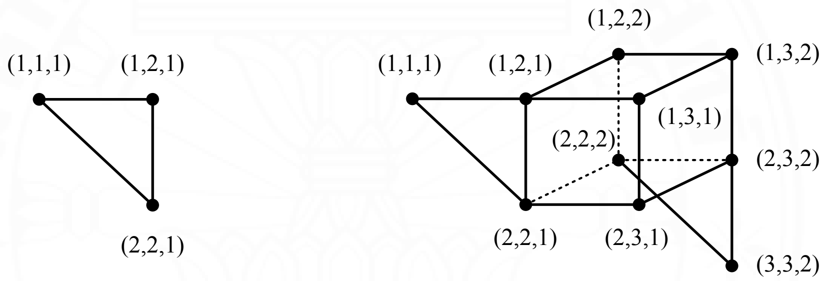

Figure 4.7 The stepgrids $SG_{2,2,1}$ (left) and $SG_{3,3,2}$ (right)

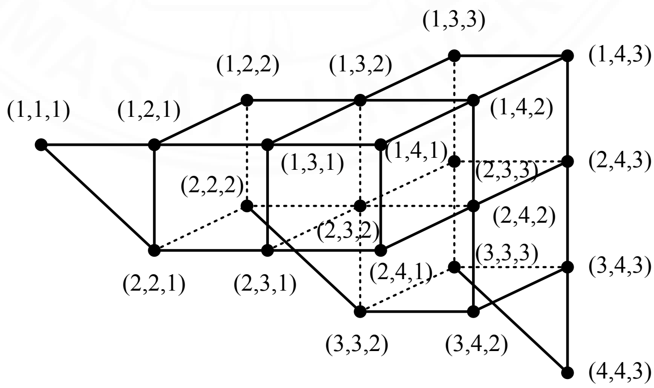

Figure 4.8 The stepgrid $SG_{4,4,3}$

Let $P_p = (v_1, v_2, \ldots, v_p)$ be the path with p vertices. In [15], the authors determined the γ-paired dominating graphs of paths and gave the following results.

Lemma 4.3.9 ([15]). *Let $k \geq 1$ be an integer. Then there is only one $\gamma_{pr}(P_{4k-1})$-set containing the pair $\{v_{4k-2}, v_{4k-1}\}$, and there is only one $\gamma_{pr}(P_{4k-1})$-set containing the pair $\{v_1, v_2\}$.*

Lemma 4.3.10 ([15]). *Let $k \geq 2$ be an integer. Then all $\gamma_{pr}(P_{4k-2})$-sets containing the pair $\{v_{4k-3}, v_{4k-2}\}$ form a path with k vertices, say $A_1, A_2, \ldots, A_k$, in $PD_\gamma(P_{4k-2})$ as an induced subgraph, where A_1 and A_k are of degree two, the others are of degree three, and A_k has a neighbor of degree two. Moreover, A_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and the others contain the pair $\{v_{4k-7}, v_{4k-6}\}$. The similar results also hold for the $\gamma_{pr}(P_{4k-2})$-sets containing the pair $\{v_1, v_2\}$.*

Lemma 4.3.11 ([15]). *Let $k \geq 3$ be an integer. Then all $\gamma_{pr}(P_{4k-3})$-sets containing the pair $\{v_{4k-4}, v_{4k-3}\}$ form a stepgrid $SG_{k,k-1}$ (see Figure 4.9) in $PD_\gamma(P_{4k-3})$ as an induced subgraph, where $B_{1,1}, B_{2,1}, B_{1,k-1}$ are of degree three, $B_{2,k-1}, B_{3,k-1}, \ldots, B_{k-1,k-1}$ are of degree four, and $B_{k,k-1}$ is of degree two. Moreover, $B_{1,k-1}, B_{2,k-1}, \ldots, B_{k-1,k-1}$ contain the pair $\{v_{4k-7}, v_{4k-6}\}$, and $B_{k,k-1}$ contains the pair $\{v_{4k-6}, v_{4k-5}\}$. The similar results also hold for the $\gamma_{pr}(P_{4k-3})$-sets containing the pair $\{v_1, v_2\}$.*

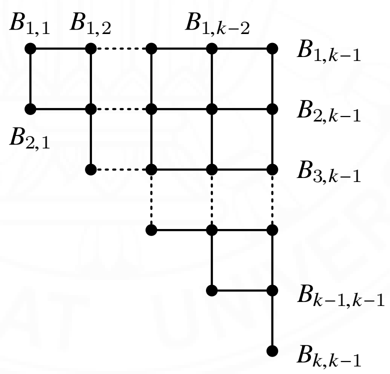

Figure 4.9 The stepgrid $SG_{k,k-1}$

Theorem 4.3.12 ([15]). *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k}) \cong P_1$.*

Theorem 4.3.13 ([15]). *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-1}) \cong P_{k+1}$.*

Theorem 4.3.14 ([15]). *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$.*

Theorem 4.3.15 ([15]). *Let $k \geq 2$ be an integer. Then $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$.*

From the proofs of Lemma 4.3.10 and Theorem 4.3.14, we can derive the following result.

Corollary 4.3.16. *Let $k \geq 2$ be an integer and $A_{x,y}$ the $\gamma_{pr}(P_{4k-2})$-set at the position (x, y) in $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$ (see Figure 4.10) for all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$. If $A_{x,k}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$, then we get the following properties.*

(A1) *If $y = k$, then $A_{x,y}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$; otherwise, it contains the pair $\{v_{4k-4}, v_{4k-3}\}$.*

(A1.1) *$A_{x,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$ for all $x \in \{1, 2, \ldots, k-1\}$, and $A_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$.*

(A2) *If $x = 1$, then $A_{x,y}$ contains the pair $\{v_1, v_2\}$; otherwise, it contains the pair $\{v_2, v_3\}$.*

(A2.1) *$A_{1,1}$ contains the pairs $\{v_1, v_2\}$, $\{v_4, v_5\}$, and $A_{1,y}$ contains the pairs $\{v_1, v_2\}$, $\{v_5, v_6\}$ for all $y \in \{2, 3, \ldots, k\}$.*

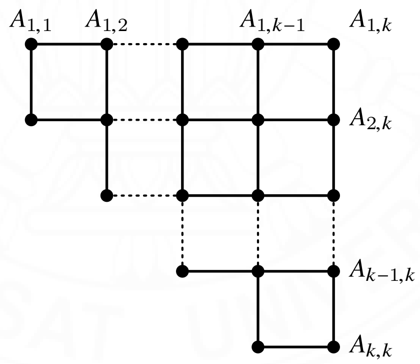

Figure 4.10 The stepgrid $SG_{k,k}$

The following result can be obtained from the proofs of Lemma 4.3.11 and Theorem 4.3.15.

Corollary 4.3.17. *Let $k \geq 3$ be an integer and $B_{x,y,z}$ the $\gamma_{pr}(P_{4k-3})$-set at the position (x, y, z) in $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$ (see Figure 4.11) for all $x, y \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$ with $x \leq y$, $z \leq y$, $x - z \leq 1$. If $B_{x,k,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$, then we get the following properties.*

(B1) *If* $y = k$*, then* $B_{x,y,z}$ *contains the pair* $\{v_{4k-4}, v_{4k-3}\}$*; otherwise, it contains the pair* $\{v_{4k-5}, v_{4k-4}\}$.

(B1.1) $B_{x,k,k-1}$ *contains the pairs* $\{v_{4k-7}, v_{4k-6}\}, \{v_{4k-4}, v_{4k-3}\}$ *for all* $x \in \{1, 2, \ldots, k-1\}$*, and* $B_{k,k,k-1}$ *contains the pairs* $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-4}, v_{4k-3}\}$.

(B1.2) $B_{x,k,z}$ *contains the pairs* $\{v_{4k-8}, v_{4k-7}\}, \{v_{4k-4}, v_{4k-3}\}$ *for all* $z \neq k-1$.

(B2) *If* $x = 1$*, then* $B_{x,y,z}$ *contains the pair* $\{v_1, v_2\}$*; otherwise, it contains the pair* $\{v_2, v_3\}$.

(B2.1) $B_{1,1,1}$ *contains the pairs* $\{v_1, v_2\}, \{v_3, v_4\}$*, and* $B_{1,y,1}$ *contains the pairs* $\{v_1, v_2\}$, $\{v_4, v_5\}$ *for all* $y \in \{2, 3, \ldots, k\}$.

(B2.2) $B_{1,y,z}$ *contains the pairs* $\{v_1, v_2\}, \{v_5, v_6\}$ *for all* $z \neq 1$.

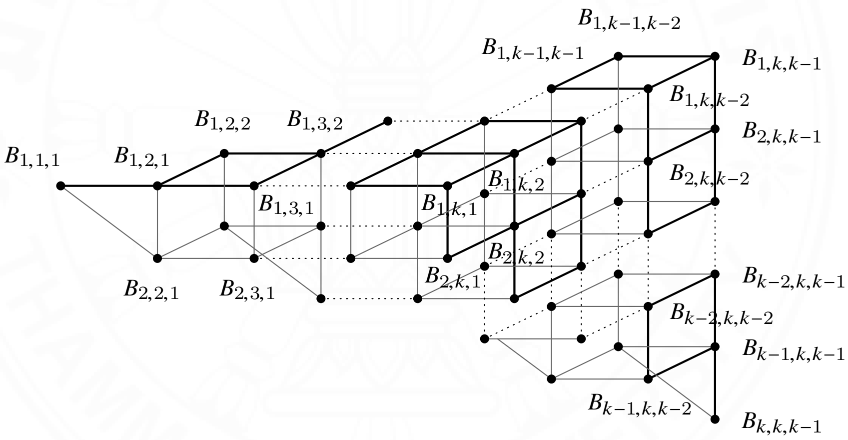

Figure 4.11 The stepgrid $SG_{k,k,k-1}$

We now present the γ-paired dominating graphs of cycles. Throughout this section, we let $C_p = (v_0, v_1, \ldots, v_{p-1})$ to be the cycle with p vertices. We first consider the γ-paired dominating graph of C_{4k}, as provided in the following theorem.

Theorem 4.3.18. *Let* $k \geq 1$ *be an integer. Then*

$$PD_\gamma(C_{4k}) \cong \begin{cases} C_4 & \text{if } k = 1; \\ 4P_1 & \text{if } k \geq 2. \end{cases}$$

Proof. Obviously, $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_0\}$ are the only $\gamma_{pr}(C_4)$-sets, and thus $PD_\gamma(C_4) \cong C_4$. Let $k \geq 2$. By Lemma 3.0.3, we have $\gamma_{pr}(C_{4k}) = 2k$. It is easy to check that $\{v_{4i}, v_{4i+1} : 0 \leq i \leq k-1\}$, $\{v_{4i+1}, v_{4i+2} : 0 \leq i \leq k-1\}$, $\{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\}$, $\{v_{4i+3}, v_{4i+4} : 0 \leq i \leq k-1\}$ are the only $\gamma_{pr}(C_{4k})$-sets, and they are not adjacent. Thus, $PD_\gamma(C_{4k}) \cong 4P_1$. □

Before proving the γ-paired dominating graph of a cycle with $4k+3$ vertices, we need the following lemma.

Lemma 4.3.19. *Let $k \geq 0$ be an integer and D a $\gamma_{pr}(C_{4k+3})$-set. Then there is exaclty one vertex not in D dominated by two vertices of D.*

Proof. We can easily get that the lemma holds for $k = 0$. Let $k \geq 1$. Note that $|D| = 2k+2$, so we can write $D = \bigcup_{x=1}^{k+1} D_x$, where D_x's are pairwise disjoint sets of paired vertices. Clearly, $|N[D_x]| = 4$ for all $x \in \{1, 2, 3, \ldots, k+1\}$, and $V(C_{4k+3}) = \bigcup_{x=1}^{k+1} N[D_x]$. If $N[D_x]$'s are pairwise disjoint sets, then $4k+3 = |V(C_{4k+3})| = \sum_{x=1}^{k+1} |N[D_x]| = 4k+4$, a contradiction. Therefore, there are exactly two disjoint sets, without loss of generality, D_1 and D_2 such that $|N[D_1] \cap N[D_2]| = 1$. Thus, this common vertex is the only vertex not in D dominated by two vertices of D. □

Theorem 4.3.20. *Let $k \geq 0$ be an integer. Then $PD_\gamma(C_{4k+3}) \cong C_{4k+3}$.*

Proof. For convenience, we omit the modulo $4k+3$ in the subscript of each vertex; for example, we write v_{x+1} instead of $v_{(x+1) \pmod{4k+3}}$. For each $x \in \{0, 1, \ldots, 4k+2\}$, let $D_x = \{v_{x+4i+1}, v_{x+4i+2} : 0 \leq i \leq k\}$ as shown in Figure 4.12, where D_x contains the black vertices. It is easy to check that D_x is a $\gamma_{pr}(C_{4k+3})$-set such that $v_x \notin D_x$ is the only vertex dominated by two vertices of D_x. Hence, $D_0, D_1, \ldots, D_{4k+2}$ are all distinct. Similarly, we omit the modulo $4k+3$ in the subscript of each $\gamma_{pr}(C_{4k+3})$-set.

We claim that $D_0, D_1, \ldots, D_{4k+2}$ are the only $\gamma_{pr}(C_{4k+3})$-sets. Let D be any $\gamma_{pr}(C_{4k+3})$-set. By Lemma 4.3.19, there is a unique vertex $v_x \notin D$, for some $x \in \{0, 1, \ldots, 4k+2\}$, dominated by two vertices of D, so $D = D_x$.

Let $x \in \{0, 1, \ldots, 4k+2\}$. To find all neighbors of D_x in $PD_\gamma(C_{4k+3})$, we can only substitute v_{x+1} with v_{x+3}, or v_{x-1} with v_{x-3} since v_x is the only vertex dominated by v_{x+1} and v_{x-1} of D_x. Thus, $(D_x \setminus \{v_{x+1}\}) \cup \{v_{x+3}\}$ and $(D_x \setminus \{v_{x-1}\}) \cup \{v_{x-3}\}$ are the only two neighbors of D_x in $PD_\gamma(C_{4k+3})$. Note that $(D_x \setminus \{v_{x+1}\}) \cup$

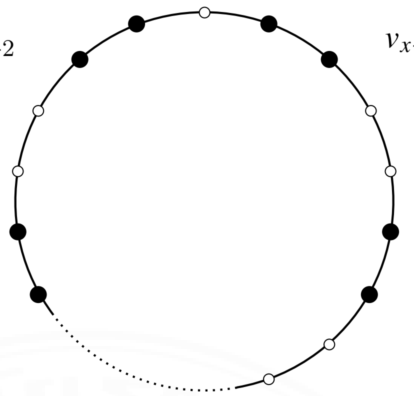

Figure 4.12 The $\gamma_{pr}(C_{4k+3})$-set D_x

$\{v_{x+3}\} = D_{x+4}$ since v_{x+4} is the only vertex dominated by two dominating vertices. Similarly, $(D_x \setminus \{v_{x-1}\}) \cup \{v_{x-3}\} = D_{x-4}$. Therefore, D_0, $D_4, \ldots, D_{4k-4}$, D_{4k}, D_1, $D_5, \ldots, D_{4k-3}, D_{4k+1}, D_2, D_6, \ldots,$ D_{4k-2}, D_{4k+2}, D_3, $D_7, \ldots, D_{4k-1}, D_0$ form a cycle with $4k + 3$ vertices. This completes the proof. □

Before we determine the γ-paired dominating graph of a cycle with $4k + 2$ vertices, we define some notations and a new graph called a *loopgrid*.

Let p and q be positive integers such that $p < q$, and i be a nonnegative integer. Let $P_p(v_i : v_{i+p-1})$ be the subgraph of the cycle C_q induced by the vertices $v_{i \bmod q}, v_{(i+1) \bmod q}, \ldots, v_{(i+p-1) \bmod q}$. Then $P_p(v_i : v_{i+p-1})$ is the path with p vertices.

Let $G_1 = (u_1, u_2, \ldots, u_{2k-1})$ and $G_2 = (v_1, v_2, \ldots, v_{3k-1})$ be two paths with $2k - 1$ and $3k - 1$ vertices, respectively, where k is a positive integer. We define a *loopgrid* of size k, denoted by LG_k, as the graph satisfying the following conditions:

- It is the subgraph of $G_1 \square G_2$ induced by $\{(u_x, v_y) \in V(G_1 \square G_2) : 0 \leq y - x \leq k\}$.
- It has additional edges $(u_1, v_y)(u_{2k-1}, v_{y+2k-1})$ for all $y \in \{1, 2, \ldots, k\}$.

Figure 4.13 illustrates the loopgrids LG_1 and LG_2, where we use (x, y) as (u_x, v_y).

Lemma 4.3.21. *Let $k \geq 2$ be an integer.*

1. *Each $\gamma_{pr}(C_{4k+2})$-set cannot contain any six or more consecutive vertices.*
2. *For any fixed four consecutive vertices in C_{4k+2}, there is exactly one $\gamma_{pr}(C_{4k+2})$-set containing them.*

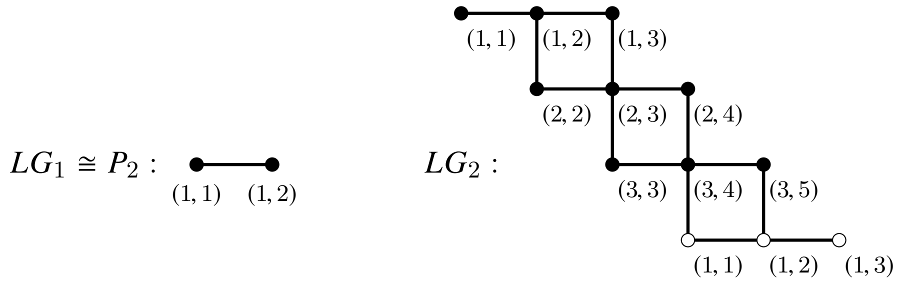

Figure 4.13 The loopgrids LG_1 (left) and LG_2 (right)

Proof. We prove the first claim by a contradiction. Suppose that there is a $\gamma_{pr}(C_{4k+2})$-set D containing $l \geq 6$ consecutive vertices of C_{4k+2}. Without loss of generality, we may assume that these l vertices are $v_1, v_2, \ldots, v_l$. Let D' be the set obtained from D by deleting any two consecutive vertices from $\{v_3, v_4, \ldots, v_{l-2}\}$. Then D' is a paired dominating set with $|D'| < |D|$, a contradiction.

For the second claim, without loss of generality, we assume the four vertices are v_1, v_2, v_3, v_4. We find all $\gamma_{pr}(C_{4k+2})$-sets containing them. By the first claim, all such $\gamma_{pr}(C_{4k+2})$-sets cannot contain v_0 and v_5. The vertices v_1, v_2, v_3, v_4 dominate six vertices in C_{4k+2}. Note that $\gamma_{pr}(C_{4k+2}) = 2k + 2$, so the other $2k - 2$ vertices must dominate all vertices in $P_{4k-4}(v_6 : v_{4k+1})$. Since $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1})) = 2k-2$, these $2k-2$ vertices form a $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$-set. Thus, each $\gamma_{pr}(C_{4k+2})$-set containing v_1, v_2, v_3, v_4 is a union of a $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$-set and $\{v_1, v_2, v_3, v_4\}$. By Theorem 4.3.12, there is a unique $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$-set. The claim follows. □

Theorem 4.3.22. *Let $k \geq 1$ be an integer. Then*

$$PD_\gamma(C_{4k+2}) \cong \begin{cases} C_3 \square C_3 & \text{if } k = 1; \\ LG_{k+1} & \text{if } k \geq 2. \end{cases}$$

Proof. Figure 4.14 shows that $PD_\gamma(C_6) \cong C_3 \square C_3$. Let $k \geq 2$. Since each $\gamma_{pr}(C_{4k+2})$-set must dominate the vertex v_0, we get it contains either the pair $\{v_{4k}, v_{4k+1}\}$, $\{v_{4k+1}, v_0\}$, $\{v_0, v_1\}$, or $\{v_1, v_2\}$. We first find all $\gamma_{pr}(C_{4k+2})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$. By Lemma 4.3.21(1), such a $\gamma_{pr}(C_{4k+2})$-set must satisfy one of the following:

(i) it contains the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0;

(ii) it contains the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k}, v_{4k+1}\}$;

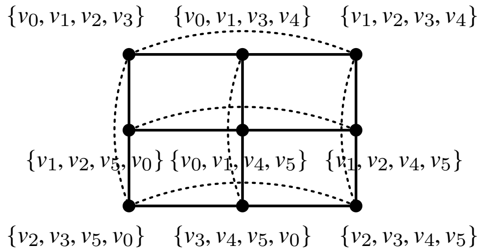

Figure 4.14 The γ-paired dominating graph of C_6

(iii) it contains the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_0, v_1\}$.

Note that each $\gamma_{pr}(C_{4k+2})$-set containing the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0 is a union of a $\gamma_{pr}(P_{4k-2}(v_1 : v_{4k-2}))$-set and $\{v_{4k}, v_{4k+1}\}$. By Theorem 4.3.14, $PD_\gamma(P_{4k-2}(v_1 : v_{4k-2})) \cong SG_{k,k}$. For all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$, let $A^{(1)}_{x,y}$ be the $\gamma_{pr}(P_{4k-2}(v_1 : v_{4k-2}))$-set at the position (x, y) in this stepgrid $SG_{k,k}$, and let

$$D^{(1)}_{x,y} = A^{(1)}_{x,y} \cup \{v_{4k}, v_{4k+1}\}.$$

Thus, $D^{(1)}_{x,y}$'s are the only $\gamma_{pr}(C_{4k+2})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0, and they form a stepgrid $SG_{k,k}$ in $PD_\gamma(C_{4k+2})$. By Lemma 4.3.10, we assume, without loss of generality, that $A_{x,k}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$ for each $x \in \{1, 2, \ldots, k\}$. By Corollary 4.3.16(A1.1), we have $A^{(1)}_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-3}, v_{4k-2}\}$. Let

$$D^{(1)}_{k+1,k} = (D^{(1)}_{k,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k-1}\}.$$

By Lemma 4.3.21(2), $D^{(1)}_{k+1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k}, v_{4k+1}\}$. By Corollary 4.3.16(A2.1), $A^{(1)}_{1,1}$ contains the pairs $\{v_1, v_2\}, \{v_4, v_5\}$. Let

$$D^{(1)}_{1,0} = (D^{(1)}_{1,1} \setminus \{v_2\}) \cup \{v_0\}.$$

By Lemma 4.3.21(2), $D^{(1)}_{1,0}$ is the only $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_0, v_1\}$. Therefore, all $D^{(1)}_{x,y}$'s form the graph, named $D^{(1)}$, in $PD_\gamma(C_{4k+2})$ as shown in Figure 4.15.

Similarly, we can construct all $\gamma_{pr}(C_{4k+2})$-sets as follows (the subscripts of all vertices are modulo $4k + 2$): for all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 0$ and for each

$i \in \{1, 2, 3, 4\}$,

$$D^{(i)}_{x,y} = A^{(i)}_{x,y} \cup \{v_{4k-1+i}, v_{4k+i}\}, \text{ where } A^{(i)}_{x,y} \text{ is a } \gamma_{pr}(P_{4k-2}(v_i : v_{4k-3+i}))\text{-set},$$
$$D^{(i)}_{k+1,k} = (D^{(i)}_{k,k} \setminus \{v_{4k-4+i}\}) \cup \{v_{4k-2+i}\}, \text{ and}$$
$$D^{(i)}_{1,0} = (D^{(i)}_{1,1} \setminus \{v_{i+1}\}) \cup \{v_{i-1}\}.$$

These $D^{(i)}_{x,y}$'s are the only $\gamma_{pr}(C_{4k+2})$-sets containing the pair $\{v_{4k-1+i}, v_{4k+i}\}$, and they form the graph $D^{(i)}$ in $PD_\gamma(C_{4k+2})$ (see Figure 4.15). By Lemma 4.3.10, without loss of generality, we assume $A^{(i)}_{x,k}$ contains the pair $\{v_{4k-4+i}, v_{4k-3+i}\}$. For all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$, we get the following properties.

(A′1) If $y = k$, then $D^{(i)}_{x,y}$ contains the pairs $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$; otherwise, it contains the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$.

(A′1.1) $D^{(i)}_{x,k}$ contains the pairs $\{v_{4k-8+i}, v_{4k-7+i}\}$, $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$ for all $x \in \{1, 2, \ldots, k-1\}$, and $D^{(i)}_{k,k}$ contains the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$.

(A′2) If $x = 1$, then $D^{(i)}_{x,y}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$; otherwise, it contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

(A′2.1) $D^{(i)}_{1,1}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+3}, v_{i+4}\}$, and $D^{(i)}_{1,y}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+4}, v_{i+5}\}$ for all $y \in \{2, 3, \ldots, k\}$.

(A′3) $D^{(i)}_{k+1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(i)}$ containing the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-3+i}, v_{4k-2+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

(A′4) $D^{(i)}_{1,0}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(i)}$ containing the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i-1}, v_i\}$, $\{v_{i+3}, v_{i+4}\}$.

Note that $D^{(1)}$ and $D^{(2)}$ cannot have any common vertices in $PD_\gamma(C_{4k+2})$; otherwise, there is a $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_{4k+1}, v_0\}$, which is impossible. Similarly, $D^{(i)}$ and $D^{(i+1)}$ do not share any vertices in $PD_\gamma(C_{4k+2})$ for all $i \in \{2, 3\}$.

We then consider all $\gamma_{pr}(C_{4k+2})$-sets that are in both $D^{(1)}$ and $D^{(3)}$. Then these sets must contain the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$. By (A′4) and (A′3), $D^{(1)}_{1,0}$ and $D^{(3)}_{k+1,k}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(1)}$ and $D^{(3)}$, respectively, containing the pairs

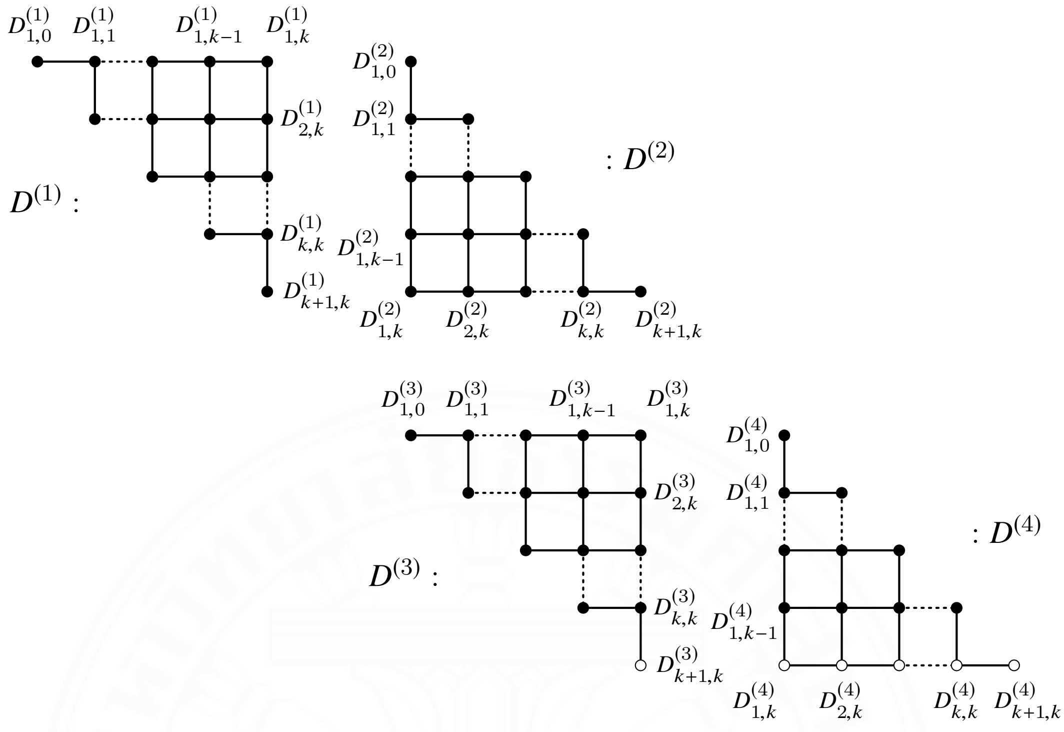

Figure 4.15 The induced subgraphs $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and $D^{(4)}$ in $PD_\gamma(C_{4k+2})$

$\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$. By Lemma 4.3.21(2), we get $D^{(1)}_{1,0} = D^{(3)}_{k+1,k}$. Similarly, $D^{(2)}_{1,0} = D^{(4)}_{k+1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set that is in both $D^{(2)}$ and $D^{(4)}$.

We next consider all $\gamma_{pr}(C_{4k+2})$-sets which are in both $D^{(1)}$ and $D^{(4)}$. These sets must contain the pairs $\{v_{4k}, v_{4k+1}\}, \{v_1, v_2\}$. By (A′2), $D^{(1)}_{1,1}, D^{(1)}_{1,2}, \ldots, D^{(1)}_{1,k}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(1)}$ containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and they form a path with k vertices. Then they also form a path in $D^{(4)}$. By (A′1), $D^{(4)}_{1,k}, D^{(4)}_{2,k}, \ldots, D^{(4)}_{k,k}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(4)}$ containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and they form a path with k vertices. To show that $D^{(1)}_{1,y} = D^{(4)}_{y,k}$ for each $y \in \{1, 2, \ldots, k\}$, it suffices to show that $D^{(1)}_{1,1} = D^{(4)}_{1,k}$. By (A′2.1), $D^{(1)}_{1,1}$ contains the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, $\{v_4, v_5\}$. By (A′1) and (A′2), $D^{(4)}_{1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(4)}$ containing these three pairs, and hence $D^{(1)}_{1,1} = D^{(4)}_{1,k}$.

Next, we consider all edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. We first show that $D^{(1)}_{1,0}$ has no neighbors in $D^{(2)}$. By (A′4), $D^{(1)}_{1,0}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$. Since each set in $D^{(2)}$ contains the pair $\{v_{4k+1}, v_0\}$, the set $D^{(1)}_{1,0}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{1,0} \setminus \{v_{4k}\}) \cup \{v_2\}$ or $(D^{(1)}_{1,0} \setminus \{v_1\}) \cup \{v_{4k-1}\}$ is a $\gamma_{pr}(C_{4k+2})$-set. It is easy to check that $D^{(1)}_{1,0}$ is not adjacent to any sets in $D^{(2)}$. By (A′3), $D^{(1)}_{k+1,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$, so $D^{(1)}_{k+1,k}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{k+1,k} \setminus \{v_{4k}\}) \cup \{v_0\}$ or $(D^{(1)}_{k+1,k} \setminus \{v_{4k-2}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$-set. We have $(D^{(1)}_{k+1,k} \setminus \{v_{4k}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$-set, but $(D^{(1)}_{k+1,k} \setminus \{v_{4k-2}\}) \cup \{v_0\}$ is not. We show that $D^{(1)}_{k+1,k}$ and $D^{(2)}_{1,k}$ are adjacent, i.e., $(D^{(2)}_{1,k} \setminus \{v_0\}) \cup \{v_{4k}\} = D^{(1)}_{k+1,k}$. By (A′1) and (A′2), $D^{(2)}_{1,k}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$, so $(D^{(2)}_{1,k} \setminus \{v_0\}) \cup \{v_{4k}\}$ is a $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$. Since $D^{(1)}_{k+1,k}$ also contains these two pairs, $(D^{(2)}_{1,k} \setminus \{v_0\}) \cup \{v_{4k}\} = D^{(1)}_{k+1,k}$ by Lemma 4.3.21(2).

We next find all neighbors in $D^{(2)}$ of the other $\gamma_{pr}(C_{4k+2})$-sets in $D^{(1)}$. We show that $D^{(1)}_{x,k}$ is adjacent to $D^{(2)}_{1,x-1}$ for all $x \in \{1, 2, \ldots, k\}$. Recall that, for all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$, $D^{(1)}_{x,y}$ contains the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0. Note that $D^{(1)}_{x,y}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{x,y} \setminus \{v_{4k}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$-set. By (A′1), if $y \neq k$, then $D^{(1)}_{x,y}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, so $(D^{(1)}_{x,y} \setminus \{v_{4k}\}) \cup \{v_0\}$ is not a $\gamma_{pr}(C_{4k+2})$-set. By (A′1) and (A′2), the set $D^{(1)}_{1,k}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and $D^{(1)}_{2,k}, D^{(1)}_{3,k}, \ldots, D^{(1)}_{k,k}$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$. For each $x \in \{1, 2, \ldots, k\}$, let $D_x = (D^{(1)}_{x,k} \setminus \{v_{4k}\}) \cup \{v_0\}$, so D_x is a $\gamma_{pr}(C_{4k+2})$-set, and these D_x's form a path with k vertices in $D^{(2)}$. Note that D_1 contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_1, v_2\}$, and $D_2, D_3, \ldots, D_k$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$. By (A′4), the set $D^{(2)}_{1,0}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_1, v_2\}$, and by (A′1) and (A′2), $D^{(2)}_{1,1}, D^{(2)}_{1,2}, \ldots, D^{(2)}_{1,k-1}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$, and they also form a path with k vertices in $D^{(2)}$. Then we conclude that, for all $x \in \{1, 2, \ldots, k\}$, $D_x = D^{(2)}_{1,x-1}$, implying that $D^{(1)}_{x,k}$ is adjacent to $D^{(2)}_{1,x-1}$. To sum up, $D^{(1)}_{x,k}$ is adjacent to $D^{(2)}_{1,x-1}$ for all $x \in \{1, 2, \ldots, k+1\}$. Likewise, for all $i \in \{2, 3\}$, we get $D^{(i)}_{x,k}$ is adjacent to $D^{(i+1)}_{1,x-1}$ for all $x \in \{1, 2, \ldots, k+1\}$.

Now, all $\gamma_{pr}(C_{4k+2})$-sets and edges form a loopgrid LG_{k+1} in $PD_\gamma(C_{4k+2})$. Then we only need to show that there is no more edge in $PD_\gamma(C_{4k+2})$. We first consider all edges between a set in $\widehat{D}^{(1)} = D^{(1)} - D^{(1)}_{1,0}$ and a set in $\widehat{D}^{(3)} = D^{(3)} - D^{(3)}_{k+1,k}$ since $D^{(1)}_{1,0} = D^{(3)}_{k+1,k}$. Note that each set in $\widehat{D}^{(1)}$ contains either the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, or the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$, while every set in $\widehat{D}^{(3)}$ contains the pair $\{v_0, v_1\}$ but not $\{v_{4k}, v_{4k+1}\}$. Hence, there is no edge between a set in $\widehat{D}^{(1)}$ and a set in $\widehat{D}^{(3)}$. Similarly, there is no edge between a set in $D^{(2)} - D^{(2)}_{1,0}$ and a set in $D^{(4)} - D^{(4)}_{k+1,k}$. Recall that

$D^{(1)}_{1,y} = D^{(4)}_{y,k}$ for all $y \in \{1, 2, \ldots, k\}$. Also, $D^{(1)}_{1,0} = D^{(3)}_{k+1,k}$ has a neighbor in $D^{(4)}$. Thus, we consider all edges between a set in $\widetilde{D}^{(1)} = D^{(1)} - \{D^{(1)}_{1,y} : 0 \le y \le k\}$ and a set in $\widetilde{D}^{(4)} = D^{(4)} - \{D^{(4)}_{y,k} : 1 \le y \le k\}$. Note that each set in $\widetilde{D}^{(1)}$ contains the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$, while each set in $\widetilde{D}^{(4)}$ contains the pair $\{v_1, v_2\}$ but not $\{v_{4k}, v_{4k+1}\}$. Hence, there is no edge between a set in $\widetilde{D}^{(1)}$ and a set in $\widetilde{D}^{(4)}$. This completes the proof. □

For any positive integer k, let $G_1 = (u_1, u_2, \ldots, u_{2k})$, $G_2 = (v_1, v_2, \ldots, v_{2k})$, and $G_3 = (w_1, w_2, \ldots, w_{2k+1})$ be three paths with $2k$, $2k$, and $2k + 1$ vertices, respectively. We next define a *loopbox* LB_k of size k as the graph satisfying the following conditions:

- It is the subgraph of $G_1 \square G_2 \square G_3$ induced by $\{(u_x, v_y, w_z) \in V(G_1 \square G_2 \square G_3) : 0 \le y - x \le k, -1 \le y - z \le k - 1, 0 \le z - x \le k\}$.
- It has additional edges $(u_1, v_1, w_1)(u_{k+1}, v_{2k}, w_{k+1})$, $(u_1, v_k, w_{k+1})(u_{2k}, v_{2k}, w_{2k+1})$, $(u_x, v_{x+k-1}, w_x)(u_x, v_{x+k}, w_{x+1})$ for all $x \in \{1, 2, \ldots, k\}$, $(u_x, v_{x+k}, w_{x+k})(u_{x+1}, v_{x+k}, w_{x+k+1})$ for all $x \in \{1, 2, \ldots, k\}$, $(u_x, v_x, w_{x+1})(u_{x+1}, v_{x+1}, w_{x+1})$ for all $x \in \{1, 2, \ldots, 2k - 1\}$, and $(u_1, v_y, w_z)(u_{z+k}, v_{2k}, w_{y+k+1})$ for all $-1 \le y - z \le k - 1$.

For example, the loopboxes of size 1, 2 and 3 are shown in Figures 4.16, 4.17, and 4.18, respectively, where we write (x, y, z) instead of (u_x, v_y, w_z).

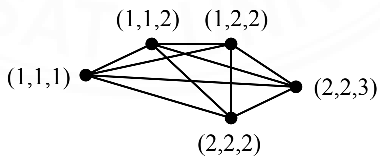

Figure 4.16 The loopbox LB_1 of size 1

Lemma 4.3.23. *Let $k \ge 2$ be an integer.*

1. *Each $\gamma_{pr}(C_{4k+1})$-set cannot contain any six or more consecutive vertices.*
2. *For any fixed four consecutive vertices in C_{4k+1}, there are k $\gamma_{pr}(C_{4k+1})$-sets that contain them.*

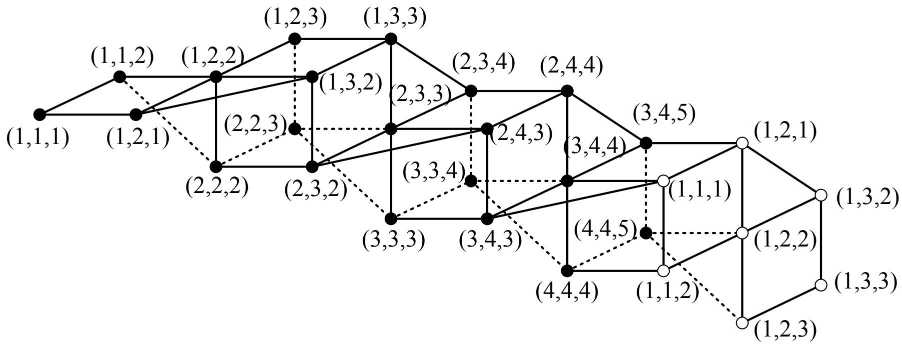

Figure 4.17 The loopbox LB_2 of size 2

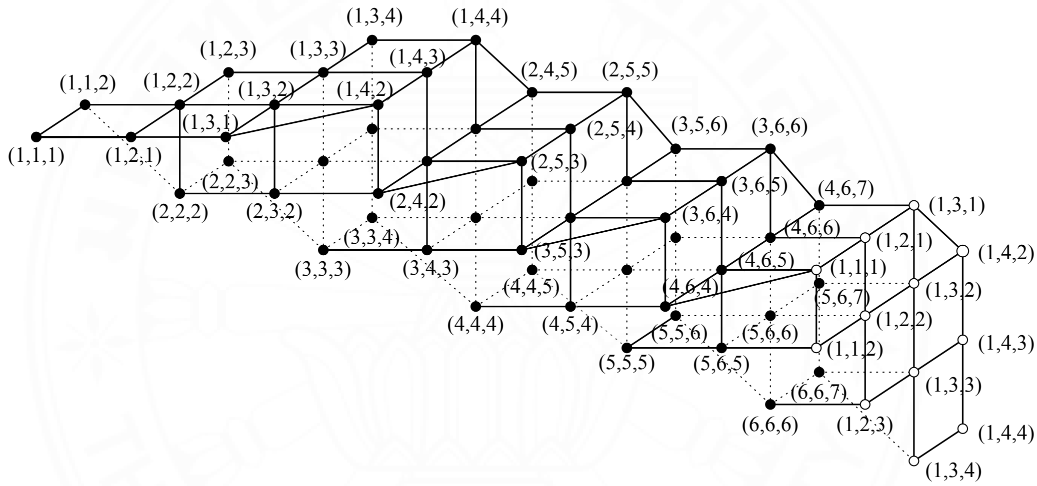

Figure 4.18 The loopbox LB_3 of size 3

Proof. Similar to Lemma 4.3.21(1), we can easily prove the first claim. Next, without loss of generality, we assume the four vertices are v_1, v_2, v_3, v_4. Then these four vertices dominate six vertices in C_{4k+1}. Note that $\gamma_{pr}(C_{4k+1}) = 2k+2$, so the other $2k-2$ vertices must dominate all vertices in $P_{4k-5}(v_6 : v_{4k})$. Since $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k})) = 2k-2$, these $2k-2$ vertices form a $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$-set. Hence, each such $\gamma_{pr}(C_{4k+1})$-set is a union of a $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$-set and $\{v_1, v_2, v_3, v_4\}$. By Theorem 4.3.13, there are k $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$-sets, so the claim follows. □

Theorem 4.3.24. *Let $k \geq 1$ be an integer. Then $PD_\gamma(C_{4k+1}) \cong LB_k$.*

Proof. We can check that $\{v_0, v_1, v_2, v_3\}$, $\{v_1, v_2, v_3, v_4\}$, $\{v_2, v_3, v_4, v_0\}$, $\{v_3, v_4, v_0, v_1\}$, and $\{v_4, v_0, v_1, v_2\}$ are the only $\gamma_{pr}(C_5)$-sets and they are all adjacent, so $PD_\gamma(C_5) \cong$

$K_5 \cong LB_1$. For $k = 2$, we have $PD_\gamma(C_9) \cong LB_2$ (see Figure 4.17), where

$(1,1,1) = \{v_0, v_1, v_2, v_3, v_5, v_6\}$, $(1,2,1) = \{v_0, v_1, v_2, v_3, v_6, v_7\}$,

$(1,1,2) = \{v_0, v_1, v_3, v_4, v_5, v_6\}$, $(1,2,2) = \{v_0, v_1, v_3, v_4, v_6, v_7\}$,

$(1,2,3) = \{v_0, v_1, v_3, v_4, v_7, v_8\}$, $(2,2,2) = \{v_0, v_1, v_4, v_5, v_6, v_7\}$,

$(2,2,3) = \{v_0, v_1, v_4, v_5, v_7, v_8\}$, $(1,3,2) = \{v_1, v_2, v_3, v_4, v_6, v_7\}$,

$(1,3,3) = \{v_1, v_2, v_3, v_4, v_7, v_8\}$, $(2,3,2) = \{v_1, v_2, v_4, v_5, v_6, v_7\}$,

$(2,3,3) = \{v_1, v_2, v_4, v_5, v_7, v_8\}$, $(3,3,3) = \{v_1, v_2, v_4, v_5, v_8, v_0\}$,

$(2,4,3) = \{v_1, v_2, v_5, v_6, v_7, v_8\}$, $(3,4,3) = \{v_1, v_2, v_5, v_6, v_8, v_0\}$,

$(2,3,4) = \{v_2, v_3, v_4, v_5, v_7, v_8\}$, $(3,3,4) = \{v_2, v_3, v_4, v_5, v_8, v_0\}$,

$(2,4,4) = \{v_2, v_3, v_5, v_6, v_7, v_8\}$, $(3,4,4) = \{v_2, v_3, v_5, v_6, v_8, v_0\}$,

$(3,4,5) = \{v_2, v_3, v_6, v_7, v_8, v_0\}$, $(4,4,4) = \{v_3, v_4, v_5, v_6, v_8, v_0\}$,

$(4,4,5) = \{v_3, v_4, v_6, v_7, v_8, v_0\}$.

Let $k \geq 3$. Since each $\gamma_{pr}(C_{4k+1})$-set must dominate the vertex v_0, it contains either the pair $\{v_{4k-1}, v_{4k}\}$, $\{v_{4k}, v_0\}$, $\{v_0, v_1\}$, or $\{v_1, v_2\}$. We first find all $\gamma_{pr}(C_{4k+1})$-sets containing the pair $\{v_{4k-1}, v_{4k}\}$. By Lemma 4.3.23(1), we get that such a $\gamma_{pr}(C_{4k+1})$-set must satisfy one of the following:

(i) it contains the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0;

(ii) it contains the pairs $\{v_{4k-3}, v_{4k-2}\}$ and $\{v_{4k-1}, v_{4k}\}$;

(iii) it contains the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_0, v_1\}$.

Note that each $\gamma_{pr}(C_{4k+1})$-set containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0 is a union of a $\gamma_{pr}(P_{4k-3}(v_1 : v_{4k-3}))$-set and $\{v_{4k-1}, v_{4k}\}$. By Theorem 4.3.15, $PD_\gamma(P_{4k-3}(v_1 : v_{4k-3})) \cong SG_{k,k,k-1}$. For all $x, y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k-1\}$ with $x \leq y$, $z \leq y$, $x - z \leq 1$, let $B^{(1)}_{x,y,z}$ be the $\gamma_{pr}(P_{4k-3}(v_1 : v_{4k-3}))$-set at the position (x, y, z) in $SG_{k,k,k-1}$, and let

$$D^{(1)}_{x,y,z} = B^{(1)}_{x,y,z} \cup \{v_{4k-1}, v_{4k}\}.$$

Therefore, $D^{(1)}_{x,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0, and they also form a stepgrid $SG_{k,k,k-1}$ in $PD_\gamma(C_{4k+1})$. By Lemma 4.3.11, without loss of generality, we may assume that $B^{(1)}_{x,k,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. By Corollary 4.3.17(B1.1), the set $B^{(1)}_{x,k,k-1}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $B^{(1)}_{k,k,k-1}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. For each $x \in \{1, 2, \dots, k\}$, let

$$D^{(1)}_{x,k,k} = (D^{(1)}_{x,k,k-1} \setminus \{v_{4k-4}\}) \cup \{v_{4k-2}\}.$$

By Lemma 4.3.23(2), these $D^{(1)}_{x,k,k}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$. By Corollary 4.3.17(B2.1), the set $B^{(1)}_{1,1,1}$ contains the pairs $\{v_1, v_2\}$, $\{v_3, v_4\}$, and $B^{(1)}_{1,y,1}$ contains the pairs $\{v_1, v_2\}$, $\{v_4, v_5\}$ for all $y \in \{2, 3, \ldots, k\}$. For each $y \in \{1, 2, \ldots, k\}$, let

$$D^{(1)}_{1,y,0} = (D^{(1)}_{1,y,1} \setminus \{v_2\}) \cup \{v_0\}.$$

By Lemma 4.3.23(2), these $D^{(1)}_{1,y,0}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$. Therefore, all $D^{(1)}_{x,y,z}$'s form the graph, named $D^{(1)}$, in $PD_\gamma(C_{4k+1})$ as shown in Figure 4.19.

Similarly, we can construct all $\gamma_{pr}(C_{4k+1})$-sets as follows (the subscripts of all vertices are modulo $4k + 1$): for all $x, y \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$ with $x \le y$, $z \le y$, $x - z \le 1$ and for each $i \in \{1, 2, 3, 4\}$,

$D^{(i)}_{x,y,z} = B^{(i)}_{x,y,z} \cup \{v_{4k-2+i}, v_{4k-1+i}\}$, where $B^{(i)}_{x,y,z}$ is a $\gamma_{pr}(P_{4k-3}(v_i : v_{4k-4+i}))$-set,
$D^{(i)}_{x,k,k} = (D^{(i)}_{x,k,k-1} \setminus \{v_{4k-5+i}\}) \cup \{v_{4k-3+i}\}$, and
$D^{(i)}_{1,y,0} = (D^{(i)}_{1,y,1} \setminus \{v_{i+1}\}) \cup \{v_{i-1}\}$.

These $D^{(i)}_{x,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pair $\{v_{4k-2+i}, v_{4k-1+i}\}$, and they form the graph $D^{(i)}$ (see Figure 4.19) in $PD_\gamma(C_{4k+1})$. By Lemma 4.3.11, without loss of generality, we may assume that $B^{(i)}_{x,k,z}$ contains the pair $\{v_{4k-5+i}, v_{4k-4+i}\}$ and then we get the following properties.

(B′1) Let $x \in \{1, 2, \ldots, k\}$ and $z \in \{0, 1, \ldots, k-1\}$ with $x - z \le 1$. If $y = k$, then $D^{(i)}_{x,y,z}$ contains the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$; otherwise, it contains the pairs $\{v_{4k-6+i}, v_{4k-5+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B′1.1) $D^{(i)}_{x,k,k-1}$ contains the pairs $\{v_{4k-8+i}, v_{4k-7+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$ for all $x \in \{1, 2, \ldots, k-1\}$, and $D^{(i)}_{k,k,k-1}$ contains the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B′1.2) If $z \ne k-1$, then $D^{(i)}_{x,k,z}$ contains the pairs $\{v_{4k-9+i}, v_{4k-8+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B′2) Let $y \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k\}$ with $z \le y$. If $x = 1$, then $D^{(i)}_{x,y,z}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$; otherwise, it contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

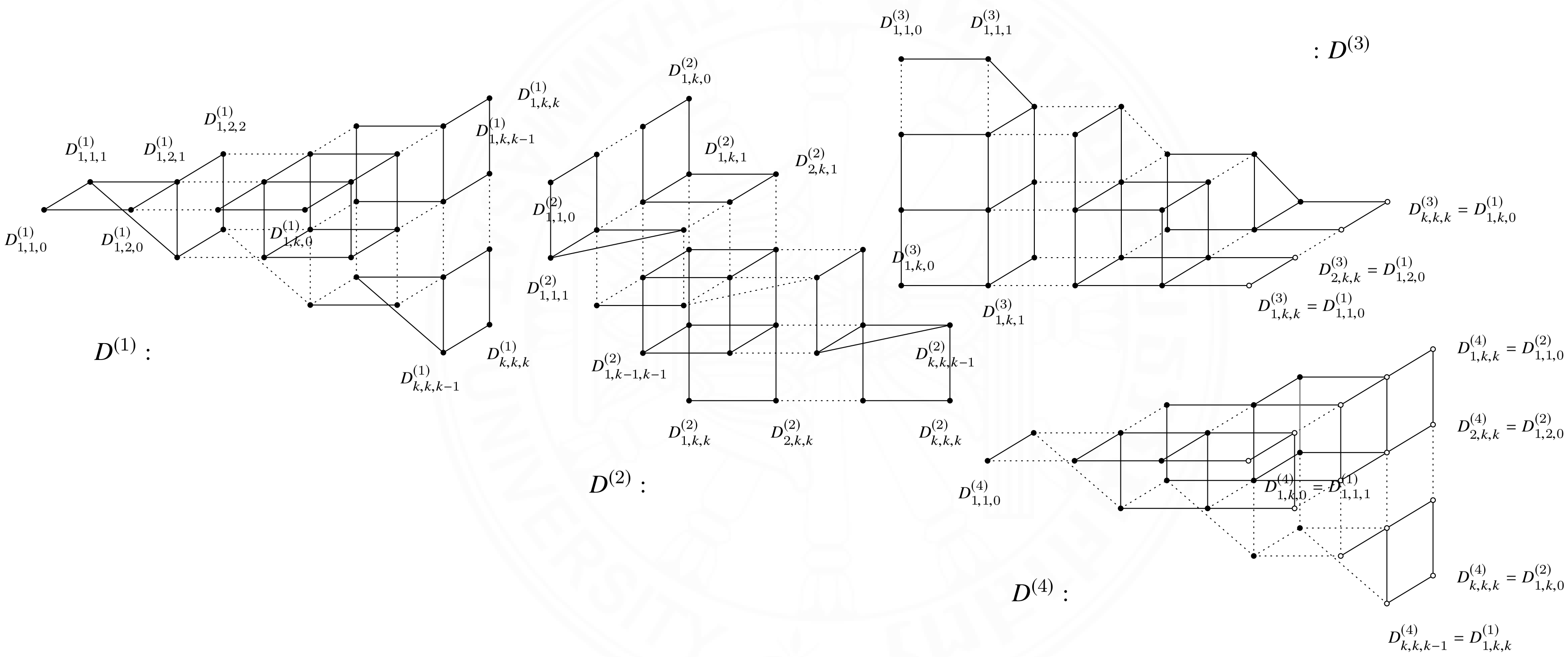

Figure 4.19 The induced subgraphs $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and $D^{(4)}$ in $PD_{\gamma}(C_{4k+1})$

(B′2.1) $D^{(i)}_{1,1,1}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+2}, v_{i+3}\}$, $D^{(i)}_{1,y,1}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+3}, v_{i+4}\}$ for all $y \in \{2, 3, \ldots, k\}$.

(B′2.2) If $z \neq 1$, then $D^{(i)}_{1,y,z}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+4}, v_{i+5}\}$.

(B′3) $D^{(i)}_{1,k,k}, D^{(i)}_{2,k,k}, \ldots, D^{(i)}_{k,k,k}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(i)}$ containing the pairs $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B′3.1) $D^{(i)}_{1,k,k}$ contains the pair $\{v_i, v_{i+1}\}$ and the others contain the pair $\{v_{i+1}, v_{i+2}\}$.

(B′3.2) $D^{(i)}_{k,k,k}$ contains the pair $\{v_{4k-7+i}, v_{4k-6+i}\}$ and the others contain the pair $\{v_{4k-8+i}, v_{4k-7+i}\}$.

(B′4) $D^{(i)}_{1,1,0}, D^{(i)}_{1,2,0}, \ldots, D^{(i)}_{1,k,0}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(i)}$ containing the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_{i-1}, v_i\}$.

(B′4.1) $D^{(i)}_{1,1,0}$ contains the pair $\{v_{i+2}, v_{i+3}\}$ and the others contain the pair $\{v_{i+3}, v_{i+4}\}$.

(B′4.2) $D^{(i)}_{1,k,0}$ contains the pair $\{v_{4k-5+i}, v_{4k-4+i}\}$ and the others contain the pair $\{v_{4k-6+i}, v_{4k-5+i}\}$.

Note that $D^{(1)}$ and $D^{(2)}$ cannot have any common vertices in $PD_\gamma(C_{4k+1})$; otherwise, there is a $\gamma_{pr}(C_{4k+1})$-set containing the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_{4k}, v_0\}$, which is impossible. Similarly, $D^{(i)}$ and $D^{(i+1)}$ do not share any vertices in $PD_\gamma(C_{4k+1})$ for all $i \in \{2, 3\}$.

We next consider all $\gamma_{pr}(C_{4k+1})$-sets that are in both $D^{(1)}$ and $D^{(3)}$. Then these sets must contain the pairs $\{v_{4k-1}, v_{4k}\}, \{v_0, v_1\}$. By (B′4) and (B′4.2), we get that $D^{(1)}_{1,1,0}, D^{(1)}_{1,2,0}, \ldots, D^{(1)}_{1,k,0}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(1)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, and particularly $D^{(1)}_{1,k,0}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. By (B′3) and (B′3.2), we have that $D^{(3)}_{1,k,k}, D^{(3)}_{2,k,k}, \ldots, D^{(3)}_{k,k,k}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(3)}$ that contain the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, and particularly $D^{(3)}_{k,k,k}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. The proof of Lemma 4.3.23(2) implies that, for each $y \in \{1, 2, \ldots, k\}$, $D^{(1)}_{1,y,0}$ and $D^{(3)}_{y,k,k}$ are unions of $\gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))$-set and $\{v_{4k-1}, v_{4k}, v_0, v_1\}$, that is,

$$D^{(1)}_{1,y,0} = T^{(1)}_{1,y,0} \cup \{v_{4k-1}, v_{4k}, v_0, v_1\}, \text{ where } T^{(1)}_{1,y,0} \text{ is a } \gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))\text{-set}$$

and

$$D^{(3)}_{y,k,k} = T^{(3)}_{y,k,k} \cup \{v_{4k-1}, v_{4k}, v_0, v_1\}, \text{ where } T^{(3)}_{y,k,k} \text{ is a } \gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))\text{-set.}$$

We know that $D^{(1)}_{1,k,0}$ and $D^{(3)}_{k,k,k}$ contain the pair $\{v_{4k-4}, v_{4k-3}\}$, so do $T^{(1)}_{1,k,0}$ and $T^{(3)}_{k,k,k}$. By Lemma 4.3.9, we get that $T^{(1)}_{1,k,0} = T^{(3)}_{k,k,k}$. By Theorem 4.3.13, for each $y \in \{1, 2, \ldots, k\}$, we have $T^{(1)}_{1,y,0} = T^{(3)}_{y,k,k}$, and hence $D^{(1)}_{1,y,0} = D^{(3)}_{y,k,k}$. Similarly, we get $D^{(2)}_{1,y,0} = D^{(4)}_{y,k,k}$ for all $y \in \{1, 2, \ldots, k\}$.

We next consider all $\gamma_{pr}(C_{4k+1})$-sets which are in both $D^{(1)}$ and $D^{(4)}$. These sets must contain the pairs $\{v_{4k-1}, v_{4k}\}, \{v_1, v_2\}$. By (B′2), for all $y, z \in \{1, 2, \ldots, k\}$ with $z \leq y$, we get that all $D^{(1)}_{1,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(1)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and they form the graph in Figure 4.20 (left). By (B′1), for all $x \in \{1, 2, \ldots, k\}$ and $z \in \{0, 1, \ldots, k-1\}$ with $x - z \leq 1$, all $D^{(4)}_{x,k,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(4)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and they form the graph in Figure 4.20 (right). To show that $D^{(1)}_{1,y,z} = D^{(4)}_{z,k,y-1}$ for all $y, z \in \{1, 2, \ldots, k\}$ with $z \leq y$, it suffices to show that $D^{(1)}_{1,k,k} = D^{(4)}_{k,k,k-1}$. By (B′3.1), we have $D^{(1)}_{1,k,k}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$. By (B′1.1), $D^{(4)}_{k,k,k-1}$ contains these three pairs as well. The proof of Lemma 4.3.23(2) implies that $D^{(1)}_{1,k,k} = T^{(1)}_{1,k,k} \cup \{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\}$ and $D^{(4)}_{k,k,k-1} = T^{(4)}_{k,k,k-1} \cup \{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\}$, where $T^{(1)}_{1,k,k}$ and $T^{(4)}_{k,k,k-1}$ are $\gamma_{pr}(P_{4k-5}(v_1 : v_{4k-5}))$-sets containing the pair $\{v_1, v_2\}$. By Lemma 4.3.9, we get $T^{(1)}_{1,k,k} = T^{(4)}_{k,k,k-1}$, and thus $D^{(1)}_{1,k,k} = D^{(4)}_{k,k,k-1}$.

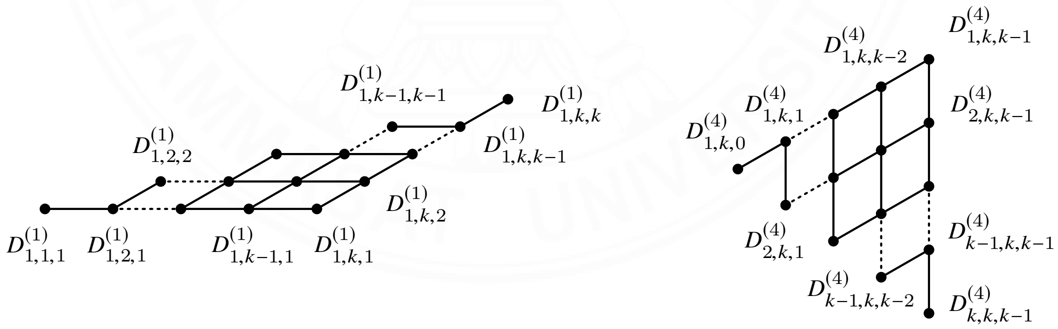

Figure 4.20 The subgraph of $D^{(1)}$ induced by $D^{(1)}_{1,y,z}$'s for all $y, z \in \{1, 2, \ldots, k\}$ with $z \leq y$ (left) and the subgraph of $D^{(4)}$ induced by $D^{(4)}_{x,k,z}$'s for all $x \in \{1, 2, \ldots, k\}$ and $z \in \{0, 1, \ldots, k-1\}$ with $x - z \leq 1$ (right)

Next, we consider all edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. We first find all neighbors of $D^{(1)}_{1,y,0}$ in $D^{(2)}$ for each $y \in \{1, 2, \ldots, k\}$. We show that $D^{(1)}_{1,1,0}$ is adjacent to $D^{(2)}_{k,k,k}$, and $D^{(1)}_{1,k,0}$ is adjacent to $D^{(2)}_{1,1,0}$. By (B′4), (B′4.1), (B′4.2), $D^{(1)}_{1,1,0}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_3, v_4\}$, the set $D^{(1)}_{1,y,0}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$ for each $y \in \{2, 3, \ldots, k-1\}$, and $D^{(1)}_{1,k,0}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$. Since each set in $D^{(2)}$ contains the pair $\{v_{4k}, v_0\}$, the set $D^{(1)}_{1,y,0}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{1,y,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ or $(D^{(1)}_{1,y,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is a $\gamma_{pr}(C_{4k+1})$-set. We have $(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ is a $\gamma_{pr}(C_{4k+1})$-set, but $(D^{(1)}_{1,y,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ is not if $y \neq 1$. Note that $(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_0\}$. By (B′3.2), $D^{(2)}_{k,k,k}$ also contains these three pairs. By Lemmas 4.3.23(2) and 4.3.9, we get $(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ and $D^{(2)}_{k,k,k}$ are unions of $\{v_{4k-2}, v_{4k-1}, v_{4k}, v_0\}$ and a unique $\gamma_{pr}(P_{4k-5}(v_2 : v_{4k-4}))$-set containing the pair $\{v_{4k-5}, v_{4k-4}\}$. Hence, $(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\} = D^{(2)}_{k,k,k}$, that is, $D^{(1)}_{1,1,0}$ is adjacent to $D^{(2)}_{k,k,k}$. Moreover, $(D^{(1)}_{1,k,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is a $\gamma_{pr}(C_{4k+1})$-set, but $(D^{(1)}_{1,y,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is not if $y \neq k$. Note that $(D^{(1)}_{1,k,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ contains the pairs $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, $\{v_4, v_5\}$. By (B′4), $D^{(2)}_{1,1,0}$ also contains these three pairs. By Lemmas 4.3.23(2) and 4.3.9, we have $(D^{(1)}_{1,k,0} \setminus \{v_{4k-1}\}) \cup \{v_2\} = D^{(2)}_{1,1,0}$, that is, $D^{(1)}_{1,k,0}$ is adjacent to $D^{(2)}_{1,1,0}$.

We next find all neighbors of $D^{(1)}_{x,k,k}$ in $D^{(2)}$ for each $x \in \{1, 2, \ldots, k\}$ by proving that $D^{(1)}_{x,k,k}$ is adjacent to $D^{(2)}_{1,k,x-1}$ for each $x \in \{1, 2, \ldots, k\}$, and $D^{(1)}_{k,k,k}$ is adjacent to $D^{(2)}_{1,k,k}$. By (B′3), (B′3.1), and (B′3.2), $D^{(1)}_{1,k,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, the set $D^{(1)}_{x,k,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ for each $x \in \{2, 3, \ldots, k-1\}$, and $D^{(1)}_{k,k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$. Note that $D^{(1)}_{x,k,k}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{x,k,k} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ or $(D^{(1)}_{x,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set. We have $(D^{(1)}_{x,k,k} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set for each $x \in \{1, 2, \ldots, k\}$, and then we let $N_x = (D^{(1)}_{x,k,k} \setminus \{v_{4k-1}\}) \cup \{v_0\}$. We note that N_1 contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and $N_2, N_3, \ldots, N_k$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form a path with k vertices in $D^{(2)}$. By (B′4.2), we have $D^{(2)}_{1,k,0}$ is the only $\gamma_{pr}(C_{4k+1})$-set in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$. By (B′1) and (B′2), we have $D^{(2)}_{1,k,1}, D^{(2)}_{1,k,2}, \ldots, D^{(2)}_{1,k,k-1}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form a path with k vertices in $D^{(2)}$. Then we can conclude that, for each $x \in \{1, 2, \ldots, k\}$, $N_x = D^{(2)}_{1,k,x-1}$, which means that $D^{(1)}_{x,k,k}$ is adjacent to $D^{(2)}_{1,k,x-1}$. Moreover, $(D^{(1)}_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set, but $(D^{(1)}_{x,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is not if $x \neq k$. Note that $(D^{(1)}_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$.

By (B′3.1), $D^{(2)}_{1,k,k}$ also contains these three pairs. By Lemmas 4.3.23(2) and 4.3.9, we get $(D^{(1)}_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\} = D^{(2)}_{1,k,k}$, that is, $D^{(1)}_{k,k,k}$ is adjacent to $D^{(2)}_{1,k,k}$.

Last but not least, we find all neighbors in $D^{(2)}$ of the other $\gamma_{pr}(C_{4k+1})$-sets in $D^{(1)}$. We prove that $D^{(1)}_{x,k,z}$ is adjacent to $D^{(2)}_{1,z,x-1}$ for all $x \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$ with $x - z \leq 1$. Recall that, for all $x, y \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$, $D^{(1)}_{x,y,z}$ contains the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0. Then $D^{(1)}_{x,y,z}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{x,y,z} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set. By (B′1), $D^{(1)}_{x,y,z}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$ for all $y \neq k$, so $(D^{(1)}_{x,y,z} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is not a $\gamma_{pr}(C_{4k+1})$-set. By (B′1) and (B′2), for all $z \in \{1, 2, \ldots, k-1\}$, we have $D^{(1)}_{1,k,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and $D^{(1)}_{x,k,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ for all $x \neq 1$. For all $x \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$ with $x - z \leq 1$, let $D_{x,z} = (D^{(1)}_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{v_0\}$, so $D_{x,z}$ is a $\gamma_{pr}(C_{4k+1})$-set in $D^{(2)}$, and these $D_{x,z}$'s form the graph in Figure 4.21. Note that, for all $z \in \{1, 2, \ldots, k-1\}$, $D_{1,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and $D_{x,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$ for all $x \neq 1$. By (B′4) and (B′4.2), $D^{(2)}_{1,1,0}, D^{(2)}_{1,2,0}, \ldots, D^{(2)}_{1,k-1,0}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and by (B′1) and (B′2), for all $y, z \in \{1, 2, \ldots, k-1\}$ with $z \leq y$, we have $D^{(2)}_{1,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form the graph in Figure 4.22. Then the graphs in Figures 4.21 and 4.22 are the same, so we can conclude that, for all $x \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$ with $x - z \leq 1$, $D_{x,z} = D^{(2)}_{1,z,x-1}$, that is, $D^{(1)}_{x,k,z}$ is adjacent to $D^{(2)}_{1,z,x-1}$.

The results about the edges between a set in $D^{(i)}$ and a set in $D^{(i+1)}$ for all $i \in \{2, 3\}$ are the same as the edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. Since $D^{(1)}_{1,1,0} = D^{(3)}_{1,k,k}$, the edges $D^{(1)}_{1,1,0}D^{(2)}_{k,k,k}$ and $D^{(2)}_{k,k,k}D^{(3)}_{1,k,k}$ are the same. Similarly, $D^{(1)}_{1,k,0}D^{(2)}_{1,1,0} = D^{(2)}_{1,1,0}D^{(3)}_{k,k,k}$ and $D^{(2)}_{1,k,0}D^{(3)}_{1,1,0} = D^{(3)}_{1,1,0}D^{(4)}_{k,k,k}$. Now, all $\gamma_{pr}(C_{4k+1})$-sets and edges form a loopbox LB_k in $PD_\gamma(C_{4k+1})$. Then we only need to show that there is no more edge in $PD_\gamma(C_{4k+1})$. Recall that $D^{(1)}_{1,y,0} = D^{(3)}_{y,k,k}$ for all $y \in \{1, 2, \ldots, k\}$, so we consider all edges between a set in $\widehat{D}^{(1)} = D^{(1)} - \{D^{(1)}_{1,y,0} : 1 \leq y \leq k\}$ and a set in $\widehat{D}^{(3)} = D^{(3)} - \{D^{(3)}_{y,k,k} : 1 \leq y \leq k\}$. Note that a set in $\widehat{D}^{(1)}$ contains either the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$ or the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$, while a set in $\widehat{D}^{(3)}$ contains the pair $\{v_0, v_1\}$ but not $\{v_{4k-1}, v_{4k}\}$. Thus, there is no edge between a set in $\widehat{D}^{(1)}$ and a set in $\widehat{D}^{(3)}$. Similarly, there is no edge

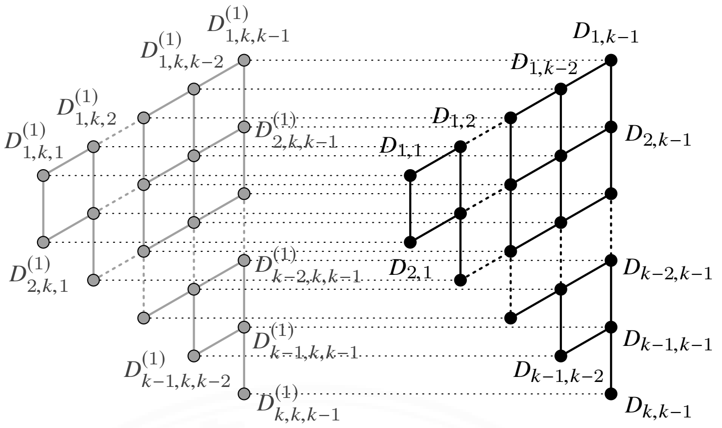

Figure 4.21 The subgraph of $D^{(2)}$ induced by $D_{x,z}$'s for all $x \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$ with $x - z \leq 1$

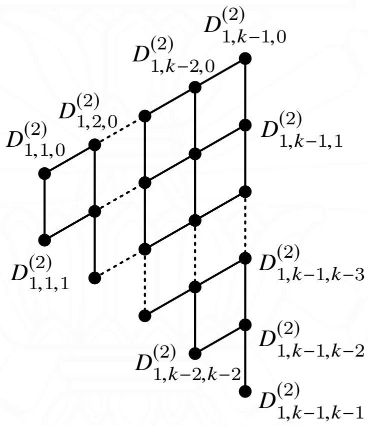

Figure 4.22 The subgraph of $D^{(2)}$ induced by $D^{(2)}_{1,y,z}$'s for all $y, z \in \{1, 2, \ldots, k-1\}$ with $z \leq y$

between a set in $D^{(2)} - \{D^{(2)}_{1,y,0} : 1 \leq y \leq k\}$ and a set in $D^{(4)} - \{D^{(4)}_{y,k,k} : 1 \leq y \leq k\}$. Recall that $D^{(1)}_{1,y,z} = D^{(4)}_{z,k,y-1}$ for all $y, z \in \{1, 2, \ldots, k\}$. Also, for all $y \in \{1, 2, \ldots, k\}$, $D^{(1)}_{1,y,0} = D^{(3)}_{y,k,k}$ has a neighbor in $D^{(4)}$. Hence, we consider all edges between a set in $\widetilde{D}^{(1)} = D^{(1)} - \{D^{(1)}_{1,y,z}, D^{(1)}_{1,y,0} : 1 \leq y, z \leq k\}$ and a set in $\widetilde{D}^{(4)} = D^{(4)} - \{D^{(4)}_{z,k,y-1} : 1 \leq y, z \leq k\}$. Note that a set in $\widetilde{D}^{(1)}$ contains the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$, while a set in $\widetilde{D}^{(4)}$ contains the pair $\{v_1, v_2\}$ but not $\{v_{4k-1}, v_{4k}\}$. Thus, there is no edge between a set in $\widetilde{D}^{(1)}$ and a set in $\widetilde{D}^{(4)}$. This completes the proof. □

4.4 Lollipop Graphs

In this section, we determine the γ-total and the γ-paired dominating graphs of lollipop graphs, which are defined in Section 3.6. We refer to the vertices of a lollipop graph as shown in Figure 3.21.

4.4.1 γ-Total Dominating Graphs of Lollipop Graphs

Before proceeding to determine the γ-total dominating graphs of lollipop graphs, we recall some useful results related to the γ-total dominating graphs of paths. We assume that the vertices of the path P_p are labelled as $P_p = (v_1, v_2, \ldots, v_p)$. From the proofs of Theorems 4.3.2, 4.3.3, and 4.3.4, we can get Corollaries 4.4.1, 4.4.2, and 4.4.3, respectively.

Corollary 4.4.1. *Let $k \geq 0$ be an integer and the vertices of the path $TD_\gamma(P_{4k+3}) \cong P_{k+2}$ be $D_1, D_2, \ldots, D_{k+2}$, where D_x is a $\gamma_t(P_{4k+3})$-set for all $x \in \{1, 2, \ldots, k+2\}$.*

(1) *If $v_{4k+3} \in D_x$, then either $x = 1$ or $x = k+2$.*

(2) *If D_{k+2} contains the vertex v_{4k+3}, then $D_{k+2} = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{v_{4k+3}\}$.*

In the next result, we think of the $\gamma_t(P_{4k+2})$-sets in $TD_\gamma(P_{4k+2}) \cong P_{k+1} \square P_{k+1}$ as the entries in a matrix.

Corollary 4.4.2. *Let $k \geq 0$ be an integer and $D_{x,y}$ the $\gamma_t(P_{4k+2})$-set at the position (x, y) (row x and column y) of $TD_\gamma(P_{4k+2}) \cong P_{k+1} \square P_{k+1}$ for all $x, y \in \{1, 2, \ldots, k+1\}$.*

(1) *If $v_{4k+2} \in D_{x,y}$, then either $x = 1$, $x = k+1$, $y = 1$, or $y = k+1$.*

(2) *If $D_{x,k+1}$ contains the vertex v_{4k+2}, then*

(2.1) $D_{x,k+1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{v_{4k+2}\}$ *for each $x \in \{1, 2, \ldots, k+1\}$,*

(2.2) $D_{k+1,k+1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}\}$, *and*

(2.3) $D_{x,y}$ *does not contain the vertex v_{4k-1} for all $x \neq k+1$.*

Corollary 4.4.3. *Let $k \geq 1$ be an integer. Then each $\gamma_t(P_{4k+1})$-set does not contain the vertex v_{4k+1}.*

We now study the γ-total dominating graph of a lollipop graph $L_{p,q}$, where p and q are both positive integers. If $q = 1$, then $L_{p,q} \cong P_{p+1}$, so we get the results on $TD_\gamma(L_{p,q})$ by Theorems 4.3.1 - 4.3.4. For $q \geq 2$, we divide the values of p into four cases. We first consider the case when $p = 4k + 2$ and then get the following theorem.

Theorem 4.4.4. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k+2,q}) \cong P_1$.*

Proof. By Theorem 3.6.1(1), we get $\gamma_t(L_{4k+2,q}) = 2k + 2$. Then there is exactly one $\gamma_t(L_{4k+2,q})$-set, which is $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k - 1\} \cup \{v_{4k+2}, u_1\}$. □

We next provide the property involving the $\gamma_t(L_{4k+1,q})$-sets and we then determine the γ-total dominating graph of $L_{4k+1,q}$.

Lemma 4.4.5. *Let $k \geq 0$ and $q \geq 2$ be integers. Then each $\gamma_t(L_{4k+1,q})$-set contains the vertex u_1.*

Proof. If $q = 2$, then u_1 is a support vertex of $L_{4k+1,2} \cong P_{4k+3}$, so this lemma follows by Observation 3.0.1. Let $q \geq 3$. Suppose, contrary to the statement, that there exists a $\gamma_t(L_{4k+1,q})$-set D that does not contain u_1. Thus, D contains exactly two vertices u_i and u_j from $\{u_2, u_3, \ldots, u_q\}$. Let $S = \{v : v \notin N(\{u_i, u_j\})\}$, and then the induced subgraph $L_{4k+1,q}[S]$ is P_{4k+1}. By Theorem 3.6.1(1), $|D| = 2k + 2$, and thus the $2k$ remaining vertices of D must dominate all vertices in $L_{4k+1,q}[S]$, which is impossible. □

Theorem 4.4.6. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k+1,q}) \cong L_{k,q}$.*

Proof. Let P^i be the subgraph of $L_{4k+1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k+1}, u_1, u_i\}$ for each $i \in \{2, 3, \ldots, q\}$, and then $P^i \cong P_{4k+3}$. By Theorem 4.3.2, for each $i \in \{2, 3, \ldots, q\}$, $TD_\gamma(P^i) \cong P_{k+2}$, say this path as $D^i_1, D^i_2, \ldots, D^i_{k+2}$, where D^i_x is a $\gamma_t(P^i)$-set for each $x \in \{1, 2, \ldots, k+2\}$. By Observation 3.0.1, we get that $u_1 \in D^i_x$ for all $x \in \{1, 2, \ldots, k+2\}$. By Corollary 4.4.1(1), without loss of generality, we may assume that D^i_{k+2} contains u_i, and D^i_x does not contain u_i for all $x \neq k + 2$. Note that if $x \neq k + 2$, then $D^i_x = D^j_x$ for all $i, j \in \{2, 3, \ldots, q\}$, so we let $D_x = D^i_x$. Next, we claim that D^i_{k+2} and D^j_{k+2} are adjacent for all $i \neq j$. By Corollary 4.4.1(2), we get $D^i_{k+2} = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_i\} = [(D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{k+2} \setminus \{u_j\}) \cup \{u_i\}$, so the claim holds.

By Lemma 3.0.2 and Theorem 3.6.1(1), $\gamma_t(P^i) = 2k + 2 = \gamma_t(L_{4k+1,q})$. We also note that every $\gamma_t(P^i)$-set is also a $\gamma_t(L_{4k+1,q})$-set for each $i \in \{2, 3, \ldots, q\}$. Hence,

$D_1, \ldots, D_{k+1}, D^2_{k+2}, \ldots, D^q_{k+2}$ are $\gamma_t(L_{4k+1,q})$-sets containing u_1. By Lemma 4.4.5, each $\gamma_t(L_{4k+1,q})$-set contains u_1, so it is a $\gamma_t(P^i)$-set for some $i \in \{2, 3, \ldots, q\}$. Therefore, $D_1, \ldots, D_{k+1}, D^2_{k+2}, \ldots, D^q_{k+2}$ are the only $\gamma_t(L_{4k+1,q})$-sets, and they also form a lollipop graph $L_{k,q}$ (see Figure 4.23). □

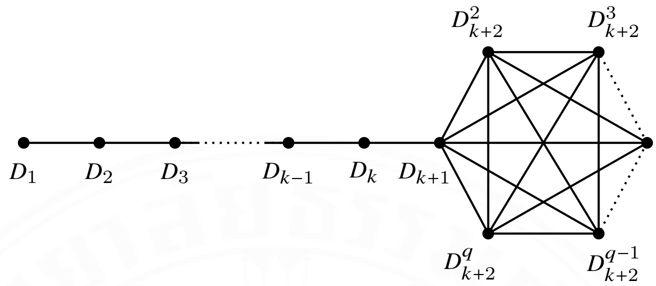

Figure 4.23 The γ-total dominating graph of $L_{4k+1,q}$

Let $L^r_{p,q} = L_{p,q} \square P_r$, where the vertices of $L^r_{p,q}$ are labeled in Figure 4.24. For convenience, we write $q-1$ vertices $v_{r,p+2}, v_{r,p+3}, \ldots, v_{r,p+q}$ of $L^r_{p,q}$ for $u_1, u_2, \ldots, u_{q-1}$, respectively. Let $JL^q_{p,r}$ denote the graph obtained from $L^r_{p,q}$ by adding the vertices $u_q, u_{q+1}, \ldots, u_{\binom{q}{2}}$ such that $u_1, u_2, \ldots, u_{q-1}, u_q, u_{q+1}, \ldots, u_{\binom{q}{2}}$ form the Johnson graph $J(q, 2)$. We illustrate the graph $JL^4_{5,4}$ in Figure 4.25.

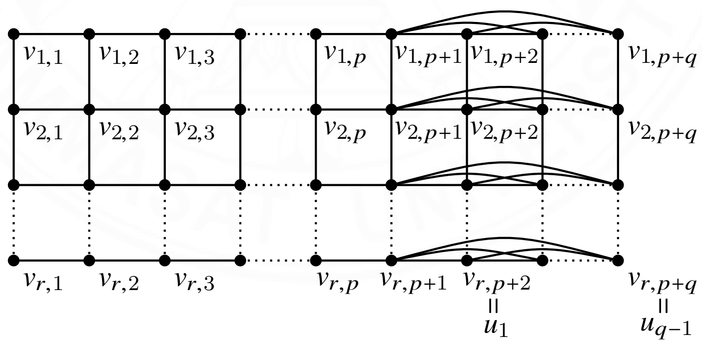

Figure 4.24 The graph $L^r_{p,q}$

Theorem 4.4.7. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k,q}) \cong JL^{k+1}_{k-1,q}$.*

Proof. Let P^i be the subgraph of $L_{4k,q}$ induced by $\{v_1, v_2, \ldots, v_{4k}, u_1, u_i\}$ for each $i \in \{2, 3, \ldots, q\}$, so $TD_\gamma(P^i) \cong TD_\gamma(P_{4k+2}) \cong P_{k+1} \square P_{k+1}$ by Theorem 4.3.1. For each $i \in \{2, 3, \ldots, q\}$ and $x, y \in \{1, 2, \ldots, k+1\}$, let $D^i_{x,y}$ be the $\gamma_t(P^i)$-set at the position

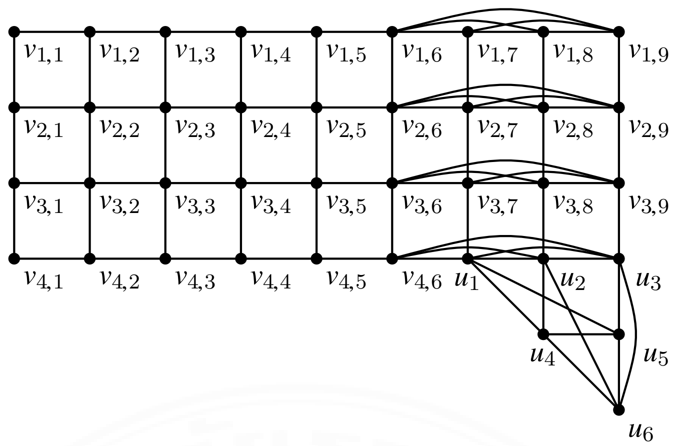

Figure 4.25 The graph $JL^4_{5,4}$

(x, y) of $TD_\gamma(P^i)$. By Corollary 4.4.2(1), without loss of generality, we may assume that $D^i_{x,k+1}$ contains u_i. If $y \neq k+1$, then $D^i_{x,y} = D^j_{x,y}$ for all $i, j \in \{2, 3, \ldots, q\}$. Hence, for all $x \in \{1, 2, \ldots, k+1\}$, let $D_{x,y} = D^i_{x,y}$ if $y \neq k+1$; otherwise, let $D^i_{x,k+1} = D_{x,k+i-1}$ for all $i \in \{2, 3, \ldots, q\}$. Note that $D_{x,k}$ is adjacent to $D_{x,k+i-1}$ for all $i \in \{2, 3, \ldots, q\}$. We next show that $D_{x,k+i-1}$ and $D_{x,k+j-1}$ are adjacent for all $i \neq j$. By Corollary 4.4.2(2.1), for each $x \in \{1, 2, \ldots, k+1\}$, we get $D_{x,k+i-1} = D^i_{x,k+1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{u_i\} = [(D_{x,k} \setminus \{v_{4k}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{x,k+1} \setminus \{u_j\}) \cup \{u_i\} = (D_{x,k+j-1} \setminus \{u_j\}) \cup \{u_i\}$, as desired.

Note that $\gamma_t(P^i) = 2k+2 = \gamma_t(L_{4k,q})$, and a $\gamma_t(P^i)$-set is a $\gamma_t(L_{4k,q})$-set containing u_1 and vice versa. Thus, all $D_{x,y}$'s with $1 \leq x \leq k+1$ and $1 \leq y \leq k+q-1$ are the only $\gamma_t(L_{4k,q})$-sets containing u_1, and they form a graph $L^{k+1}_{k-1,q}$ in $TD_\gamma(L_{4k,q})$ (see Figure 4.26).

Finally, we find all $\gamma_t(L_{4k,q})$-sets that do not contain u_1. Then such a set contains $2k$ vertices from $\{v_1, v_2, \ldots, v_{4k}\}$ and two vertices from $\{u_2, u_3, \ldots, u_q\}$. Thus, it is a union of $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\}$ and $\{u_i, u_j\}$ for some distinct $i, j \in \{2, 3, \ldots, q\}$. By Corollary 4.4.2(2.2), for each $i \in \{2, 3, \ldots, q\}$, $D_{k+1,k+i-1} = D^i_{k+1,k+1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{u_1, u_i\} = D \cup \{u_1, u_i\}$. For all $1 \leq i < j \leq q$, let $D^{i,j} = D \cup \{u_i, u_j\}$. Theorem 4.4.7 implies that all $D^{i,j}$'s form the Johnson graph $J(q, 2)$ in $TD_\gamma(L_{4k,q})$ (see Figure 4.26). Moreover, for all $2 \leq i < j \leq q$, $D^{i,j}$ is not adjacent to $D_{x,y}$ for all $y \leq k$, which does not contain $u_2, u_3, \ldots, u_q$. By Corollary 4.4.2(2.3), for each $x \neq k+1$ and $y \in \{2, 3, \ldots, q\}$, $D_{x,k+y-1} = D^y_{x,k+1}$

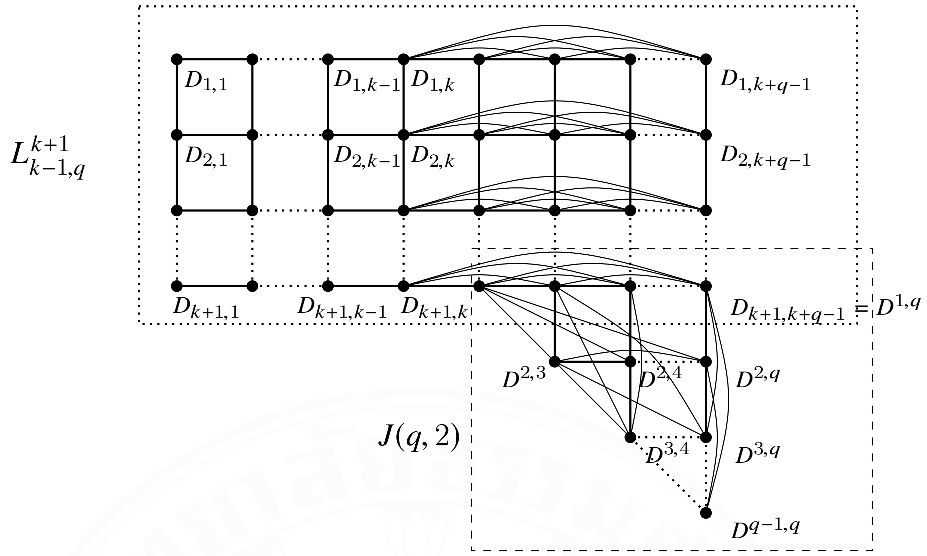

Figure 4.26 The γ-total dominating graph of $L_{4k,q}$

contains u_1 and u_y but not v_{4k-1}, so $(D_{x,k+y-1} \setminus \{u_1\}) \cup \{u_j\}$ is not a total dominating set for all $j \notin \{1, y\}$ since v_{4k} is not dominated. This means that $D_{x,k+y-1}$ with $x \neq k+1$ is not adjacent to $D^{i,j}$ for all $2 \leq i < j \leq q$. This completes the proof. □

We finally determine the γ-total dominating graph of $L_{4k-1,q}$, where $k \geq 1$ and $q \geq 2$ are both integers. To complete the result, we need the following lemma.

Lemma 4.4.8. *Let $k \geq 1$ and $q \geq 2$ be integers. Then each $\gamma_t(L_{4k-1,q})$-set does not contain the vertex u_i for all $i \in \{2, 3, \ldots, q\}$.*

Proof. Assume on contrary that there exists a $\gamma_t(L_{4k-1,q})$-set D containing u_i for some $i \in \{2, 3, \ldots, q\}$. To dominate u_i, we need at least one vertex $u_j \in D$ for some $j \in \{1, 2, \ldots, q\}$ with $j \neq i$. Let $S = \{v : v \notin N(\{u_i, u_j\})\}$. If $j = 1$, then the induced subgraph $L_{4k-1,q}[S] \cong P_{4k-2}$; otherwise, $L_{4k-1,q}[S] \cong P_{4k-1}$. Note that $|D| = 2k+1$, so Lemma 3.0.2 implies that the $2k-1$ remaining vertices of D cannot dominate all vertices in $L_{4k-1,q}[S]$, a contradiction. □

Theorem 4.4.9. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $TD_\gamma(L_{4k-1,q}) \cong P_k$.*

Proof. Let P^i be the subgraph of $L_{4k-1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k-1}, u_1, u_i\}$ for each $i \in \{2, 3, \ldots, q\}$, and then by Theorem 4.3.4, $TD_\gamma(P^i) \cong P_k$, say $D_1^i, D_2^i, \ldots, D_k^i$, where D_x^i is a $\gamma_t(P^i)$-set for all $x \in \{1, 2, \ldots, k\}$. By Corollary 4.4.3, $D_1^i, D_2^i, \ldots, D_k^i$ do not contain u_i for each $i \in \{2, 3, \ldots, q\}$. Without loss of generality, we may assume that

$D_x^i = D_x^j$ for all $i, j \in \{2, 3, \ldots, q\}$, and we let $D_x = D_x^i$. Since $\gamma_t(P^i) = 2k + 1 = \gamma_t(L_{4k-1,q})$ and every $\gamma_t(P^i)$-set is a $\gamma_t(L_{4k-1,q})$-set for all $i \in \{2, 3, \ldots, q\}$, we get $D_1, D_2, \ldots, D_k$ are $\gamma_t(L_{4k-1,q})$-sets. Lemma 4.4.8 implies that each $\gamma_t(L_{4k-1,q})$-set is also a $\gamma_t(P^i)$-set for some $i \in \{2, 3, \ldots, q\}$. Therefore, $D_1, D_2, \ldots, D_k$ are the only $\gamma_t(L_{4k-1,q})$-sets, and they form a path with k vertices in $TD_\gamma(L_{4k-1,q})$. □

4.4.2 γ-Paired Dominating Graphs of Lollipop Graphs

We now determine the γ-paired dominating graphs of lollipop graphs. To do so, we need some useful results involving the γ-paired dominating graphs of paths. Corollary 4.4.10 (respectively, Corollaries 4.4.11 and 4.4.12) can be obtained from the proofs of Lemma 4.3.9 (respectively, Lemmas 4.3.10 and 4.3.11) and Theorem 4.3.13 (respectively, Theorems 4.3.14 and 4.3.15).

Corollary 4.4.10. *Let $k \geq 0$ be an integer and the vertices of the path $PD_\gamma(P_{4k+3}) \cong P_{k+2}$ be $D_1, D_2, \ldots, D_{k+2}$, where D_x is a $\gamma_{pr}(P_{4k+3})$-set for all $x \in \{1, 2, \ldots, k+2\}$.*

(1) *If $v_{4k+3} \in D_x$, then either $x = 1$ or $x = k + 2$.*

(2) *If D_{k+2} contains the vertex v_{4k+3}, then*

(2.1) *$D_{k+2} = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{v_{4k+3}\}$ and*

(2.2) *$D_x = S_x \cup \{v_{4k+1}, v_{4k+2}\}$, where S_x is a $\gamma_{pr}(P_{4k-1})$-set for all $x \in \{1, 2, \ldots, k+1\}$ and particularly S_{k+1} contains the pair $\{v_{4k-2}, v_{4k-1}\}$, and $D_{k+2} = S_{k+1} \cup \{v_{4k+2}, v_{4k+3}\}$.*

Corollary 4.4.11. *Let $k \geq 0$ be an integer and $D_{x,y}$ the $\gamma_{pr}(P_{4k+2})$-set at the position (x, y) of $PD_\gamma(P_{4k+2}) \cong SG_{k+1,k+1}$ for all $x, y \in \{1, 2, \ldots, k+1\}$ with $x - y \leq 1$.*

(1) *If $v_{4k+2} \in D_{x,y}$, then either $x = 1$ or $y = k + 1$.*

(2) *If $D_{x,k+1}$ contains the vertex v_{4k+2}, then*

(2.1) *$D_{x,k+1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{v_{4k+2}\}$ for each $x \in \{1, 2, \ldots, k+1\}$,*

(2.2) *$D_{k+1,k+1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}\}$, and*

(2.3) *$D_{x,y}$ does not contain the vertex v_{4k-1} for all $x \neq k + 1$.*

Corollary 4.4.12. *Let $k \geq 1$ be an integer and $D_{x,y,z}$ the $\gamma_{pr}(P_{4k+1})$-set at the position (x, y, z) in $PD_\gamma(P_{4k+1}) \cong SG_{k+1,k+1,k}$ for all $x, y \in \{1, 2, \ldots, k+1\}$, $z \in \{1, 2, \ldots, k\}$ with $x - y \leq 0, x - z \leq 1, y - z \geq 0$.*

(1) *If $v_{4k+1} \in D_{x,y,z}$, then either $x = 1$ or $y = k + 1$.*

(2) *If $D_{x,k+1,z}$ contains the vertex v_{4k+1}, then*

(2.1) *$D_{x,k+1,z} = (D_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{v_{4k+1}\}$ for all $x, z \in \{1, 2, \ldots, k\}$, and $D_{k+1,k+1,k} = (D_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k+1}\}$,*

(2.2) *$D_{x,k+1,k} = D_x \cup \{v_{4k-3}, v_{4k-2}, v_{4k}, v_{4k+1}\}$, where D_x is a $\gamma_{pr}(P_{4k-5})$-set for all $x \in \{1, 2, \ldots, k\}$, D_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and $D_{k+1,k+1,k} = D_k \cup \{v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\}$,*

(2.3) *$D_{x,k+1,z}$ does not contain the vertex v_{4k-2} for all $z < k$.*

We are now in a position to determine the γ-paired dominating graph of a lollipop graph $L_{p,q}$. If $q = 1$, then we get the γ-paired dominating graph of $L_{p,q} \cong P_{p+1}$ by Theorems 4.3.12 - 4.3.15. For $q \geq 2$, we consider the values of p into four cases. If $p = 4k + 2$, then we obtain the following result.

Theorem 4.4.13. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $PD_\gamma(L_{4k+2,q}) \cong P_1$.*

Proof. By Theorem 3.6.1(2), we have $\gamma_{pr}(L_{4k+2,q}) = 2k + 2$. It is easy to check that $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k - 1\} \cup \{v_{4k+2}, u_1\}$ is the only $\gamma_{pr}(L_{4k+2,q})$-set, so the theorem holds. □

We provide some properties in the next lemma before determining the γ-paired dominating graph of $L_{4k+1,q}$ in Theorem 4.4.15.

Lemma 4.4.14. *Let $k \geq 0$ and $q \geq 2$ be integers. Then each $\gamma_{pr}(L_{4k+1,q})$-set contains the vertex u_1.*

Proof. If $q = 2$, then u_1 is a support vertex of $L_{4k+1,q}$, so by Observation 3.0.1, this lemma holds. Let $q \geq 3$ and suppose that D is a $\gamma_{pr}(L_{4k+1,q})$-set with $u_1 \notin D$. Then D must contain exactly two vertices from $\{u_2, u_3, \ldots, u_q\}$. Since $|D| = 2k + 2$, the other $2k$ vertices of D must dominate all vertices in $L_{4k+1,q}[\{v_1, v_2, \ldots, v_{4k+1}\}] \cong P_{4k+1}$, which is impossible. □

Theorem 4.4.15. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $PD_\gamma(L_{4k+1,q}) \cong L_{k,q}$.*

Proof. Let P^i be the subgraph of $L_{4k+1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k+1}, u_1, u_i\}$ for each $i \in \{2, 3, \ldots, q\}$, and then $P^i \cong P_{4k+3}$. By Theorem 4.3.13, for each $i \in \{2, 3, \ldots, q\}$, $PD_\gamma(P^i) \cong P_{k+2}$, say this path as $D^i_1, D^i_2, \ldots, D^i_{k+2}$, where D^i_x is a $\gamma_{pr}(P^i)$-set for each $x \in \{1, 2, \ldots, k+2\}$, so $u_1 \in D^i_x$ by Observation 3.0.1. By Corollary 4.4.10(1), without loss of generality, we may assume that D^i_{k+2} contains u_i. If $x \neq k+2$, then $D^i_x = D^j_x$ for all $i, j \in \{2, 3, \ldots, q\}$, so we let $D_x = D^i_x$. Next, we show that D^i_{k+2} and D^j_{k+2} are adjacent for all $i \neq j$. By Corollary 4.4.10(2.1), we obtain $D^i_{k+2} = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_i\} = [(D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{k+2} \setminus \{u_j\}) \cup \{u_i\}$, as needed.

Note that $\gamma_{pr}(P^i) = 2k + 2 = \gamma_t(L_{4k+1,q})$, and every $\gamma_{pr}(P^i)$-set is also a $\gamma_{pr}(L_{4k+1,q})$-set for each $i \in \{2, 3, \ldots, q\}$. Thus, $D_1, \ldots, D_{k+1}, D^2_{k+2}, \ldots, D^q_{k+2}$ are $\gamma_{pr}(L_{4k+1,q})$-sets containing u_1. Lemma 4.4.14 implies that each $\gamma_{pr}(L_{4k+1,q})$-set is a $\gamma_{pr}(P^i)$-set for some $i \in \{2, 3, \ldots, q\}$. We conclude that $D_1, \ldots, D_{k+1}, D^2_{k+2}, \ldots, D^q_{k+2}$ are the only $\gamma_{pr}(L_{4k+1,q})$-sets, and they form a lollipop graph $L_{k,q}$. □

Let $SL^r_{p,q}$ (respectively, $SJL^r_{p,q}$) denote the graph that is obtained from $L^r_{p,q}$ (respectively, $JL^r_{p,q}$) by deleting the vertices $v_{x,y}$ for all $x \in \{1, 2, \ldots, r\}$ and $y \in \{1, 2, \ldots, p\}$ with $x - y \geq 2$. Figure 4.27 exhibits the graphs $SL^3_{1,3}$ and $SL^4_{2,4}$, and Figure 4.28 displays the graphs $SJL^3_{1,3}$ and $SJL^4_{2,4}$.

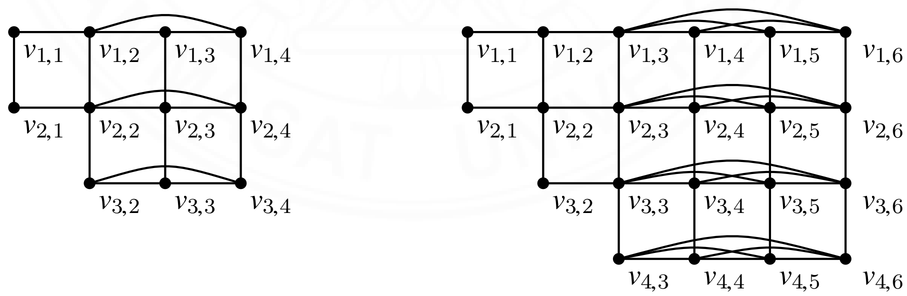

Figure 4.27 The graphs $SL^3_{1,3}$ (left) and $SL^4_{2,4}$ (right)

Theorem 4.4.16. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $PD_\gamma(L_{4k,q}) \cong SJL^{k+1}_{k-1,q}$.*

Proof. Let P^i be the subgraph of $L_{4k,q}$ induced by $\{v_1, v_2, \ldots, v_{4k}, u_1, u_i\}$ for each $i \in \{2, 3, \ldots, q\}$, so $PD_\gamma(P^i) \cong PD_\gamma(P_{4k+2}) \cong SG_{k+1,k+1}$ by Theorem 4.3.14. For each $i \in \{2, 3, \ldots, q\}$ and $x, y \in \{1, 2, \ldots, k+1\}$ with $x - y \leq 1$, let $D^i_{x,y}$ be the $\gamma_{pr}(P^i)$-set

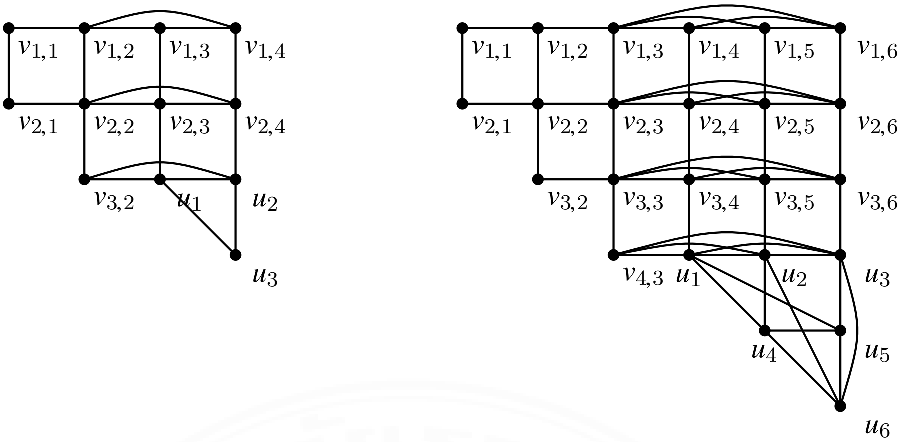

Figure 4.28 The graphs $SJL^3_{1,3}$ (left) and $SJL^4_{2,4}$ (right)

at the position (x, y) of $PD_\gamma(P^i)$. By Corollary 4.4.11(1), without loss of generality, we may assume that $D^i_{x,k+1}$ contains u_i. If $y \neq k+1$, then $D^i_{x,y} = D^j_{x,y}$ for all $i, j \in \{2, 3, \ldots, q\}$. For all $x \in \{1, 2, \ldots, k+1\}$, if $y \neq k+1$, we let $D_{x,y} = D^i_{x,y}$; otherwise, let $D^i_{x,k+1} = D_{x,k+i-1}$ for all $i \in \{2, 3, \ldots, q\}$. It is obvious that $D_{x,k}$ is adjacent to $D_{x,k+i-1}$ for all $i \in \{2, 3, \ldots, q\}$. Next, we show that $D_{x,k+i-1}$ and $D_{x,k+j-1}$ are adjacent for all $i \neq j$. By Corollary 4.4.11(2.1), for each $x \in \{1, 2, \ldots, k+1\}$, we have $D_{x,k+i-1} = D^i_{x,k+1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{u_i\} = [(D_{x,k} \setminus \{v_{4k}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{x,k+1} \setminus \{u_j\}) \cup \{u_i\} = (D_{x,k+j-1} \setminus \{u_j\}) \cup \{u_i\}$, as needed.

Observe that $\gamma_{pr}(P^i) = 2k+2 = \gamma_{pr}(L_{4k,q})$, and a $\gamma_{pr}(P^i)$-set is a $\gamma_{pr}(L_{4k,q})$-set containing u_1 and vice versa. Hence, all $D_{x,y}$'s with $1 \leq x \leq k+1$ and $1 \leq y \leq k+q-1$ are the only $\gamma_{pr}(L_{4k,q})$-sets containing u_1, and they form a graph $SL^{k+1}_{k-1,q}$ in $PD_\gamma(L_{4k,q})$ (see Figure 4.29).

Finally, we find all $\gamma_{pr}(L_{4k,q})$-sets that do not contain u_1. It is easy to check that such a set is a union of $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\}$ and $\{u_i, u_j\}$ for some distinct $i, j \in \{2, 3, \ldots, q\}$. By Corollary 4.4.11(2.2), for each $i \in \{2, 3, \ldots, q\}$, $D_{k+1,k+i-1} = D^i_{k+1,k+1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{u_1, u_i\} = D \cup \{u_1, u_i\}$. For all $1 \leq i < j \leq q$, let $D^{i,j} = D \cup \{u_i, u_j\}$. Theorem 4.1.2 implies that all $D^{i,j}$'s form the Johnson graph $J(q, 2)$ in $PD_\gamma(L_{4k,q})$ (see Figure 4.29). Clearly, $D^{i,j}$ with $2 \leq i < j \leq q$ is not adjacent to $D_{x,y}$ for all $y \leq k$, which does not contain the vertices $u_2, u_3, \ldots, u_q$. By Corollary 4.4.11(2.3), for each $x \neq k+1$ and $y \in \{2, 3, \ldots, q\}$, $D_{x,k+y-1} = D^y_{x,k+1}$ contains u_1 and u_y but not v_{4k-1}, so $(D_{x,k+y-1} \setminus \{u_1\}) \cup \{u_j\}$ is not a paired dominating

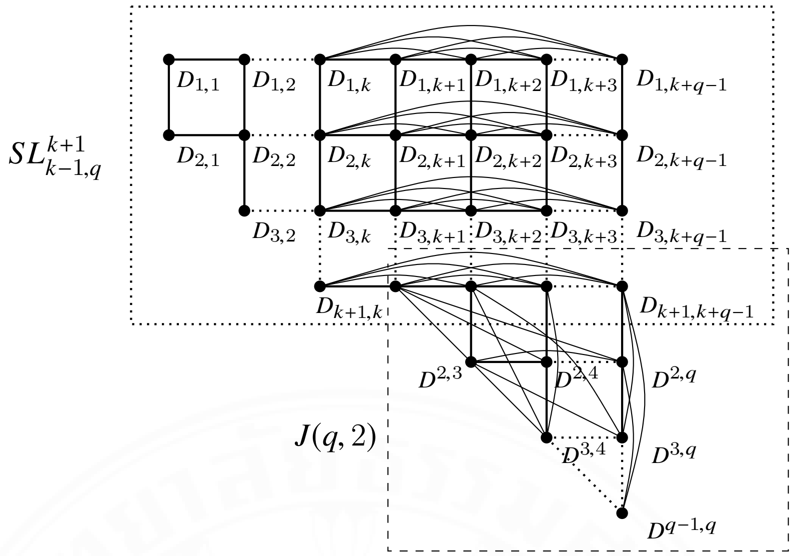

Figure 4.29 The γ-paired dominating graph of $L_{4k,q}$

set for all $j \notin \{1, y\}$, implying that $D_{x,k+y-1}$ with $x \neq k+1$ is not adjacent to $D^{i,j}$ for all $2 \leq i < j \leq q$. This completes the proof. □

Let p, q and r be positive integers. We define $A_{p,q,r}$ as the graph with the vertex set $V(A_{p,q,r}) = V(SG_{p,q,r})$ and the edge set

$$E(A_{p,q,r}) = E(SG_{p,q,r}) \cup \{(u_x, v_y, w_z)(u_x, v_{y'}, w_z) : r+2 \leq y+2 \leq y' \leq q\} \cup \{(u_r, v_r, w_r)(u_{r+1}, v_{y'}, w_r) : r+2 \leq y' \leq q\}.$$

The graphs $A_{4,5,3}$ and $A_{3,5,2}$ are shown in Figures 4.30 and 4.31, respectively, where we write (x, y, z) instead of (u_x, v_y, w_z).

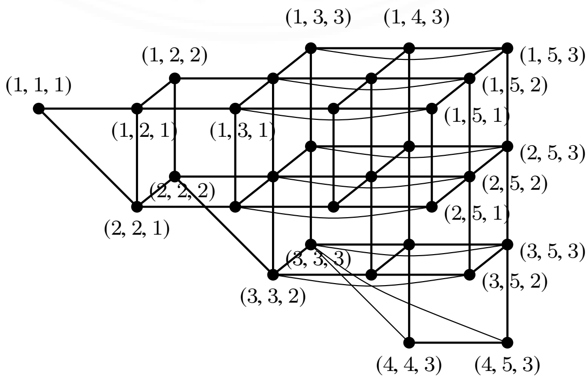

Figure 4.30 The graph $A_{4,5,3}$

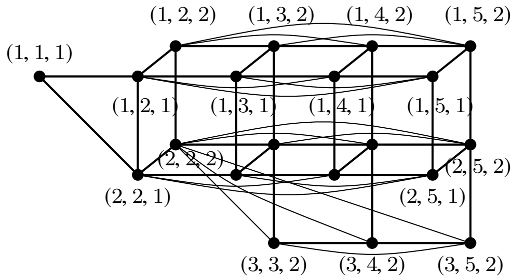

Figure 4.31 The graph $A_{3,5,2}$

Let $B_{p,q,r}$ be the graph with the vertex set $V(B_{p,q,r}) = V(A_{p,q,r}) \cup \{(u_x, v_y, w_z) : 1 \le x \le p, r+1 \le z < y \le q\}$ and the edge set

$$\begin{aligned} E(B_{p,q,r}) \;=\; & E(A_{p,q,r}) \cup \{(u_x, v_y, w_z)(u_x, v_y, w_{z'}) : r+2 \le y \le q, r \le z < z' \le y-1\} \cup \\ & \{(u_x, v_y, w_z)(u_x, v_{y'}, w_z) : r+1 \le z \le q-2, z+1 \le y < y' \le q\} \cup \\ & \{(u_x, v_y, w_z)(u_x, v_{y'}, w_y) : r \le z < y < y' \le q\} \cup \\ & \{(u_x, v_y, w_z)(u_{x+1}, v_y, w_z) : r < z < q\}. \end{aligned}$$

The graphs $B_{4,5,3}$ and $B_{3,5,2}$ are shown in Figures 4.32 and 4.33, respectively, where we write (x, y, z) for (u_x, v_y, w_z). Note that if $q = r$ or $q = r+1$, then $B_{p,q,r} \cong SG_{p,q,r}$.

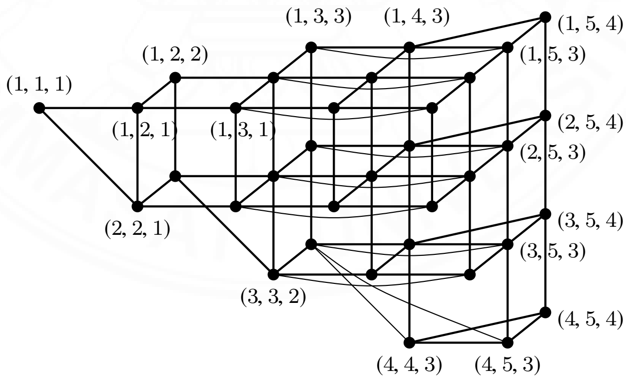

Figure 4.32 The graph $B_{4,5,3}$

Theorem 4.4.17. *Let $k \ge 1$ and $q \ge 2$ be integers. Then $PD_\gamma(L_{4k-1,q}) \cong B_{k+1,k+q-1,k}$.*

Proof. If $q = 2$, then $L_{4k-1,q} \cong P_{4k+1}$, so $PD_\gamma(L_{4k-1,2}) \cong SG_{k+1,k+1,k} \cong B_{k+1,k+1,k}$ by Theorem 4.3.15. Let $q \ge 3$. We first find all $\gamma_{pr}(L_{4k-1,q})$-sets containing the vertex u_1. For each $i \in \{2, 3, \ldots, q\}$, let P^i be the subgraph of $L_{4k-1,q}$ induced by

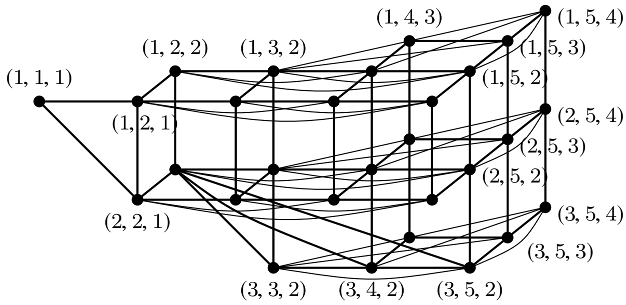

Figure 4.33 The graph $B_{3,5,2}$

$\{v_1, v_2, \ldots, v_{4k-1}, u_1, u_i\}$, and then $PD_\gamma(P^i) \cong SG_{k+1,k+1,k}$ by Theorem 4.3.15. For all $x, y \in \{1, 2, \ldots, k+1\}$, $z \in \{1, 2, \ldots, k\}$ with $x - y \leq 0$, $x - z \leq 1$, $y - z \geq 0$ and for each $i \in \{2, 3, \ldots, q\}$, let $D^i_{x,y,z}$ be the $\gamma_{pr}(P^i)$-set at the position (x, y, z) in $SG_{k+1,k+1,k}$. By Corollary 4.4.12(1), without loss of generality, we may assume that $D^i_{x,k+1,z}$ contains u_i and $D^i_{x,y,z}$ does not contain u_i for all $y \neq k+1$. Note that, for $y \neq k+1$, we have $D^i_{x,y,z} = D^j_{x,y,z}$ for all $i, j \in \{2, 3, \ldots, q\}$. For all $x \in \{1, 2, \ldots, k+1\}$ and $z \in \{1, 2, \ldots, k\}$, let $D_{x,y,z} = D^i_{x,y,z}$ if $y \neq k+1$; otherwise, let $D_{x,k+i-1,z} = D^i_{x,k+1,z}$ for each $i \in \{2, 3, \ldots, q\}$. We observe that $D_{x,k,z}$ is adjacent to $D_{x,k+i-1,z}$ for all $i \in \{2, 3, \ldots, q\}$. Next, we show that $D_{x,k+i-1,z}$ is adjacent to $D_{x,k+j-1,z}$ for all $i \neq j$. By Corollary 4.4.12(2.1), for $x, z \in \{1, 2, \ldots, k\}$, $D_{x,k+i-1,z} = D^i_{x,k+1,z} = (D_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{u_i\} = [(D_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{x,k+1,z} \setminus \{u_j\}) \cup \{u_i\} = (D_{x,k+j-1,z} \setminus \{u_j\}) \cup \{u_i\}$, and $D_{k+1,k+i-1,k} = D^i_{k+1,k+1,k} = (D_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{u_i\} = [(D_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{k+1,k+1,k} \setminus \{u_j\}) \cup \{u_i\} = (D_{k+1,k+j-1,k} \setminus \{u_j\}) \cup \{u_i\}$. The claim holds. Note that $\gamma_{pr}(P^i) = 2k + 2 = \gamma_{pr}(L_{4k-1,q})$, and a $\gamma_{pr}(P^i)$-set is also a $\gamma_{pr}(L_{4k,q})$-set containing u_1 and vice versa. Therefore, all $D_{x,y,z}$'s with $1 \leq x \leq k+1$, $1 \leq y \leq k+q-1$, $1 \leq z \leq k$ are the only $\gamma_{pr}(L_{4k-1,q})$-sets containing u_1, and they form a graph $A_{k+1,k+q-1,k}$ in $PD_\gamma(L_{4k-1,q})$ (see Figure 4.30 for $k = 3$ and $q = 2$).

We next find all $\gamma_{pr}(L_{4k-1,q})$-sets that do not contain the vertex u_1. Then such a $\gamma_{pr}(L_{4k-1,q})$-set is a union of a $\gamma_{pr}(P_{4k-1})$-set and $\{u_i, u_j\}$ for some distinct $i, j \in \{2, 3, \ldots, q\}$. By Theorem 4.3.13, $PD_\gamma(P_{4k-1}) \cong P_{k+1}$, say $D_1, D_2, \ldots, D_{k+1}$, where D_x is a $\gamma_{pr}(P_{4k-1})$-set for all $x \in \{1, 2, \ldots, k+1\}$. By Corollary 4.4.10(1), without loss of generality, we may assume that D_{k+1} contains v_{4k-1}. For all $x \in \{1, 2, \ldots, k+1\}$ and $2 \leq i < j \leq q$, let $D^{i,j}_x = D_x \cup \{u_i, u_j\}$. Thus, for each pair of i and j, the sets $D^{i,j}_1, D^{i,j}_2, \ldots, D^{i,j}_{k+1}$ are the only $\gamma_{pr}(L_{4k-1,q})$-sets containing the pair $\{u_i, u_j\}$, and they form a path in $PD_\gamma(L_{4k-1,q})$. By Corollary 4.4.10(2.2), for all $x \in \{1, 2, \ldots, k\}$ and $2 \le i < j \le q$,

$$D_x^{i,j} = D_x \cup \{u_i, u_j\} = S_x \cup \{v_{4k-3}, v_{4k-2}, u_i, u_j\},$$

where S_x is a $\gamma_{pr}(P_{4k-5})$-set and particularly S_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and

$$D_{k+1}^{i,j} = D_{k+1} \cup \{u_i, u_j\} = S_k \cup \{v_{4k-2}, v_{4k-1}, u_i, u_j\}.$$

For all $x \in \{1, 2, \ldots, k+1\}$ and $i \in \{2, 3, \ldots, q\}$, let $D_x^{1,i} = D_{x,k+i-1,k} = D^i_{x,k+1,k}$. By Corollary 4.4.12(2.2), for all $x \in \{1, 2, \ldots, k\}$ and $i \in \{2, 3, \ldots, q\}$,

$$D_x^{1,i} = D^i_{x,k+1,k} = S'_x \cup \{v_{4k-3}, v_{4k-2}, u_1, u_i\},$$

where S'_x is a $\gamma_{pr}(P_{4k-5})$-set and particularly S'_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and

$$D_{k+1}^{1,i} = D^i_{k+1,k+1,k} = S'_k \cup \{v_{4k-2}, v_{4k-1}, u_1, u_i\}.$$

By Lemma 4.3.9, we get $S_k = S'_k$. Theorem 4.3.13 shows that $S_x = S'_x$ for all $x \in \{1, 2, \ldots, k\}$. Therefore, for each $x \in \{1, 2, \ldots, k+1\}$, all $D_x^{i,j}$'s with $1 \le i < j \le q$ form the Johnson graph $J(q, 2)$ in $PD_\gamma(L_{4k-1,q})$ (see Figure 4.34).

Let $D = \{D_x^{i,j} : 1 \le x \le k+1, 2 \le i < j \le q\}$. Note that $D_{x,y,z}$ with $y \le k$ does not contain $u_2, u_3, \ldots, u_q$, so it is not adjacent to any set in D. By Corollary 4.4.12(2.3), for each $i \in \{2, 3, \ldots, q\}$, $D_{x,k+i-1,z} = D^i_{x,k+1,z}$ with $z < k$ does not contain v_{4k-2}, so $(D_{x,k+i-1,z} \setminus \{u_1\}) \cup \{u_j\}$ is not a paired dominating set for all $j \neq 1$. This implies that $D_{x,k+i-1,z}$ is not adjacent to any set in D. Hence, all $\gamma_{pr}(L_{4k-1,q})$-sets form a graph $B_{k+1,k+q-1,k}$. □

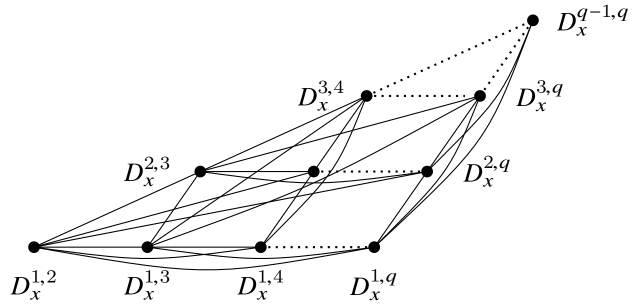

Figure 4.34 The Johnson graph $J(q, 2)$ formed by $D_x^{i,j}$'s for all $1 \le i < j \le q$

4.5 Umbrella Graphs and Coconut Graphs

Umbrella graphs and coconut graphs are both defined in Section 3.6. We also refer the vertices of umbrella graphs and coconut graphs as shown in Figures 3.22 and 3.23, respectively. The γ-total and the γ-paired dominating graphs of these two graphs are studied in this section.

4.5.1 γ-Total Dominating Graphs of Umbrella and Coconut Graphs

Let p and q be positive integers. If $q = 1$, then $U_{p,q} \cong P_{p+1} \cong C_{p,q}$, and thus $TD_\gamma(U_{p,q})$ and $TD_\gamma(C_{p,q})$ can be obtained from Theorems 4.3.1 - 4.3.4. For $q = 2$, we provide the results on $TD_\gamma(U_{p,q})$ and $TD_\gamma(C_{p,q})$ in Theorem 4.5.1 by the following discussions.

If $p = 4k + 2$ for some $k \geq 0$, then we can verify that $\{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_p, u_1\}$ is the only $\gamma_t(U_{p,q})$-set and the only $\gamma_t(C_{p,q})$-set, so $TD_\gamma(U_{p,q}) \cong P_1 \cong TD_\gamma(C_{p,q})$.

Similar proof of Lemma 4.4.5 provides that u_1 is in every $\gamma_t(U_{4k+1,q})$-set. Observation 3.0.1 also tells that u_1 is in every $\gamma_t(C_{4k+1,q})$-set. Then we follow the steps in the proof of Theorem 4.4.6, so we get $TD_\gamma(U_{4k+1,q}) \cong L_{k,q} \cong TD_\gamma(C_{4k+1,q})$.

If $q \in \{2, 3\}$, then $U_{4k,q} \cong L_{4k,q}$, so we get that $TD_\gamma(U_{4k,q}) \cong JL^{k+1}_{k-1,q}$ by Theorem 4.4.7. We note that every $\gamma_t(U_{4k,q})$-set is a $\gamma_t(L_{4k,q})$-set, but the converse is not necessarily true. From the proof of Theorem 4.4.7, we know that $D^{i,j} = \{v_{4l+2}, v_{4l+3} : 0 \leq l \leq k-1\} \cup \{u_i, u_j\}$ is a $\gamma_t(L_{4k,q})$-set for $2 \leq i < j \leq q$. If $q = 4$, then $D^{2,4}$ is a $\gamma_t(L_{4k,q})$-set but not a $\gamma_t(U_{4k,q})$-set, and thus $TD_\gamma(U_{4k,q}) \cong TD_\gamma(L_{4k,q}) - \{D^{2,4}\}$. Similarly, for $q = 5$, $TD_\gamma(U_{4k,q}) \cong TD_\gamma(L_{4k,q}) - \{D^{2,3}, D^{2,4}, D^{2,5}, D^{3,5}, D^{4,5}\}$. Note that u_1 is in every $\gamma_t(U_{4k,q})$-set for all $q \geq 6$ and in every $\gamma_t(C_{4k,q})$-set for all $q \geq 2$, so $TD_\gamma(U_{4k,q}) \cong L^{k+1}_{k-1,q}$ for all $q \geq 6$, and $TD_\gamma(C_{4k,q}) \cong L^{k+1}_{k-1,q}$ for all $q \geq 2$ by following the first two paragraphs in the proof of Theorem 4.4.7.

Similar to Lemma 4.4.8, each $\gamma_t(U_{4k-1,q})$-set and each $\gamma_t(C_{4k-1,q})$-set do not contain the vertices u_i for all $i \in \{2, 3, \ldots, q\}$. Then we follow the steps in the proof of Theorem 4.4.9, so $TD_\gamma(U_{4k-1,q}) \cong P_k \cong TD_\gamma(C_{4k-1,q})$.

Theorem 4.5.1. *Let p and q be positive integers. Then*

$$TD_\gamma(U_{p,q}) \cong \begin{cases} P_1 & \textit{if } p = 4k+2, q \geq 2; \\ L_{k,q} & \textit{if } p = 4k+1, q \geq 2; \\ L^{k+1}_{k-1,q} & \textit{if } p = 4k, q \geq 6; \\ P_k & \textit{if } p = 4k-1, q \geq 2; \end{cases}$$

and

$$TD_\gamma(C_{p,q}) \cong \begin{cases} P_1 & \textit{if } p = 4k+2, q \geq 2; \\ L_{k,q} & \textit{if } p = 4k+1, q \geq 2; \\ L^{k+1}_{k-1,q} & \textit{if } p = 4k, q \geq 2; \\ P_k & \textit{if } p = 4k-1, q \geq 2. \end{cases}$$

4.5.2 γ-Paired Dominating Graphs of Umbrella and Coconut Graphs

Let p and q be positive integers. If $q = 1$, then Theorems 4.3.12 - 4.3.15 give the results on $PD_\gamma(U_{p,q})$ and $PD_\gamma(C_{p,q})$. If $q \geq 2$, then $PD_\gamma(U_{p,q})$ and $PD_\gamma(C_{p,q})$ are determined in Theorem 4.5.2 by the following discussions.

It is easy to check that $\{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_{4k+2}, u_1\}$ is the only $\gamma_{pr}(U_{4k+2,q})$-set and the only $\gamma_{pr}(C_{4k+2,q})$-set, which means that $PD_\gamma(U_{4k+2,q}) \cong P_1 \cong PD_\gamma(C_{4k+2,q})$.

Similar to Lemma 4.4.14, we can prove that each $\gamma_{pr}(U_{4k+1,q})$-set contains the vertex u_1. Observation 3.0.1 gives that each $\gamma_{pr}(C_{4k+1,q})$-set contains the vertex u_1. We follow the steps in the proof of Theorem 4.4.15, and we then get that $PD_\gamma(U_{4k+1,q}) \cong L_{k,q} \cong PD_\gamma(C_{4k+1,q})$.

If $q \in \{2, 3\}$, then $U_{4k,q} \cong L_{4k,q}$, and hence $PD_\gamma(U_{4k,q}) \cong SJL^{k+1}_{k-1,q}$ by Theorem 4.4.16. Let $q \geq 4$. Note that every $\gamma_{pr}(U_{4k,q})$-set is a $\gamma_{pr}(L_{4k,q})$-set, but the converse need not be true for some $\gamma_{pr}(L_{4k,q})$-set that does not contain u_1. From the proof of Theorem 4.4.16, we get that a $\gamma_{pr}(L_{4k,q})$-set that does not contain u_1 is of the form $D^{i,j} = D \cup \{u_i, u_j\}$, where D is a $\gamma_{pr}(P_{4k})$-set and $2 \leq i < j \leq q$. Similarly, a $\gamma_{pr}(U_{4k,q})$-set that does not contain u_1 is also of the form $D \cup \{u_i, u_j\}$ for some $2 \leq i < j \leq q$. For $q = 4$, we have $D^{2,4}$ is a $\gamma_{pr}(L_{4k,q})$-set but not a $\gamma_{pr}(U_{4k,q})$-set, so $PD_\gamma(U_{4k,q}) \cong PD_\gamma(L_{4k,q}) - \{D^{2,4}\}$. For $q = 5$, only $D^{3,4}$ is a $\gamma_{pr}(U_{4k,q})$-set among all

$\gamma_{pr}(L_{4k,q})$-sets containing the pair $\{u_i, u_j\}$ where $2 \leq i < j \leq q$, and thus $PD_\gamma(U_{4k,q}) \cong PD_\gamma(L_{4k,q}) - \{D^{2,3}, D^{2,4}, D^{2,5}, D^{3,5}, D^{4,5}\}$. If $q \geq 6$, then it is easy to verify that u_1 is in every $\gamma_{pr}(U_{4k,q})$-set, and we then follow the first two paragraphs of the proof in Theorem 4.4.16 to get that $PD_\gamma(U_{4k,q}) \cong SL^{k+1}_{k-1,q}$. Note that u_1 is in every $\gamma_{pr}(C_{4k,q})$-set for all $q \geq 2$. Again, we follow the first two paragraphs of the proof in Theorem 4.4.16, so $PD_\gamma(C_{4k,q}) \cong SL^{k+1}_{k-1,q}$ for all $q \geq 2$.

If $q \in \{2, 3\}$, then we get that $PD_\gamma(U_{4k-1,q}) \cong PD_\gamma(L_{4k-1,q}) \cong B_{k+1,k+q-1,k}$ by Theorem 4.4.17. Now, we let $q \geq 4$. In the proof of Theorem 4.4.17, we know that $D^{i,j}_1, D^{i,j}_2, \ldots, D^{i,j}_{k+1}$ are the only $\gamma_{pr}(L_{4k-1,q})$-sets containing the pair $\{u_i, u_j\}$ where $2 \leq i < j \leq q$. Note that $D^{2,4}_1, D^{2,4}_2, \ldots, D^{2,4}_{k+1}$ are not $\gamma_{pr}(U_{4k-1,4})$-sets, so $PD_\gamma(U_{4k-1,4}) \cong PD_\gamma(L_{4k-1,4}) - \{D^{2,4}_x : 1 \leq x \leq k+1\}$. Among all $\gamma_{pr}(L_{4k-1,5})$-sets containing the pair $\{u_i, u_j\}$ for $2 \leq i < j \leq 5$, only $D^{3,4}_1, D^{3,4}_2, \ldots, D^{3,4}_{k+1}$ are $\gamma_{pr}(U_{4k-1,5})$-sets, so we get that $PD_\gamma(U_{4k-1,5}) \cong PD_\gamma(L_{4k-1,5}) - \{D^{2,3}_x, D^{2,4}_x, D^{2,5}_x, D^{3,5}_x, D^{4,5}_x : 1 \leq x \leq k+1\}$. We can easily check that u_1 is contained in every $\gamma_{pr}(U_{4k-1,q})$-set for all $q \geq 6$ as well as every $\gamma_{pr}(C_{4k-1,q})$-set for all $q \geq 2$. We obtain that $PD_\gamma(U_{4k-1,q}) \cong A_{k+1,k+q-1,k}$ for all $q \geq 6$, and $PD_\gamma(C_{4k-1,q}) \cong A_{k+1,k+q-1,k}$ for all $q \geq 2$ by following the steps of proof in Theorem 4.4.17 (first paragraph).

Theorem 4.5.2. *Let p and q be positive integers. Then*

$$PD_\gamma(U_{p,q}) \cong \begin{cases} P_1 & \text{if } p = 4k+2, q \geq 2; \\ L_{k,q} & \text{if } p = 4k+1, q \geq 2; \\ SL^{k+1}_{k-1,q} & \text{if } p = 4k, q \geq 6; \\ A_{k+1,k+q-1,k} & \text{if } p = 4k-1, q \geq 6; \end{cases}$$

and

$$PD_\gamma(C_{p,q}) \cong \begin{cases} P_1 & \text{if } p = 4k+2, q \geq 2; \\ L_{k,q} & \text{if } p = 4k+1, q \geq 2; \\ SL^{k+1}_{k-1,q} & \text{if } p = 4k, q \geq 2; \\ A_{k+1,k+q-1,k} & \text{if } p = 4k-1, q \geq 2. \end{cases}$$

CHAPTER 5
CONCLUSIONS

In this chapter, we provide a summary that arises from this dissertation and propose some open problems for future consideration.

This dissertation presents the total and the paired domination numbers of some families of graphs in Chapter 3. Section 3.1 specifically covers wheel graphs, helm graphs, flower graphs, and sunflower graphs. In Section 3.2, we revise some values of the total domination numbers of Jahangir graphs, which were originally provided by Mtarneh *et al.* [48], and subsequently present the paired domination numbers of Jahangir graphs. In Section 3.3, we provide the total and the paired domination numbers of $P_p \square C_q$ for $p \in \{2, 3, 4\}$ and $q \geq 5$, which extend the results in [30] showing the case for $p \geq 2$ and $q \in \{3, 4\}$. We also present some of their upper and lower bounds for the other values of p and q. To find the exact values for the other cases, we propose the folowing problem.

Problem 5.0.1. *Determine the total and the paired domination numbers of* $P_p \square C_q$ *for* $p, q \geq 5$.

Section 3.4 presents the total and the paired domination numbers of some closed helm graphs and their upper bounds for the other cases. Additionally, we provide the total and the paired domination numbers of some web graphs, along with their upper bounds in the other cases. We next provide the problem aimed at determining lower bounds and exact values for these two classes of graphs.

Problem 5.0.2. *Determine lower bounds and exact values for the total and the paired domination numbers of* $CH_{p,q}$ *for* $p, q \geq 5$ *and* $W_{p,q}$ *for* $p \geq 7$ *and* $q \geq 5$.

Furthermore, we compute the total and the paired domination numbers for windmill class of graphs in Section 3.5. We close Chapter 3 by providing the total and the paired domination numbers of lollipop graphs, umbrella graphs, and coconut graphs in Section 3.6. There is another graph that is similar in structure to the latter three graphs, which is a *tadpole graph* $T_{p,q}$ obtained by joining an endpoint of a path P_p to a vertex of a cycle C_q. The following problem is an intriguing one to address.

Problem 5.0.3. *Determine the total and the paired domination numbers of tadpole graphs.*

In Chapter 4, we determine γ-total and γ-paired dominating graphs of some classes of graphs. We start this chapter by considering double stars, complete graphs, complete bipartite graphs, and fan graphs in Section 4.1. As stated above, the total and paired domination numbers of wheel graphs, helm graphs, flower graphs, and sunflower graphs are presented in Section 3.1. However, we only investigate the γ-total and the γ-paired dominating graphs of the first three classes of graphs in Section 4.2, while leaving the ones of sunflower graphs in the next problem.

Problem 5.0.4. *Determine the γ-total and the γ-paired dominating graphs of sunflower graphs.*

Section 4.3 presents the γ-paired dominating graphs of cycles, which extend the results of [15] and [67]. We then provide the γ-total and the γ-paired dominating graphs of lollipop graphs in Section 4.4, and the ones of umbrella graphs and coconut graphs in Section 4.5.

The answer of Problem 5.0.3 provides the total and the paired domination numbers of tadpole graphs, which are useful to the next problem.

Problem 5.0.5. *Determine the γ-total and the γ-paired dominating graphs of tadpole graphs.*

We can notice that Chapter 4 has not yet discussed the γ-total and the γ-paired dominating graphs of Jahangir graphs, cylinders, closed helm graphs, web graphs, and windmill class of graphs, which have been mentioned in Chapter 3. We let their determination be a problem as follows.

Problem 5.0.6. *Determine the γ-total and the γ-paired dominating graphs of Jahangir graphs, cylinders, closed helm graphs, web graphs, and windmill class of graphs.*

Finally, we propose three more fascinating questions. The first question is to ask which graph has the property that its γ-total and γ-paired dominating graphs are isomorphic. According to the results in Chapter 4, we obtain that

- $TD_\gamma(S_{p,q}) \cong PD_\gamma(S_{p,q})$ when $p, q \geq 1$;

- $TD_\gamma(K_p) \cong PD_\gamma(K_p)$ when $p \geq 2$;
- $TD_\gamma(K_{p,q}) \cong PD_\gamma(K_{p,q})$ when $p, q \geq 1$;
- $TD_\gamma(F_{p,q}) \cong PD_\gamma(F_{p,q})$ when $p, q \geq 1$;
- $TD_\gamma(W_p) \cong PD_\gamma(W_p)$ when $p \geq 3$;
- $TD_\gamma(H_p) \cong PD_\gamma(H_p)$ when $p \geq 3$ is even;
- $TD_\gamma(Fl_p) \cong PD_\gamma(Fl_p)$ when $p \geq 3$;
- $TD_\gamma(P_p) \cong PD_\gamma(P_p)$ when $p = 2, p = 6$, or $p \equiv 0, 3 \pmod 4$;
- $TD_\gamma(C_p) \cong PD_\gamma(C_p)$ when $p = 6$ or $p \equiv 0, 3 \pmod 4$;
- $TD_\gamma(L_{p,q}) \cong PD_\gamma(L_{p,q})$ when $p = 4$ and $q \geq 2$, or $p \equiv 1, 2 \pmod 4$ and $q \geq 1$;
- $TD_\gamma(U_{p,q}) \cong PD_\gamma(U_{p,q})$ when $p = 4$ and $q \geq 2$, or $p \equiv 1, 2 \pmod 4$ and $q \geq 1$;
- $TD_\gamma(C_{p,q}) \cong PD_\gamma(C_{p,q})$ when $p = 4$ and $q \geq 2$, or $p \equiv 1, 2 \pmod 4$ and $q \geq 1$.

The first question leads us to propose the following problem.

Problem 5.0.7. *Characterize the graph G for which $TD_\gamma(G) \cong PD_\gamma(G)$.*

The second question is to ask which graph is isomorphic to its γ-total (γ-paired) dominating graph. As appeared in Theorems 4.3.5, 4.3.6, and 4.3.8, we obtain that $TD_\gamma(C_p) \cong C_p$ when $p = 4$ or $p \equiv 1, 3 \pmod 4$. Similarly, as mentioned in Theorems 4.3.18 and 4.3.20, we know $PD_\gamma(C_p) \cong C_p$ when $p = 4$ or $p \equiv 3 \pmod 4$. We then provide two more problems as follows.

Problem 5.0.8. *Characterize the graph G for which $TD_\gamma(G) \cong G$.*

Problem 5.0.9. *Characterize the graph G for which $PD_\gamma(G) \cong G$.*

For the last question, it asks which graph has the property that its γ-total (γ-paired) dominating graph is connected or disconnected. Among all graphs that are considered in this dissertation, by Theorems 4.3.5 and 4.3.18, there is only the cycle C_p with $p \equiv 0 \pmod 4$ and $p \geq 8$ such that $TD_\gamma(C_p)$ and $PD_\gamma(C_p)$ are disconnected. Thus, the following problems are worth considering.

Problem 5.0.10. *Determine conditions on the graph G under which $TD_\gamma(G)$ is connected or disconnected.*

Problem 5.0.11. *Determine conditions on the graph G under which $PD_\gamma(G)$ is connected or disconnected.*

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